1st European Mathematical Olympiad

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Problem 1. Determine all positive integers n with the following property: the set $\{1, 2, \ldots, 2n-1, 2n\}$ can be partitioned into two disjoint sets $\mathcal{A}$ and $\mathcal{B}$ with n elements each, such that the sum of the elements of $\mathcal{A}$ divides the sum of the elements of $\mathcal{B}$.

Solution. We claim that this can be done for all $n \not\equiv 5 \pmod 6$.

Let S_A, S_B denote the sums of the elements of $\mathcal{A}$ and $\mathcal{B}$ respectively. Since $S_A \mid S_B$, let $\frac{S_B}{S_A} = k$ for an integer $k \geq 1$. Therefore, $(k+1)S_A = S_A + S_B = \frac{2n(2n+1)}{2}$ as it is the sum of the integers $\{1, 2, ..., 2n-1, 2n\}$.

Now we bound k by observing that the smallest possible value for S_A is obtained by taking the first n elements, so $S_A \geq \frac{n(n+1)}{2}$. Therefore:

$$k + 1 \leq \frac{\frac{2n(2n+1)}{2}}{\frac{n(n+1)}{2}} = \frac{4n+2}{n+1} < 4$$

which implies $k < 3$, so $k \in \{1, 2\}$. In particular, since

$$k + 1 \mid \frac{2n(2n+1)}{2}$$

which implies $2 \mid \frac{2n(2n+1)}{2} \iff 2 \mid n$ or $3 \mid \frac{2n(2n+1)}{2} \iff 3 \mid n(2n+1)$. That in turn implies for $n \equiv 5 \pmod 6$ there is no solution.

We now prove that in every other case we can indeed split the set into sets $\mathcal{A}$ and $\mathcal{B}$.

We know that $k + 1 \mid \frac{2n(2n+1)}{2}$, which is the sum of all the numbers in the original set. If we can find $\mathcal{A}$ such that $S_A = \frac{2n(2n+1)}{2(k+1)}$ then the proof is complete.

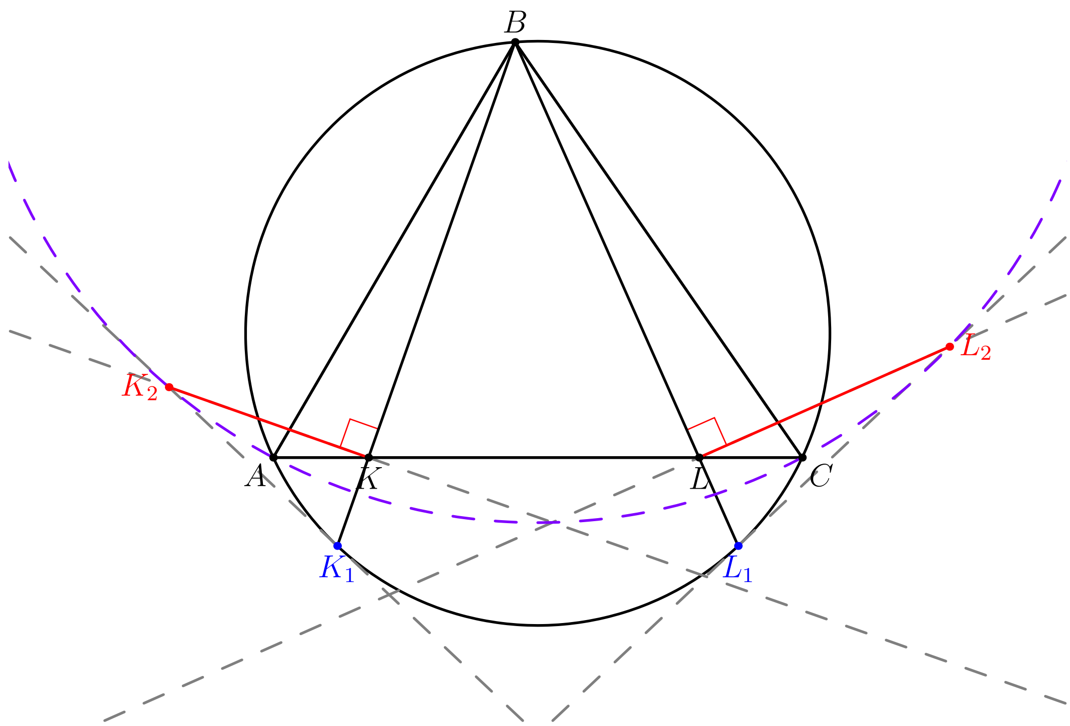
This can be achieved, starting with $\mathcal{A} = \{1, 2, \ldots, n\}$ and then iteratively increasing elements one by one. Observe that we can choose $\mathcal{A}$ such that S_A takes any value between $\frac{n(n+1)}{2}$ and $\frac{2n(2n+1)}{2} - \frac{n(n+1)}{2} = \frac{n(3n+1)}{2}$. Since $\frac{2n(2n+1)}{2(k+1)}$ is between those two numbers we can thus construct $\mathcal{A}$ with exactly that sum.

Remark 1: For even n there is an easier construction by pairing x and $2n - x$ together into $n = 2k$ pairs, each with sum $2n$ and then taking k of these pairs in both A and B. For even n, there is also an inductive proof.

Remark 2: There are various ways to construct $\mathcal{A}$ and $\mathcal{B}$, most simply using the greedy algorithm, but also by listing the elements explicitly. For $n = 6k + 1$ one can take $\mathcal{A} = \{k + 1, \ldots, 7k + 1\}$ and for $n = 6k + 3$ we can take $\mathcal{A} = \{k + 1, \ldots 5k + 2, 5k + 4, \ldots 7k + 4\}$.

Problem 2. Given a triangle ABC, let K and L be distinct points on side AC such that $\angle ABK = \angle CBL$. Rays BK and BL are not orthogonal to AC, and intersect the circumcircle of triangle ABC for the second time at points K_1 and L_1, respectively. Points K_2 and L_2 lie on the tangents to the circumcircle of triangle ABC at points K_1 and L_1, respectively, such that $\angle BKK_2 = \angle BLL_2 = 90°$.
Prove that points A, C, K_2, and L_2 lie on a circle.

Solution. As $\angle ABK_1 = \angle ABK = \angle CBL = \angle CBL_1$, arcs AK_1 and CL_1 are equal, and $K_1L_1 \parallel AC$. Let I be the intersection of lines K_2K, L_2L. Since $\angle IKB = 90°$ and $\angle ILB = 90°$, $BKIL$ is cyclic. Now, $\angle(BI, IK_2) = \angle(BI, IK) = \angle(BL, LK) = \angle(BL_1, L_1K_1) = \angle(BK_1, K_1K_2)$, so points B, I, K_1, K_2 lie on a circle. This means that $K_2K \times KI = BK \times KK_1 = AK \times KC$, so points K_2, A, I, C also lie on a circle. Similarly, L_2, A, I, C lie on a circle, so all five of A, I, C, K_2, L_2 lie on a circle, concluding the problem.



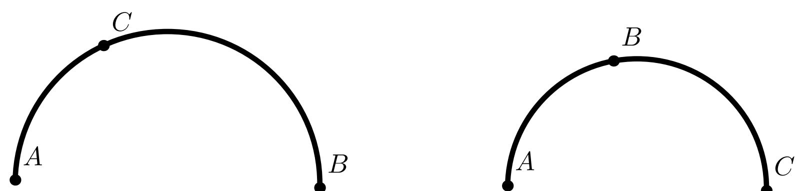
Problem 3. Let $n \geqslant 2$ be an integer. Euroland has n cities, with direct flights connecting every pair of cities in both directions. For each pair of cities, the emperor assigns a positive price, which is the same in each direction. For two distinct cities A and B, let $D(A, B)$ be the number of flights in the cheapest journey between them; if there are multiple such journeys, then $D(A, B)$ is defined by the longest one.
For each value of n, find the largest possible average value of $D(A, B)$ over all pairs of distinct cities (A, B), that the emperor can achieve.

Solution 1. The required minimum is $c = \frac{1}{3}(n+1)$. Consider the obvious graph interpretation: vertices represent cities and edges represent flights, having costs as weights. As there are $\binom{n}{2}$ pairs of cities, it is sufficient to prove that the sum of $D(A, B)$ is at most $\binom{n+1}{3}$.

To prove $c \geq \frac{1}{3}(n+1)$, consider a graph on vertices $V_1, V_2, \ldots, V_n$. Assign weight 1 to every edge $V_i V_{i+1}$ and weight n to every other edge. The cheapest paths are precisely those whose edges are all assigned weight 1, so $\mathcal{D}(V_i, V_j) = |i-j|$. Note that $k = |i-j|$ occurs exactly $n-k$ times over all pairs, so

$$\begin{aligned} \sum_{i<j} \mathcal{D}(V_i, V_j) = \sum_{i<j} |i-j| &= \sum_{k=1}^{n-1} k(n-k) = n \sum_{k=1}^{n-1} k - \sum_{k=1}^{n-1} k^2 \\ &= \frac{1}{2} n(n-1) - \frac{1}{6} n(n-1)(2n-1) = \binom{n+1}{3}. \end{aligned}$$

To prove $c \leq \frac{1}{3}(n+1)$, assign each path of length k, say, $A_0, A_1, \ldots, A_k$, the collection of $k-1$ sets $\{A_0, A_i, A_k\}, i = 1, \ldots, k-1$.



Path 1 Path 2

We now show that no two minimal cost paths share three pairwise distinct vertices. Suppose two such paths share vertices A, B, C. Note that an interior vertex of a minimal-cost path splits that minimal path into two minimal-cost sub-paths. Without loss of generality, suppose that A and B are the endpoints of one path and that A and C are the endpoints of the other. Let c_1 and c_2 denote costs along the two paths. The remark on paths and sub-paths implies that:

$$\begin{aligned} c_1(A, B) &= c_1(AC) + c_1(C, B) = c_2(AC) + c_1(C, B) = c_2(A, B) + c_2(B, C) + c_1(C, B) \\ &> c_2(A, B) = c_1(A, B) \end{aligned}$$

and we reach a contradiction. Finally, count the the number of 3 -vertex sets assigned to get

$$\binom{n}{3} \geq \sum_{i<j} (\mathcal{D}(C_i, C_j) - 1) = \sum_{i<j} \mathcal{D}(C_i, C_j) - \binom{n}{2}.$$

Obvious manipulations yield $\sum_{i<j} \mathcal{D}(C_i, C_j) \leq \binom{n+1}{3}$, so $c \leq \frac{1}{3}(n+1)$. This completes the proof and completes the solution.

Solution 2. We provide an alternative proof that the sum of all $\binom{n}{2}$ numbers of the form $\mathcal{D}(A, B)$ is at most $\binom{n+1}{3}$. By a cheapest path we always mean a path of maximal length among the cheapest ones.

We argue by induction on n, with a base case $n = 2$. So it remains to present the inductive step. Notice first that, if the cheapest path from A to B moves through C, then its sub-paths are the cheapest ones from A to C and from C to B.

Now choose any city V. We draw the tree of cheapest paths from V as follows.

Initiate the tree with a single vertex, V. Now, at every step, choose a city X which is not yet in the tree. Choose a cheapest path from V to X; let Y be the last city on this path which is in the tree. Then append to the tree the remainder of the path from Y to X, including all the intermediate cities and the edges connecting them. The new tree still contains a cheapest path from V to X. Continue this process until the tree contains every city, and call the final tree T.

Denote by $\mathcal{T}(A, B)$ the cost of the unique path from A to B in the constructed tree T. Now let X and Y be vertices in the tree T such that the path from X to Y in T does not pass through V (i.e., X and Y are in one branch of the tree). In this case it follows that: $\mathcal{T}(X,Y) < \mathcal{T}(X,V) + \mathcal{T}(Y,V) = \mathcal{D}(X,V) + \mathcal{D}(Y,V)$ so the cheapest path from X to Y cannot pass through V. Now remove the root V from T, and for any two remaining cities A and B, re-define the price of the direct flight between them as the lesser of the original price, and the cost of the two-flight route from X to Y via V. If this new cost equals the cost of the two-flight route, say that the flight is *dangerous*. By the inductive hypothesis, under the new price schedule the sum of values of $\mathcal{D}(A, B)$ is at most $\binom{n}{3}$. In comparison to the previous price schedule, this sum will increase, as a maximum, by (1) the sum of all values of $\mathcal{D}(V, X) = \mathcal{T}(V, X)$, and also (2) the number of pairs (A, B) such that the cheapest route between them uses a dangerous flight (by the above argument, such X and Y must lie in different branches of T).

Number the edges of the tree connected to the root V as $1, 2, \ldots k$ and let $a_1, a_2, \ldots, a_k$ be the number of vertices in the respective branches. Then the increment due to (1) is at most $\sum_i (1 + 2 + \cdots + a_i) = \sum_i a_i (a_i + 1)/2$. As we have seen, the increment due to (2) is at most $\sum_{i<j} a_i a_j$. So the total increment is at most

$$\sum_i \frac{a_i (a_i + 1)}{2} + \sum_{i<j} a_i a_j = \frac{1}{2} \left(\left(\sum_i a_i \right)^2 + \sum_i a_i \right) = \frac{n(n-1)}{2},$$

so the sum of the $\mathcal{D}(A, B)$ does not exceed $\binom{n}{3} + \binom{n}{2} = \binom{n+1}{3}$, as desired.

Problem 4. Let $\mathbb{N}$ be the set of positive integers. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ that simultaneously satisfy the following properties:

(i) $f(mn) = f(m)f(n)$ for all positive integers m and n;

(ii) There exists a positive integer c such that $f(n) \leqslant n^c$ for all positive integers n;

(iii) The numbers $f(n) + m$ and $f(m) + n + 1$ are coprime for all positive integers m and n.

Solution. All solutions start with the following lemma.

Lemma 1. For every prime p, there is a nonnegative integer $a \le c$ such that $f(p) = p^a$.

Proof. Assume that there exists a prime r such that $r \mid f(p)$ and $r \neq p$. Let us consider $n = ps$ and $m = r$. Note that $f(n) + m = f(ps) + r = f(p)f(s) + r$ and thus $r \mid f(n) + m$. We will now show that we can choose s such that $r \mid f(m) + n + 1$. This is equivalent to $ps \equiv -f(r) - 1 \pmod r$. Since $p \neq r$ we have that p is invertible modulo r and thus s can be chosen with this property. Thus $r \mid f(m) + n + 1$ for this choice of s, which contradicts condition (i) of the problem. We conclude that $f(p) = p^a$, and that $a \le c$ because of (ii).

Note that $f(n) = n$ for all positive integers n is obviously a solution. We now present several ways to show that there are no other solutions.

We know that for every prime there exists a nonnegative integer l_p such that $f(p) = p^{l_p}$. For $0 \le l \le c$, let A_l be the set of primes p such that $f(p) = p^l$.

Lemma 2. There exists $0 \le t \le c$ such that for any prime p, there exists a prime $q \in A_t$ which is a primitive root modulo p.

Proof. For the sake of contradiction, assume for every l, there exists a prime p_l for which such $q \in A_l$ does not exist. Pick a number g such that $g \pmod{p_l}$ is a primitive root modulo p_l for all l, which exists by the Chinese remainder theorem. By Dirichlet's theorem, there exists a prime $q \equiv g \pmod{p_0 p_1 \dots p_c}$, which has to lie in some A_l, a contradiction.

Let A_t have the above property. For some fixed n, we find an integer m and prime q such that both $f(n)+m, f(m)+n+1$ are divisible by q. Assume $\gcd(f(n), q) = 1$. Then there exist some $p \in A_t$ and k such that $q \mid f(n) + p^k$, since p may be chosen to be a primitive root modulo q. Pick $m = p^k$. Then $f(m) = p^{tk}$, and we need $q \mid (p^k)^t + n + 1$. Because $p^k \equiv -f(n) \pmod q$, we need $q \mid (-f(n))^t + n + 1$. It follows that there cannot exist an integer n such that $(-f(n))^t + n + 1$ has a prime divisor q coprime with $f(n)$, or else we would reach a contradiction.

Assume $l_p = 0$ for some prime p. Then for any k, we have $f(p^k) = 1$, and taking $n = p^k$, we have that $(-f(p^k))^t + p^k + 1 = (-1)^t + p^k + 1$ has to be ± 1 for all k, which is clearly impossible. Thus $l_p > 0$ for all p, which implies $n \mid f(n)$ for all n. In addition, the set A_0 is empty, so $t > 0$. This means $(-f(n))^t + n + 1$ is coprime with n (and with $f(n)$), so it must be ± 1. Then by using complete multiplicativity we have $(-f(n^k))^t + n^k + 1 = (-1)^t f(n)^{kt} + n^k + 1 = \pm 1$ for all n and k as well. If $f(n) > n$, $(-1)^t f(n)^{kt}$ grows faster in absolute value than n^k (as a function of k), and thus this expression cannot be ± 1 for all k.

Solution 1. Assume that for some prime p, we have $f(p) = p^k$ and $k \neq 1$. Initially let M and R both be the empty set. For all $l \in \{0, 1, \dots, c\}$, let $P_l(x) = (-1)^l x^{kl} + x + 1$. Starting with $l = 0$, for each l we can choose s_l a power of p such that $P_l(s_l)$ has a prime factor not in M; this prime will be denoted by q_l. Namely, since M is finite at each step, consider $A_M = \prod_{m \in M} m$. Let $s_l = p^{6\varphi(A_M)}$. Then using Euler's theorem we have $P_l(s_l) \equiv 1 \text{ or } 3 \pmod M$. Thus, for $m \neq 3$, we have $m \nmid P_l(s_l)$. For $m = 3$ we have $P_l(s_l) \equiv 3 \pmod 9$ using again Euler's theorem for $p \neq 3$, and for $p = 3$ we have $P_l(s_l) \equiv 1 \pmod 3$. Consider the congruence $R_l : x \equiv -s_l^k \pmod{q_l}$. At each step add q_l to M and R_l to R, until we obtain $q_0, q_1, \dots, q_c$ in M with corresponding congruences $R_0, R_1, \dots, R_c$.

Lemma 3. If a prime t satisfies congruence R_l, then we cannot have $f(t) = t^l$.

Proof. If $f(t) = t^l$, taking $n = t$ and $m = s$ gives that $t + f(s) = t + s^k$ is a multiple of q_l, but so is $P_l(s)$ by definition of q_l, which means that so is $f(t) + s + 1$, a contradiction.

But now by Dirichlet's theorem there exist a prime q satisfying all of the congruences, whereas by Lemma 3 we cannot have $f(q) = q^l$ for any $0 \le l \le c$, a contradiction.

Solution 2. We start with the following lemma.

Lemma 4. For any arithmetic progression that contains infinitely many primes, there exists a prime q in the arithmetic progression such that $f(q) = q$.

Proof. Let us consider a prime s such that $\gcd(s-1, (3c^3)!) = 2$ and s is coprime to the common difference of the arithmetic progression being considered; this can be easily ensured using Dirichlet's theorem for primes in arithmetic progression. Consider now a primitive root modulo s, call it s_0. Using Dirichlet's theorem again we can find three primes p, q such that $p, q \equiv s_0 \pmod{s}$, where s will be appropiately chosen and q also lying in the arithmetic progression being considered.

Let us consider $n = p^x$ and $m = q^y$. We will argue that the positive integers x, y can be chosen such that $s \mid f(n) + m$ and $s \mid f(m) + n + 1$. Note that from Lemma 1 there are positive integers $\alpha, \beta \leq c$ such that $f(p) = p^\alpha$, $f(q) = q^\beta$. Assume $\beta > 1$. Thus $f(n) + m = f(p^x) + q^y = p^{\alpha x} + q^y$ and $f(m) + n + 1 = f(q^y) + p^x + 1 = q^{\beta y} + p^x + 1$.

Modulo s, the numbers p^x, q^y a run over all nonzero residues, so denoting $X = p^x$, $Y = q^y$, our congruences boil down to finding a nonzero solution $X^\alpha + Y \equiv 0 \pmod{s}$ and $Y^\beta + X + 1 \equiv 0 \pmod{s}$.

By the first congruence we have $Y \equiv -X^\alpha$ and substituting in the second we obtain $\pm X^{\beta\alpha} + X + 1 \equiv 0 \pmod{s}$. For $i > 1$ and $j > 1$ let $f_{i,j} = X^{ij} + X + 1$ and $g_{i,j} = -X^{ij} + X + 1$. Due to Nagell's theorem we can find infinitely many primes s such that all these polynomials have a root modulo s and we are done.

Lemma 5. For any prime p we have $f(p) = p$.

Proof. Suppose the contrary and let p be a prime $f(p) = p^a$ where $a > 1$. Pick m a positive integer such that $p^{am} - p^m - 1 \geq 3$. We will choose a prime q appropriately such that $f(q) = q$, and moreover, $f(p^m) + q$ and $f(q) + p^m + 1$ are not coprime. This is equivalent $p^{am} + q$ and $q + p^n + 1$ not to be coprime. Let s be an odd prime such that $s \mid p^{am} - p^m - 1$; such a prime always by our choice. Then according to Dirichlet's theorem and Lemma 4 we can find a prime $q \equiv -p - 1 \pmod{s}$ with $f(q) = q$.

We have thus finished the proof of the lemma.

Solution 3. For $0 \leq i \leq c$ define

$$M_i = \{p \text{ prime} : \text{there exists } w \in \mathbb{Z} \text{ s.t. } (-w^i)^i + w + 1 \equiv 0 \pmod{p}\}.$$

Lemma 6. If q is a prime that is a primitive root modulo p with $p \in M_i$, then $f(q) \neq q^i$.

Proof. Assume the contrary. Since $p \in M_i$ we can find $w \in \mathbb{Z}$ with $(-w^i)^i + w + 1 \equiv 0 \pmod{p}$. Also, q is a primitive root modulo p, thus there exist integers x, y such that $q^x \equiv w \pmod{p}$ and $q^y \equiv -w^i \pmod{p}$ (note that w is clearly nonzero modulo p). This implies $f(q^x) = q^{ix} \equiv w^i$ and $f(q^y) \equiv (-w^i)^i$. Setting $n = q^x$ and $m = q^y$ yields a contradiction to condition (i).

Lemma 7. For any $i \neq 1$, the set of primes p such that $f(p) = p^i$, has density[1] 0.

Proof. By the Siegel–Walfisz theorem (quantiative version of Dirichlet's theorem), the density of primes congruent to $n' \pmod{n}$ is $\frac{1}{\varphi(n)}$ as long as $\gcd(n', n) = 1$. For a prime number p there are $\varphi(p-1)$ primitive roots modulo p. It follows that the density of primes which are not primitive roots modulo any prime in M_i is bounded by

$$\prod_{\substack{p \in M_i \\ p \leq N}} \left(1 - \frac{\varphi(p-1)}{p-1}\right)$$

for any N. By using Lemma 3, we want to show this quantity converges to zero as $N \to \infty$. For this, it is enough to show that

$$\sum_{p \in M_i} \frac{\varphi(p-1)}{p-1} = \infty.$$

Using the well-known fact that there exists a constant $c_1 > 0$ such that $\varphi(x) \geq c_1 \cdot \frac{x}{\log\log x}$ for all large enough x, it suffices to show that $\sum_{p \in M_i} \frac{1}{\log\log(p-1)} = \infty$ where the sum is over large enough primes p.

Let $P_i(w) = (-w^i)^i + w + 1$ for $i > 1$ (or any non-constant polynomial with integer coefficients). There is a positive constant $c_2 > 0$ such that for $|w| \leq N$ and large enough N, the polynomial P_i yields $\geq c_2 N$ distinct images. If all of these images are of the form $\pm p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ for some primes p_j and nonnegative integers α_j, note that there exists a constant $c_3 > 0$ such that $\alpha_j \leq c_3 \log N$ for all $1 \leq j \leq k$, and there

[1] That is, we have $\lim_{N \to \infty} |\{p \leq N \text{ prime} : f(p) = p^i\}| / \pi(N) = 0$, where $\pi(N)$ is the number of primes less than N.

are at most $2(c_3 \log N)^k$ such numbers. Therefore we must have $2(c_3 \log N)^k \geq c_2 N$, which means that $k \geq c_4\sqrt{\log N}$ for some constant $c_4 > 0$ and large enough N. It follows that for large enough N, for each $1 < i \leq c$ the number of primes in M_i which are less than N is at least $c_4\sqrt{\log N}$. From here we have $\sum_{\substack{p\in M_i \\ p\leq N}} \frac{1}{\log\log(p-1)} \geq \frac{c_4\sqrt{\log N}}{\log\log N} \to \infty$ as $N \to \infty$, so that the sum diverges.

It follows that for density 1 subset of the primes we have $f(p) = p$. Now, if there exists a prime $q \neq 3$ such that $f(q) = q^m$ with $m \neq 1$ or $f(3) = 3^m$ with $m > 1$, then take any prime $q' \mid q^m + 1 - q$ and set n to be any prime p which is congruent to $-q \pmod{q'}$, contradicting condition (i) (with $m = q$). So we have $f(p) = p$ for all primes p, except possibly $f(3) = 1$. But if $f(3) = 1$ then taking $m = 2, n = 3$ gives that $1 + 2$ and $2 + 3 + 1$ are not coprime, so we must have $f(3) = 3$ also.

Remark: In Solution 3, the divergence of $\sum_{p\in M_i} \frac{1}{\log\log p}$ follows directly by the pole at $s = 1$ of the associated Dedekind zeta-function, or even the Chebotarev density theorem.

The Chebotarev density theorem has a wide range of applications. Here, we provide a brief formulation of it, let $k \subset \mathbb{C}$ be the splitting field of polynomial $f(X)$ with integer coefficients over $\mathbb{Q}$ (the smallest extension of $\mathbb{Q}$ containing all roots of f), with Galois group $G = \mathrm{Aut}(K/\mathbb{Q})$. Then, for each conjugacy class C of G, there exist infinitely many primes p such that $\pi_p \in C$. In fact the proportion of primes p satisfying $\pi_p \in C$ is $\frac{|C|}{|G|}$, that is $\lim_{N\to\infty} \frac{\#\{p\leq N : \pi_p\in C\}}{\pi(N)} = \frac{|C|}{|G|}$. This theorem has a wide (even somehow far-reaching) range of applications, for example, we can deduce that for a non-constant irreducible polynomial $f(X)$ with integer coefficients, there are infinitely many primes that do not divide the elements of $f(\mathbb{Z})$. More interestingly, we can deduce that if f is a polynomial with rational coefficients such that it has a root modulo p for every large prime p, then f has a real root.

Problem 5. Let $n \geqslant 4$ be a positive integer. Find all positive real numbers $x_1, x_2, \ldots, x_n$ such that

$$\begin{cases} x_1 + x_2 &= x_2x_3 + 1 \\ x_2 + x_3 &= x_3x_4 + 1 \\ &\vdots \\ x_{n-1} + x_n &= x_nx_1 + 1 \\ x_n + x_1 &= x_1x_2 + 1. \end{cases}$$

Solution 1. We claim that for odd n the only solution is $x_i = 1 \forall i$ and for even n the only solutions are of the form $x_i = 1$ for all even i and $x_j = c$ for all odd j and any $c > 0$ and vice versa.

We first rearrange the equalities to $x_i - 1 = (x_{i+2} - 1)x_{i+1}$, where we always take the indices $\mod n$.

Suppose there is i s.t. $x_i = 1$. Then $x_{i-2} - 1 = (1 - 1)x_{i-1} = 0$ so $x_{i-2} = 1$.

If there does not exist such i, then we can multiply everything together and cancel $\prod(x_i - 1)$ from both sides to obtain $\prod x_i = 1$. This also means that none of the x_i is equal to 0. By looking at the sign of $x_i - 1$ and $(x_{i+2} - 1)x_{i+1}$ we see that x_i and x_{i+2} are on the same side of the number 1.

Now suppose $n = 1$ is odd. If $x_i = 1$ for some i then we get that all $x_j = 1$ since we know that $x_{i-2} = 1$ and we can keep applying that to cycle through all the numbers. If however there is no x_i equal to 1, we can see that means that all of x_i are on the same side of 1, so either all are bigger or all are smaller, but this can't hold since the product has to be equal to 1. So the only solution when n is odd is when all the x_i are equal to 1 and we can easily see it is indeed a solution.

Now suppose n is even. Again suppose $x_i = 1$ for some i and WLOG let $i = 1$. This means that all the x_j with odd indices will be equal to 1 and by looking at $x_{2k} - 1 = (x_{2k+2} - 1)x_{2k+1}$ we see that $x_{2k} = x_{2k+2}$ so we see that all the x_j with even indices will be equal amongst themselves as well. We can see that we can set them equal to any constant $c > 0$ and the solution will still be valid. Since we assumed $i = 1$ we need to extend the solution to the even case as well.

But if all $x_i \neq 1$ then since $\prod x_i = 1$, there has to exist some $x_i > 1$. Again WLOG $i = 1$. We now see that since all odds are on the same side of 1 they must all be > 1 and we must have all the evens be < 1. However this is impossible since we can now take the product of all the equations $x_{2k+1} - 1 = (x_{2k+3} - 1)x_{2k+2}$ and we see that the odds cancel out and the product of the evens has to be equal to 1, which is impossible.

So the only solutions in the even case are $x_{2k+1} = 1, x_{2k} = c, \forall k$ and $x_{2k} = 1, x_{2k+1} = c, \forall k$ for any constant $c > 0$.

Solution 2. Let us define $sgn(t)$ to be 1 for $t > 0$, -1 for $t < 0$, and 0 when $t = 0$. Let us also refer to all indices $\mod n$, and rearrange all the equations into the form $(x_i - 1) = (x_{i+2} - 1)x_{i+1}$.

Since $x_{i+1} > 0$, it follows that $sgn(x_i - 1) = sgn(x_{i+2} - 1)$: either they are non-zero (and so $x_i \neq 1$) and of same sign, or both equal to zero.

If n is odd, then $sgn(x_i - 1)$ is the same for all i, since $\{x_{i+2}, x_{i+4}, \ldots, x_{i+2n}\}$ covers entire set of variables. We can verify that case $sgn(x_i - 1) = 0$, i.e. $x_i = 1\ \forall i$ works. Otherwise, we can multiply all equations together and divide by (non-zero) $\prod(x_i - 1)$ to obtain $\prod x_i = 1$. But this implies that x_i are all either above 1, or below 1 - yet they multiply to 1 exactly; contradiction.

If n is even, let us consider odd and even indices separately.

Suppose one of the elements, WLOG $x_1 = 1$. Then by $sgn(x_i - 1) = sgn(x_{i+2} - 1)$, all odd-indexed elements are 1, and we can see that $x_{2k} - 1 = x_{2k+1}(x_{2k+2} - 1) = x_{2k+2} - 1$, i.e. all remaining elements are equal. We can the verify that for any positive c, the equations hold. Analogously, we obtain the solution of $\{x_{2k} = 1, x_{2k+1} = c\}$ for any positive c.

Now suppose all $x_i \neq 1$. Then, we take every second equation (of form $x_{2k+1} - 1 = (x_{2k+3} - 1)x_{2k+2}$) and multiply them together to obtain $\prod x_{2k} = 1$ (non-zero elements of odd indices cancel out). But since all evens are on the same side of 1, they can't multiply to 1 - contradiction.

Hence we get the solutions of $x_{2k} = 1, x_{2k+1} = c > 0$ and $x_{2k} = c > 0, x_{2k+1} = 1$ for even n, and $x_i = 1$ as the only solution for odd n.

Solution 3. Rewrite the equations as $x_{i-1} - x_{i+1} = (x_i - 1)(x_{i+1} - 1)$. Suppose that there exists $x_i = 1$. Then we easily see that this implies $x_{i-1} = x_{i+1}$ and by shifting our equation by one we also get that $x_{i-2} = x_i = 1$ and by that we can continue to either get $x_j = 1 \forall j$ if n is odd or $x_j = 1$ for all $j \equiv i \pmod 2$ and $x_j \equiv c$ for all $j \equiv i + 1 \mod 2$.

Now suppose $x_i \neq 1 \forall i$. Denote by $a_i = x_i - 1 > -1$. We then have that $a_i > 0 \iff x_i > 1$ and since $x_i \neq 1$ then $a_i \neq 0$. The equation can then be rewritten as $a_{i-1} - a_{i+1} = a_i a_{i+1}$ or $\frac{a_{i-1}}{a_{i+1}} - 1 = a_i$. From that we see that a_{i-1} and a_{i+1} must have the same sign, as $a_i > -1$. For odd n we see that would mean all of them have the same sign. Now we can see different possibilities for signs of a_2, a_3 and obtain contradiction from all the choices.

Suppose $a_2, a_3 > 0$, then $a_1 - a_3 > 0$ so $a_1 > a_3 > 0$ and we can continue this, but since we have a finite sequence we can't always strictly increase the terms and we will loop back, a contradiction. Same contradiction with both of them being < 0.

In the case of n even we also get the possibility of $a_1 > 0$ and $a_2 < 0$ or vice versa, but in that case the product of two consecutive terms is always negative and we now obtain a strictly decreasing looping sequence, a contradiction.

Solution 4. Substract the second equation from the first to obtain

$$x_1 - x_3 = x_3(x_2 - x_4).$$

Now we see that since $x_i > 0$ if $x_1 > x_3$, then $x_2 > x_4$ and we can continue to obtain $x_3 > x_5$ and at some point we arrive back at x_1 in the chain, so $x_3 > x_1$, which is a contradiction. Same holds if $x_1 < x_3$. So we must have $x_1 = x_3$, which implies $x_2 = x_4$ and so on, so all terms of the same parity must be equal. If n is odd we obtain that all are equal, and the only solution will satisfy $x + x = x^2 + 1 \iff x^2 - 2x + 1 = 0 \iff x = 1$. For even n we obtain an equation $x + y = xy + 1 \iff (x-1)(y-1) = 0$ so at least one of those two values has to be 1 and the other can actually be any positive real number and we are done.

Problem 6. Determine all positive integers $n \geqslant 2$ with the following property: for every positive divisor d of n, the product of all the other positive divisors of n is a perfect power.
A perfect power is a number of the form a^b for some integers $a \geqslant 1$ and $b \geqslant 2$.

Solution. Write $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ in its canonical prime factorisation. The p_i-adic valuation of the product of all divisors of n is equal to

$$\frac{d(n)}{\alpha_i + 1} \cdot (1 + 2 + \cdots + \alpha_i) = d(n) \cdot \frac{\alpha_i}{2},$$

where $d(n) = \prod_{i=1}^{k}(\alpha_i + 1)$ is the number of divisors of n. We have that $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ is powerful if and only if we have

$$\gcd\left\{\frac{\alpha_i d(n)}{2} - a_i : 1 \leq i \leq k\right\} > 1$$

for all $0 \leq a_i \leq \alpha_i$. If $k = 1$, prime powers $n = p^\alpha$ are obviously powerful, henceforth let $k \geq 2$. In particular we must have

$$G := \gcd(\alpha_i d(n) - 2a_i, \alpha_j d(n) - 2a_j) > 2$$

for all $i \neq j$.

Let $g = \gcd(\alpha_i, \alpha_j)$. Then G also divides $\frac{\alpha_j \alpha_i}{g} d(n) - 2\frac{\alpha_j}{g} a_i - \frac{\alpha_i \alpha_j}{g} d(n) + 2\frac{\alpha_i}{g} a_j = 2\left(\frac{\alpha_i}{g} a_j - \frac{\alpha_j}{g} a_i\right)$. By Bézout's lemma, we may find $0 \leq a_i \leq \frac{\alpha_i}{g} \leq \alpha_i$ and $0 \leq a_j \leq \frac{\alpha_j}{g} \leq \alpha_j$ such that $\frac{\alpha_i}{g} a_j - \frac{\alpha_j}{g} a_i = \gcd\left(\frac{\alpha_i}{g}, \frac{\alpha_j}{g}\right) = 1$, thus G would have to divide 2, a contradiction to $G > 2$.

Problem 7. Let ABC be an acute triangle with $AB < AC$. Let M be the midpoint of segment BC. Let E and F be points on segments AC and AB, respectively, such that the circumcircle of triangle MEF is tangent to BC. The circumcircles of triangles AEF and ABC intersect at a point $P \neq A$. Let Q be a point on the circumcircle of triangle AEF such that AQ is perpendicular to BC.
Prove that PQ passes through the circumcentre of triangle MEF.

Solution 1. Define N as the midpoint of EF, O as the circumcenter of ω and H as the intersection of lines EF and BC.
First note that $\angle QPE = \angle QAE = 90° - \angle ACB$, hence it's enough to show that $\angle OPE = 90° - \angle ACB$.
Since ω is tangent to BC at M, $\angle OMH = 90°$. Also, $OE = OF$ and N is the midpoint of EF, so ON is an altitude in an isosceles triangle and $\angle ONH = 90°$. Now $\angle ONH = \angle OMH$ and that makes O, M, N, H lie on a circle, with OH as diameter.
By having multiple circles we have:

$$\angle(EF, FP) = \angle(EA, AP) = \angle(CA, AP) = \angle(CB, BP)$$

$$\angle(FP, PE) = \angle(FA, AE) = \angle(BA, AC) = \angle(BP, PC)$$

showing that $\triangle PEF \sim \triangle PCB$. That allows a spiral similarity centered at P to exist, taking E to C, F to B, and consequently the midpoint N (of EF) to the midpoint M (of CB). This gives us another set of similar triangles $\triangle PNM \sim \triangle PFB$, which gives us

$$\angle(NP, PM) = \angle(FP, PB) = \angle(FP, PA) + \angle(AP, PB) =$$

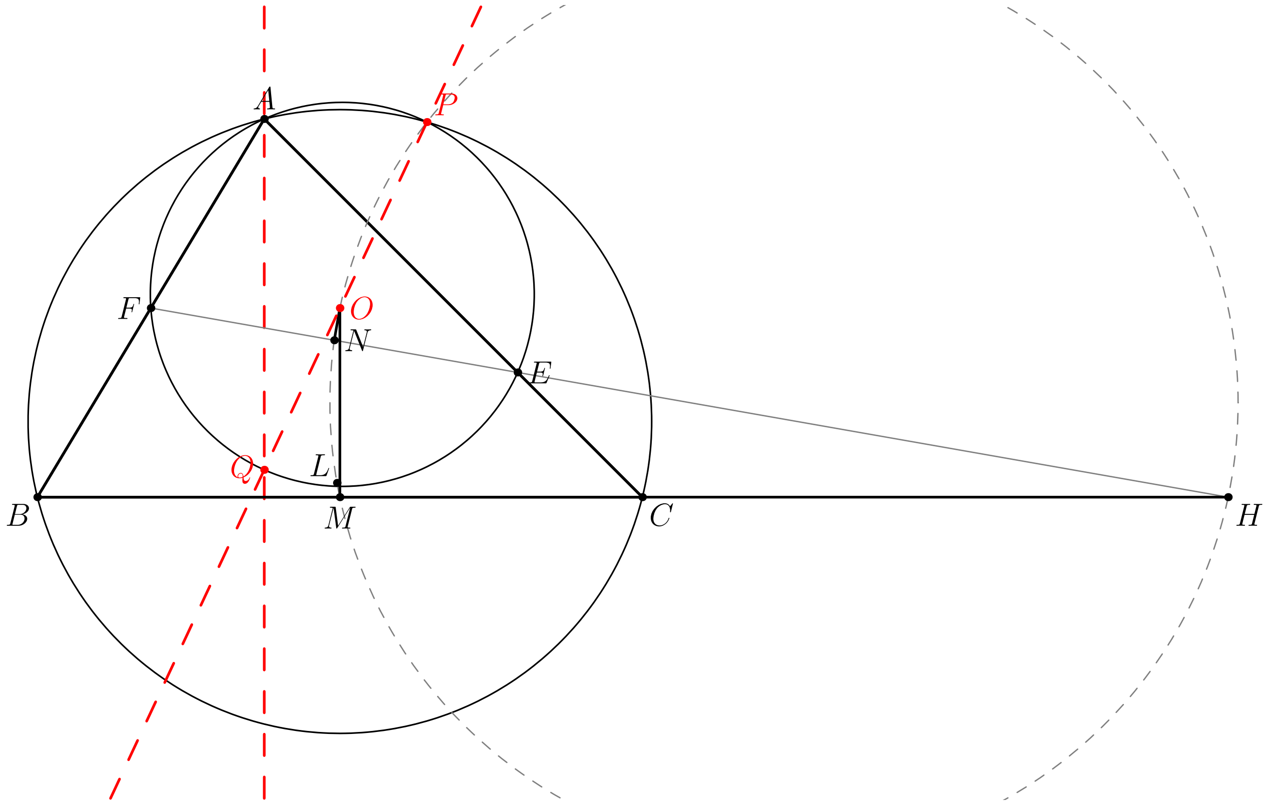
$$\angle(FE, EA) + \angle(AC, CB) = \angle(NH, CA) + \angle(AC, HM) = \angle(NH, HM)$$

showing that H, M, N, P (and therefore O from earlier) lie on a circle.
The spiral similarity also gives $\triangle PEC \sim \triangle PNM$, giving $\angle(EP, PC) = \angle(NP, PM) = \angle(NH, HM) = \angle(EH, HC)$, showing that P, H, C, E lie on a circle.
Now, $\angle OPH = 90°$ and $\angle EPH = \angle ACB$, thus $\angle OPE = 90° - \angle ACB$, as desired.
Alternative finish:
Let L be the intersection of AM and the circumcircle of OPH. We have that $\angle(PL, LA) = \angle(PL, LM) = \angle(PH, HM) = \angle(PH, HC) = \angle(PE, EC) = \angle(PE, EA)$, showing that A, E, F, L lie on a circle.
Finally,

$$\angle(QP, PL) = \angle(QA, AL) = \angle(QA, HM) + \angle(HM, AL) = 90° + \angle(HM, ML) =$$

$$\angle(OL, LH) + \angle(HO, OL) = \angle(HO, LH) = \angle(OH, HL) = \angle(OP, PL)$$

proving that P, O, Q are collinear.



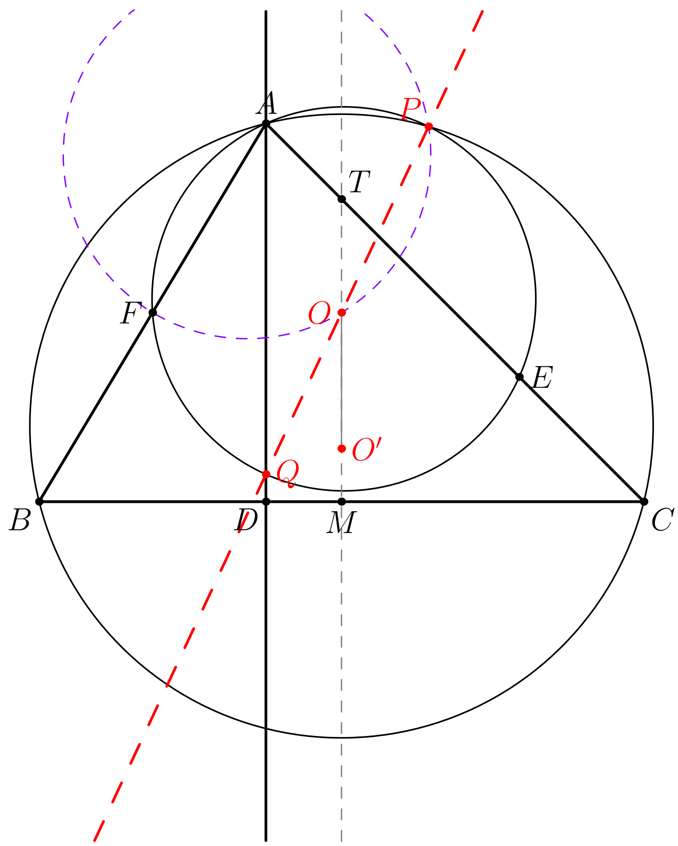
Solution 2. Suppose without loss of generality that $AB > AC$. Let O be the center of ω. The main idea is to consider $T = OM \cap AC$. Now, consider the following claim:

Claim: $POFT$ is cyclic.

Proof: Since P is the center of the spiral similarity that sends $EF \to BC$, consider O' the image of O in such transformation. Because O is in the perpendicular bisector of EF, O' is in the perpendicular bisector of BC, which is OM.

However, notice P is also the center of the spiral similarity sending $FC \to OO'$ and thus it must be in the circle passing through F, O and $OO' \cap FC = T$, proving the claim.

To finish off, consider the angle chase: $\angle OPF = \angle OTF = \angle QAF = \angle QPF$ The first equality following from the claim, the second from the fact that $OT \parallel AQ$ and the third from $(AFQP)$ being cyclic. Therefore $\angle OPF = \angle QPF$, proving that O, P and Q are collinear and finishing the problem.



Solution 3. Let O, U and V be the centers of $(ABC), \omega, (AEF)$, respectively. Let the intersection of AP and BC be point H.

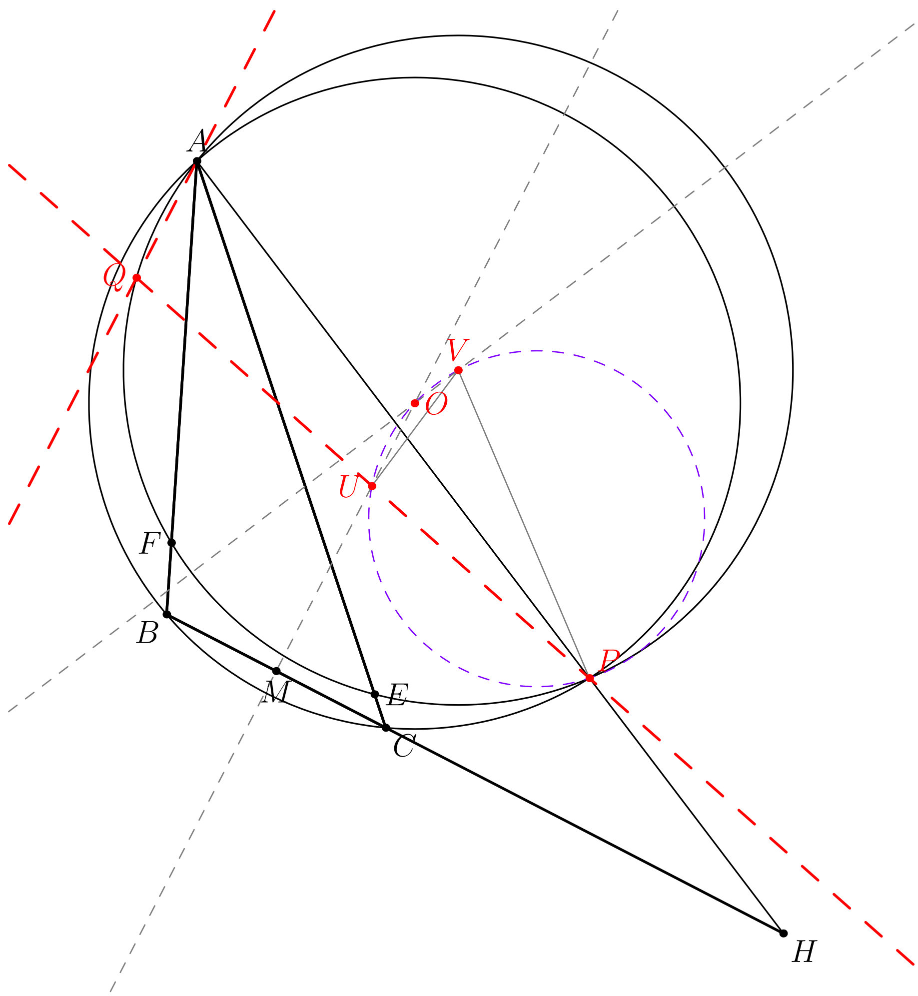
Claim: $PVOU$ is cyclic.

Proof: Notice that P is the center of the spiral similarity mapping FE to BC so it maps the (PEF) to (PBC) and therefore it maps V to O. Let's say that this spiral similarity maps U to G. Since U lies on the perpendicular bisector of FE, then G lies on the perpendicular bisector of BC. Since U also lies on the perpendicular bisector of BC (tangency to BC condition), we have that O, U, M, G are collinear. From these we get that $\angle PVO = \angle PUG \implies PVOU$ is cyclic.

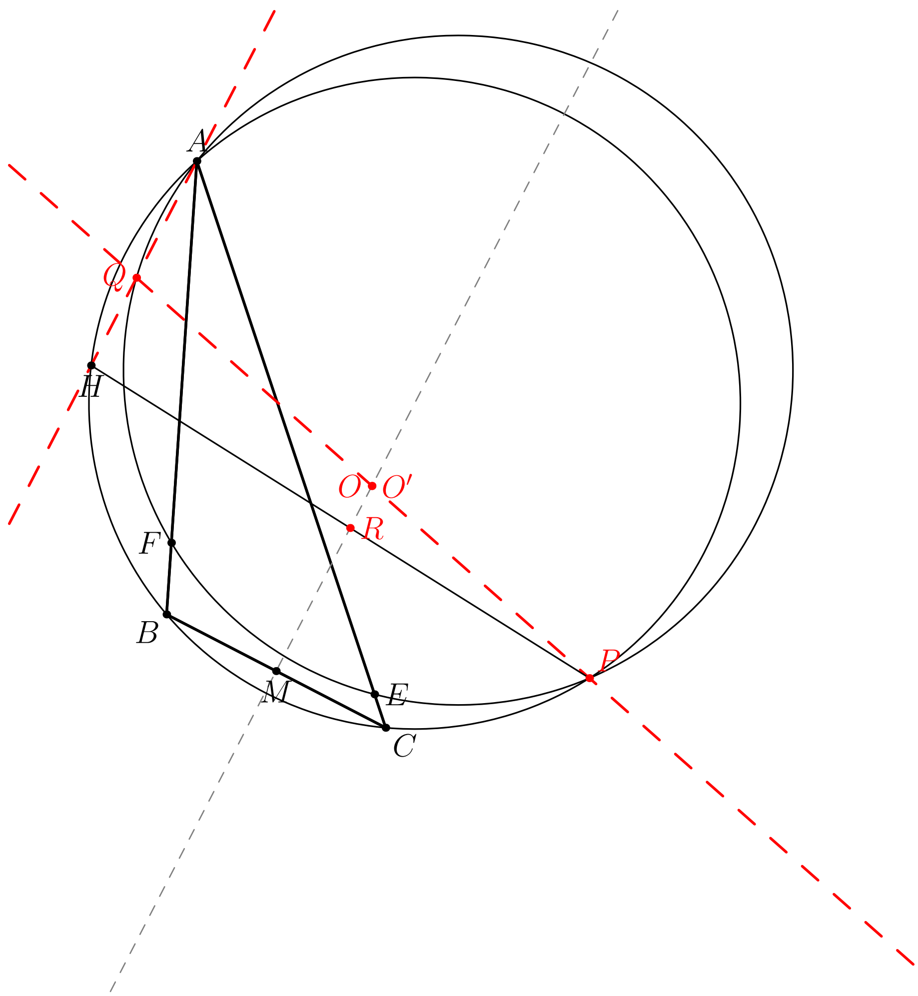
Claim: $P - U - D$ are collinear.

Proof: Notice that $OU \perp BC$ and $OV \perp AP$. This means that $\angle AHM = 180^\circ - \angle VOU$. Now, $\angle VPD = 90^\circ - \angle DAP = \angle AHM = 180^\circ - \angle VOU = \angle VPU$, implying that P, U, D are collinear.

Hence the problem statement is proved.



Solution 4. Consider the point H which is the intersection of AQ and circumcircle of $\triangle ABC$ and let $R = PH \cap OM$, where O is a center of ω. Let O' be the intersection of PQ with perpendicular bisector of FE. It is enough to show that $O'R||QH$ which would imply that $O' = O$. Since $\angle PEF = \angle PAB = \angle PCB$ and $\angle BPC = \angle BAC = \angle FPE$ we have that $\triangle PFE \sim \triangle PBC$. Since $\angle FPQ = \angle FAQ = \angle BPH$ and O' and R are both defined as intersections with the corresponding perpendicular bisectors, we have $\frac{PQ}{PO'} = \frac{PH}{PR}$ and therefore $O'R||QH$ which concludes the proof.


A
Q
H
O
O'
R
F
B
M
E
C
P

Problem 8. For a convex polygon $\mathcal{P}$, let $\mathcal{B}$ be the set of points on the boundary of $\mathcal{P}$. A function $f\colon \mathcal{B} \to \mathcal{B}$ is *European* if it satisfies the following properties:

(i) $f(f(X)) = X$ for all points $X \in \mathcal{B}$;

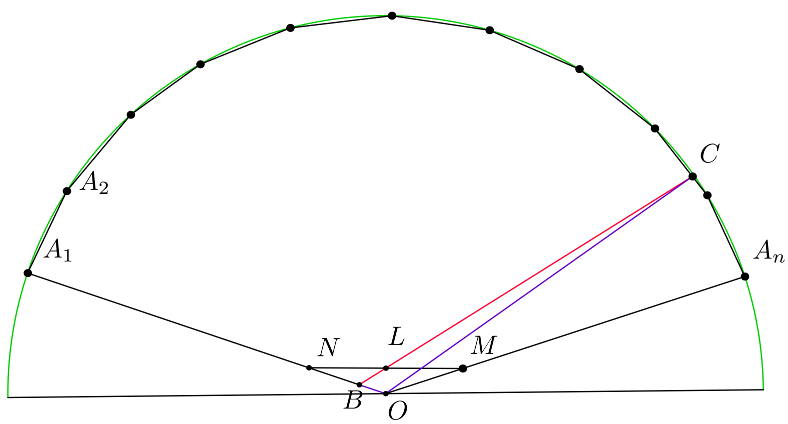
(ii) Line segments $Yf(Y)$ and $Zf(Z)$ have a common point strictly inside the polygon, for all points $Y, Z \in \mathcal{B}$.

What is the largest real number c such that for any convex polygon $\mathcal{P}$ and European function f, there is a point $W \in \mathcal{B}$ such that the length of line segment $Wf(W)$ is at least c times the perimeter of $\mathcal{P}$?

Solution 1. We will prove that $c = \frac{1}{\pi+2}$. Suppose we have a convex polygon $OA_1A_2\dots A_n$ such that $OA_i = 1$ for all $1 \le i \le n$. We can take points M and N on segments OA_n and OA_1 such that $OM = ON = \varepsilon$ with $0 < \varepsilon < 1$, and let L be the midpoint of segment MN. We can define a *good* function f in the following way:

For a point $A \in \mathcal{B}$ define $f(A)$ as the second intersection of AL with $\mathcal{B}$. It's obvious that this is a *good* function. Consider points $B, C \in \mathcal{B}$ such that $f(B) = C$. We will prove that $|BC| < 1 + \varepsilon$. If $\{B, C\} = \{M, N\}$, $|MN| < |OM| + |ON| = 2\varepsilon < 1 + \varepsilon$. Else, exactly one of B and C must lie on the same side of MN as O. We can consider that to be B, so we get $|OB| < \varepsilon$. Since $\mathcal{B}$ is contained in the circle with center O and radius 1, $|OC| \le 1$, so we are done, as $|BC| \le |OB| + |OC|$.

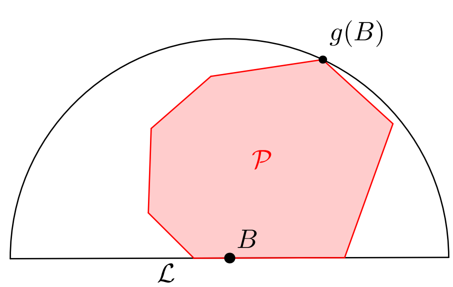
Now, as a semicircle of radius 1 has the perimeter $\pi + 2$, and we can approximate it as well as possible with the polygon $OA_1A_2\dots A_n$, we can get the perimeter as close to $\pi + 2$ as we want. Since the perimeter approaching $\pi + 2$ is independent of ε, if a constant c works, then $1 + \varepsilon \ge c(\pi + 2)$ for any $0 < \varepsilon < 1$. This implies that any constant c that works is at most $\frac{1}{\pi+2}$.



We define a function $g : \mathcal{B} \to \mathcal{B}$ mapping each boundary point to the farthest boundary point (arbitrarily chosen if multiple exist). Notably, g maps exclusively to the polygon's vertices. Otherwise we would have four distinct points A, B, C, D, such that D is on the segment BC and $g(A) = D$. But WLOG we can assume that $\angle ADB \ge 90^\circ$, which would imply $|AB| > |AD|$, contradiction.

Lemma 1. Let p denote the perimeter of $\mathcal{P}$. Then for every point $B \in \mathcal{B}$, holds $|Bg(B)| \ge \frac{1}{\pi+2}p$.

Proof. Let $\mathcal{L}$ be a line through B that does not intersect the interior of $\mathcal{P}$, so $\mathcal{P}$ lies in a half-plane bounded by $\mathcal{L}$. Since $\mathcal{P}$ is also contained in a circle of radius $|Bg(B)|$ centered at B, it lies within one of the semicircles determined by $\mathcal{L}$. The perimeter of a convex polygon contained in a convex shape is at most that of the latter. Hence, the result follows, as the semicircle's perimeter is $(\pi + 2)|Bg(B)|$. $\square$



Let X and Y be 2 points on the boundary that maximize $|XY|$. If $f(X) = Y$, we are trivially done, so assume this is not the case. Let $\overline{X}$ denote the points of $\mathcal{B}$ in the path between X and $f(X)$, which does not include Y. We also assume that $f(X) \in \overline{X}$, but $X \notin \overline{X}$. Denote $\overline{Y} = \mathcal{B}/\overline{X}$.

Let $v \in \overline{Y}$ be a vertex. We claim that the map $v \mapsto g(f(v))$ sends v to another vertex in $\overline{Y}$. Since $vf(v)$ intersects $Xf(X)$, it follows that $f(v) \in \overline{X}$. Assume $g(f(v)) \in \overline{X}$. Then $f(v)g(f(v))$ and XY do not intersect. If $X, Y, f(v), g(f(v))$ form a convex quadrilateral, the triangle inequality leads to a contradiction:

$$|Xf(v)| + |Yg(f(v))| > |XY| + |f(v)g(f(v))|.$$

This is impossible since $|f(v)g(f(v))| \geq |Xf(v)|$ and $|XY| \geq |Yg(f(v))|$. By the same argument, in the other case we would get

$$|Xg(f(v))| + |f(v)Y| > |XY| + |f(v)g(f(v))|$$

however this can't be true since $|XY| \geq |Xg(f(v))|$ and $|f(v)g(f(v))| \geq |f(v)Y|$. Thus $g(f(v)) \in \overline{Y}$.

Now if we consider a graph on the vertices of $\overline{Y}$, where $v \to g(f(v))$, it must contain a cycle. So, we have distinct points $A_1, A_2 \dots A_k$ in $\overline{Y}$ and points $B_1, B_2 \dots B_k$ in $\overline{X}$, such that $A_{i+1} = g(B_i)$ and $B_i = f(A_i)$, where $A_{k+1} = A_1$. The B_is are distinct because f is injective and A_is are distinct.

Between all matchings between sets $\{A_i\}$ and $\{B_i\}$, the one which maximizes the sum of lengths of segments between paired points, must satisfy that any two segments intersect. Otherwise there would exist segments $A_{i_1}B_{j_1}$ and $A_{i_2}B_{j_2}$ that don't intersect. But then $A_{i_1}B_{j_2}$ and $A_{i_2}B_{j_1}$ would intersect at point D. Then by triangle inequality:

$$|A_{i_1}B_{j_2}| + |A_{i_2}B_{j_1}| = (|A_{i_1}D| + |DB_{j_1}|) + (|A_{i_2}D| + |DB_{j_2}|) > |A_{i_1}B_{i_1}| + |A_{i_2}B_{i_2}|$$

So replacing these this pair of segments by $A_{i_1}B_{j_2}$ and $A_{i_2}B_{j_1}$ would increase the sum of lengths.

Now, since $\{A_i\}$ and $\{B_i\}$ are $2k$ distinct points on the boundary of a convex polygon, there exists exactly one matching where all segments intersect, specifically the one where each segment splits the remaining vertices into 2 sets of equal size. So the maximal matching is unique.

Since $A_iB_i = A_if(A_i)$ satisfies that every two segments intersect, it must be the maximal matching. Hence:

$$\max_i(|A_iB_i|) \geq \frac{1}{k}\sum_{i=1}^{k}|A_iB_i| \geq \frac{1}{k}\sum_{i=1}^{k}|A_{i+1}B_i| \geq \min_i(|A_{i+1}B_i|).$$

But then

$$\max_i(|A_if(A_i)|) \geq \min_i(|B_ig(B_i)|) \geq \frac{1}{\pi+2}p$$

as needed.

Solution 2. If $f(a) = a$ for some a, then the second condition yields $f(x_0) = a$ for all x. Otherwise, f is bijective by the first condition, and now the second condition yields easily that f is continuous. We present a different proof of the lower bound $c \geq \frac{1}{\pi+2}p$. We also define the function $g : \lfloor \to \mathcal{B}$ mapping each point to the point of $\mathcal{B}$ farthest from it, resolving the ambiguities arbitrarily. We notice that $g(x)$ is always a vertex. Also, we implement Lemma 1 from the first solution. We intend to prove that $xf(x) = xg(x)$ for some $x \in \mathcal{B}$, arguing indirectly.

Notice that the ambiguities in the definition of g may appear only at a finite number of points on the boundary: those are intersections of $\mathcal{B}$ with the perpendicular bisectors of the segments joining the vertices. Call a point where g could be defined in more than one way ambiguous.

Now choose two vertices a and b with maximal distance; we may (and will) assume that $b = g(a0$ and $a = g(b)$. Without loss of generality, assume that the points $a, f(a)$, and b follow each other in this cyclic order while traversing $\mathcal{B}$ clockwise. We show that for every x, the points $x, f(x)$, and $g(x)$ also follow each other in this cyclic order; this will deliver a desired contradiction, as this is not true for $x = b$, where those points are b, a, and some point on the arc $af(a)$ not containing b (as $af(a)$ and $bf(b)$ intersect).

Let now x move along $\mathcal{B}$ clockwise, starting from a. We show that the stated condition holds. While $g(x)$ remains constant, this holds by continuity. Now let us check what happens when x goes through an ambiguous point v; let u and w be two points on $\mathcal{B}$ sufficiently close to v, u before v and w after v (while moving clockwise). Then, since u and w are close to each other, $g(w)$ cannot lie on the clockwise are uw;

it also cannot lie on $wg(u)$, as otherwise $ug(u) + wg(w) < ug(w) + wg(u)$, which contradicts the definition of g. Hence $g(w)$ lies on the clockwise arc $g(u)u$, while $f(v)$ still lies on the arc $ug(u)$, by continuity. This yields that our claim still holds for $x = w$, as desired.

Remark: In general, it is not true that any two segments $ug(u)$ and $wg(w)$ intersect; but this holds for sufficiently close u and w, and we use this statement in such (correct) form.