Mathematical induction, Geometric and Arithmetic sequence

Qno.1

a. $r = -\frac{1}{3}$ *(A1)*

$S_\infty = \frac{27}{1+\frac{1}{3}}$ ***M1***

$S_\infty = \frac{81}{4}$ $(= 20.25)$ ***A1*** ***N1***

[3 marks]

b. Attempting to show that the result is true for $n = 1$ ***M1***

LHS $= a$ **and** RHS $= \frac{a(1-r)}{1-r} = a$ ***A1***

Hence the result is true for $n = 1$

Assume it is true for $n = k$

$a + ar + ar^2 + \ldots + ar^{k-1} = \frac{a(1-rk)}{1-r}$ ***M1***

Consider $n = k + 1$:

$a + ar + ar^2 + \ldots + ar^{k-1} + ar^k = \frac{a(1-r^k)}{1-r} + ar^k$ ***M1***

$= \frac{a(1-r^k)+ar^k(1-r)}{1-r}$

$= \frac{a-ar^k+ar^k-ar^{k+1}}{1-r}$ ***A1***

Note: Award ***A1*** for an equivalent correct intermediate step.

$= \frac{a-ar^{k+1}}{1-r}$

$= \frac{a(1-r^{k+1})}{1-r}$ ***A1***

Note: Illogical attempted proofs that use the result to be proved would gain ***M1A0A0*** for the last three above marks.

The result is true for $n = k \Rightarrow$ it is true for $n = k + 1$ **and** as it is true for $n = 1$, the result is proved by mathematical induction. ***R1*** ***N0***

Note: To obtain the final ***R1*** mark a reasonable attempt must have been made to prove the $k + 1$ step.

[7 marks]

Qno.2

let $\mathrm{P}(n)$ be the proposition that $n(n^2+5)$ is divisible by 6 for $n \in \mathbb{Z}^+$

consider P(1):

when $n=1,\ n(n^2+5)=1\times(1^2+5)=6$ and so P(1) is true ***R1***

assume $\mathrm{P}(k)$ is true *ie*, $k(k^2+5)=6m$ where $k,\ m \in \mathbb{Z}^+$ ***M1***

Note: Do not award ***M1*** for statements such as "let $n=k$".

consider $\mathrm{P}(k+1)$:

$(k+1)\left((k+1)^2+5\right)$ ***M1***

$=(k+1)(k^2+2k+6)$

$=k^3+3k^2+8k+6$ ***(A1)***

$=(k^3+5k)+(3k^2+3k+6)$ ***A1***

$=k(k^2+5)+3k(k+1)+6$ ***A1***

$k(k+1)$ is even hence all three terms are divisible by 6 ***R1***

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and P(1) is true, so $\mathrm{P}(n)$ is true for $n \in \mathbb{Z}^+$ ***R1***

Note: To obtain the final ***R1***, four of the previous marks must have been awarded.

[8 marks]

Qno.3

An 81 metre rope is cut into n pieces of increasing lengths that form an arithmetic sequence with a common difference of d metres. Given that the lengths of the shortest and longest pieces are 1.5 metres and 7.5 metres respectively, find the values of n and d.

Markscheme

$81 = \frac{n}{2}(1.5 + 7.5)$ ***M1***

$\Rightarrow n = 18$ ***A1***

$1.5 + 17d = 7.5$ ***M1***

$\Rightarrow d = \frac{6}{17}$ ***A1*** ***N0***

[4 marks]

Qno.4

. (a) $S_6 = 81 \Rightarrow 81 = \frac{6}{2}(2a + 5d)$ ***M1A1***

$\Rightarrow 27 = 2a + 5d$

$S_{11} = 231 \Rightarrow 231 = \frac{11}{2}(2a + 10d)$ ***M1A1***

$\Rightarrow 21 = a + 5d$

solving simultaneously, $a = 6$, $d = 3$ ***A1A1***

[6 marks]

(b) $a + ar = 1$ ***A1***

$a + ar + ar^2 + ar^3 = 5$ ***A1***

$\Rightarrow (a + ar) + ar^2(1 + r) = 5$

$\Rightarrow 1 + ar^2 \times \frac{1}{a} = 5$

obtaining $r^2 - 4 = 0$ ***M1***

$\Rightarrow r = \pm 2$

$r = 2$ (since all terms are positive) ***A1***

$a = \frac{1}{3}$ ***A1***

[5 marks]

(c) AP r^{th} term is $3r + 3$ ***A1***

GP r^{th} term is $\frac{1}{3}2^{r-1}$ ***A1***

$3(r + 1) \times \frac{1}{3}2^{r-1} = (r + 1)2^{r-1}$ ***M1AG***

[3 marks]

Total [14 marks]

d. prove: $P_n : \sum_{r=1}^{n} (r + 1)2^{r-1} = n2^n,\ n \in \mathbb{Z}^+$.

show true for $n = 1$, *i.e.*

$\text{LHS} = 2 \times 2^0 = 2 = \text{RHS}$ ***A1***

assume true for $n = k$, *i.e.* ***M1***

$\sum_{r=1}^{k} (r + 1)2^{r-1} = k2^k,\ k \in \mathbb{Z}^+$

consider $n = k + 1$

$\sum_{r=1}^{k+1} (r + 1)2^{r-1} = k2^k + (k + 2)2^k$ ***M1A1***

$= 2^k(k + k + 2)$

$= 2(k + 1)2^k$ ***A1***

$= (k + 1)2^{k+1}$ ***A1***

hence true for $n = k + 1$

P_{k+1} is true whenever P_k is true, and P_1 is true, therefore P_n is true ***R1*** for $n \in \mathbb{Z}^+$

[7 marks]

Qno.5

a. **EITHER**

the first three terms of the geometric sequence are 9, $9r$ and $9r^2$ ***(M1)***

$9 + 3d = 9r (\Rightarrow 3 + d = 3r)$ and $9 + 7d = 9r^2$ ***(A1)***

attempt to solve simultaneously ***(M1)***

$9 + 7d = 9\left(\frac{3+d}{3}\right)^2$

OR

the 1^{st}, 4^{th} and 8^{th} terms of the arithmetic sequence are

$9,\ 9 + 3d,\ 9 + 7d$ ***(M1)***

$\frac{9+7d}{9+3d} = \frac{9+3d}{9}$ ***(A1)***

attempt to solve ***(M1)***

THEN

$d = 1$ ***A1***

[4 marks]

b. $r = \frac{4}{3}$ ***A1***

Note: Accept answers where a candidate obtains d by finding r first. The first two marks in either method for part (a) are awarded for the same ideas and the third mark is awarded for attempting to solve an equation in r.

[1 mark]

Qno.6

for the first series $\frac{a}{1-r} = 76$ ***A1***

for the second series $\frac{a}{1-r^3} = 36$ ***A1***

attempt to eliminate a *e.g.* $\frac{76(1-r)}{1-r^3} = 36$ ***M1***

simplify and obtain $9r^2 + 9r - 10 = 0$ ***(M1)A1***

Note: Only award the ***M1*** if a quadratic is seen.

obtain $r = \frac{12}{18}$ and $-\frac{30}{18}$ ***(A1)***

[redacted]

$r = \frac{12}{18}\left(= \frac{2}{3} = 0.666\ldots\right)$ ***A1***

Note: Award ***A0*** if the extra value of r is given in the final answer.

Total [7 marks]