ALGEBRAIC PROOF

GENMATHS RESOURCES

Task 1

1) Prove that for any positive integer n, $3n + 18 + n + 6$ is always a multiple of 4.

Proof

$3n + 18 + n + 6$

$= 4n + 24$

$= 4(n + 6)$

$\therefore$ Always a multiple of 4

2) Prove that for any positive integer n, $5n^2 + 16 + n^2 - 4 + 6n$ is always a multiple of 6.

Proof

$5n^2 + 16 + n^2 - 4 + 6n$

$= 6n^2 + 6n + 12$

$= 6(n^2 + n + 2)$

$\therefore$ Always a multiple of 6

3) Prove that for any positive integer n, $16n + 5 + 4n + 25$ is always a multiple of 10.

Proof

$16n + 5 + 4n + 25$

$= 20n + 30$

$= 10(2n + 3)$

$\therefore$ Always a multiple of 10

4) Prove that for any integer n, $9n + 19 + 12n + 3$ is always one more than a multiple of 3.

Proof

$9n + 19 + 12n + 3$

$= 21n + 22$

$= 21n + 21 + 1$

$= 3(7n + 7) + 1$

$\therefore$ Always one more than a multiple of 3

5) Prove that for any positive integer n, $3(6n + 7) + 2(3n - 1)$ is always one more than a multiple of 6

Proof

$3(6n + 7) + 2(3n - 1)$

$= 18n + 21 + 6n - 2$

$= 24n + 19$

$= 24n + 18 + 1$

$= 6(4n + 3) + 1$

$\therefore$ Always one more than a multiple of 6

Task 2

6) Prove that the sum of two even integers is always even.

Proof

Let $2m$ and $2n$ be even integers

$2m + 2n$

$= 2(m + n)$

$\therefore$ Always even

7) Prove that the product of two odd integers is always odd.

Proof

Let $2m + 1$ and $2n + 1$ be odd integers

$(2m + 1)(2n + 1)$

$= 4mn + 2m + 2n + 1$

$= 2(2mn + m + n) + 1$

$\therefore$ Always odd

8) Prove that $4n^2 + 4n + 1$ is always a square number, for all positive integer values of n.

Proof

$4n^2 + 4n + 1$

$= (2n + 1)(2n + 1)$

$= (2n + 1)^2$

$\therefore$ Always a square number

9) Prove that $(2n+1)^2 - (2n-1)^2$, is always a multiple of 8, for all positive integer values of n.

Proof

$(2n+1)(2n+1) - (2n-1)(2n-1)$

$= 4n^2 + 2n + 2n + 1 - (4n^2 - 2n - 2n + 1)$

$= 4n^2 + 4n + 1 - (4n^2 - 4n + 1)$

$= 8n$

∴ Always a multiple of 8

10) Prove that $(3n+4)^2 - (3n-4)^2$ is always a multiple of 16, for all positive integer values of n.

Proof

$= (3n+4)(3n+4) - (3n-4)(3n-4)$

$= 9n^2 + 12n + 12n + 16 - (9n^2 - 12n - 12n + 16)$

$= 9n^2 + 24n + 16 - (9n^2 - 24n + 16)$

$= 48n$

$= 16(3n)$

∴ Always a multiple of 16

11) Prove that the sum of two consecutive integers is always odd.

Proof

Let n and $n+1$ be consecutive integers

$n + n + 1$

$= 2n + 1$

∴ Always odd

12) Prove that the sum of three consecutive integers is always a multiple of 3.

Proof

Let n, $n+1$ and $n+2$ be consecutive integers

$n + n + 1 + n + 2$

$= 3n + 3$

$= 3(n+1)$

∴ Always a multiple of 3

13) Prove that the difference between two odd integers is always even.

Proof

Let $2n+1$ and $2m+1$ be odd integers

$= 2n + 1 - (2m + 1)$

$= 2n - 2m$

$= 2(n - m)$

∴ Always even

14) Prove that the sum of two consecutive even integers is always even.

Proof

Let $2n$ and $2n+2$ be consecutive even integers

$2n + 2n + 2$

$= 4n + 2$

$= 2(2n + 1)$

∴ Always even

15) Prove that the difference between two consecutive square numbers is always odd.

Proof

Let n^2 and $(n+1)^2$ be consecutive square numbers

$(n+1)^2 - n^2$

$= (n+1)(n+1) - n^2$

$= n^2 + n + n + 1 - n^2$

$= 2n + 1$

∴ Always odd

16) Prove that the product of two consecutive integers is always even.

Proof

Let n and $n+1$ be consecutive integers

$n(n+1)$

One of these numbers must be even.

Even × odd = even

Odd × even = even

∴ Always even

17) Prove that the square of any multiple of 3 is also a multiple of 9.

Proof

A multiple of 3 can be written as $3n$

$(3n)^2$

$= 9n^2$

$\therefore$ Also a multiple of 9

18) Prove that the square of any odd integer is also odd.

Proof

Let $2n + 1$ be an odd integer

$(2n + 1)^2$

$= (2n + 1)(2n + 1)$

$= 4n^2 + 2n + 2n + 1$

$= 4n^2 + 4n + 1$

$= 2(2n^2 + 2n) + 1$

$\therefore$ Also odd

19) Prove that the square of any even integer is a multiple of 4.

Proof

Let $2n$ be an even integer

$(2n)^2$

$= 4n^2$

$\therefore$ A multiple of 4

20) Show that the difference between the squares of two consecutive integers equals the sum of those integers.

Proof

Let n and $n + 1$ be consecutive integers

Difference between the squares:

$(n + 1)^2 - n^2$

$= (n + 1)(n + 1) - n^2$

$= n^2 + n + n + 1 - n^2$

$= 2n + 1$

Sum:

$n + n + 1 = 2n + 1$

$\therefore$ The difference between the squares of two consecutive integers equals the sum of those integers.

21) Prove that the sum of the squares of two consecutive integers is always odd.

Proof

Let n and $n + 1$ be consecutive integers

$(n + 1)^2 + n^2$

$= (n + 1)(n + 1) + n^2$

$= n^2 + n + n + 1 + n^2$

$= 2n^2 + 2n + 1$

$= 2(n^2 + n) + 1$

$\therefore$ The sum of the squares of two consecutive integers is always odd.

22) Prove that $2(x^2 + y^2 + xy) - (x^2 + y^2)$, is a square number for all positive integer values of n.

Proof

$2(x^2 + y^2 + xy) - (x^2 + y^2)$

$2x^2 + 2y^2 + 2xy - x^2 - y^2$

$= x^2 + 2xy + y^2$

$= (x + y)(x + y)$

$= (x + y)^2$

$\therefore$ A square number

Challenge

23) Prove that the difference between the cubes of two consecutive integers can be written as $(3n(n + 1) + 1)$

Proof

Let n and $n + 1$ be consecutive integers

$(n + 1)^3 - n^3$

$= (n + 1)(n + 1)(n + 1) - n^3$

$= (n + 1)(n^2 + n + n + 1) - n^3$

$= (n + 1)(n^2 + 2n + 1) - n^3$

$= n^3 + 2n^2 + n + n^2 + 2n + 1 - n^3$

$= 3n^2 + 3n + 1$

$= 3n(n + 1) + 1$ ✓

24) Prove that $n^3 - n$ is always a multiple of 6, for any positive integer value of n.

Proof

$n^3 - n$

$= n(n^2 - 1)$

$= n(n+1)(n-1)$

The numbers are three consecutive integers. Therefore, one must be a multiple of 3 and there is at least one even number (if a number is divisible by 2 and 3, then it is also divisible by 6).

∴ Always a multiple of 6

25) Prove that the product of two consecutive even integers is a multiple of 8.

Proof

Let $2n$ and $2n+2$ be consecutive even integers

$2n(2n+2)$

$= 4n^2 + 4n$

$= 4n(n+1)$*

$= 4(2k)$

$= 8k$

*n and $n+1$ are consecutive integers, and therefore their product is even and can be written as $2k$

∴ Always a multiple of 8