Sample Linear Program

Problem Statement: Consider the following situation. A manufacturing company makes two products P_1 and P_2 using machines M_1 and M_2. One unit of P_1 needs 4 hours on M_1 and 2 hours on M_2 whereas one unit of P_2 needs 3 hours on M_1 and 1 hour on M_2. Further, a unit of P_1 yields a revenue of $7 and a unit of P_2 yields a revenue of $5. Given that the total available time on machines M_1 and M_2 is 240 hours and 100 hours, how many products of each type should the company make so as to maximize its total revenue.

Solution: We can model this problem as a linear program as follows. We first define the decision variables. Let x_1 be the no. of manufactured units of product P_1 and x_2 be the no. of manufactured units of product P_2.

Since each unit of P_1 yields a revenue of $7, x_1 units of P_1 will yield a revenue of $7*x_1$. Similarly, x_2 units of P_2 will yield a revenue of $5*x_2$. The total revenue is therefore given by "$7*x_1 + 5*x_2$", which becomes our objective function.

Since we are making x_1 units of P_1, we will need $4*x_1$ hours on M_1 and $2*x_1$ hours on M_2. Similarly, x_2 units of P_2 will need $3*x_2$ hours on M_1 and $1*x_2$ hours on M_2. Therefore, the total number of hours required for M_1 will be $4*x_1 + 3*x_2$, and the total number of hours required for M_2 will be $2*x_1 + 1*x_2$.

Since the total available time for M_1 and M_2 is 240 hours and 100 hours, we need to ensure that "$4*x_1 + 3*x_2 \leq 240$" and "$2*x_1 + 1*x_2 \leq 100$", which become our resource constraints. Also, the number of products manufactured cannot be negative. So, we need to ensure that "$x_1, x_2 \geq 0$", which become our non-negativity constraints.

Here is the linear programming model for this problem.

$$\text{Maximize } 7*x_1 + 5*x_2 \quad (1)$$

Subject to

$$4*x_1 + 3*x_2 \leq 240 \quad (2)$$

$$2*x_1 + 1*x_2 \leq 100 \quad (3)$$

$$x_1, x_2 \geq 0 \quad (4)$$

The origins of linear programming go back to the period after World War II, when the US Air Force was optimizing its logistical operations. The word programming here refers to a "plan" that could be followed by the military.

Since this is a two-dimensional problem, we can solve it graphically. As shown in Figure 1, every point in the shaded area satisfies all the constraints and corresponds to a feasible solution.

We want to find the point or points for which the value of the objective function is the greatest, which we will call the optimal solution.

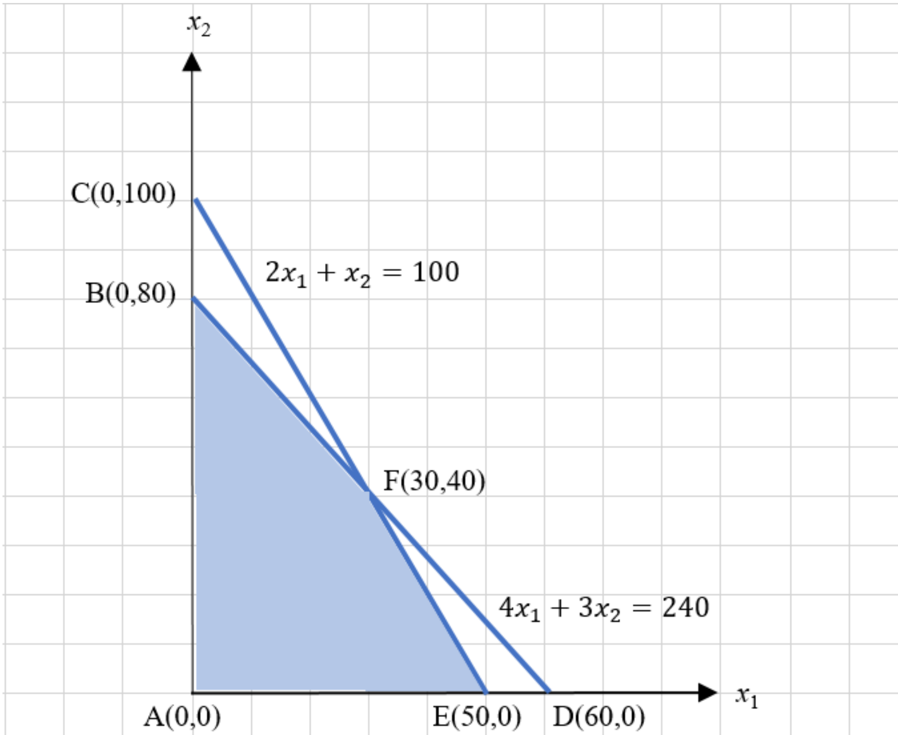

Figure 1: Feasible region for the linear program

To get an intuition, we can start with an initial estimate of the objective function and then modify it based on where it appears in the graph. Let us start by considering an objective value of 140 and plot the line $7*x_1 + 5*x_2 = 140$.

As shown in Figure 2, this line lies entirely within the feasible region and so we can infer that an objective value of 140 is feasible.

Let us now increase the objective value to 350. As shown in Figure 3, the line $7*x_1 + 5*x_2 = 350$ also lies within the feasible region. We also observe that the line with a constant objective function value is moving to the north-east side of the feasible region.

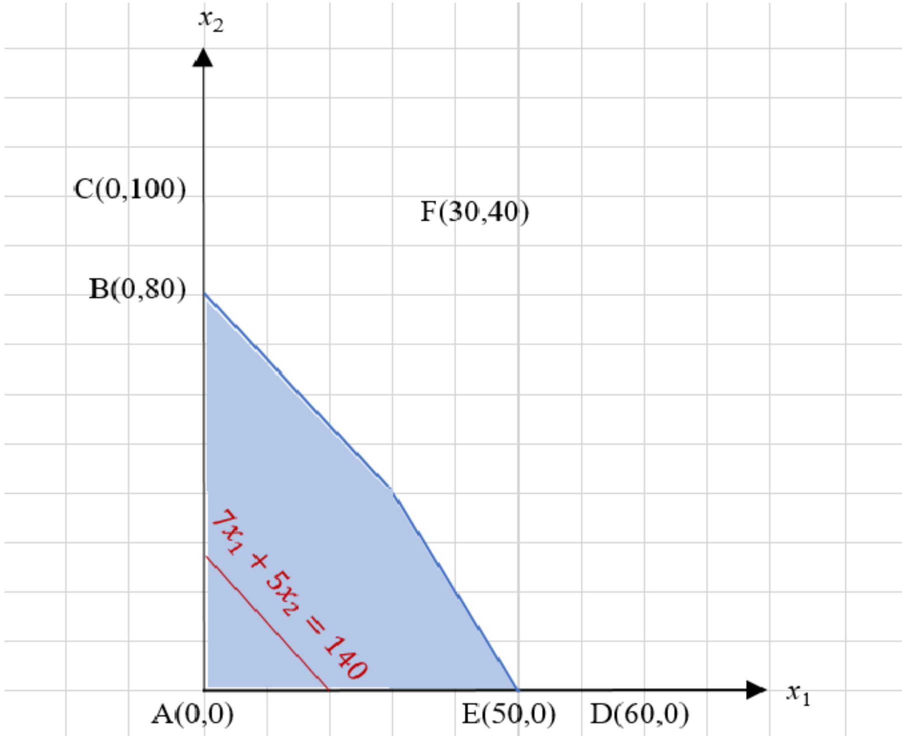

Figure 2: Checking for an objective function value of 140

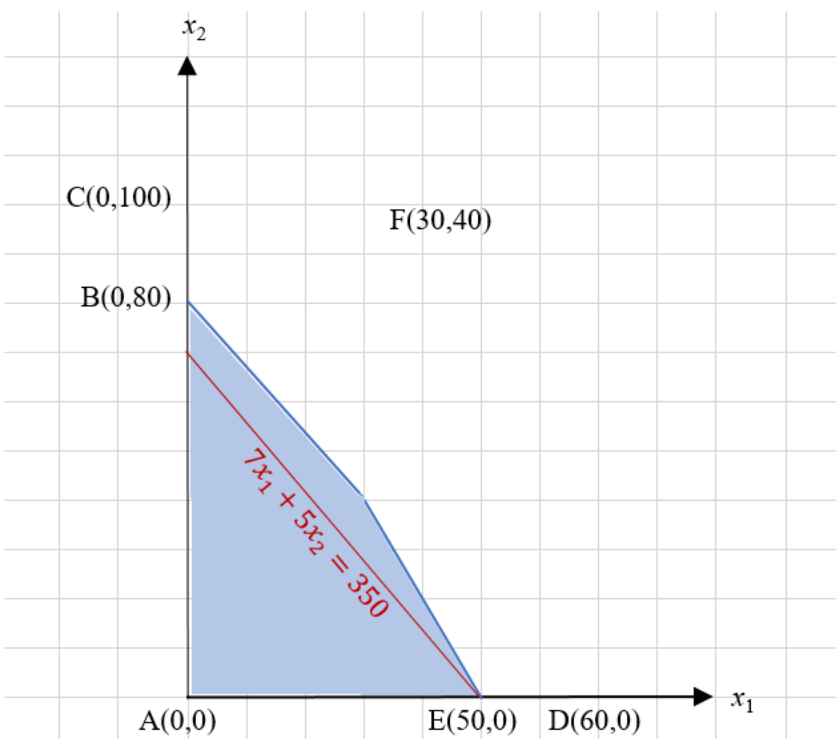

Figure 3: Checking for an objective function value of 350

Since we have some room for further increasing the objective value, we change it to 420 and observe the effect. As shown in Figure 4, the line $7*x_1 + 5*x_2 = 420$ touches a corner point at the boundary of the feasible region. If we were to increase the objective value any further, then the line will go outside the feasible region and therefore will not correspond to a feasible solution. Thus, we can be sure that we have the optimal solution to the linear program.

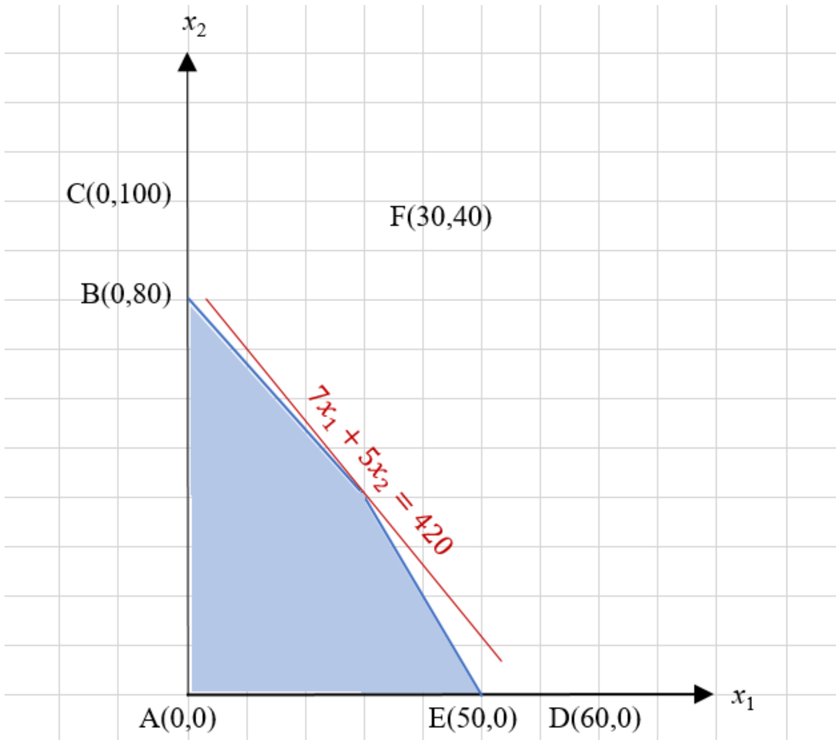

Figure 4: Checking for an objective function value of 420

The Fundamental Theorem of Linear Programming states that the optimal solution to a linear program always occurs at a corner point. Thus, instead of checking the objective value at every point in the feasible region (which contains an infinite number of points), we can simply check the value at the corner points and pick the best. For problems with hundreds of thousands of constraints, the number of corner points can be very large and so it is computationally prohibitive to check the objective value at each corner point. An algorithm called the Simplex Method is used, which typically checks the objective value at only a subset of points before arriving at the optimal solution.