PHY 180: Testing the Accuracy of Mathematic Models Using a Homemade Pendulum

Introduction

The purpose of this exploration was to understand how closely mathematical theory could predict the behaviours of a homemade pendulum. Aspects of mathematical theory that were tested include a) the correlation between amplitude(A) and period(T) b) the behaviour of the Q factor in a pendulum and c) the impact length(L) has on the pendulum's period and Q factor.

Pendulums are structures which consist of a mass connected to a string hung on a fixed point. The mass, when applied to an external force, oscillates. In theory, due to Newton's law of Inertia and energy conservation, the motion of a pendulum remains constant until external forces are applied, regardless of mass and length. This is called "simple harmonic motion and is modelled using the equation:

$$x(t) = Acos(\omega t) \quad (1)$$

Where $x(t)$ is displacement, ω is angular speed (rad s^{-1}).

(The Pendulum, n.d) However, factors like friction, air resistance and pendulum length impede its motion, creating "damped harmonic motion", which should follow the equation

$$\theta(t) = \theta_0 \, e^{\frac{-t}{\tau}} \cos(2\pi \, \frac{t}{T} + \varphi_0). \quad (2)$$

Where θ(rad) and τ is in seconds.

The model provided by Professor Brian Wilson states that period and amplitude are independent. The period is approximately:

$$T \sim 2\sqrt{L} \quad (3)$$

(Wilson, Brian. PHY 180 Lab Project 2024).

From my results, the period is constant and independent of the 18 different amplitudes at which the pendulum is released, supporting the model. It followed the quadratic equation, in the form:

$$T = T_o \, (1 + B\theta_o + C\theta_o^2) \quad (4)$$

Where T and T_o is the period (s). Since $(1 + B\theta_o + C\theta_o^2)$ is unitless, B and C have units rad $^{-1}$.

The derived equation is $y = 0.005(\pm 0.003)x^2 - 0.1(\pm 0.1)x + 1.60(\pm 0.01)$

The small C and B values of $0.005(\pm 0.003)x^2 - 0.1(\pm 0.1)\,x$ indicate a flat parabola, alluding zero correlation between amplitude and period. These values also prove this is a symmetric pendulum. The period was 1.2 $\pm$ 0.04s, far from the literature value of 0.9s (string length of 0.22 m).

The Q factor was determined using 1) the counting method and 2) a provided Python code. The graph showed exponential decay in the form ($y=\theta_0 e^{-cx}$) as $y = 1.4(\pm 0.02)e^{-0.02x(\pm 0.001)}$. By correlating

$$\theta_0 e^{-cx} = \theta_0 e^{-\frac{t}{\tau}} \quad (5)$$

and using the equation

$$Q = \pi \frac{\tau}{T} \quad (6)$$

the Q factor for the pendulum was $160 \pm 8.5\%$ (160 ± 20). Since the range fits in with the range for the counting method (140 ± 10), a Q factor of 140 $\pm$ 20 was chosen (lower uncertainty).

The model stated that the data correlation between period and length should follow a power law function in the form

$$T = kL^n \quad (7)$$

where k = 2 and n = 0.5.

Empirically, the period varied with length. The power law function gave the equation 3.4(±0.3) $x^{0.44(\pm 0.07)}$. The Q factor follows a natural logarithmic trend in the equation with the equation $y = 150(\pm 50)ln(x) - 240(\pm 90)$. This report details the methodology and analysis and how the results compare with the predicted models.

Experimental Setup

<u>Figure 1: Experiment Setup.</u> *A nylon string (0.22$\pm$ 0.005m length) is attached to a constant mass (12.9493$\pm$0.0001 g). A clear protractor was put behind the pendulum to ensure it rested at $\frac{\pi}{2}$ radians (1.571 rad) and allowed for the angle on Tracker to be calibrated and more accurate. The string hangs from a small knob screwed back of a keyboard tray. The string is looped around this knob and can be adjusted by turning the knob.*

In *Figure 1*, a 1 m of nylon string is strung around a particular screwable knob which is put backwards into a screw hole that was pre-made on my desk. This allowed the pendulum to hang freely and for the string to be adjusted by rotating the screw knob. To control the impact of the load on the pendulum, a small lock with a mass of 12.9493 ± 0.0001g is tied to the bottom of the string. This load creates minor tension, allowing for better oscillations.

A screw was chosen over taping the pendulum because it is more secure, allowing for a controlled experiment.

The camera of a *Samsung* phone has grid lines and angle lines ensuring that the phone is straight. It was set 1m away on a clear, flat surface, so the camera remains stationary. A protractor is put behind the pendulum to help calibrate the video tracker, and so it can be easily determined what a right angle in real life looks like on video. To avoid systematic errors associated with stopwatches, the camera was set to 60 frames per second to capture a slower and more accurate trajectory of the mass, giving a default time uncertainty of $\pm$0.01s.

Methodology

Lab 1

I predicted that as amplitude decreases, the period will decrease since the total distance the pendulum travels decreases.

A Physics *Tracker* app (Brown et al., *Tracker* 2024) was used to test this idea. The pendulum was dropped at each angle, and a video was taken to see how long it took to reach its initial angle again. To lower Type A uncertainties, five trials were conducted for each angle and the average uncertainty was taken using the equation:

$$u(\overline{x}) = \frac{\sigma}{\sqrt{n}} \qquad (8)$$

To find the Q factor, the software is calibrated in terms of length and angle by using the protractor in the setup. The data was manually tracked by clicking on the mass throughout the cycle. This method was chosen because the auto-tracker was unable to accurately capture all the points while oscillating due to the similar background and mass colour. Using the data, the local and absolute maxima values from the graph are analyzed using a Python code provided by Professor Brian Wilson, and an exponential line of best fit is applied. The parameters and uncertainties are applied to *Equation (5) and (6)* to calculate Q factors.

The second method to find the Q factor is using the counting method. To ensure accuracy, a video is taken of the pendulum as it begins oscillating until it reaches roughly 5% of its oscillation. Since all oscillations started at 1.57rad(π/2) from the physical protractor, the oscillations should end when the angle is around 5%. As it is difficult to estimate exactly 0.080 radians, the termination of the video was estimated when the pendulum had minimum oscillations, and the actual termination time was determined through the videos.

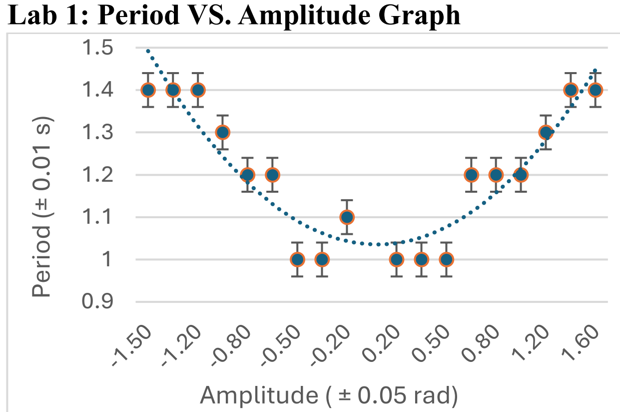
Lab 2:

Using the same video-capturing methods in Lab 1, the period was calculated three times for each length. The variable lengths of the pendulum were 0.10±0.005m, 0.15±0.005m, 0.20±0.005 m, 0.25±0.005m, 0.35±0.005m, 0.40±0.005m and 0.50±0.005m. From the results in Lab 1 (pg. 4), the counting method for the Q factor was used. This provides less uncertainty and is more convenient.

The parameter uncertainty is found using Curve Fit and MATLAB's 95% confidence interval (CI). The upper bound CI is subtracted from the parameter, and divided by 2, as 95% represents the standard deviation of 2σ.

Results

Lab 1: Period VS. Amplitude Graph



<u>Figure 2: Period VS. Amplitude of a Pendulum.</u> *A quadratic series is applied (where y = ax² + bx + c relates to T = To (1+Bθo+Cθo²). The equation of the curve of best fit is y* $= 0.005(\pm 0.003)x^2 - 0.1(\pm 0.1)\,x\, +$

$1.60(\pm 0.01)$ *where the parameters are* $A = 0.005(\pm 0.003), B = -0.1 \pm 0.1, C = 1.60(\pm 0.01)s$. *The A and B parameters of 0.0054 and 0.1096 determine the symmetry of the pendulum. The C parameter of 1.5964 provides the largest measured period. A quadratic model was chosen over other models because it provided the largest* R^2 *value of 0.404, and thus, a stronger correlation.*

Uncertainty Analysis

The pendulum is damped and will not always make its angle mark before starting another revolution, making this a Type A error in calculating the period, of ± 0.05 radians.

The online protractor measures angles with a precision of ± 00.1 radians, allowing the Type B uncertainty of the protractor to be propagated to be half the increment value, of ± 0.05 rad. The period uncertainty corresponds to ± 0.01s.

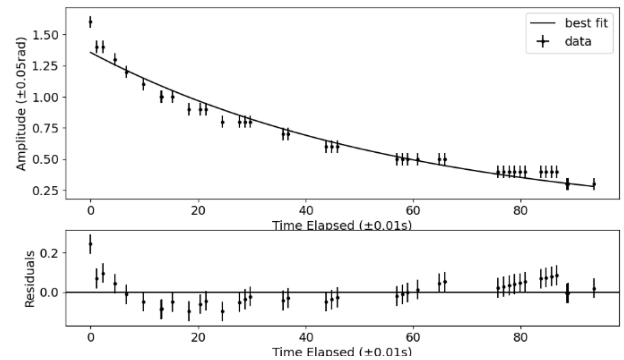
The calculated average uncertainty is 0.04. To account for errors and better distribution, the maximum of 0.01s and 0.04s is taken as the final uncertainty for the period.

Data Analysis

The period remains constant, especially for small angles, while increasing slightly when the angle is more than ± 0.50 angles change more rapidly when they are bigger. This might be due to small angle approximations. To confirm the symmetry of the pendulum, a quadratic series with 18-period data was plotted. The equation was $0.005(\pm 0.003)x^2 - 0.1(\pm 0.1)\,x + 1.60(\pm 0.01)$. The C and B parameters of $0.0054\ and\ 0.1096$ determine the symmetry of the pendulum. A symmetric pendulum should have parameters equal to 0. Since the C value of 0.005 is less than the period uncertainty of ± 0.01s the value encompasses a range of values close to 0. The variation can be due to the small differences that may occur in each video. Ultimately, this pendulum is symmetrical, with an average period of 1.2 $\pm$ 0.04s. It is worth noting that the pendulum's mass, the experimental setup and length of the string can cause fluctuations in time and period.

Determining the Q Factor of a Pendulum

Using *Tracker,* 2976 frames were analyzed, and the local maxima values were plotted to find Tau and Q factors. This graph is available in *Appendix A.*



<u>Figure 3: Amplitude VS. Time Elapsed for a Pendulum</u>. *Showing the local maxima points plotted using a Python code to understand better where the points lie and what type of fit they make with the graph. The error bars, show an uncertainty of ±0.05 in the angle and an uncertainty of ±0.01 in time elapsed. A natural exponential line of best fit most accurately fits the set of data. The equation of the line (form* $y = ae^{-bx}$ $y = 1.4(\pm 0.02)e^{-0.02(\pm 0.001)x}$ *where the parameter uncertainty of A is 1.4*$\pm 0.02\ rad$ *The value -0.02 corresponds to* $\frac{t}{\tau}$*.and the value of x (time) is 94.5* $\pm$*0.01.* $\tau\ is\ 60 \pm\ 1.6\ (rounded\ to\ 2.0)$.

Uncertainty Analysis

Since the time elapsed depends on the time of the video, there might be a lag in the time between when the video was captured, and the pendulum was released. This uncertainty is estimated to be $\pm$ 0.5s and is found by looking at the differences in peak height from adjacent points (seen in *Appendix A)*. This equation fails to account that oscillations stop when it is 4% of its original amplitude, and eventually, the function will approach 0. The residuals graph shows that these error bars are large, as the points are closer to the residual line. Since only 1 trial was conducted, the Type A uncertainties are larger than the Type B uncertainty, justifying a large error bar.

While there is a default uncertainty of ± 0.05 radians, there can be a computer error regarding whether the angle was tracked from the centre of mass each time, increasing the uncertainty to $\pm$ 0.1 rad. The graph shows a gradual decrease where adjacent peaks increase and decrease concerning each other but follow a decrease over a larger scale of time. The inconsistent peaks can represent uncertainty as the point mass may be clicked in a different position from the last time, or the angle may not have reached its peak before oscillating again. This corresponds to an uncertainty of 0.1seconds, since that is the distance between a peak and its adjacent points, whether they are higher or lower than the peak being assessed. These results can be found in *Figure 4*.

The Q factor shows the dampening effect of the pendulum and is modelled by the functions:

$$\theta(t) = \theta_0 e^{\frac{-t}{\tau}} \cos(2\pi \frac{t}{T} + \varphi_0) \text{ And } Q = \pi \frac{\tau}{T}$$

A natural exponent line of best fit was deemed the best fit, where $\theta_0 e^{-cx} = \theta_0 e^{-\frac{t}{\tau}}$. t is the total elapsed time, in this case 94.5 seconds (1min 34s). Based on this, of τ is calculated to be 60 $\pm$20s, resulting in a Q factor of 160 $\pm$20. Since the points that were believed to correspond to the local maxima were all manually picked, a Type B uncertainty could have been the omission of points that could have otherwise skewed the data and curve of best fit. In the data, there were often three or four adjacent points with similar angles, creating almost a flat line before decreasing in angle again. This can be a setup limitation, as the auto tracker may not have been able to distinguish between the pendulum bob and the background. To reduce discrepancies, these were omitted. They could have reduced the number of trials upon inputting them into Python, resulting in a high uncertainty for Tau. To confirm this value, the number of oscillations was cross-referenced by manually counting the local maxima points. The counted Q factor is 140 $\pm$ 10. The uncertainty of 10 counts is type A uncertainty as the local maximum was counted three times. This is because many points overlap, causing human error in counting. The redundancy allows for accurate collection.

Analysis:

$$\tau = 60 \pm 2 \; seconds$$

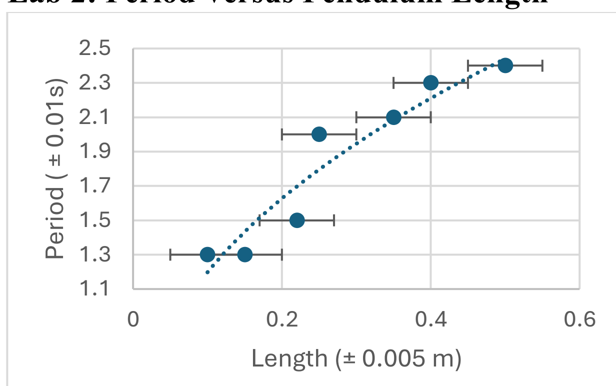
$$Q = \pi \frac{\tau}{T}$$

Take the largest percent uncertainty.

$Q = 155 \pm 8.5\%$ ($\sim$160 $\pm$ 20) counts

The Q factor was determined to be 137 $\pm$ 10 counts upon using the counting method, and 155 $\pm$ 20 using the equation. This creates a range of Q values from 135 - 175. This range includes the counted Q factor of 137 $\pm$ 10 counts, meaning the two values are similar, even though the Q factor derived from the equation is lower than the Q factor determined through counting. The type A uncertainties are greater in the equation method than in the counting method, as only one video trial was taken to analyze the points, and the values derived from the equation were automatically plugged in, whereas the points in the graph were counted three times to ensure consistency and accuracy in the obtained value. This results in lower average uncertainty. Additionally, a limitation is that the curve of best fit from *Figure 4* fails to account for the absolute maximum value of the function (at $\theta = 1.6$ rad) and the last angle displacement values (at $\theta = 0.3$ rad). If these numbers are used in the calculations instead, the Q factor would have been 146 $\pm$10. This value would provide the range between 136 – 156, encompassing Q factors 140 $\pm$ 10 and 160 $\pm$ 20. The residuals graph shows that the error bars are too large, as 79% of the data is near the residual line the data. Although the error bars show a Type A uncertainty of too few points, it is a Type B uncertainty of 60 $\pm$20 because the values were manually picked out, causing the largest uncertainty.

Lab 2: Period Versus Pendulum Length



<u>Figure 4: Period VS Length of a Pendulum</u>. *This is fitted to a power series in the form* $y = ax^b$*, with the equation* $y = 3.3(\pm 0.3)\, x^{0.42(\pm 0.07)}$*. The correlation of the power series in the data was 0.9105, proving this is a good fit. This result tells us that as length increases, the period also increases, but the rate at which this occurs slows down.*

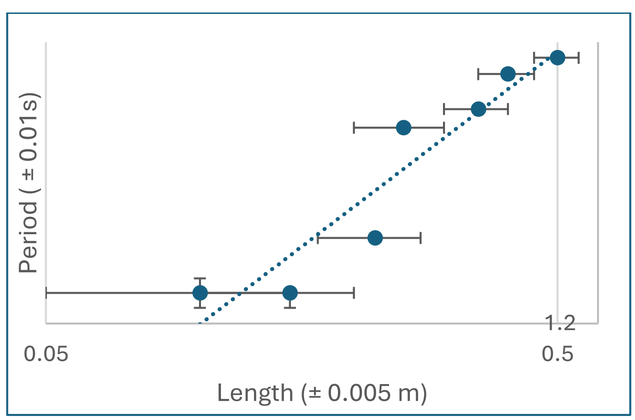


Figure 5: Period VS Length of a Pendulum (log-log plot). *The graph has the equation y =* ***3.4(±0.3) x^0.44(±0.07)***. *The line of best fit is qualitatively linear when plotted against a logarithmic scale, proving that the power series is a strong function. The average uncertainty was ±0.06s.*

Uncertainty Analysis

To lower type A uncertainties, the period trials were conducted three times for each trial, and the average uncertainty was ±0.06s, which was larger than the period uncertainty.

The log of the error bars (from Figure 4) is taken to find the error bar uncertainties. They are found by dividing the uncertainty by the respective x or y values (Error Bars in Log Plots 2017). (they were found taking the logarithm of the original uncertainty)

The Type B uncertainty of period and length are consistent as shown in *Figure 2.*

Data Analysis:

The graph shows a clear correlation between the length and the period of the pendulum. The parameter uncertainty of $y = 3.4(\pm0.3)\ x^{0.44(\pm0.07)}$ proves that this is likely a strong correlation, satisfying the prediction provided. The gradual decrease in the rate of period change can be due to how much friction is added when the length increases. Other comparable functions with a high 95% confidence interval and correlation were quadratic and quartic functions (R^2 0.92 and 0.94 respectively)

Q Factor VS. Length of a Pendulum

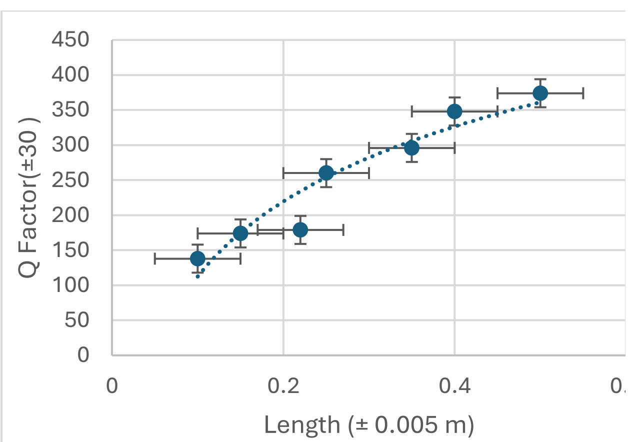


Figure 6: Q Factor VS. Length of a Pendulum. *The trend follows a natural logarithmic function in the form* $y = A\ln(x) + b$, *with equation y = 150(±30) ln(x) + 460(±40). There is a 0.9116 correlation, making this a strong fit.*

Uncertainty Analysis

There is a Type B uncertainty of ±0.005m for the length, measured from the ruler, and a Q factor uncertainty of ±30. The parameter uncertainty of the equation y = 150(±30) ln(x) + 460(±40) means that 95% of Q factor values range between 120 to 500, which is a large range that encompasses the data, making it a suboptimal fit.

For smaller angles, it was possible to conduct three trials because oscillations and Q factor were not very large, but for the larger lengths, one trial was taken by counting all the oscillations, and the other two were counted by counting the first half of the oscillations (up until it reaches an angle of 0.698 rad, which was thought to be appropriate because of the small angle approximations), and multiplied by 2. This should result in a maximum uncertainty of 20.

There is type A uncertainty of the Q factor because it was counted three times for the smaller angles and was counted halfway three times for the larger lengths and then multiplied by 2. Each length's standard deviation differed, resulting in average uncertainties ranging from 8 to 15. Although theoretically, Q factor should be the same regardless of when you experiment, if it depends on string length. The standard deviation should remain constant throughout all lengths. There can be experimental errors like counting, friction or how the pendulum was released, giving this range. Therefore, the maximum average uncertainty, rounded to 20 was used.

Conclusion

The period of the pendulum with length 0.220± 0.005 was found to mostly be independent of amplitude, which supports the model. This is more obvious for smaller angles, than for larger angles, which increase lightly, and over larger angle changes. This result supports the prediction that period and amplitude are generally independent unless the angle at which they are released increases significantly. This might be due to the extra length the pendulum has to travel and the speed of the rope.

The Q factor was determined to be 140 $\pm$ 10 upon using the counting method, and 160 $\pm$ 20 using the equation, making them similar values. There was a higher uncertainty in using the Python code method than counting. This means that the equation model predicted in *Equation 2* is an accurate model for this experiment. The correlation between the period and the length of the pendulum did follow a power series, with equation y=3.4(±0.3) $x^{0.44(\pm 0.07)}$. closely. The correlation between the Q factor and the length followed a logarithmic trend, with the equation y = 150(±50) ln(x) + 470(±80).

The mathematical theory that the function $T = k\,L^n$ will provide an accurate fit aligns with our observation. This is accurate because when you increase the degree of a polynomial from $T = k\,L^{0.44}$, to $T = k\,L^4$, the correlation increases to 0.94. This occurs because there are only seven data points, giving a quartic function a higher variance and more parameters to fit each point. As the n value approaches the number of data points, the correlation gets closer to 1.00. From Lab 1, the initial largest source of error in the experiment was the period of the graph before being changed by using a camera instead. The largest source of error is from the Q factor in part 1 where the oscillations were counted manually and omitted manually as well. This method was kept because it did provide significance in that it accurately showed the length and trajectory of the curve, but this can cause fewer data points. If this experiment was redone, instead of manually choosing coordinates I could have tracked the angle at only the max displacements of the oscillations.

The largest source of error from *Lab 2* is when finding the Q factor for longer lengths. Only calculating half of the Q factor caused the uncertainty to double. If this lab were redone, and if possible, an oscilloscope or attempt to 'excite' the pendulum could have been used.

Q factor describes the dampening of oscillators or resonators. Outside of the scope of pendulums, Q-factors have applications in electric circuits. Tufts University tested a method using Gravitational Wave Interferometry to analyze Q-factors in pendulums (Wielgus, n.d). Connecting resonance, a pendulum's resonance frequency can be 'excited' using a constant higher voltage. Resonance frequency is usually the initial frequency of the pendulum, when it is at its highest amplitude, and is measured using $T = \frac{1}{2}\pi\sqrt{\frac{L}{g}}$. Increasing the resonance, using high voltage allows to have an increased frequency and decreases the amplitude. Measuring from an oscilloscope, this should be a much quicker and accurate process than manually counting. A similar model is by using auto resonance, where the resonance frequency of the pendulum is altered such that, it matches with the 'driving frequency' (frequency of a pendulum from external influences) and can also be measured through an oscilloscope. However, a limitation to using this method requires a higher-fidelity pendulum to allow for high voltage wire influence, and are more effective in pendulums with a higher Q factor, simulated in a vacuum

From these learnings, Type A and Type B uncertainties are drastically corrected. The largest source of uncertainty is a Type B uncertainty of only tracking the pendulum halfway for two trials in the longer pendulum lengths.

If Lab 2 was redone, it would be more beneficial to increase the increments of length to 10-15 cm, and to cover a range until 1m. Since the length of the string was quite small until 0.40m, it followed a linear trend, while becoming logarithmic near the end. At small lengths, drag force does not significantly contribute to the speed of the pendulum. However, as length increases, the surface area of the string increases. In viscous drag, which is where fluids, including air, as the length increases, the pendulum must endure greater resistance, becoming more influential (*Physics, n.d*). At high speeds, a higher drag coefficient would cause velocity to decrease, assuming all other factors are held constant. Therefore, I hypothesize that if the lengths were increased even further to near 1m, the drag force would dominate, and the Q factor rate would plateau.

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Appendix A: Tracker Graph of All the Amplitudes as Time Elapsed (Measured Through 3000 frames)

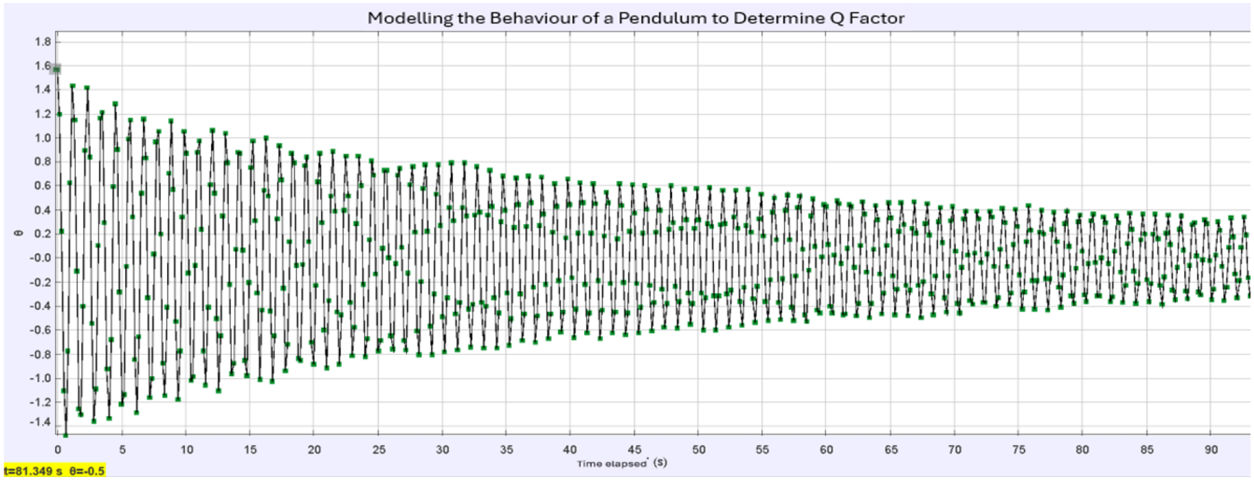

Figure 7: Modelling the Behaviour of A Pendulum to Determine the Q Factor. *Made through Tracker outlining the position of the mass over 3000 frames (measured through a protractor and cross-referenced with the protractor built within my pendulum setup). It can be observed that the amplitude generally decreases as time elapses. However, the decrease is not fully consistent as peaks are followed by lower peaks. There is a variance of 0.1rad from the counted peak to its adjacent points which are local maxima, but a sinusoidal function does not fit the general decaying trend.*