TRIGONOMETRIC RATIO & IDENTITIES

Leonhard Euler

Euler (1707 – 1783) served as a medical lieutenant in the Russian navy from 1727 to 1730. Euler became professor of physics at the Academy in 1730. In 1733 Euler was appointed to senior chair of mathematics. By 1740 Euler had a very high reputation, having won the Grand Prize of the Paris Academy in 1738 and 1740.

He wrote books on the calculus of variations; on the calculation of planetary orbits; on artillery and ballistics; on analysis; on shipbuilding and navigation; on the motion of the moon; lectures on the differential calculus. Euler's work in mathematics is so vast that an article of this nature cannot but give a very superficial account of it. He made large bounds forward in the study of modern analytic geometry and trigonometry where he was the first to consider sin, cos etc. as functions rather than as chords.

He made decisive and formative contributions to geometry, calculus and number theory. We owe to Euler the notation $f(x)$ for a function, e for the base of natural logs, i for the square root of –1, □ for pi, □□for summation, the notation for finite differences □y and □2y and many others.

Perhaps the result that brought Euler the most fame was to find a closed form for the sum of the infinite series □(2) = □(1/n^2) .Euler gave the formula $e^{ix} = \cos x + i \sin x$.

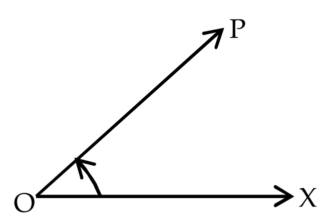
Analytic functions of a complex variable were investigated by Euler in a number of different contexts. Euler made substantial contributions to differential geometry, investigating the theory of surfaces and curvature of surfaces.

Introduction

The word 'Trigonometry" is derived from the Greek words (i) Trigonon means a triangle (ii) metron means a measure. Hence trigonometry means science of measuring triangles.

□ **Angle**

"It may be defined as the amount of revolution undergone by a revolving line in a plane."



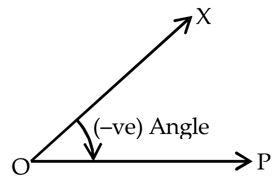
Let a revolving line, starting from its Initial position OX to the terminal position OP, Then $\angle XOP$ is to have been traced out.

Here OX is called Initial side and OP as terminal side, where 'O' is called the vertex.

Rules for Signs of Angles

(i) *The* angle $\angle XOP$ is regarded as positive, If it is traced out in the anticlockwise direction.

(ii) The angle XOP is regarded as negative. If it is traced out in the clockwise direction.



System of Measurement of Angle

There are three system for measurement of angles.

Sexagesimal system

The principal unit in this system is degree (°). One right angle is divided into 90 equal parts and each part is called one degree (1°). One degree is divided into 60 equal parts and each part is called one minute. Minute is denoted by (1'). One minute is equally divided into 60 equal parts and each part is called one second (1").

In Mathematical form

- One right angle
  $= 90°$

$1° = 60'$

$1' = 60''$

Centesignal system

In this system, a right angle is divided into 100 equal parts and each part is called one grades. Each grade is divided into 100 equal parts called minutes. Each minute is further divided into 100 equal parts called seconds.

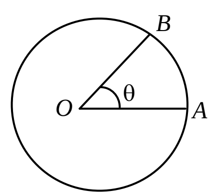
1 right angle = 100^g

$1^g = 100'$

$1' = 100''$

Circular System

(1^c) One radian is the measure of an angle subtended at the centre of a circle by an arc of length equal to the radius of the circle.



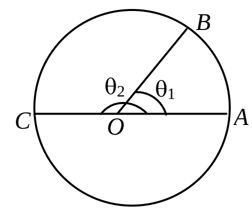
$Q\ \theta$ is $= 1^c$, if $\overset{\frown}{AB} = OA = OB =$ radius

Relation Between Radian and Degree

Consider a circle of radius r having centre at O and arc AC divided in two part $\overset{\frown}{AB}$ and $\overset{\frown}{BC}$.

Since the angles at the centre of a circle are proportional to the arc subtending them.

i.e. $\frac{\theta_1}{\theta_2} = \frac{\overset{\frown}{AB}}{\overset{\frown}{AC}}$



…(i)

Let $\theta_1 = 1^c$, $\overset{\frown}{AB} = r$

$\theta_2 = 180°$, $\overset{\frown}{AC} = \frac{2\pi r}{2} = \pi r$

from (i) $\frac{1^c}{180°} - \frac{r}{\pi r}$ ∴ $1^c = \frac{180°}{\pi} = 57°19'27''$ (approximately)

∴ $\pi^c = 180°$

Note :

The unit radian is denoted by c (circular measure) and it is customary to omit this symbol c. Thus, when an angle is denoted as $\frac{\pi}{2}$, it means that the angle is $\frac{\pi}{2}$ radians where p is the number with approximate value 3.14159.

∴ **Relation Between Three System**

As 1 right angle = $90° = 100^g$ …(i)

and $\pi^c = 180°$ …(ii)

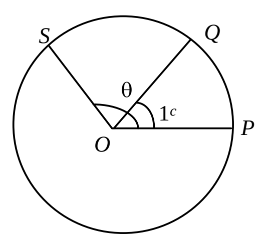
from (i) and (ii)

$180° = 200^g = \pi^c$

∴ $\frac{D}{180°} = \frac{G}{200} = \frac{R}{\pi}$, where D, G and R denotes degree, grades and radians respectively.

∴ **Length of an arc of a circle:**

Consider an arc PS of a circle which subtends an angle q radians.



Choose an arc such that $\overline{PQ} = r$,

∴ $\angle PQR = 1^c$

length of the arc PS = l,

Since the angle at the centre of a circle are proportional to

$\frac{\theta}{1} = \frac{l}{r}$

$\Rightarrow \boxed{\theta = \frac{l}{r}}$

Note: θ is always taken in radian

ILLUSTRATIONS

***Illustration* 1**

Change $\left(\frac{15}{8}\right)^o$ in degree, minute & second.

Solution

$$\left(\frac{15}{8}\right)^o = 1^o + \left(\frac{7}{8}\right)^\circ$$

$$= 1^\circ + \left(\frac{7}{8}\right)^\circ$$

$$= 1^\circ + \left(\frac{105}{2}\right)'$$

$$= 1^\circ + 52' + \left(\frac{1}{2}\right)'$$

$$= 1^\circ + 52' + \left(\frac{1}{2} \times 60\right)''$$

$$= 1^\circ + 52' + 30''$$

$$\Rightarrow \left(\frac{15}{8}\right)^o = 1^\circ\ 52'\ 30'$$

***Illustration* 2**

The minute hand of a clock is 5 cm long. How far does the tip of the hand move in 15 minutes ?

Solution

The minute hand moves through $2\pi \times \frac{15}{60}$ or $\frac{\pi}{2}$ radians. Since the length of the minute hand is 5 cm, the distance moved by the tip of the hand is given by the formula

$$l = r\theta = 5 \times \frac{\pi}{2} = \frac{5\pi}{2} cm..$$

***Illustration* 3**

A rail road curve is to be laid out on a circle. What radius should be used if the track is to change direction by 25° in a distance of 40 metres?

Solution

The angle in radian measure = $\frac{25}{180}\pi = \frac{5}{36}\pi$

If r is the radius of the circle, using □ = l/r, we have

$$r = \frac{l}{\theta} = \frac{40}{5\pi/\ 36} = \frac{288}{\pi} \text{ or } \frac{288 \times 7}{22} = 91.636 \text{ m}$$

PRACTICE EXERCISE

1. The circular measure of an angle of a triangle is $\frac{3\pi}{10}$. The number of grades in the second angle is 70. Find the number of degree in the third angle.

2. The angle's of a triangle are in A.P. and the number of degrees in the least is to the number of radians in the greatest as 60 : π. Find the angle's in degrees and radians.
3. In a circle of diameter 42cm; the length of a chord is 21 cm. Find the length of minor arc formed by chord.
4. A cow is tied to post by a rope. If the cow moves along a circular path always keeping the rope tight and describes 44 metres. When it has traced out 72° at the centre, Find the length of the rope.
5. Kartarpur is 64 km. from Amritsar. Find to the nearest second the angle subtended at the centre of the earth by an arc joining these two towns, earth being regarded as a sphere of 6400 Kms. Radius.
6. If the angular diameter of the moon be (30)'. How far from the eye a coin of 2.2cm. diameter be kept to hide the moon completely.
7. A person of normal eye sight can read point at such a distance that the letters subtend an angle (5)' at the eye. Find the height of the letters (in cm.) he can read at a distance of 420m.

Answers

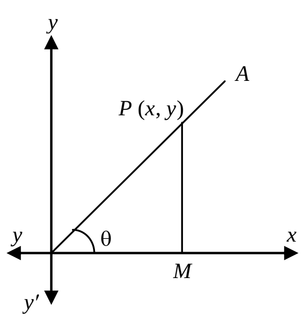
1. (63°) **2.** (30°, 60°, 90° or $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$) **3.** 22 cm

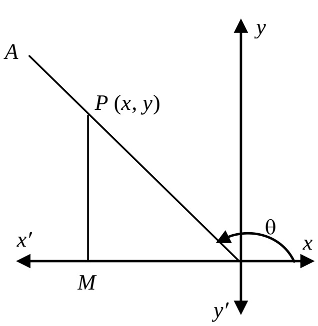
4. 35 cm **5.** 34', 22'' app. **6.**. 114.5 cm

7.. 61.1 cm

4.1 Trigonometric Ratios

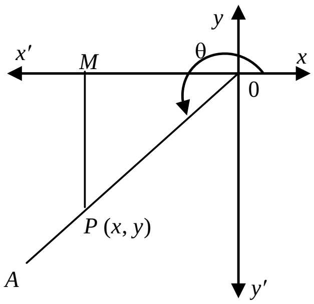
Consider an angle $\theta = \angle XOA$ as shown in figure. *P* be any point other than *O* on its terminal side *OA* and let *PM* be perpendicular from *P* on *x*-axis. Let length *OP* = *r*, *OM* = *x* and *MP* = *y*. We take the length *OP* = *r* always positive while *x* and *y* can be positive or negative depending upon the position of the terminal side *OA* of $\angle XOA$

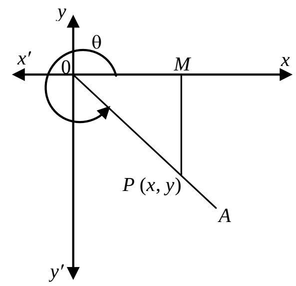




In the right angled triangle *OMP*, we have

Base = *OM* = *x*, perpendicular = *PM* = *y* and, Hypotenuse = *OP* = *r*.





We define the following trigonometric ratios which are also known as trigonometric functions.

$$\sin\theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{y}{r}, \qquad \cos\theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{x}{r}$$

$$\tan\theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{y}{x}, \qquad \text{cosec}\,\theta = \frac{\text{Hypotaneuse}}{\text{Perpendicular}} = \frac{r}{y}$$

$$\sec\theta = \frac{\text{Hypotaneuse}}{\text{Base}} = \frac{r}{x}, \qquad \cot\theta = \frac{\text{Base}}{\text{Perpendicular}} = \frac{x}{y}$$

4.2 Signs of Trigonometric Ratios

In first quadrant: $x > 0, y > 0$

□ sin □ = $\frac{y}{r} > 0$, cosec □ = $\frac{r}{x} > 0$,

cos □ = $\frac{x}{r} > 0$, sec □ = $\frac{r}{x} > 0$,

tan □ = $\frac{y}{x} > 0$, and cot □ = $\frac{x}{y} > 0$

Thus. in the first quadrant all trigonometric functions are positive.

In second quadrant: $x < 0, y > 0$

□ sin □ = $\frac{y}{r} > 0$ cos □ = $\frac{x}{r} < 0$,

tan □ = $\frac{y}{x} < 0$ cosec □ = $\frac{r}{y} > 0$,

sec □ = $\frac{r}{x} < 0$ and, cot □ = $\frac{x}{y} < 0$

Thus, in the second quadrant sine and cosecant functions are positive and all others are negative.

In third quadrant: $x < 0, y < 0$

□ sin □ = $\frac{y}{r} < 0$, cos □ = $\frac{x}{r} < 0$,

tan □ = $\frac{y}{x} > 0$, cosec □ = $\frac{x}{y} < 0$,

sec □ = $\frac{r}{x} < 0$ and cot □ = $\frac{x}{y} > 0$

Thus, in the third quadrant all trigonometric functions are negative except tangent and cotangent. In fourth quadrant: $x > 0$, $y < 0$

∴ $\sin\theta = \frac{y}{r} > 0$, $\cos\theta = \frac{x}{r} > 0$,

$\tan\theta = \frac{y}{x} > 0$, $\text{cosec}\,\theta = \frac{r}{y} < 0$

$\sec\theta = \frac{r}{x} > 0$ and $\cot\theta = \frac{x}{y} < 0$

Thus, in the fourth quadrant all trigonometric functions are negative except cosine and secant.

□ Range of Trigonometric Ratios

The range of the trigonometric ratios in the four quadrants are depicted in the following table.

In the second quadrant	**In the first quadrant**
sine decreases from 1 to 0	sine decreases from 0 to 0
cosine decreases from 0 to –1	cosine decreases from 1 to 0
tangent increases from –∞ to 0	tangent increases from 0 to ∞
cotangent decreases from to 0 to –∞	cotangent decreases from to ∞ to 0
secant increases from – ∞ to –1	secant increases from 1 to ∞
cosecant increases from 1 to ∞	cosecant increases from ∞ to 1
In the third quadrant	**In the fourth quadrant**
sine decreases from 0 to –1	sine increases from –1 to 0
cosine decreases from –1 to 0	cosine increases from 0 to 1
tangent increases from 0 to ∞	tangent increases from – ∞ to 0
cotangent decreases from to ∞ to 0	cotangent decreases from to 0 to – ∞
secant increases from –1 to – ∞	secant decreases from ∞ to 1
cosecant increases from –∞ to –1	cosecant decreases from –1 to – ∞

4.3 Trigonometric Ratios of Some Standard Acute Angle

Trigonometric Ratios of Standard Angles					
Angle (θ) → T-Ratio ↓	0°	30°	45°	60°	90°
sin θ	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1
cos θ	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0
tan θ	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞
cot θ	∞	$\sqrt{3}$	1	$1/\sqrt{3}$	0
sec θ	1	$2/\sqrt{3}$	$\sqrt{2}$	2	∞
cosec θ	∞	2	$\sqrt{2}$	$2/\sqrt{3}$	1

4.4 Trigonometric Ratios of Allied Angles

Two angles are said to be allied when their sum or difference is either zero or a multiple of 90°.

The angle –□, 90° ± □, 180° ± □, 360° ± □ etc are angles allied to the angle □ is measured in degrees. However, if □ is measured in radians, then the angles allied to □ are – □, $\frac{\pi}{2}$ ± □, □ + □, 2□ ± □ etc.

Trigonometric Ratios of Allied Angle are Depicted in the Following Table

	– □	90 °– □	90° + □	180° – □	180° + □	270° – □	270° + □	360° – □	360° + □
sin □	– sin □	cos □	cos □	sin □	– sin □	– cos □	– cos □	– sin □	sin □
cos □	cos □	sin □	– sin □	– cos □	– cos □	– sin □	sin □	cos □	cos □
tan □	– tan □	cot □	– cot □	– tan □	tan □	cot □	– cot □	– tan □	tan □
cot □	– cot □	tan □	– tan □	– cot □	cot □	tan □	– tan □	– cot □	cot □
sec □	sec □	cosec □	– cosec □	– sec □	– sec □	– cosec □	cosec □	sec □	sec □
cosec □	– cosec □	sec □	sec □	cosec □	– cosec □	– sec □	– sec □	– cosec □	cosec □

□ **Some Important Fact**

(i) sin □, cos □, tan □ are the reciprocal of cosec , sec □ and cot □.

(ii) sin²□ + cos²□ = 1, 1 + tan²□ = sec²□, 1 + cot²□ = cosec² □

(iii) sec □ – tan □ = $\frac{1}{\sec\theta + \tan\theta}$ and cosec □ – cot □ = $\frac{1}{\cos ec\theta + \cot\theta}$

(iv) |sin □| □ 1 □ –1 □ sin □ □ 1

(v) |cos □| □ 1 □ – 1 □ cos □ □ 1

(vi) |sec □| □ 1 □ sec □ □ – 1 or sec □ □ 1

(vii) |cosec □| □ 1 □ cosec □ □ – 1 or cosec □ □ 1

(viii) tan □ □ *R* or – □ < tan □ < + □

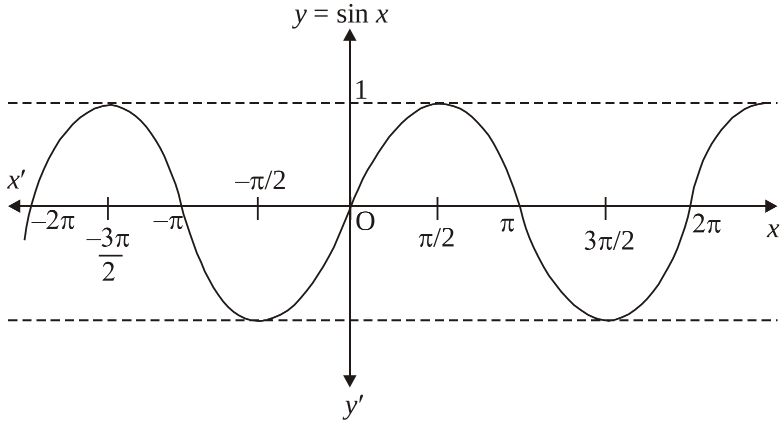
(ix) cot □ □ *R* or – □ < cot □ < + □ ‘

4.5 Graphs of Different Trigonometric Ratios

From the knowledge of trigonometry we compute the following table.

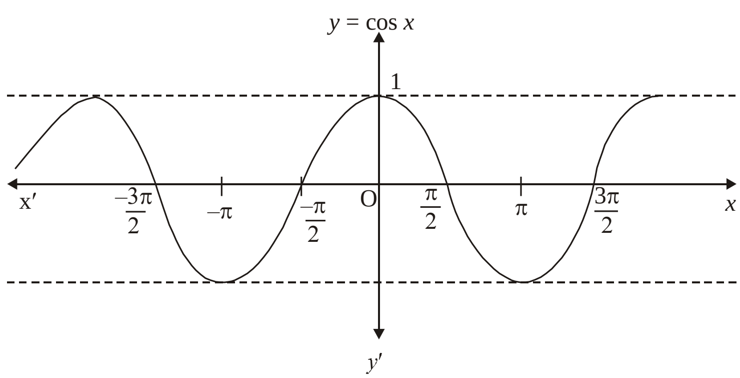
$y = \sin x$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	□
$\sin x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0



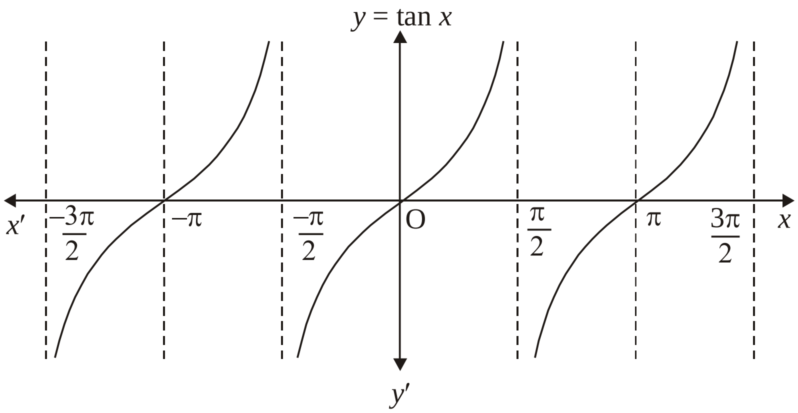
$y = \cos x$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	□
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	–1



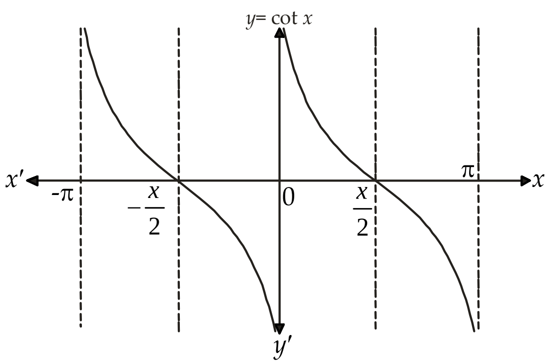
$y = \tan x$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	□
$\tan x$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	□	$-\sqrt{3}$	–1	$-\frac{1}{\sqrt{3}}$	0



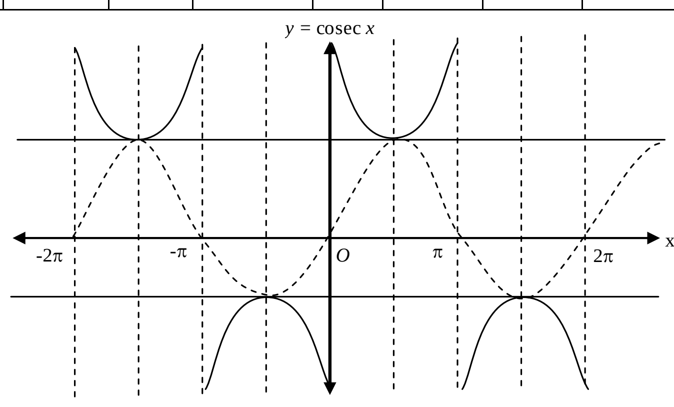
y = cot x

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	□
cot x	□	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	–1	$-\sqrt{3}$	– □



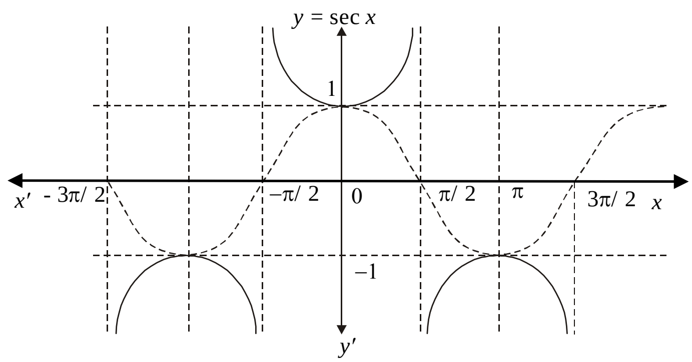
y = cosec x

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	□
cosec x	□	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	□



y = sec x

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	□
sec x	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	□	–2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	–1



ILLUSTRATIONS

***Illustration* 4**

Evaluate : sin (1560°)

Solution

Dividing 1560° = 17 × 90° +30°

Since, quotient is 17 (i.e. odd), so sine changes its co-function (i.e. co-sine) and 1560° lies in second quadrant, we know. In IInd quadrant sine is positive.

Hence sin 1560° = cos 30° = $\frac{\sqrt{3}}{2}$

***Illustration* 5**

Evaluate : cos (–2910°)

Solution

= cos (2910°) (Q cos (–q) = cos q)

= cos (32 × 90° + 30°)

= cos 30° [Q 32 is even & 2910° lies in first quadrant]

= $\sqrt{3/2}$

***Illustration* 6**

Prove that, $\cos 510° \cos 330° + \sin 390° \cos 120° = -1$

Solution

L.H.S. = $\cos 510° \cos 330° + \sin 390° \cos 120°$

= $\cos(5 \times 90° + 60°) \cos(360° - 30°) + \sin(360+30) \cos(90+30)$

$= -\left[\cos^2 30° + \sin^2 30°\right]$

= -1 = R.H.S

PRACTICE EXERCISE

8. If $\tan^2\theta = 1 - e^2$, prove that $\sec\theta + \tan^3\theta\,\text{cosec}\,\theta = (2 - e^2)^{\frac{3}{2}}$

9. If $m = \tan\theta + \sin\theta$ and $n = \tan\theta - \sin\theta$ prove that $m^2 - n^2 = 4\sqrt{mn}$

10. If $\sin^4 A + \sin^2(A) = 1$, prove that

(i) $\frac{1}{\tan^4 A} + \frac{1}{\tan^2 A} = 1$ (ii) $\tan^4 A - \tan^2 A = 1$

11. If sin □ + sin² □ + sin³ □ = 1 then prove that cos⁶ □ – 4 cos⁴ □ + 8 cos ² □ = 4

4.6 Compound Angles

□ **Sum Or Difference Of The Angle**

The algebraic sums of two or more angles are generally called compound angles and the angles are known as the constituent angles.

e.g. If *A*, *B*, *C* are three angles then A + B , A + B + C , are compound angles.

□ **Sum and Difference of Cosine of Two Angles**

(i) cos (*A* – *B*) = cos *A* cos *B* + sin *A* sin *B*

(ii) cos (*A* + *B*) = cos *A* cos *B* – sin *A* sin *B*

for all angles *A* and *B*.

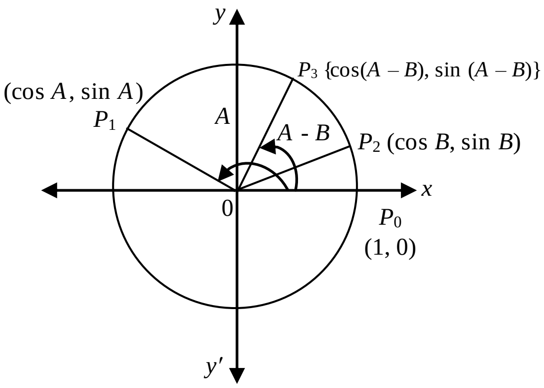
Let *X* □ *OX* and *YOY* □ be the coordinate axes. Consider a unit circle with *O* as the centre.

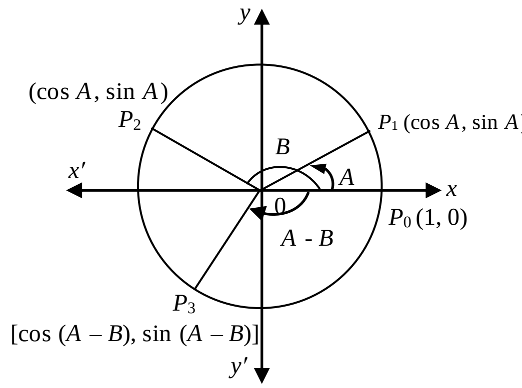
Let P_1, P_2 and P_3 be three points on the circles such that □$XOP_1 = A$, □$XOP_2 = B$ and □$XOP_3 = A - B$.

The terminal side of any angle intersects the circle with centre at *O* and unit radius at a point whose coordinates are respectively the cosine and sine of the angle. Therefore, coordinates of P_1, P_2 and P_3 are (cos *A*, sin *A*), (cos *B*, sin *B*) and (cos (*A* – *B*), sin (*A* – *B*)) respectively.

We know that equal chords of a circle make equal angles at its centre. Since chords P_0P_3 and P_1P_2 subtend equal angles at *O*. Therefore,

Chord P_0P_3 = Chord P_1P_2





□ $\sqrt{\{\cos(A-B)-1\}^2+\{\sin(A-B)-0\}^2} = \sqrt{(\cos B-\cos A)^2+(\sin B-\sin A)^2}$

□ $\{\cos (A - B) - 1]^2 + \sin^2 (A - B) = (\cos B - \cos A)^2 + (\sin B - \sin A)^2$

□ $\cos^2 (A - B) - 2 \cos (A - B) + 1 + \sin^2 (A - B) = \cos^2 B + \cos^2 A - 2 \cos A \cos B$
$+ \sin^2 B + \sin^2 A - 2 \sin A \sin B$

□ $2 - 2 \cos (A - B) = 2 - 2 \cos A \cos B - 2 \sin A \sin B$

□ $\cos (A - B) = \cos A \cos B + \sin A \sin B$

Hence $\cos (A - B) = \cos A \cos B + \sin A \sin B$ …(i)

We have

$\cos (A + B)$

$= \cos (A - (- B))$

$= \cos A \cos (- B) + \sin A \sin (- B)$ [Using (i)]

$= \cos A \cos B - \sin A \sin B$

Hence, $\cos (A + B) = \cos A \cos B - \sin A \sin B$

□ **Sum and Difference of Sine of Two Angle**

Prove that

(i) $\sin (A - B) = \sin A \cos B - \cos A \sin B$

(ii) $\sin (A + B) = \sin A \cos B + \cos A \sin B$

for all values of A and B

(i) We have

$\sin (A - B)$

$= \cos (90° - (A - B))$ [Q cos (90° – □) = sin □]

$= \cos ((90° - A) + B)$

$= \cos (90° - A) \cos B - \sin (90° - A) \sin B$

$= \sin A \cos B - \cos A \sin B$

(ii) $\sin (A + B)$

$= \sin (A - (- B))$

= $\sin A \cos(-B)$ $\cos A \sin(-B)$ [Using (i)]

= $\sin A \cos B + \cos A \sin B$

 Tangent of the Difference and Sum of Two Angles

Prove that

(i) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(ii) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(i) We have

$\tan(A + B)$

$= \dfrac{\sin(A+B)}{\cos(A+B)}$

$= \dfrac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$

$= \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ [On dividing the numerator and denominator by]

(ii) We have,

$\tan(A - B)$

= $\tan A + (-B))$

$= \dfrac{\tan A + \tan(-B)}{1 - \tan A \tan(-B)}$

$= \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

Similarly, it can be proved that

$\cot(A + B) = \dfrac{\cot A \cot B - 1}{\cot B + \cot A}$

and $\cot(A - B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

 Other Useful Results

(i) $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(ii) $\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

(iii) $\sin(A + B + C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$

(iv) $\cos(A + B + C) = \cos A \cos B \cos C - \sin A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C$

(v) $\tan(A+B+C) = \dfrac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A} = \dfrac{s_1-s_3}{1-s_2}$

where S_r = sum of product of tangents taken r at a time

ILLUSTRATIONS

***Illustration* 7**

If $\tan A = \dfrac{1}{3}$ and $\tan B = \dfrac{1}{7}$ the value of 2A + B is

(a) 30° (b) 60° (c) 45° (d) 145°

Solution

$$\tan(2A+B) = \tan(A+A+B)$$

$$= \frac{\tan A+\tan A+\tan B-\tan A\cdot\tan A\tan B}{1-\tan A\cdot\tan A-\tan A\cdot\tan B-\tan B\tan A}$$

$$= \frac{2\tan A+\tan B-\tan^2 A\tan B}{1-\tan^2 A-2\tan A\tan B}$$

$$= \frac{2\times\frac{1}{3}+\frac{1}{7}-\left(\frac{1}{3}\right)^2\times\frac{1}{7}}{1-\left(\frac{1}{3}\right)^2-2\times\frac{1}{3}\times\frac{1}{7}}$$

$$= \frac{42+9-1}{63-7-6} = \frac{50}{50} = 1$$

$$\Rightarrow 2A+B = 45°$$

***Illustration* 8**

Prove that $\tan 13\theta - \tan 9\theta - \tan 4\theta = \tan 13 \tan 9\theta \tan 4\theta$

Solution

In this problem, note that one of three angle 13θ is sum of other two angles 9θ, 4θ

i.e. $13\theta = 9\theta + 4\theta$

$\Rightarrow$ $\tan 13\theta = \tan(9\theta + 4\theta)$

$\Rightarrow$ $\tan 13\theta = \dfrac{\tan 9\theta+\tan 4\theta}{1-\tan 9\theta\tan 4\theta}$

$\Rightarrow$ $\tan 13\theta\,(1-\tan 9\theta\tan 4\theta) = \tan 9\theta + \tan 4\theta$

$\Rightarrow$ $\tan 13\theta - \tan 9\theta - \tan 4\theta = \tan 13 \cdot \tan 9\theta \tan 4\theta$

PRACTICE EXERCISE

12. Prove that tan70° = 2tan50° + tan20°

13. If $\cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos(\alpha-\beta)=-\dfrac{3}{2}$, prove that

$$\cos\alpha+\cos\beta+\cos\gamma=\sin\alpha+\sin\beta+\sin\gamma=0$$

14. If $m\tan(\theta-30°)=n\tan(\theta+120°)$, show that $\cos 2\theta=\dfrac{m+n}{2(m-n)}$

15. Prove thattan 8□ – tan 6□ – tan 2□ = tan 8□ tan 6□ tan 2□

16. If tan (□ cos □) = cot (□ sin □), prove that cos $\left(\theta-\dfrac{\pi}{4}\right)=\dfrac{1}{2\sqrt{2}}$.

17. If □ and □ are the solutions of *a* cos □ + *b* sin □ = *c*, then show that

4.7 Transformation Formulae

□ **Expressing the product of Trigonometric Ratio into Sum or Difference**

(a) $2\sin A\cos B=\sin(A+B)+\sin(A-B)$

(b) $2\cos A\sin B=\sin(A+B)-\sin(A-B)$

(c) $2\cos A\cos B=\cos(A+B)+\cos(A-B)$

(d) $2\sin A\sin B=\cos(A-B)-\cos(A+B)$

ILLUSTRATIONS

***Illustration* 9**

Prove that

$$2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}=0$$

Solution

We have

LHS = $2\cos\dfrac{\pi}{13}\cos\dfrac{9\pi}{13}+\cos\dfrac{3\pi}{13}+\cos\dfrac{5\pi}{13}$

$= \cos\left(\dfrac{9\pi}{13}+\dfrac{\pi}{13}\right)+\cos\left(\dfrac{9\pi}{13}-\dfrac{\pi}{13}\right)+\cos\dfrac{3\pi}{13}+\cos\dfrac{5\pi}{13}$

$= \cos\dfrac{10\pi}{13}+\cos\dfrac{8\pi}{13}+\cos\dfrac{3\pi}{13}+\cos\dfrac{5\pi}{13}$

$= \cos\left(\pi-\dfrac{3\pi}{13}\right)+\cos\left(\pi-\dfrac{5\pi}{13}\right)+\cos\dfrac{3\pi}{13}+\cos\dfrac{5\pi}{13}$

$= -\cos\dfrac{3\pi}{13}-\cos\dfrac{5\pi}{13}+\cos\dfrac{3\pi}{13}+\cos\dfrac{5\pi}{13}$

= 0 = RHS.

□ Expressing the Sum or Difference of Trigonometric Ratios into Product

Let $A + B = C$ and $A - B = D$ then $A = \dfrac{C+D}{2}$ and $B = \dfrac{C-D}{2}$

Substituting in (i)

(a) $\sin C + \sin D = 2\sin\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C+D}{2}\right)$

(b) $\sin C - \sin D = 2\cos\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$

(c) $\cos C - \cos D = 2\sin\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{D-C}{2}\right)$

(d) $\cos C + \cos D = 2\sin\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{D-C}{2}\right)$

ILLUSTRATIONS

***Illustration* 10**

$\dfrac{\sin 3\theta + \sin 5\theta + \sin 7\theta + \sin 9\theta}{\cos 3\theta + \cos 5\theta + \cos 7\theta + \cos 9\theta} =$

(a) tan 6θ (b) tan 3θ (c) cot 2θ (d) cot 6θ

Solution

$\dfrac{\sin 3\theta + \sin 5\theta + \sin 7\theta + \sin 9\theta}{\cos 3\theta + \cos 5\theta + \cos 7\theta + \cos 9\theta}$

$\dfrac{(\sin 3\theta + \sin 7\theta) + (\sin 5\theta + \sin 9\theta)}{(\cos 3\theta + \cos 7\theta) + (\cos 5\theta + \cos 9\theta)}$

$$= \frac{2\sin\left(\frac{3\theta+7\theta}{2}\right)\cos\left(\frac{7\theta-3\theta}{2}\right) + 2\sin\left(\frac{5\theta+9\theta}{2}\right)\cos\left(\frac{9\theta-5\theta}{2}\right)}{2\cos\left(\frac{7\theta+3\theta}{2}\right)\cos\left(\frac{7\theta-3\theta}{2}\right) + 2\cos\left(\frac{9\theta+5\theta}{2}\right)\cos\left(\frac{9\theta-5\theta}{2}\right)}$$

$$= \frac{\sin 5\theta\cos 2\theta + \sin 7\theta\cos 2\theta}{\cos 5\theta\cos 2\theta + \cos 7\theta\cos 2\theta}$$

$$= \frac{\sin 5\theta + \sin 7\theta}{\cos 5\theta + \cos 7\theta} = \frac{2\sin 6\theta\cos\theta}{2\cos 6\theta\cos\theta}$$

$= \tan 6\theta$

PRACTICE EXERCISE

18. If $\sin\alpha-\sin\beta=\frac{1}{3} and \cos\beta-\cos\alpha=\frac{1}{2}$, show that $\cot\frac{\alpha+\beta}{2}=\frac{2}{3}$

19. Prove that $\frac{\sin A+\sin 3A+\sin 5A+\sin 7A}{\cos A+\cos A+\cos 5A+\cos 7A}=\tan 4A$

20. Prove that sin20° sin40° sin60° sin80° = $\frac{3}{16}$

21. Prove that $\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)$

$= 4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}$

22. Prove that $\frac{\cos 2A\cos 3A-\cos 2A\cos 7A+\cos A\cos 10A}{\sin 4A\sin 3A-\sin 2A\sin 5A+\sin 4A\sin 7A}=\cot 6A\cot 5A$

23. If sin $(y+z-x)$, sin $(z+x-y)$, sin $(x+y-z)$ are in A.P. prove that tan x, tan y, tan z are also in A.P.

24. If □, □, □ □ $\left(0,\frac{\pi}{2}\right)$, prove that sin □ + sin □ + sin □ > sin (□ + □ + □)

4.8 Trigonometric Ratio of Multiple and Sub-Multiple Angles

□ **Multiple of 2 A**

(i) $\sin 2A = 2\sin A\cos A$

(ii) $\sin 2A = \frac{2\tan A}{1+\tan^2 A}$

(iii) $\cos 2A = \cos^2 A-\sin^2 A$

(iv) $\cos A = 2\cos^2 A-1$ or $1+\cos 2A=2\cos^2 A$

(v) $\cos 2A = 1-2\sin^2 A$ or $1-\cos 2A=2\sin^2 A$

(vi) $\cos 2A \ \frac{1-\tan^2 A}{1+\tan^2 A}$

(vii) $\tan 2A=\frac{2\tan A}{1-\tan^2 A}$

(viii) $\cot 2A=\frac{\cot^2 A-1}{2\cot A}$

If A be replaced by $\frac{A}{2}$, these formula reduce to sub-multiple angles. Above result can be proved by using the formulae sin $(A+B)$, cos $(A+B)$, tan $(A+B)$ cot $(A+B)$ after replacing B by A.

(i) $\sin A=2\sin\frac{A}{2}\cos\frac{A}{2}=\frac{2\tan A/2}{1+\tan^2 A/2}$

(ii) $\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2} = 2\cos^2 \frac{A}{2} - 1 = 1 - 2\sin^2 \frac{A}{2} = \frac{1 - \tan^2 A/2}{1 + \tan^2 A/2}$

(iii) $\tan A = \frac{2\tan A/2}{1 - \tan^2 A/2}$

□ **Multiple of 3 A**

(i) $\sin 3A = 3\sin A - 4\sin^3 A$

$\sin 3A = \sin(2A + A) = \sin 2A \cos A + \cos 2A \cdot \sin A$

$= 2\sin A\cos^2 A + \sin A(1 - 2\sin^2 A) = 3\sin A - 4\sin^3 A$

(ii) $\cos 3A = 4\cos^3 A - 3\cos A$

$\cos 3A = \cos(2A + A) = \cos 2A \cos A - \sin 2A \sin A$

$= (2\cos^2 A - 1)\cos A - 2\sin^2 A\cos A$

$= (2\cos^2 A - 1)\cos A - 2(1 - \cos^2 A)\cos A$

$= 4\cos^3 A - 3\cos A$

(iii) $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

$$\tan 3A = \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A}$$

$$= \frac{\frac{2\tan A}{1 - \tan^2 A} + \tan A}{1 - \frac{2\tan^2 A}{1 - \tan^2 A}}$$

$$= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

□ **Triple Angle Formulae**

(i) 4 sin □ sin (60° – □) sin (60° + □) = sin 3□

(ii) 4 cos □ cos (60° – □) cos (60° + □) = cos 3□

(iii) tan □ tan (60° – □) tan (60° + □) = tan 3□

(iv) cot □ cot (60° – □) cot (60° + □) = cot 3□

ILLUSTRATIONS

***Illustration* 11**

Prove that: cos 20° cos40° cos60° cos80° = $\frac{1}{16}$

Solution

L.H.S. = cos 20° cos40° cos60° cos80°

$$= \frac{1}{2}\cos 20° \cos(60° - 20°)\cos(60° + 20°)$$

$$= \frac{1}{2}\cos 20° [\cos^2 60° - \sin^2 20°]$$

$$= \frac{1}{2}\cos 20° \left[\frac{1}{4} - \left(1 - \cos^2 20°\right)\right]$$

$$= \frac{1}{2}\cos 20° \left[\cos^2 20° - \frac{3}{4}\right]$$

$$= \frac{1}{8}\left[4\cos^3 20° - 3\cos 20°\right]$$

$$= \frac{1}{8}\cos(3 \times 20°)$$

$$= \frac{1}{8}\cos 60^{\circ}$$

$$= \frac{1}{16} = \textbf{R.H.S.}$$

Some Important Values

$$\sin 18^{\circ} = \frac{(\sqrt{5}-1)}{4} = \cos 72^{\circ}$$

$$\cos 18^{\circ} = \frac{\sqrt{10+2\sqrt{5}}}{4} = \sin 72^{\circ}$$

$$\cos 36^{\circ} = \frac{(\sqrt{5}+1)}{4} = \sin 54^{\circ}$$

$$\sin 36^{\circ} = \frac{\sqrt{10-2\sqrt{5}}}{4} = \cos 54^{\circ}$$

$$\sin 22\frac{1^{\circ}}{2} = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$$

$$\cos 22\frac{1^{\circ}}{2} = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$

$$\tan 22\frac{1^{\circ}}{2} = \sqrt{2}-1$$

$\cot 22\frac{1°}{2} = \sqrt{2}+1$

PRACTICE EXERCISE

25. If $\tan\theta = -\frac{3}{4} \, and \, \frac{\pi}{2} < \theta < \pi$. Find the values of $\sin\theta, \cos\theta \, and \cot\theta$

26. Prove that $\frac{\tan 2^n\theta}{\tan\theta} = (1+\sec 2\theta)(1+\sec 2^2\theta)(1+\sec 2^3\theta)...(1+\sec 2^n\theta)$

27. Show that $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x) = 13$

28. If $\sin\alpha + \sin\beta = a \, and \cos\alpha + \cos\beta = b$, prove that

(i) $\sin(\alpha+\beta) = \frac{2ab}{a^2+b^2}$

(ii) $\cos(\alpha-\beta) = \frac{1}{2}(a^2+b^2-2)$

29. Prove that $\sin\frac{\pi}{14}.\sin\frac{3\pi}{14}\sin\frac{5\pi}{14} = \frac{1}{8}$

30. Prove that $\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{1}{2}$

Answers

25. $\frac{3}{5}, \frac{-4}{5}, -\frac{4}{3}$

□ **Some Other Useful Results**

(i) sin □ + sin (□ + □) + sin (□ + 2□) + … … to *n* terms = $\frac{\sin\left[\alpha + \frac{(n-1)\beta}{2}\right]\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

(ii) cos □ + cos (□ + □) + cos (□ + 2□) + …… + …… to *n*

term = $\frac{\cos\left[\alpha + \frac{(n-1)\beta}{2}\right]\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

(iii) $\cos A \cos 2A \cos 2^3 A$ ______ $\cos 2^{n-1} A = \frac{\sin 2^n A}{2^n \sin A}$

(iv) $\sin\frac{A}{2} + \cos\frac{A}{2} = \pm\sqrt{1+\sin A}$

(v) $\sin\frac{A}{2} - \cos\frac{A}{2} = \pm\sqrt{1-\sin A}$

(vi) $\sin A \pm \cos A = \sqrt{2}\sin\left(\frac{\pi}{2} \pm A\right) = \sqrt{2}\cos\left(A \mp \frac{\pi}{4}\right)$

ILLUSTRATIONS

***Illustration* 12**

Prove that

$$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15} = \frac{1}{128}$$

Solution

We have

$$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}$$

$$= \frac{1}{2}\left\{\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{7\pi}{15}\right\}\left\{\cos\frac{3\pi}{15}\cos\frac{6\pi}{15}\right\}$$

$$= -\frac{1}{2}\left\{\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\right\}\left\{\cos\frac{3\pi}{15}\cos\frac{6\pi}{15}\right\}$$

$$= -\frac{1}{2}\left\{\frac{\sin\frac{2^4\pi}{15}}{2^4\sin\frac{\pi}{15}}\right\}\times\left\{\frac{\sin\left(\frac{2^2\times 3\pi}{15}\right)}{2^2\sin\frac{3\pi}{15}}\right\}$$

$$= -\frac{1}{2}\left\{\frac{\sin\frac{16\pi}{15}}{16\sin\frac{\pi}{15}}\right\}\times\left\{\frac{\sin\frac{4\pi}{5}}{4\sin\frac{\pi}{5}}\right\}$$

$$= -\frac{1}{128}\left\{\frac{-\sin\frac{\pi}{15}}{\sin\frac{\pi}{15}}\right\}\times\left\{\frac{\sin\frac{\pi}{5}}{\sin\frac{\pi}{5}}\right\}$$

$$= \frac{1}{128}.$$

4.9 Conditional Identities

When the angles involved satisfy a given relation, the identity is called conditional identity. To prove these identities we require properties of complementary and supplementary angles.

□ **Some Important Conditional Identities**

If $A + B + C = \pi$, then

(i) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(ii) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

(iii) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(iv) $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

(v) $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C$

(vi) $\sin^2 A + \sin^2 B + \sin^2 C = 2 (1 + \cos A \cos B \cos C)$

(vii) $\sin A + \sin B + \sin C = 4 \cos A/2 \cos B/2 \cos C/2$

(viii) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(ix) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}.\tan\frac{A}{2} = 1$

(x) $\cot A/2 + \cot B/2 + \cot C/2 = \cot A/2 \cot B/2 \cot C/2$

ILLUSTRATIONS

***Illustration* 13**

If A + B + C = 2□, prove that $\cos^2 B + \cos^2 C - \sin^2 A = 2 \cos A \cos B \cos C$

Solution

LHS = $\cos^2 B + \cos^2 C - \sin^2 A$

$= \frac{1}{2}$ [cos 2B + 1 + cos 2C + 1 + cos 2A – 1]

$= \frac{1}{2}$ [cos 2A + cos 2B + cos 2C + 1]

$= \frac{1}{2}$ [2 cos (A + B) cos (A – B) + 2 $\cos^2$ C]

= cos (2p – C) cos (A – B) + $\cos^2$ C

= cos C [Cos (A – B) + cos (2p – (A + B))]

= cos C [cos (A – B) + cos (A + B)]

= 2 cos A cos B cos C

***Illustration* 14**

If A + B + C = 0, show that sin A + sin B + sin C = $-4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Solution

$\sin A + \sin B + \sin C = \sin A + 2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)$

$= 2\sin\frac{A}{2}\cos\frac{A}{2} - 2\sin\frac{A}{2}\cos\frac{B-C}{2} \quad \left(\text{since } \frac{B+C}{2} = -\frac{A}{2}\right)$

$= -2\sin\frac{A}{2}\left[\cos\frac{B-C}{2} - \cos\frac{A}{2}\right]$

$$= -2\sin\frac{A}{2}\left[\cos\frac{B-C}{2}-\cos\frac{B+C}{2}\right]$$

$$= -4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

PRACTICE EXERCISE

31. If A+B+C = $\frac{\pi}{2}$, show that

(i) $\sin^2 A + \sin^2 B + \sin^2 C = 1 - 2\sin A \sin B \sin C$

(ii) $\cos^2 A + \cot^2 B + \cos^2 C = 2 + 2\sin A \sin B \sin C$

32. If A+B+C = $\frac{\pi}{2}$, show that

(i) cot A + cot B + cot C = cot A cot B cot C

(ii) tan A tan B + tan B tan C + tan C tan A = 1

33. If x + y + z = $\frac{\pi}{2}$, prove that

$$\cos(x-y-z)+\cos(y-z-x)+\cos(z-x-y)-4\cos x\cos y\cos z=0$$

34. Show that sin (x-y) + sin(y-z) + sin (z-x) $+4\sin\frac{x-y}{2}\sin\frac{y-z}{2}\sin\frac{z-x}{2}=0$

35. If A+B+C = □, prove that

$$\sin(B+2C)+\sin(C+2A)+\sin(A+2B) = 4\sin\frac{B-C}{2}\sin\frac{C-A}{2}\sin\frac{A-B}{2}$$

4.10 Maximum and minimum values of a cos □ + b sin □:

Consider a point (*a*, *b*) on the cartesian plane, let its distance from origin be *r* and the line joining the point and the origin make an angle a with the positive direction of *x* axis.

then *a* = *r* cos □ and *b* = *r* sin □

Squaring and adding $r=\sqrt{a^2+b^2}$

So, *a* cos □ + *b* sin □ = *r* [cos □ cos □ + sin □ sin □]

= *r* cos (□ – □)

but, – 1 □ cos (□ – □)□□ 1

□ – *r* □ *a* cos □ + *b* sin □ □ *r*

So maximum value is $\sqrt{a^2+b^2}$ and minimum value is $-\sqrt{a^2+b^2}$

ILLUSTRATIONS

***Illustration* 16**

Find the maximum value of $1+\sin\left(\frac{\pi}{4}+\theta\right)+2\cos\left(\frac{\pi}{4}-\theta\right)$ for all real values of □.

Solution

$$= 1+\sin\left(\frac{\pi}{4}+\theta\right)+2\cos\left(\frac{\pi}{4}-\theta\right)$$

$$= 1+\frac{1}{\sqrt{2}}(\sin\theta+\cos\theta)+\sqrt{2}(\cos\theta+\sin\theta)$$

$$= 1+\left(\frac{1}{\sqrt{2}}+\sqrt{2}\right)(\cos\theta+\sin\theta)$$

= maximum is $1+\left(\frac{1}{\sqrt{2}}+\sqrt{2}\right)\sqrt{2} = 4$

PRACTICE EXERCISE

38. Find a and b such that $a \le 3\cos x + 5\sin\left(x-\frac{\pi}{4}\right) \le b \text{ for all } x$

39. Prove that

$(2\sqrt{3}+3)\sin x + 2\sqrt{3}\cos x$ lies between $-(2\sqrt{3}+\sqrt{15})$ and $(2\sqrt{3}+\sqrt{15})$

40. Show that the value of $\frac{\tan x}{\tan 3x}$ wherever defined never lies between $\frac{1}{3}$ and 3.

41. Show that $\frac{\sec^2\theta-\tan\theta}{\sec^2\theta+\tan\theta}$ lies between $\frac{-1}{3}$ and 3

Answers

38. $\sqrt{34-15\sqrt{2}}$ and $\sqrt{34-15\sqrt{2}}$

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MISCELLANEOUS PROBLEMS

OBJECTIVE TYPE QUESTIONS

Example 1

If $\frac{\tan 3\theta}{\tan \theta} = 4, \textit{then} \frac{\sin 3\theta}{\sin \theta}$ equals

(a) $\frac{3}{5}$ (b) $\frac{4}{5}$ (c) $\frac{3}{4}$ (d) none of these.

Solution

Given $\frac{\tan 3\theta}{\tan \theta} = 4$ □ $\frac{3 - \tan^2 \theta}{1 - 3\tan^2 \theta} = 4$

$\Rightarrow \tan^2 \theta = \frac{1}{11}$

$\Rightarrow \sin^2 \theta = \frac{1}{12}$

Now, $\frac{\sin 3\theta}{\sin \theta} = 3 - 4\sin^2 \theta$

$3 - 4 \times \frac{1}{12}$

$= \frac{8}{3}$

□ Ans. (d)

Example 2

For any real □, the maximum value of $\cos^2(\cos\theta) + \sin^2(\sin\theta)$ is

(a) 1 (b) $1 + \sin^2 1$ (c) $1 + \cos^2 1$ (d) does not exist

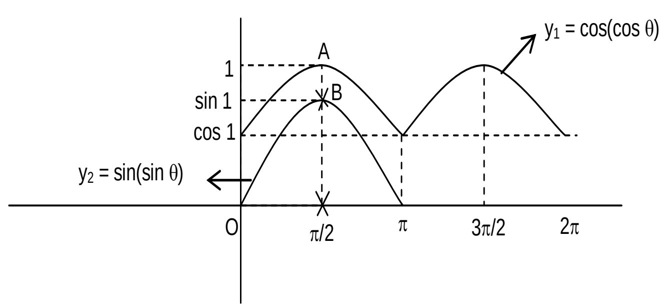
Solution

Draw graph $y = \cos^2(\cos\theta) + \sin^2(\sin\theta)$

let $y_1 = \cos(\cos\theta), y_2 = \sin(\sin\theta)$

so maximum value is $AC + BC = 1 + \sin^2 1$

□ **Ans. (b)**



Example 3

The value of $\sin\frac{\pi}{18}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}$ is

(a) $\frac{1}{8}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $\frac{1}{16}$

Solution

We have $\sin\frac{\pi}{18}\cdot\sin\frac{5\pi}{18}\cdot\sin\frac{7\pi}{18}$

$$=\cos\left(\frac{\pi}{2}-\frac{\pi}{18}\right)\cdot\cos\left(\frac{\pi}{2}-\frac{5\pi}{18}\right)\cdot\cos\left(\frac{\pi}{2}-\frac{7\pi}{18}\right)$$

$$=\cos\frac{4\pi}{9}\cdot\cos\frac{2\pi}{9}\cdot\cos\frac{\pi}{9}$$

$$=\frac{1}{2\sin\frac{\pi}{9}}\cdot\sin\frac{2\pi}{9}\cdot\cos\frac{2\pi}{9}\cdot\cos\frac{4\pi}{9}$$

$$=\frac{1}{2^2\sin\frac{\pi}{9}}\sin\frac{4\pi}{9}\cdot\cos\frac{4\pi}{9}$$

$$=\frac{1}{2^3\sin\frac{\pi}{9}}\cdot\sin\frac{\pi}{9}=\frac{1}{8}$$

□ **Ans. (a)**

Example 4

If $A=\cos^2\theta+\sin^4\theta$, then for all values of □,

(a) 1 □ A □ 2 (b) $\frac{13}{16}\le A\le 1$ (c) $\frac{3}{4}\le A\le\frac{13}{16}$ (d) $\frac{3}{4}\le A\le 1$

Solution

We have $A=\cos^2\theta+\sin^4\theta$

$$=\cos^2\theta+\left(1-\cos^2\theta\right)^2$$

$$=\cos^4\theta-\cos^2+1$$

$$=\left(\cos^2\theta-\frac{1}{2}\right)^2+\frac{3}{4}$$

□ Maximum value of $A=\frac{3}{4}$

Also $\sin^4\theta\le\sin^2\theta$

$\therefore\ A\le\cos^2\theta+\sin^2\theta=1$

□ maximum value of A = 1

$\therefore\ \frac{3}{4} \le A \le 1$

□ **Ans. (d)**

Example 5

If A > 0, B > 0 and $A+B=\frac{\pi}{3}$, then the maximum value of tan$A$ tan$B$ is

(a) $\frac{1}{3}$ (b) 1 (c) $\sqrt{3}$ (d) $\frac{1}{\sqrt{3}}$

Solution

Given $A+B=\frac{\pi}{3} \Rightarrow B=\frac{\pi}{3}-A$

Let k = tan A tan B $=\tan A\cdot\tan\left(\frac{\pi}{3}-A\right) \quad =\tan A\cdot\frac{\sqrt{3}-\tan A}{1+\sqrt{3}\tan A}$

$\Rightarrow \tan^2 A+\sqrt{3}(k-1)\tan+k=0$

since tan A is real,

$$3(k-1)^2-4k\ge 0$$

□ (3k – 1) (k – 3) □ 0

$\Rightarrow k\le\frac{1}{3}$ or k □ 3, But k cannot be greater than 3, since $A+B=\frac{\pi}{3}$.

□ Maximum value of tan A tan B is $\frac{1}{3}$

□ **Ans. (a)**

Example 6

If $\frac{2\sin\alpha}{1+\cos\alpha+\sin\alpha}=y$, then $\frac{1-\cos\alpha+\sin\alpha}{1+\sin\alpha}$ is

(a) $\frac{1}{y}$ (b) y (c) 1 – y (d) 1 + y

Solution

$$\frac{2\sin\alpha}{1+\cos\alpha+\sin\alpha}=y$$

$$\Rightarrow \frac{4\sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}+2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}=y$$

$$\Rightarrow \frac{2\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}=y$$

Now, $\dfrac{1-\cos\alpha+\sin\alpha}{1+\sin\alpha}=\dfrac{2\sin^2\dfrac{\alpha}{2}+2\sin\dfrac{\alpha}{2}\cdot\cos\dfrac{\alpha}{2}}{\left(\cos\dfrac{\alpha}{2}+\sin\dfrac{\alpha}{2}\right)^2}$

□ **Ans. (b)**

Example 7

If 4n□ = □, then cot □ . cot 2□ . cot 3□ ....cot (2n –1)□ is equal to

(a) (b) 1 (c) □ (d) none of these

Solution

Given 4n□ = □

Now, $\cot\alpha\cdot\cot 2\alpha\cdot\cot 3\alpha........\cot(2n-3)\alpha\cdot\cot(2n-2)\alpha\cdot\cot(2n-1)\alpha$

$$=\cot\alpha\cdot\cot 2\alpha\cdot\cot 3\alpha.......\cot\left(\frac{\pi}{2}-3\alpha\right)\cdot\cot\left(\frac{\pi}{2}-2\alpha\right)\cdot\cot\left(\frac{\pi}{2}-\alpha\right)$$

$$=\cot\alpha\cdot\cot 2\alpha\cdot\cot 3\alpha........\tan 3\alpha\cdot\tan 2\alpha\cdot\tan\alpha$$

= 1

□ **Ans. (a)**

Example 8

If tan □ = a □ 0, tan 2□ = b □ 0 and tan □ + tan 2□ = tan 3□ then

(a) a = b (b)ab = 1 (c) a + b = 0 (d) b = 2a

Solution

Given tan □ + tan 2□ = tan 3□

$$\Rightarrow \tan\theta+\tan 2\theta=\frac{\tan\theta+\tan 2\theta}{1-\tan\theta\cdot\tan 2\theta}$$

$$\Rightarrow (\tan\theta+\tan 2\theta)\left(1-\frac{1}{1-\tan\theta\tan 2\theta}\right)=0$$

□ tan □ + tan 2□ = 0 or tan □ tan 2□ = 0

□ a + b = 0 $[Q\ ab\neq 0]$

□ **Ans. (c)**

Example 9

If $\sin x+\sin^2 x=1$ then the value of $\cos^2 x+\cos^4 x+\cot^4 x-\cot^2 x$ is equal to

(a) 1 (b) 0 (c) 2 (d) none of these

Solution

Given $\sin x+\sin^2 x=1$

$\Rightarrow \sin x=\cos^2 x$

$\Rightarrow \sin^2 x = \cos^4 x$

Now, $\cos^2 x + \cos^4 x + \cot^4 x - \cot^2 x$

$= \cos^2 x + \sin^2 x + \cos ec^2 x - \cot^2 x$

= 2

 Ans. (c)

Example 10

The least value of $3\tan^2\theta + 9\cot^2\theta + 18$ is

(a) 18 (b) $6\sqrt{3}$ (c) $6\sqrt{3}+8$ (d) none of these

Solution

We have $\dfrac{3\tan^2\theta + 9\cot^2\theta}{2} \ge (27)^{1/2}$

$\Rightarrow 3\tan^2\theta + 9\cot^2\theta \ge 6\sqrt{3}$

$\Rightarrow 3\tan^2\theta + 9\cot^2\theta + 18 \ge 6\sqrt{3} + 18$

 Least value of $3\tan^2\theta + 9\cot^2\theta + 18$ is $6\sqrt{3}+18$

 Ans. (c)

Example 11

If $A + B = \dfrac{\pi}{3}$, where $A, B \in R^+$, then the minimum value of sec A + sec B is equal to

(a) $\dfrac{2}{\sqrt{3}}$ (b) $\dfrac{4}{\sqrt{3}}$ (c) $2\sqrt{3}$ (d) none of these

Solution

We know that $\dfrac{\sec A + \sec B}{2} \ge (\sec A \cdot \sec B)^{1/2}$

minimum value of sec A + sec B occus when

sec A = sec B

$\Rightarrow A = B = \dfrac{\pi}{6}$

minimum $(\sec A + \sec B) = \sec\dfrac{\pi}{6} + \sec\dfrac{\pi}{6} = \dfrac{4}{\sqrt{3}}$

 Ans. (b)

Example 12

Let p(□) = cos□ (cos□ + cos3□), then which of the following is true

(a) p(□) □ 0 for all □ □ R

(b) p(□) > 0 for all □ □ (0, □)

(c) p(□) > 0 for all $\theta \in \left(0, \frac{\pi}{4}\right)$

(d) none of these

Solution

Given p(□) = cos □(cos □ + cos 3□)

$$= 2\cos^2\theta \cdot \cos 2\theta > 0$$

when $\cos 2\theta > 0 \Rightarrow \theta \in \left(0, \frac{\pi}{4}\right)$

□ **Ans. (c)**

Example 13

Let n be a positive integer such that $\sin\frac{\pi}{2^n} + \cos\frac{\pi}{2^n} = \frac{\sqrt{n}}{2}$, then

(a) 6 □ n □ 8

(b) 4 < n □ 8

(c) 4 □ n < 8

(d) 4 < n < 8

Solution

Given $\sin\frac{\pi}{2^n} + \cos\frac{\pi}{2^n} = \frac{\sqrt{n}}{2}$

we have $\sin\frac{\pi}{2^n} + \cos\frac{\pi}{2^n} = \sqrt{2}\sin\left(\frac{\pi}{4} + \frac{\pi}{2^n}\right)$, which lies in $\left[-\sqrt{2}, \sqrt{2}\right]$;

therefore $\frac{\sqrt{n}}{2} \in \left[-\sqrt{2}, \sqrt{2}\right]$, $\quad \Rightarrow \quad \frac{\sqrt{n}}{2} \le \sqrt{2}$

$\Rightarrow \sqrt{n} \le 2\sqrt{2} \quad \Rightarrow \quad n \le 8$

we note that n = 1 does not satisfy the given equation and for n □ 1,

$$\frac{\pi}{2} \ge \frac{\pi}{4} + \frac{\pi}{2^n} > \frac{\pi}{4} \qquad \Rightarrow \quad \sin\left(\frac{\pi}{4} + \frac{\pi}{2^n}\right) > \sin\frac{\pi}{4}$$

$$\Rightarrow \sqrt{2}\sin\left(\frac{\pi}{4} + \frac{\pi}{2^n}\right) > 1 \quad \Rightarrow \quad \frac{\sqrt{n}}{2} > 1$$

□ n □ 4 hence 4 □ n □ 8

□ **Ans. (b)**

Example 14

sin 47 + sin 61 – sin 11 – sin 25 is equal to

(a) sin 36

(b) cos 36

(c) sin 7

(d) cos 7

Solution

We have sin 47□ + sin 61□ - sin 11□ - sin 25□

= 2sin 54° cos 7° - 2sin 18° cos 7°

= 2cos 7° [sin 54° - sin 18°]

= 2cos 7° 2cos 36° sin 18°

$$= 4\cos 7^o \cdot \frac{\sqrt{5}+1}{4} \times \frac{\sqrt{5}-1}{4}$$

= cos 7°

∴ **Ans. (d)**

Example 15

If tanθ = n tan ϕ, then maximum value of $\tan^2(\theta-\phi)$ is equal to

(a) $\frac{(n-1)^2}{4n}$ (b) $\frac{(n+1)^2}{4n}$ (c) $\frac{(n+1)}{2n}$ (d) $\frac{(n-1)}{2n}$

Solution

We have $\tan(\theta-\phi) = \frac{\tan\theta - \tan\phi}{1+\tan\theta\cdot\tan\phi}$

$$= \frac{(n-1)\tan\phi}{1+n\tan^2\phi} = \frac{n-1}{\cot\phi + n\tan\phi}$$

$$\Rightarrow \tan^2(\theta-\phi) = \frac{(n-1)^2}{(\cot\phi+n\tan\phi)^2} = \frac{(n-1)^2}{(\cot\phi - n\tan\phi)^2+4n}$$

$$\Rightarrow \tan^2(\theta-\phi) \le \frac{(n-1)^2}{4n}$$

∴ Maximum value of $\tan^2(\theta-\phi)$ is $\frac{(n-1)^2}{4n}$

∴ **Ans. (a)**

Example 16

If $a \le 16\sin x\cos x + 12\cos^2 x - 6 \le b$ for all x ∈ R then

(a) a = -5, b = 5 (b) a = -4, b = 4 (c) a = -10, b =10 (d) none of these

Solution

We have $16\sin x\cdot\cos x + 12\cos^2 x - 6$

= 8sin 2x + 6cos 2x

Now, $-\sqrt{8^2+6^2} \le 8\sin 2x + 6\cos 2x \le \sqrt{8^2+6^2}$

∴ $-10 \le 8\sin 2x + 6\cos 2x \le 10$

∴ a = -10, b = 10

∴ **Ans. (c)**

Example 17

If $x, y \in \left(0,\ \dfrac{\pi}{2}\right)$, then the expression $\sin x + \cos y + \tan^2 y + \cot^2 x + 5$ is always greater than

(a) 7 (b) 8 (c) 9 (d) none of these

Solution

Given expression $= \sin x + \cos y + \tan^2 y + \cot^2 x + 5$

$$= \sin x + \cos y + \sec^2 y + \operatorname{cosec}^2 x + 3$$

Now, $\sin x + \dfrac{1}{\sin^2 x} \geq \sin x + \dfrac{1}{\sin x} \geq 2$

and $\cos y + \dfrac{1}{\cos^2 y} \geq \cos y + \dfrac{1}{\cos y} \geq 2$

□ Given expression □ 7.

□ **Ans. (a)**

Example 18

If sinx + cosx + tanx + cot x + secx + cosecx = 7 and $\sin 2x = a - b\sqrt{7}$, then ordered pair (a, b) can be,

(a) (6, 2) (b) (8, 3) (c) (22, 8) (d) (11, 4)

Solution

Given sin x + cos x + tan x + cot x + sec x + cosec x = 7

$\Rightarrow\ (\sin x + \cos x) + \dfrac{1}{\sin x \cdot \cos x} + \dfrac{\sin x + \cos x}{\sin x \cdot \cos x} = 7$

$\Rightarrow\ (\sin x + \cos x)\left(1 + \dfrac{1}{\sin x \cdot \cos x}\right) = 7 - \dfrac{1}{\sin x \cos x}$

$\Rightarrow\ (1 + \sin 2x)\left(1 + \dfrac{2}{\sin 2x}\right)^2 = \left(7 - \dfrac{2}{\sin 2x}\right)^2$

$\Rightarrow\ (1+t)(t+2)^2 = (7t-2)^2$, where t = sin 2x

$\Rightarrow\ t^3 - 44t^2 + 36t = 0$

$\Rightarrow\ t^2 - 44t + 36 = 0$ [as sin 2x □ 0]

$$t = \frac{44 \pm \sqrt{44^2 - 4 \times 36}}{2} = 22 \pm 8\sqrt{7}$$

$\therefore\ \sin 2x = 22 - 8\sqrt{7}$

□ **Ans. (c)**

Example 19

If $\cos\alpha = \frac{2\cos\beta - 1}{2 - \cos\beta}$, $(0 < \alpha < \pi,\ 0 < \beta < \pi)$, then $\tan\frac{\alpha}{2}\cot\frac{\beta}{2}$ is equal to

(a) 1 (b) $\sqrt{2}$ (c) $\sqrt{3}$ (d) none of these

Solution

Given $\cos\alpha = \frac{2\cos\beta - 1}{2 - \cos\beta}$

$$\Rightarrow \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \frac{2\cdot\frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}}{2-\frac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}}$$

$$\Rightarrow \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \frac{1-3\tan^2\frac{\beta}{2}}{1+3\tan^2\frac{\beta}{2}}$$

$\Rightarrow \tan^2\frac{\alpha}{2} = 3\tan^2\frac{\beta}{2}$ [By componendo and dividendo]

$\Rightarrow \tan\frac{\alpha}{2}\cdot\cot\frac{\beta}{2} = \sqrt{3}$

 Ans. (c)

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SUBJECTIVE TYPE

Example 1

Prove that 2 $\sin^2$ q + 4 cos (q + a) sin a sin q + cos 2 (q + a) is independent of q.

Solution

2 $\sin^2$ q + 4 cos (q + a) sin a sin q + cos 2 (a + q)

= 2 $\sin^2$ q + 2 (cos (q + a))(2 sin a sin q) + cos 2 (q + a)

= 2 $\sin^2$ q + 2 cos (q + a) [cos (q – a) – cos (q + a)] + cos 2 (q + a)

= 2 $\sin^2$ q + 2 cos (q + a) cos (q – a) – 2 $\cos^2$ (q + a) + cos 2 (q + a)

= 2 $\sin^2$ q + 2 ($\cos^2$ q – $\sin^2$ a) – 2 $\cos^2$ (q + a) + cos2 (q + a)

= 2 $\sin^2$ q + 2 $\cos^2$ q – 2 $\sin^2$ a – 1

= 1 – 2 $\sin^2$ a

= $\cos^2$ □ which is independent of □

Example 2

Prove the identity (cos A + cos B) (cos 2A + cos 2B) (cos 2^2 A + cos 2^2 B) …… (cos 2^{n-1} A + cos 2^{n-1} B)

$$= \frac{\cos 2^n A - \cos 2^n B}{2^n (\cos A - \cos B)}$$

Solution

(cos A – cos B) (cos A + cos B) = $\cos^2$ A – $\cos^2$ B

= $\frac{1}{2}$ [(1 + cos 2A) – (1 + cos 2B)]

= $\frac{1}{2}$ [cos 2A – cos 2B]

□□ $\frac{1}{2}$ (cos A – cos B) (cos A + cos B) (cos 2A + cos 2B)

= $\frac{1}{2}$ (cos 2A – cos 2B) (cos 2A + cos 2B)

= $\frac{1}{2^2}$ [cos 2^2 A – cos 2^2 B]

Proceeding in this manner, we get

(cos A – cos B) (cos A + cos B) (cos 2A + cos 2B) …… (cos 2^{n-1} A + cos 2^{n-1} B)

= $\frac{1}{2^n}$ (cos 2^n A + cos 2^n B)

Example 3

If A + B + C = p and $\tan\left(\frac{A+B-C}{4}\right)\tan\left(\frac{B+C-A}{4}\right)\tan\left(\frac{C+A-B}{4}\right)=1$

Prove that sin A + sin B + sin C + sin A sin B sin C = 0

Solution

$$\tan\left(\frac{A+B-C}{4}\right)=\tan\left(\frac{\pi-2C}{4}\right)=\tan\left(\frac{\pi}{4}-\frac{C}{2}\right)=\frac{1-\tan\frac{C}{2}}{1+\tan\frac{C}{2}}$$

$$=\frac{\left(\cos\frac{C}{2}-\sin\frac{C}{2}\right)^2}{\cos^2\frac{C}{2}-\sin^2\frac{C}{2}}=\frac{1-\sin C}{\cos C}=\frac{\cos C}{1+\sin C}$$

Similarly, $\tan\left(\frac{B+C-A}{4}\right)=\frac{1-\sin A}{\cos A}=\frac{\cos A}{1+\sin A}$

and $\tan\left(\frac{C+A-B}{4}\right)=\frac{1-\sin B}{\cos B}=\frac{\cos B}{1+\sin B}$

The given condition implies

$$\left(\frac{1-\sin A}{\cos A}\right)\left(\frac{1-\sin B}{\cos B}\right)\left(\frac{1-\sin C}{\cos C}\right)=1 \qquad \ldots \text{(i)}$$

as well as

$$\left(\frac{\cos A}{1+\sin A}\right)\left(\frac{\cos B}{1+\sin B}\right)\left(\frac{\cos C}{1+\sin C}\right)=1 \qquad \ldots \text{(ii)}$$

from eq. (i) and (ii), we get

cos A cos B cos C = (1 – sin A) (1 – sin B) (1 – sin C) = (1 + sin A) (1 + sin B) (1 + sin C) $\frac{1-\sin A}{1+\sin A}$

□□ $1-\sum\sin A+\sum\sin A\sin B-\sin A\sin B\sin C = 1+\sum\sin A+\sum\sin A\sin B+\sin A\sin B\sin C$

□ $\sum\sin A+\sin A\sin B\sin C=0$

Example 4

Show that cos (sin q) > sin (cos q) for all q belonging to the interval $\left[0,\frac{\pi}{2}\right]$

Solution

We have to show that $\sin\left(\frac{\pi}{2}-\sin\theta\right)>\sin(\cos\theta)$

Now the first quadrant is from 0 to $\frac{\pi}{2}\left(\approx 1.57^C\right)$, the sine function increases from 0 to 1 in it.

Here the angles which are compared are $\left(\frac{\pi}{2}-\sin\theta\right)$ and cos q, both of which also have values lying in $\left[0,\frac{\pi}{2}\right]$ because 0 £ sin q £ 1 and 0 £ cos q £ 1 for $\theta\in\left[0,\frac{\pi}{2}\right]$, that is we have to show that $\frac{\pi}{2}-\sin\theta>\cos\theta$ or sin q + cos q < $\frac{\pi}{2}$ which is true since the maximum value of sin q + cos q is $\sqrt{1^2+1^2}=\sqrt{2}<\frac{\pi}{2}$.

Example 5

If 0 □ *a* □ 3, 0 □ *b* □ 3 and the equation $x^2 + 4 + 3\cos(ax+b) = 2x$ has at least one solution then find the value of (*a* + *b*)

Solution

$x^2 - 2x + 4 = -3\cos(ax+b)$

$(x-1)^2 + 3 = -3\cos(ax+b)$ … (i)

as – 1□□□cos (*ax* + *b*) □ 1 and$(x-1)^2$ □ 0

eq. (i) is only possible if

cos (*ax* + *b*) = – 1 and *x* – 1 = 0

so, *a* + *b* = p, 3p, 5p ……

but, 3p > 6, where as *a* + *b* □ 6

□ *a* + *b* = p

Example 6

Show that $2^{\sin x} + 2^{\cos x}$□□ $2^{1-\frac{1}{\sqrt{2}}}$ for all real *x*.

Solution

Clearly, $2^{\sin x}$ and $2^{\cos x}$ are positive,

So their A.M. □ G.M.

$$\frac{2^{\sin x}+2^{\cos x}}{2}\ge\sqrt{2^{\sin x}2^{\cos x}}=\sqrt{2^{\sin x+\cos x}} \quad \text{… (i)}$$

as we know

sin *x* + cos *x* □□ $-\sqrt{2}$

□ $2^{\sin x + \cos x}$ 3 $2^{-\sqrt{2}}$

$$\sqrt{2^{\sin x+\cos x}}\ge 2^{-\frac{\sqrt{2}}{2}}$$

or $\sqrt{2^{\sin x+\cos x}}\ge 2^{-\frac{1}{\sqrt{2}}}$ … (ii)

from (i) and (ii)

$$\frac{2^{\sin x}+2^{\cos x}}{2} \geq \sqrt{2^{\sin x+\cos x}} \geq 2^{-\frac{1}{\sqrt{2}}}$$

$$2^{\sin x}+2^{\cos x} \geq 2 \,.\, 2^{-\frac{1}{\sqrt{2}}}$$

or $\quad 2^{\sin x}+2^{\cos x} \geq 2^{1-\frac{1}{\sqrt{2}}}$ for all values of x.

Example 7

In a $\triangle ABC$, prove that $1 < \cos A + \cos B + \cos C \leq \frac{3}{2}$.

Solution

We have,

$\cos A + \cos B + \cos C$

$= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}+1-2\sin^2\frac{C}{2}$

$= 2\sin\frac{C}{2}\left\{\cos\frac{A-B}{2}-\sin\frac{C}{2}\right\}+1$

$\leq 2\sin\frac{C}{2}\left\{1-\sin\frac{C}{2}\right\}+1 \qquad \left[\because \text{ Max. value of } \cos\frac{A-B}{2} \text{ is } 1\right] \qquad$ ...(i)

$= 1-2\left\{\sin^2\frac{C}{2}-\sin\frac{C}{2}\right\}$

$= 1-2\left\{\left(\sin\frac{C}{2}-\frac{1}{2}\right)^2-\frac{1}{4}\right\}$

$= \frac{3}{2}-2\left(\sin\frac{C}{2}-\frac{1}{2}\right)^2$

Thus, we have

$\cos A + \cos B + \cos C \leq \frac{3}{2}-2\left(\sin\frac{C}{2}-\frac{1}{2}\right)^2 \leq \frac{3}{2} \qquad$ …(ii)

It is evident from (i) and (ii) that

$\cos A + \cos B + \cos C = \frac{3}{2}$,

if $\quad \cos\frac{A-B}{2}=1$ and $\sin\frac{C}{2}-\frac{1}{2}=0$

$\Rightarrow \quad A - B = 0$ and $C = 60°$

$\Rightarrow \quad A = B = C = 60°$

$\Rightarrow \quad \triangle ABC$ is equilateral.

$\because \quad \cos A + \cos B + \cos C = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}+1$

and,

$$4 \sin A/2 \sin B/2 \sin C/2 > 0$$

$\Rightarrow \quad \cos A + \cos B + \cos C > 1$

Hence, $1 < \cos A + \cos B + \cos C \leq \frac{3}{2}$.

Example 8

In a $\triangle ABC$, prove that $1 \sin A \sin B \sin C \leq \frac{3\sqrt{3}}{8}$

Solution

We have,

$\sin^2 A + \sin^2 B + \sin^2 C$

$= \quad \frac{1-\cos 2A}{2} + \frac{1-\cos 2B}{2} + \sin^2 C$

$= \quad 1 - \frac{1}{2}\{\cos 2A + \cos 2B\} + \sin^2 C$

$= \quad 1 - \cos(A+B)\cos(A-B) + \sin^2 C$

$= \quad 1 + \cos C \cos(A-B) + 1 - \cos^2 C$

$= \quad 2 - \cos^2 C + \cos C \cos(A-B)$

$\leq \quad 2 - \cos^2 C + \cos C \; [\because \cos(A-B) \leq 1]$

$= \quad 2 - (\cos^2 C - \cos C)$

$= \quad 2 - \left(\cos C - \frac{1}{2}\right)^2 + \frac{1}{4}$

$= \quad \frac{9}{4} - \left(\cos C - \frac{1}{2}\right)^2 \leq \frac{9}{4}$

$\Rightarrow \quad \sin^2 A + \sin^2 B + \sin^2 C \leq \frac{9}{4}$ …(i)

Now,

A.M. $\geq$ G.M.

$\Rightarrow \quad \frac{\sin^2 A + \sin^2 B + \sin^2 C}{3} \geq (\sin^2 A \sin^2 B \sin^2 C)^{1/3}$

$\Rightarrow \quad (\sin^2 A \sin^2 B \sin^2 C)^{1/3} \leq \frac{3}{4}$ [Using (i)]

$\Rightarrow \quad \sin A \sin B \sin C \leq \left(\frac{3}{4}\right)^{3/2}$

$\Rightarrow \quad \sin A \sin B \sin C \leq \frac{3\sqrt{3}}{8}$.

Example 9

In a □*ABC*, $\angle B = \frac{\pi}{3}$ and sin *A* sin *C* = □. Find the set of all possible values of □.

Solution

We have,

$$\angle B = \frac{\pi}{3}$$

□ $A + B + C =$ □

□ $A + C = \frac{2\pi}{3}$ □ $C = \frac{2\pi}{3} - A$

Now,

sin *A* sin *C* = □

□ $\sin A \sin\left(\frac{2\pi}{3} - A\right) = \lambda$

□ $\sin A\left(\frac{\sqrt{3}}{2}\cos A + \frac{1}{2}\sin A\right) = \lambda$

□ $\frac{\sqrt{3}}{4}\sin 2A + \frac{1}{2}\sin^2 A = \lambda$

□ $\frac{\sqrt{3}}{4}\sin 2A + \frac{1}{4}(1 - \cos 2A) = \lambda$

□ $\frac{\sqrt{3}}{4}\sin 2A - \frac{1}{4}\cos 2A = \lambda - \frac{1}{4}$

Now,

$$-\sqrt{\frac{3}{16} + \frac{1}{16}} \le \frac{\sqrt{3}}{4}\ \sin\ -\frac{1}{4}\cos 2A \le \sqrt{\frac{3}{16} + \frac{1}{16}}$$

□ $-\frac{1}{2} \le \frac{\sqrt{3}}{4}$ sin 2*A* – $\frac{1}{4}$ cos 2*A* □ $\frac{1}{2}$

□ $-\frac{1}{2} \le \lambda - \frac{1}{4} \le \frac{1}{2}$

□ $-\frac{1}{4} \le \lambda \le \frac{3}{4}$

□ □ □ [–1/4, 3/4].

Example 10

If an a □ *ABC*, $\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ are in H.P., then find the minimum value of cot $\frac{B}{2}$.

Solution

In □ *ABC*, We have,

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} \qquad \ldots\text{(i)}$$

It is given that

$\tan\frac{A}{2},\tan\frac{B}{2},\tan\frac{C}{2}$ are in H.P.

□ $\cot\frac{A}{2},\cot\frac{B}{2}\cot\frac{C}{2}$ are in A.P.

□ $2\cot\frac{B}{2}=\cot\frac{A}{2}+\cot\frac{C}{2} \qquad \ldots\text{(ii)}$

From (i) and (ii), we have

$$\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}=3\cot\frac{B}{2}$$

□ $\cot\frac{A}{2}\cot\frac{C}{2}=3$

Now, A.M. □ G.M.

□ $\dfrac{\cot\frac{A}{2}+\cot\frac{C}{2}}{2}\geq\sqrt{\cot\frac{A}{2}\cot\frac{C}{2}}$

□ $\cot\frac{B}{2}\geq\sqrt{3}$.

Hence, the minimum value of $\cot\frac{B}{2}$ is $\sqrt{3}$.

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Exercise - I

OBJECTIVE TYPE QUESTIONS

Multiple choice questions with ONE option correct

1. The equation $\sec^2\theta = \frac{4xy}{(x+y)^2}$ is possible only if

(a) x = y (b) x < y

(c) x > y (d) None of these

2. If $\sin\theta + \text{cosec}\,\theta = 2$, the value of $\sin^{10}\theta + \text{cosec}^{10}\theta$ is

(a) 10 (b) 2^{10} (c) 2^9 (d) 2

3. If $\sin\theta = -\sqrt{\frac{3}{5}}$ and q lies in third quadrant, then cos q =

(a) $\frac{1}{\sqrt{5}}$ (b) $-\frac{1}{\sqrt{5}}$ (c) $\sqrt{\frac{2}{5}}$ (d) $-\sqrt{\frac{2}{5}}$

4. The value of $e^{[\log_{10}\tan 1° + \log_{10}\tan 2° + \log_{10}\tan 3° + ... + \log_{10}\tan 89°]}$ is

(a) 0 (b) – 1 (c) 1/e (d) 1

5. The value of $2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1$ is

(a) 2 (b) 0 (c) 4 (d) 2

6. If $\sin x + \sin^2 x = 1$, then the value of $\cos^{12} x + 3\cos^{10} x + 3\cos^8 x + \cos^6 x - 2$ is equal to

(a) 0 (b) 1 (c) –1 (d) 2

7. $\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{2\pi}{5}\right)\cos\left(\frac{4\pi}{5}\right)\cos\left(\frac{8\pi}{5}\right) =$

(a) 1/16 (b) 0 (c) –1/8 (d) –1/16

8. $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right) =$

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{8}$ (d) $\frac{1}{16}$

9. If $x = y\cos\frac{2\pi}{3} = z\cos\frac{4\pi}{3}$, then $xy + yz + zx$ is equal to

(a) –1 (b) 0 (c) 1 (d) 2

10. The value of the expression

$1 - \frac{\sin^2 y}{1+\cos y} + \frac{1+\cos y}{\sin y} - \frac{\sin y}{1-\cos y}$ is equal to

(a) 0 (b) 1 (c) sin y (d) cos y

11. If $\cos(\theta-\alpha)=a$ and $\sin(\theta-\beta)=b$, then $\cos^2(\alpha-\beta)+2ab\sin(\alpha-\beta)$ is equal to

(a) $4a^2b^2$ (b) a^2-b^2 (c) a^2+b^2 (d) $-a^2b^2$

12. If $\tan\frac{\pi}{9}, x$ and $\tan\frac{5\pi}{18}$ are in A.P. and $\tan\frac{\pi}{9}, y$ and $\tan\frac{7\pi}{18}$ are also in A.P., then

(a) 2x = y (b) x > y (c) x = y (d) none of these

13. If $\cos 5\theta = a\cos^5\theta + b\cos^3\theta + c\cos\theta$, then c is equal to

(a) – 5 (b) 1 (c) 5 (d) none of these

14. $\sum_{r=1}^{n-1}\cos^2\frac{r\pi}{n}$ is equal to

(a) $\frac{n}{2}$ (b) $\frac{n-1}{2}$ (c) $\frac{n}{2}-1$ (d) none of these

15. If $\tan(\pi\cos x)=\cot(\pi\sin x)$, then $\cos\left(x-\frac{\pi}{4}\right)$ is

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2\sqrt{2}}$ (c) 0 (d) none of these

Multiple choice questions with MORE THAN ONE option correct

1. If $\tan\alpha, \tan\beta$ are the roots of the equation $x^2+px+q=0\ (p\neq 0)$, then

(a) $\sin^2(\alpha+\beta)+p\sin(\alpha+\beta)\cos(\alpha+\beta)+q\cos^2(\alpha+\beta)=q$

(b) $\tan(\alpha+\beta)=\frac{p}{q-1}$

(c) $\cos(\alpha-\beta)=1-q$

(d) $\sin(\alpha+\beta)=-p$

2. If sin b is the the geometric mean between sin a and cos a, then cos 2b is equal to

(a) $2\sin^2\left(\frac{\pi}{4}-\alpha\right)$ (b) $2\cos^2\left(\frac{\pi}{4}-\alpha\right)$ (c) $2\cos^2\left(\frac{\pi}{4}+\alpha\right)$ (d) $2\sin^2\left(\frac{\pi}{4}+\alpha\right)$

3. In a triangle tan *A* + tan *B* + tan *C* = 6 and tan *A* tan *B* = 2, then the value of tan *A*, tan *B* and tan *C* are

(a) 1, 2, 3 (b) 2, 1, 3 (c) 1, 2, 0 (d) None of these

4. If $\cos(A-B)=\frac{3}{5}$ and $\tan A\cdot\tan B=2$, then

(a) $\cos A\cos B=1/5$ (b) $\sin A\ \sin B=-2/5$

(c) $\cos(A+B)=-1/5$ (d) None of these

5. For a positive integer n, let $f_n(\theta) = \tan\frac{\theta}{2}(1+\sec\theta)(1+\sec 2\theta)(1+\sec 4\theta)..........(1+\sec 2^n\theta)$, then

(a) $f_2\left(\frac{\pi}{16}\right) = 1$ (b) $f_3\left(\frac{\pi}{32}\right) = 1$ (c) $f_4\left(\frac{\pi}{64}\right) = 1$ (d) $f_5\left(\frac{\pi}{128}\right) = 1$

6. If $\frac{\sin A}{\sin B} = p$, $\frac{\cos A}{\cos B} = q$, then

(a) $\tan A = \frac{p}{q}\sqrt{\frac{q^2-1}{1-p^2}}$ (b) $\tan A = -\frac{p}{q}\sqrt{\frac{q^2-1}{1-p^2}}$

(c) $\tan B = \sqrt{\frac{q^2-1}{1-p^2}}$ (d) $\tan B = -\sqrt{\frac{q^2-1}{1-p^2}}$

7. $\sin\frac{15\pi}{32}.\sin\frac{7\pi}{16}.\sin\frac{3\pi}{8}$ is equal to

(a) $\frac{1}{8\sqrt{2}\cos\frac{15\pi}{32}}$ (b) $\frac{1}{8\sin\frac{\pi}{32}}$ (c) $\frac{1}{4\sqrt{2}}\operatorname{cosec}\frac{\pi}{16}$ (d) $\frac{1}{8\sqrt{2}}\operatorname{cosec}\frac{\pi}{32}$

8. If $a = \frac{1}{5\cos x + 12\sin x}$ then for all real x

(a) the least positive value of a is (b) the greatest negative value of a is $-\frac{1}{13}$

(c) $a \le \frac{1}{13}$ (d) $-\frac{1}{13} \le a \le \frac{1}{13}$

9. If 7 cos x – 24 sin x = □ cos (x + □), 0 < □ < $\frac{\pi}{2}$, be true for all x □ R then

(a) □ = 25 (b) □ = $\sin^{-1}\frac{24}{25}$ (c) □ = – 25 (d) □ = $\cos^{-1}\frac{7}{25}$

10. If $A + B = \frac{\pi}{3}$ and $\cos A + \cos B = 1$ then

(a) $\cos(A - B) = \frac{1}{3}$ (b) $|\cos A - \cos B| = \sqrt{\frac{2}{3}}$

(c) $\cos(A - B) = -\frac{1}{3}$ (d) $|\cos A - \cos B| = \frac{1}{2\sqrt{3}}$

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Exercise - II

ASSERTION & REASON , COMPREHENSIVE & MATCHING TYPE

Assertion & Reason Type

In the following question, a statement of Assertion (A) is given which is followed by a corresponding statement of reason (R). Mark the correct answer out of the following options/codes.

(a) If both (A) and (R) are true and (R) is the correct explanation of (A).

(b) If both (A) and (R) are true but (R) is not correct explanation of (A).

(c) If (A) is true but (R) is false.

(d) If both (A) and (R) are false.

(e) If (A) is false but (R) is true.

1. A: $\sin^2(\theta) + \sin^2(\theta + 60) + \sin^2(\theta - 60) = \frac{3}{2}$

   R: $\cos\alpha + \cos(120 + \alpha) + \cos(120 - \alpha) = 0$

2. A: In $\Delta ABC, \sum \frac{\cot A + \cot B}{\tan A + \tan B} = 2$

   R: If $\alpha + \beta + \gamma = 180°$ then $\sum \cot\alpha \cot\beta = 1$

3. A: If $\tan\alpha, \tan\beta$ are the roots of $x^2 + px + q = 0$ then tan $(\alpha + \beta) = \frac{p}{1-q}$

   R: If α, β are the roots $ax^2 + bx + c = 0$ then $\alpha + \beta = \frac{-b}{a}$ and $\alpha\beta = \frac{c}{a}$

4. A: If $x\cos\alpha + y\sin\alpha = 2a,\ x\cos\beta + y\sin\beta = 2a\ (\alpha \neq \beta)$ then $\cos\alpha\cos\beta = \frac{4a^2 + y^2}{x^2 + y^2}$

   R: If $a\alpha^2 + b\alpha + c = 0$ and $a\beta^2 + b\beta + c = 0\ (\alpha \neq \beta)$, then □□□□ are roots of the equation $ax^2 + bx + c = 0$

5. A: If $f(\theta) = \tan\frac{\theta}{2}(1 + \sec\theta)(1 + \sec 2\theta)\ldots\ldots\ldots..(1 + \sec 2^5\theta)$ then $f(\frac{\pi}{128})$ =2

   R: $\tan\frac{\theta}{2}(1 + \sec\theta)(1 + \sec 2\theta)\ldots\ldots...(1 + \sec 2^n\theta) = \tan(2^n\theta)$

Passage Based Questions

Passage – I

If A + B + C = 180° then

(i) $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$

(ii) $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$

(iii) $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(iv) $\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(v) $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

(vi) $\cot B\cot C+\cot C\cot A+\cot A\cot B=1$

(vii) $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

Read the above passage carefully and mark the correct answers:

1. If tan A + tan B + tan C = tan A. tan B. tan C then

   (a) A ,B ,C must be angles of a triangle

   (b) the sum of any two A, B, C is equal to the third.

   (c) A + B +C must be an integral multiple of π (d) none of these.

2. In a triangle ABC, whose angles are acute and +ve such that A + B + C = π and $\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}=k$, then

   (a) $k\le 3$ (b) $k\le 3\sqrt{3}$ (c) $k\ge 3\sqrt{3}$ (d) none of these

3. If A, B, C are acute positive angels such that A + B + C = π and cot A cot B cot C = K, then

   (a) $k\le\frac{1}{3\sqrt{3}}$ (b) $k\ge\frac{1}{3\sqrt{3}}$ (c) $k<\frac{1}{9}$ (d) $k>\frac{1}{3}$

Passage – II

We have

(i) $$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+....+\cos(\alpha+(n-1)\beta)=\frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}\sin\left[\frac{2\alpha+(n-1)\beta}{2}\right]$$

$$\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+.....+\cos(\alpha+(n-1)\beta)=\frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}\cos\left[\frac{2\alpha+(n-1)\beta}{2}\right]$$

$[\sin\frac{\beta}{2}\ne 0]$

Mark the correct answers:

1. $\sum_{r=1}^{n-1}\cos^2\left(\frac{r\pi}{n}\right)=......$

   (a) $\frac{n}{2}+1$ (b) $\frac{n}{2}-1$ (c) $\frac{n}{2}$ (d) none of these

2. $\sin\frac{\pi}{n}+\sin\frac{3\pi}{n}+\sin\frac{5\pi}{n}+.......$ to n term =

(a) 1 (b) 0 (c) – 1 (d) none of these

3. $\sqrt{1+\cos\alpha}+\sqrt{1+\cos 2\alpha}+\sqrt{1+\cos 3\alpha}+...to\ n\ terms=$

(a) $\frac{\sin\left(\frac{n\alpha}{4}\right)}{\sin\left(\frac{\alpha}{4}\right)}\cos\left(\frac{n\alpha}{4}\right)$ (b) $\frac{\sin\left(\frac{n\alpha}{4}\right)}{\sin\left(\frac{\alpha}{4}\right)}\cos\left(\frac{(n+1)\alpha}{4}\right)$

(c) $\sqrt{2}\frac{\sin\left(\frac{n\alpha}{4}\right)}{\sin\left(\frac{\alpha}{4}\right)}\cos\left(\frac{(n+1)\alpha}{4}\right)$ (d) none of these

Matching Type Questions

1. **Column I** **Column II**

(A) $\sin 20°\sin 40°\sin 60°\sin 80°$ (P) $\frac{1}{16}$

(B) $\cos 20°\cos 40°\cos 60°\cos 80°$ (Q) $-\frac{1}{8}$

(C) $\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}$ (R) $\frac{3}{16}$

(D) $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}$ (S) $\frac{1}{8}$

(a) A-R, B-P, C-S, D-Q (b) A-P, B-Q, C-R, D-S

(c) A-S, B-Q, C-P, D-R](d)A-Q, B-R, C-S, D-P

2. **Column I** **Column II**

(A) Range of $2\cos^2 x + \sin^2 x$ (P) [1,3]

(B) Range of $1+8\sin^2(x^2)\cos^2(x^2)$ (Q) [3/4,1]

(C) Range of $\cos^2 x + \sin^4 x$ (R) [3,4]

(D) Range of $3\sin^2 x + 4\cos^2 x$ (S) [1,2]

(a) A-R, B-P, C-S, D-Q (b) A-S, B-P, C-Q, D-R

(c) A-S, B-Q, C-P, D-R (d) A-Q, B-R, C-S, D-P

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Exercise - III

SUBJECTIVE TYPE

1. Prove that $\dfrac{\sec A - \tan A}{\sec A + \tan A} = 1 - 2\sec A + \tan A + 2\tan^2 A$

2. Prove that $\dfrac{\tan A}{1-\cot A} + \dfrac{\cot A}{1-\tan A} = \sec A \ \ \text{cosec} A + 1$

3. If $\dfrac{\sin^4 \alpha}{a} + \dfrac{\cos^4 \alpha}{b} = \dfrac{1}{a+b}$ prove that

   $\dfrac{\sin^8 \alpha}{a^3} + \dfrac{\cos^8 \alpha}{b^3} = \dfrac{1}{(a+b)^3}$

4. Prove that $2\sec^2 \theta + 3\,\text{cosec}^2 \theta \geq 5 + 2\sqrt{6} \ \ \forall \theta$

5. Prove that the roots of the equation

   $8x^3 - 4x^2 - 4x + 1 = 0$ are $\cos\dfrac{\pi}{7}, \cos\dfrac{3\pi}{7}, \cos\dfrac{5\pi}{7}$ and hence show that

6. Show that there is no real x for which

   $e^{\sin x} - e^{-\sin x} = 4$

7. Show that $\dfrac{\sec^2 \theta - \tan \theta}{\sec^2 \theta + \tan \theta}$ lies between $\dfrac{1}{3}$ and 3 for all real q.

8. Find the maximum and minimum values of $\sin^6 x + \cos^6 x$

9. If $A + B + C = \pi$, prove that

   $\tan B \tan C + \tan C \tan A + \tan A \tan B = 1 + \sec A . \sec B . \sec C$

10. If $\tan\left(\dfrac{\pi}{4} + \dfrac{y}{2}\right) = \tan^3\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)$, prove that $\dfrac{\sin y}{\sin x} = \dfrac{3 + \sin^2 x}{1 + 3\sin^2 x}$

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Exercise - IV

IIT – JEE PROBLEMS

1. If $A = \cos^2 \theta + \sin^4 \theta,$ then for all values of □

(a) $1 \le A \le 2$ (b) $\frac{13}{16} \le A \le 1$ (c) $\frac{3}{4} \le A \le \frac{13}{16}$ (d) $\frac{3}{4} \le A \le 1$

2. The maximum values of $\sin\left(x+\frac{\pi}{6}\right)+\cos\left(x+\frac{\pi}{6}\right)$ in the interval $\left(0, \frac{\pi}{2}\right)$ is attained at

(a) $\frac{\pi}{12}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

3. The number of integral values of k for which the equation 7 cos x + 5 sin x = 2k +1 has a solution is

(a) 4 (b) 8 (c) 10 (d) 12

4. If $\alpha+\beta=\frac{\pi}{2}$ and $\beta+\gamma=\alpha,$ then $\tan\alpha$ equals

(a) $2(\tan\beta+\tan\gamma)$ (b) $\tan\beta+\tan\gamma$

(c) $\tan\beta+2\tan\gamma$ (d) $2\tan\beta+\tan\gamma$

5. The maximum value of $\cos\alpha_1 \cdot \cos\alpha_2 \cdot \cos\alpha_3 ... \cos\alpha_n$ under the restriction $0 \le \alpha_1, \alpha_2, ... \alpha_n \le \frac{\pi}{2}$ and $\cot\alpha_1 \cdot \cot\alpha_2 ...... \cot\alpha_n = 1$ is

(a) $\frac{1}{2^{n/2}}$ (b) $\frac{1}{2^n}$ (c) $\frac{-1}{2^n}$ (d) 1

6. The minimum value of the expression $\sin\alpha+\sin\beta+\sin\gamma,$ where α, β, γ are real numbers satisfying $\alpha+\beta+\gamma=\pi$ is

(a) positive (b) zero (c) negative (d) – 3

7. Which of the following number (s) is / are rational?

(a) sin 15º (b) cos 15º (c) sin 15º cos 15º (d) sin 15º cos 75º

8. Let n be an odd integer. If $\sin n\theta = \sum_{r=0}^{n} b \sin^r \theta$ for every value of □, then

(a) $b_0 = 1, b_1 = 3$ (b) $b_0 = 0, b_1 = n$

(c) $b_0 = -1, b_1 = n$ (d) $b_0 = 0, b_1 = n^2 - n + 3$

9. The graph of the function $\cos x \cos(x+2) - \cos^2(x+1)$ is

(a) a straight line passing through (0, $\sin^2 1$) with slope 2
(b) a straight line passing through (0, 0)
(c) a parabola with vertex (1, $-\sin^2 1$)
(d) a straight line passing through the point $\left(\frac{\pi}{2}, -\sin^2 1\right)$ and parallel to the x-axis

10. $\sec^2\theta = \frac{4xy}{(x+y)^2}$ is true if and only if

(a) $x + y \neq 0$ (b) $x = y, x \neq 0$ (c) $x = y$ (d) $x \neq 0, y \neq 0$

11. For $0 < \phi < \frac{\pi}{2}$ if $x = \sum_{n=0}^{\infty} \cos^{2n}\phi$, $y = \sum_{n=0}^{\infty} \sin^{2n}\phi$ and $z = \sum_{n=0}^{\infty} \cos^{2n}\phi \sin^{2n}\phi$, then

(a) $xyz = xz + y$ (b) $xyz = xy + z$ (c) $xyz = x + y + z$ (d) $xyz = yz + x$

12. If $\frac{\pi}{2} < \alpha < \pi, \pi < \beta < \frac{3\pi}{2}$ $\sin(\alpha) = \frac{15}{17}$ and $\tan\beta = \frac{12}{5}$, the value of $\sin(\beta - \alpha)$ is

(a) $-\frac{21}{221}$ (b) $\frac{21}{221}$ (c) $\frac{-171}{221}$ (d) $\frac{171}{221}$

13. If $\tan\alpha = \frac{x}{x+1}$ and $\tan\beta = \frac{1}{2x+1}$, then $\alpha + \beta$ is

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

14. The value of the expression $\frac{\sin^3 x}{1+\cos x} + \frac{\cos^3 x}{1+\sin x}$ is /are

(a) $\sqrt{2}\cos\left(\frac{\pi}{4} - x\right)$ (b) $\sqrt{2}\cos\left(\frac{\pi}{4} + x\right)$
(c) $\sqrt{2}\sin\left(\frac{\pi}{4} - x\right)$ (d) $\sqrt{2}\sin\left(\frac{\pi}{4} + x\right)$

15. If $\alpha\cos^2 3\theta + \beta\cos^4\alpha = 16\cos^6\theta + 9\cos^2\theta$ is an identity, then

(a) $\alpha = 1, \beta = 18$ (b) $\alpha = 1, \beta = 24$ (c) $\alpha = 3, \beta = 24$ (d) $\alpha = 4, \beta = 2$

16. If $\operatorname{cosec}\theta + \cot\theta = 5/2$, then the value of $\tan\theta$ is

(a) $\frac{15}{16}$ (b) $\frac{21}{20}$ (c) $\frac{15}{21}$ (d) $\frac{20}{21}$

17. The value of the expression $(\sqrt{3}\sin 75^\circ - \cos 75^\circ)$ is

(a) $2\sin 15^\circ$ (b) $1+\sqrt{3}$ (c) $2\sin 105^\circ$ (d) $\sqrt{2}$

18. Given that $\frac{\pi}{2} < x < \pi,$ then the value of the expression $\sqrt{\frac{1-\sin x}{1+\sin x}} + \sqrt{\frac{1+\sin x}{1-\sin x}}$ is

(a) $\frac{2}{\cos x}$ (b) $\frac{1}{\sin x}$ (c) $-\frac{2}{\cos x}$ (d) non-existing

19. The value of the expression $\frac{1-4\sin 10^{o}\sin 70^{o}}{2\sin 10^{o}}$ is

(a) 1/2 (b) 1 (c) 2 (d)none of these

20. In a triangle ABC, the value of the expression cosec A (sin B cos C + cos B sin C) is

(a) 1 (b) c/a (c) a/c (d)none of these

21. In an equilateral triangle, 3 coins of radii 1 unit each are kept so that they touch each other and also the sides of the triangle. Area of the triangle is

(a) $4+2\sqrt{3}$ (b) $6+4\sqrt{3}$ (c) $12+\frac{7\sqrt{3}}{4}$ (d) $3+\frac{7\sqrt{3}}{4}$

*****

Answers

Exercise - I

Only One Option is correct

1.	(a)	2.	(d)	3.	(d)	4.	(d)	5.	(b)
6.	(c)	7.	(d)	8.	(c)	9.	(b)	10.	(b)
11.	(c)	12.	(a)	13.	(c)	14.	( )	15.	( )

More Than One Choice Correct

1.	(a,b)	2.	(a, c)	3.	(a, b)	4.	(a, c)	5.	(a, b, c, d)
6.	(a, b, c, d)	7.	(a, d)	8.	(a, b)	9.	(a, b, d)	10.	(b, c)

Exercise - II

Assertion and Reason

1.	(a)	2.	(b)	3.	(d)	4.	(a)	5.	(d)

Passage – I

1. (c) 2. (a)

Passage – II

1. (b) 2. (b) 3. (c)

Matching Type Questions

1. (a) 2. (b)

Exercise - IV

IIT-JEE Level Problem

1.	(d)	2.	(a)	3.	(b)	4.	(c)	5.	(a)
6.	(c)	7.	(c)	8.	(b)	9.	(d)	10.	(b)
11.	(b)	12.	(d)	13.	(b)	14.	(a)	15.	(b)
16.	(d)	17.	(d)	18.	(c)	19.	(b)	20.	(a)
21.	(b)								

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TRIGONOMETRIC EQUATIONS

JOHN NAPIER

Napier's study of mathematics was only a hobby and in his mathematical works he writes that he often found it hard to find the time for the necessary calculations between working on theology. He is best known, however, for his invention of logarithms but his other mathematical contributions include a mnemonic for formulae used in solving spherical triangles, two formulae known as "Napier's analogies" used in solving spherical triangles and an invention called "Napier's bones" used for mechanically multiplying dividing and taking square roots and cube roots. Napier also found exponential expressions for trigonometric functions, and introduced the decimal notation for fractions.

Unlike the logarithms used today, Napier's logarithms are not really to any base although in our present terminology it is not unreasonable (but perhaps a little misleading) to say that they are to base $1/e$. Certainly they involve a constant 107 which arose from the construction in a way that we will now explain. Napier did not think of logarithms in an algebraic way, in fact algebra was not well enough developed in Napier's time to make this a realistic approach.

Napier suggested to Briggs the new tables should be constructed with base 10 and with log 1 = 0, and indeed Briggs did construct such tables. In fact Briggs spent a month with Napier on his first visit of 1615, made a second journey from London to Edinburgh to visit Napier again in 1616 and would have made yet a third visit the following year but Napier died in the spring before the planned summer visit.

Napier presented a mechanical means of simplifying calculations. He described a method of multiplication using "numbering rods" with numbers marked off on them.

TRIGONOMETRIC EQUATION

INTRODUCTION

Trigonometric equation can be divided into two categories: Identities and conditional equations:

Identities are true for any angle, where as conditional equations are true for certain angles. Identities can be tested, checked, and created using knowledge of fundamental identities.

5.1 SOLUTION OF A TRIGONOMETRIC EQUATION

A value of the unknown angle that satisfies the trigonometric equation is called its solution.

Since all trigonometric ratios are periodic in nature, generally a trigonometric equation has more than one solution or an infinite number of solutions.

These are basically three type of solutions:

(i) Particular solution:

A specific value of unknown angle satisfying the equation.

(ii) Principle solution:

Smallest numerical value of the unknown angle satisfying the equation (numerically smallest particular solution)

(iii) General solution:

Complete set of values of the unknown angle satisfying the equation. It contains all particular solutions as well as principle solution.

 Method To Find Principle Value

(i) **First draw a trigonometric circle and mark the quadrant in which the angle may lie.**

(ii) **Select anticlockwise direction for 1st and 2nd quadrants and select clockwise for 3rd and 4th quadrants.**

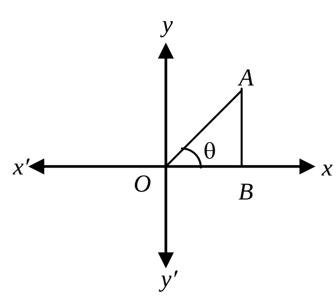
(iii) **Find the angle in the first rotation.**

(iv) **Select the numerically least angle from these two values, the angle thus found will be the principle value.**

(v) **In case, two angles (one with positive sign and the other with negative sign) qualify for the numerically least angle then we select the angle with positive sign as the principle value.**

5.2 General Solution of Trigonometric Equations

(i) **$\sin\theta = 0$**



$\Rightarrow \dfrac{AB}{OA} = 0 \Rightarrow AB = 0$

$\Rightarrow$ **OA coincide with x axis**

$\therefore\ \theta = 0, \pi, 2\pi, \ldots\ldots$ **(anticlockwise) ….(i)**

$\theta = -\pi, -2\pi, -3\pi \ldots\ldots$ **(clockwise) …(ii)**

Combining (i) and (ii)

$\theta = n\pi$, where $n \in I$

(ii) **$\cos\theta = 0$**

⇒ $\frac{OB}{OA}=0$ ⇒ ***OB* = 0**

⇒ ***OA* coincide with *y* axis**

∴ **θ = π/2, 3π/2,……(anticlockwise) …(iii)**

θ = –π/2, –3π/2….(clockwise) …(iv)

Combining (iii) & (iv)

θ = (2*n*+1)π/2, where *n* ∈ *I*

(iii) **tan θ = 0**

⇒ $\frac{\sin\theta}{\cos\theta}=0$

⇒ **sin θ = 0**

∴ **θ = *n*π, where *n* ∈ *I***

(iv) **if sin = sin then** $\theta = n\pi + (-1)^n \alpha$ **, where** $n \in I$

we have, sin θ = sin α

⇒ **sin θ – sin α = 0**

⇒ $2\cos\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right)=0$

⇒ $\cos\left(\frac{\theta+\alpha}{2}\right)=0$ or $\sin\left(\frac{\theta-\alpha}{2}\right)=0$

⇒ $\left(\frac{\theta+\alpha}{2}\right)=(2m+1)\frac{\pi}{2},\ m\in I$ **or** $\left(\frac{\theta-\alpha}{2}\right)=m\pi,\ m\in I$

⇒ **θ = (2*m* + 1) – , *m*∈ *I* or θ = (2*m*) + , *m* ∈ *I*.**

⇒ **θ = *n* + (–1)*ⁿ* , where *n*∈ *I* and ∈** $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

(v) **if cos = cos then** $\theta = 2n\pi \pm \alpha$ **, where** $n \in I$

we have, cos θ = cos α

⇒ **cos θ – cos α = 0**

⇒ $-2\sin\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right)=0$

⇒ $\sin\left(\frac{\theta+\alpha}{2}\right)=0$ or $\sin\left(\frac{\theta-\alpha}{2}\right)=0$

(EMBED Equation.DSMT4 or EMBED Equation.DSMT4

((= 2n – , or (= 2n + , n (I.

((= 2n ± , where n(I and ([0,].

(vi) if tan = tan then EMBED Equation.DSMT4 , where EMBED **Equation.DSMT4** $n \in I$

We have, tan θ = tan α, where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\Rightarrow \quad \frac{\sin\theta}{\cos\theta} = \frac{\sin\alpha}{\cos\alpha}$

$\Rightarrow$ **sin θ cos α – cos θ sin α = 0**

$\Rightarrow$ **sin (θ – α) = 0**

$\Rightarrow \quad \theta - \alpha = n\square, n \in I$

$\Rightarrow \quad \theta = n\square + \alpha, n \in I$

$\Rightarrow \quad \theta = n\,\square + \square$, **where** $n \in I$ **and** $\square \in (-\square/2, \square/2)$

(vii) **if** $\sin^2 \square = \sin^2 \square$ **then** $\square = n\square \pm \square\square$, **where** $n \in I\square$

$\Rightarrow \quad \frac{1-\cos 2\theta}{2} = \frac{1-\cos 2\alpha}{2}$

$\Rightarrow$ **cos 2θ = cos 2α**

$\Rightarrow \quad 2\theta = 2n\square \pm 2\square, n \in I.$

$\Rightarrow \quad \theta = n\,\square \pm \square, n \in I.$

(viii) **if** $\cos^2 \square = \cos^2 \square$ **then** $\square = n\square \pm \square\square$, **where** $n \in I$

$\Rightarrow \quad \frac{1+\cos 2\theta}{2} = \frac{1+\cos 2\alpha}{2}$

$\Rightarrow$ **cos 2θ = cos 2α**

$\Rightarrow \quad 2\theta = 2n\square \pm 2\square, n \in I.$

$\Rightarrow \quad \theta = n\,\square \pm \square, n \in I.$

(ix) **if** $\tan^2 \square = \tan^2 \square$ **then** $\square = n\square \pm \square\square$, **where** $n \in I$

$\Rightarrow \quad \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$

$\Rightarrow$ **cos 2θ = cos 2α**

$\Rightarrow \quad 2\theta = 2n\square \pm 2\square, n \in I.$

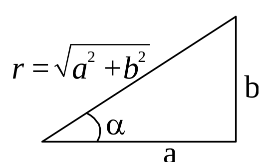
$\Rightarrow \quad \theta = n\,\square \pm \square, n \in I.$

5.3 Solution of Trigonometric Equations

□ Equation of the Form $a \cos\theta + b \sin\theta = c$,

if $|c| \le \sqrt{a^2 + b^2}$ **then the solution of the given equation is obtained by putting** $a = r\cos\alpha$ **and** $b = r\sin\alpha$

where $r = \frac{c}{\sqrt{a^2+b^2}}$ **and** $\square = \tan^{-1}\left(\frac{b}{a}\right)$



then the equation becomes

$r(\cos\alpha\cos\theta + \sin\alpha\sin\theta) = c$

(EMBED Equation.DSMT4

 – = 2n ±

(= 2n ± + , **where tan = *b*/*a* is the general solution.**

Alternatively, putting a = r sin and b = r cos where = $r = \sqrt{a^2 + b^2}$

$\Rightarrow \quad \sin(\theta + \alpha) = \frac{c}{\sqrt{a^2 + b^2}} = \sin\gamma$ (say),

$\Rightarrow \theta = n\pi + (-1)^n \gamma - \alpha$ **then ,** $\tan\alpha = \frac{a}{b}$ **is the general solution**

both the methods will give the same set of values of .

ILLUSTRATIONS

***Illustration* 1**

Solve, $\cos^2 x - \sin x - \frac{1}{4} = 0$

Solution

Replacing $\cos^2 x$ by $1 - \sin^2 x$,we get a quadratic in sine the form

$4\sin^2 x + 4\sin x - 3 = 0$

$(2\sin x + 3)(2\sin x - 1) = 0$

$\sin x \neq -\frac{3}{2}$ since $-1 \le \sin x \le 1$

If $\sin x = \frac{1}{2} \Rightarrow$ Principle solution is $x = \frac{\pi}{6}$

General solution is $x = n\pi + (-1)^n \frac{\pi}{6}$, $n \in I$

Illustration 2

Solve the equation $\tan 6\theta - \tan 4\theta = 0$

Solution

Given $\tan 6\theta = \tan 4\theta$ (from $\tan\theta = \tan\alpha$)

$\Rightarrow \quad 6\theta = n\pi + 4\theta$

$2\theta = n\pi$

$\theta = \frac{n\pi}{2}, \quad n \in I$

Illustration 3

Find the number of distinct solution of sec *x* + tan *x* = $\sqrt{3}$, where 0 ≤ x (3.

Solution

Here,

$\text{Sec } x + \tan x = \sqrt{3}$

$\Rightarrow$ $1 + \sin x = \sqrt{3} \cos x$

or $\sqrt{3} \cos x - \sin x = 1$

dividing both sides by $\sqrt{a^2 + b^2}$ i.e., $\sqrt{4}$ = 2, we get

$\Rightarrow$ $\cos\left(x + \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right)$

(EMBED Equation.DSMT4

or EMBED Equation.DSMT4

(when x = 2n□ + EMBED Equation.DSMT4 there are solution for n = 0, 1

and when x = $2n$□ – $\frac{\pi}{2}$ there are no solution.

$\therefore$ Total number of solutions are 2.

Illustration 4

Solve $\tan\theta + \tan 2\theta + \tan\theta \cdot \tan 2\theta = 1$

Solution

We rewrite the equation as

$$\tan\theta + \tan 2\theta = 1 - \tan\theta \tan 2\theta$$

$$\frac{\tan\theta + \tan 2\theta}{1 - \tan\theta - \tan 2\theta} = 1$$

$$\tan 3\theta = 1 = \tan \pi/4$$

$$3\theta = n\pi + \frac{\pi}{4}$$

$$\theta = \frac{n\pi}{3} + \frac{\pi}{12},\ n \in I$$

***Illustration* 5**

Find the general solution of $|\text{cosec } (\theta + \pi/4)| = \frac{2}{\sqrt{3}}$

Solution

Given equation is same as

$$\text{cosec}^2\left(\theta + \frac{\pi}{4}\right) = \frac{4}{3} = \text{cos}ec^2\ \frac{\pi}{3}$$

$$\theta + \frac{\pi}{4} = n\pi \pm \frac{\pi}{3}$$

$$\theta = n\pi + \frac{\pi}{12} \quad \text{or} \quad n\pi + \frac{7\pi}{12}$$

***Illustration* 6**

Solve the equations

(i) $\sin 9\theta = \sin\theta$ *(ii)* $\sin 5x = \cos 2x$

Solution

(i) Given equation is $\sin 9\theta = \sin\theta$

or $\sin 9\theta - \sin\theta = 0$

or $2\cos\left(\frac{9\theta+\theta}{2}\right)\cdot\sin\left(\frac{9\theta-\theta}{2}\right) = 0$

or $\cos 5\theta \sin 4\theta = 0$

either $\cos 5\theta = 0$ or $\sin 4\theta = 0$

$5\theta = (2n+1)\frac{\pi}{2}$ or $4\theta = k\pi$

$\Rightarrow \theta = \frac{k\pi}{4}$ or $(2n+1)\frac{\pi}{10}$

where, $n, k = 0, \pm 1, \pm 2, ...$

(ii) Given equation is

$\sin 5x = \cos 2x$

or $\cos 2x = \cos\left(\frac{\pi}{2} - 5x\right)$

$\Rightarrow 2x = 2n\pi \pm \left(\frac{\pi}{2} - 5x\right)$

Taking positive sign

$$2x = 2n\pi + \frac{\pi}{2} - 5x$$

$$7x = 2n\pi + \frac{\pi}{2}$$

$$x = (4n+1)\frac{\pi}{14}$$

Taking negative sign

$$2x = 2n\pi - \frac{\pi}{2} + 5x$$

$$-3x = (4n-1)\frac{\pi}{2},\ x = -(4n-1)\frac{\pi}{6}$$

Hence, $x = (4n+1)\frac{\pi}{14}$ or $-(4n-1)\frac{\pi}{6}$ where $n = 0, \pm 1, \pm 2, ...$

- Equation of the Form

$a_0 \sin^n x + a_1 \sin^{n-1} x \cos x + a_2 \sin^{n-2} x \cos^2 x + \ldots + a_n \cos^n x = 0$,

where $a_0, a_1, \ldots a_n$ are real numbers and the sum of the exponents in sin x and cos x in each term is equal to n, are said to be homogeneous with respect to sin x and cos x . For cos $x \neq 0$, above equation can be written as,

$a_0 \tan^n x + a_1 \tan^{n-1} x + \ldots + a_n = 0$

- A Trigonometric Equation of the Form

R (sin kx, cos nx, tan mx, cot lx) = 0 …(i)

where R is a rational function of the indicated arguments and (k, l, m, n are natural numbers) can be reduced to a rational equation with respect to the arguments sin x, cos x, tan x and cot x by means of the formulae for trigonometric functions of the sum of angles (in particular, the formulae for double and triple angles) and then reduce equation (i) to a rational equation with respect to the unknown, $t = \tan \frac{x}{2}$ by means of the formulas,

$$\sin x = \frac{2\tan \frac{x}{2}}{1+\tan^2 \frac{x}{2}}, \cos x = \frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}},$$

$$\tan x = \frac{2\tan \frac{x}{2}}{1-\tan^2 \frac{x}{2}}, \cot x = \frac{1-\tan^2 \frac{x}{2}}{2\tan \frac{x}{2}}$$

- Equation of the Form

R (sin x + cos x, sin x . cot x) = 0

Where R is a rational function of the arguments in brackets,

Put sin x + cos x = t …(i) and use the following identity

$(\sin x + \cos x)^2 = \sin^2 x + \cos^2 x + 2 \sin x \cos x$

$= 1 + 2 \sin x \cos x$

$\Rightarrow \sin x \cos x = \frac{t^2 - 1}{2}$

Taking (i) and (ii) into account, we can reduce given equation into;

$$R\left(t, \frac{t^2-1}{2}\right) = 0$$

Similarly,

By the substitution (sin x – cos x) = t, we can reduce the equation of the form;

R (sin x – cos x, sin x cos x) = 0 to an equation;

$$R\left(t,\ \frac{1-t^2}{2}\right)=0\ .$$

ILLUSTRATIONS

***Illustration* 7**

Solve the equation; $5\sin^2 x - 7\sin x\cos x + 16\cos^2 x = 4$

Solution

To solve this equation; we use the fundamental formula trigonometric identity,

$$\sin^2 x + \cos^2 x = 1$$

writing the equation in the form,

$$5\sin^2 x - 7\sin x\ .\ \cos x + 16\cos^2 x = 4(\sin^2 x + \cos^2 x)$$

$$\Rightarrow\quad \sin^2 x - 7\sin x\cos x + 12\cos^2 x = 0$$

dividing by $\cos^2 x$ on both side we get,

$$\tan^2 x - 7\tan x + 12 = 0$$

Now it can be factorized as;

$$(\tan x - 3)(\tan x - 4) = 0$$

$$\Rightarrow\quad \tan x = 3, 4$$

i.e., $\tan x = \tan(\tan^{-1} 3)$ or $\tan x = \tan(\tan^{-1} 4)$

$\Rightarrow\quad x = n□ + \tan^{-1} 3$ or $x = n□ + \tan^{-1} 4$.

***Illustration* 8**

Solve the equation; $(\cos x - \sin x)\left(2\tan x + \frac{1}{\cos x}\right) + 2 = 0$

Solution

Let $t = \tan x/2$, and using the formula. We get,

$$\left\{\frac{1-\tan^2 x/2}{1+\tan^2 x/2} - \frac{2\tan x/2}{1+\tan^2 x/2}\right\}\left\{\frac{4\tan x/2}{1-\tan^2 x/2} + \frac{1+\tan^2 x/2}{1-\tan^2 x/2}\right\} + 2 = 0$$

$$\left(\frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2}\right)\left(\frac{4t}{1-t^2} + \frac{1+t^2}{1-t^2}\right) + 2 = 0$$

$$\Rightarrow\quad \frac{3t^2 + 6t^3 + 8t^2 - 2t - 3}{(t^2+1)(1-t^2)} = 0$$

its roots are; $t_1 = \frac{1}{\sqrt{3}}$ and $t_2 = -\frac{1}{\sqrt{3}}$.

Thus the solution of the equation reduces to that of two elementary equations,

$\tan\frac{x}{2}=\frac{1}{\sqrt{3}}$, $\tan\frac{x}{2}=-\frac{1}{\sqrt{3}}$

$\Rightarrow\ \frac{x}{2}=n\pi\pm\frac{\pi}{6}$

$\Rightarrow\ x = 2n\square \pm \frac{\pi}{3}$, is required solution.

***Illustration* 9**

Solve the equation sin x + cos x – $2\sqrt{2}$ sin x cos x = 0

Solution

Let (sin x + cos x) = t and using the equation

sin x . cos x = $\frac{t^2-1}{2}$, we get

$$t-2\sqrt{2}\left(\frac{t^2-1}{2}\right)=0$$

$$\Rightarrow\ \sqrt{2}t^2-t-\sqrt{2}=0$$

The numbers $t_1=\sqrt{2}, t_2=-\frac{1}{\sqrt{2}}$ are roots of this quadratic equation.

Thus the solution of the given equation reduces to the solution of two trigonometric equations

sin x + co x = $\sqrt{2}$ and sin x + cos x = $-\frac{1}{\sqrt{2}}$

$\Rightarrow\ \sin\left(x+\frac{\pi}{4}\right)=1$ and $\sin\left(x+\frac{\pi}{4}\right)=-\frac{1}{2}$

$\Rightarrow\ x=2n\pi+\frac{\pi}{4}$ and $x=n\pi+(-1)^n\frac{\pi}{6}-\frac{\pi}{4}$

5.4 Solving Simultaneous Equations

Find the solution of two equations satisfied simultaneously. We may divide the problem into two categories.

(i) Two equations is one 'unknown' satisfied simultaneously.

(ii) Two equations in two 'unknowns' satisfied simultaneously**.**

ILLUSTRATIONS

***Illustration* 10**

If tan ($A - B$) = 1 and sec ($A + B$) = $\frac{2}{\sqrt{3}}$, find the smallest positive values of A and B and their most general values.

Solution

For the smallest positive values, find $A + B$ and $A - B$ between 0 and 2□ from the given equations.

Since A and B are positive angles, $A + B > A - B$. Solve the two get A and B.

For the most general values, find he general values of $A - B$. Solve the two get A and B. For the most general values, find the general values of $A - B$ and $A + B$ by solving the given equations separately. Solve the two to get A and B.

$$\tan(A - B) = 1 \Rightarrow A - B = \frac{\pi}{4} \text{ or } \frac{5\pi}{4} \qquad \ldots\text{(i)}$$

Also sec ($A + B$) $\Rightarrow$ $A + B = \frac{\pi}{6}$ or $\frac{11\pi}{6}$

Since $A + B > A - B$,

$$A + B = \frac{11\pi}{6}$$

Solving (i) & (ii), we get

$$A = \frac{25\pi}{4}, B = \frac{19\pi}{24} \quad \text{or} \quad A = \frac{37\pi}{24}, B = \frac{7\pi}{24}$$

For the most general values,

$$\tan(A - B) = 1 \Rightarrow A - B = n□ + \frac{\pi}{4} \qquad \ldots\text{(ii)}$$

$$\sec(A + B) = \frac{2}{\sqrt{3}}$$

$$\Rightarrow \cos(A + B) = \frac{\sqrt{3}}{2}$$

$$\Rightarrow A + B = 2m□ \pm \frac{\pi}{6} \qquad \ldots\text{(iii)}$$

Solving (ii) and (iii), we get

$$A = \frac{1}{2}\left[(2m + n)\pi + \frac{\pi}{4} \pm \frac{\pi}{6}\right]$$

$$B = \frac{1}{2}\left[(2m - n)\pi - \frac{\pi}{4} \pm \frac{\pi}{6}\right] \text{ where } m, n \in I.$$

***Illustration* 11**

Solve the system of equations

$x + y = 2□/3$ and $\frac{\sin x}{\sin y} = 2$

Solution

Let us reduce the second equation of the system to the form,

$\sin x = 2 \sin y$

using $x + y = 2□/3$ we get,

$\sin x = 2 \sin (2□/3 – x)$

$\Rightarrow \quad \sin x = 2\left(\sin\frac{2\pi}{3}.\cos x - \cos\frac{2\pi}{3}.\sin x\right)$

$$= 2\left(\frac{\sqrt{3}}{2}.\cos x + \frac{1}{2}.\sin x\right)$$

$\Rightarrow \quad \sin x = \sqrt{3}\cos x + \sin x$

$\Rightarrow \quad \cos x = 0$

$\Rightarrow \quad x = (2n + 1)\frac{\pi}{2}$

$\Rightarrow \quad x = \frac{\pi}{2} + n\pi$

Substituting in $x + y = 2□/3$. We get, $y = – n□ + □/6$

$\therefore \; x = \frac{\pi}{2} + n\pi, y = \frac{\pi}{6} - n\pi,$ where $n \in$ integer.

***Illustration* 12**

If $r > 0$, $– □ \le □ \le □$ and r, □ satisfy $r \sin □ = 3$ and $r = 4 (1 + \sin □)$ then find the possible solutions of the pair (r, θ).

Solution

Here,

$r = 4 (1 + \sin\theta)$ and $r \sin\theta = 3$

eliminating θ from above equations

$r = 4\left(1 + \frac{3}{r}\right)$

$\Rightarrow \quad r^2 = 4r – 12 = 0$

$r = 6$ and $r = – 2$

$r \sin\theta = 3$

$\sin\theta = \frac{1}{2}$ and $\sin\theta = -\frac{3}{2}$

neglecting $\sin □ = –3/2$. We get, $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

(r, □) = (6, □/6) and (6, 5□/6) are the required pairs.

PRACTICE EXERCISE

1. Solve for x, $(-\pi \le x \le \pi)$, the equation $2\left[\cos x + \cos 2x\right] + \sin 2x\ (1 + 2\cos x) = 2\sin x$

2. Find the range of y such that the equation y cos x = sin x has a real solution. Also find x for $y = 1$ such that $0 < x < 2\pi$.

3. If $\sin(\pi\cos\theta) = \cos(\pi\sin\theta)$, then prove that $\cos\left(\theta \pm \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$

4. Solve $\sqrt{3}\cos\theta + \sin\theta = 1$ for $-2\pi < \theta < 2\pi$

5. Find the solution of $\sin x > -\frac{1}{2}$

6. Solve the equation $\sec^2 x = 1 - y^2$

7. Find all values of θ lying between 0 and 2π, satisfying the equations

   $r\sin\theta = \sqrt{3}$ ....(i)

   $r + 4\sin\theta = 2(\sqrt{3} + 1)$ …(ii)

8. Find the value of θ for which the following system has a non trivial solution:

   $(\sin 3\theta)\, x - y + z = 0$

   $(\cos 2\theta)\, x + 4y + 3z = 0$

   $2x + 7y + 7z = 0$

9. Solve the inequality $\frac{5}{4}\sin^2 x + \sin^2 x \,.\, \cos^2 x > \cos 2x$

10. Solve the inequality, $\sin x \cos x + \frac{1}{4}\tan x \ge 1$

Answers

1. $x = \pi, -\pi, -\frac{\pi}{3}, -\frac{\pi}{2}$

2. $x = \frac{\pi}{2}, \pi$

4. $-\frac{3\pi}{2}, -\frac{\pi}{6}, -\frac{\pi}{2}, \frac{11\pi}{6}$

5. $2\pi n - \frac{\pi}{6} < \theta < (2n+1)\pi + \pi/6$

6. $x = n\pi$ and $y = 0$

7. $\frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}$ and $\frac{5\pi}{6}$

8. $\theta = 4\pi$ or $n\pi + (-1)^n \frac{\pi}{6}$

9. $n\pi + \frac{\pi}{6} < x < n\pi + \frac{5\pi}{6}$

10. $x \in \left[n\pi + \frac{\pi}{4}, n\pi + \frac{\pi}{2}\right]$.