STRAIGHT LINE

René Descartes

René Descartes (1596 –1650), also known as Renatus Cartesius, was a highly influential French philosopher, mathematician, scientist, and writer.

The Cartesian coordinate system used in plane geometry and algebra being named after him, and he was one of the key figures in the Scientific Revolution.

As the inventor of the Cartesian coordinate system, Descartes founded analytic geometry, that bridge between algebra and geometry crucial to the invention of calculus and analysis.

Descartes's theory provided the basis for the calculus of Newton and Leibniz, by applying infinitesimal calculus to the tangent problem, thus permitting the evolution of that branch of modern mathematics.

Descartes invented analytic geometry, and discovered the law of conservation of momentum. He outlined his views on the universe in his Principles of Philosophy.

Descartes also made contributions in the field of optics; for instance, he showed by geometrical construction using the Law of Refraction that the angular radius of a rainbow is 42°.

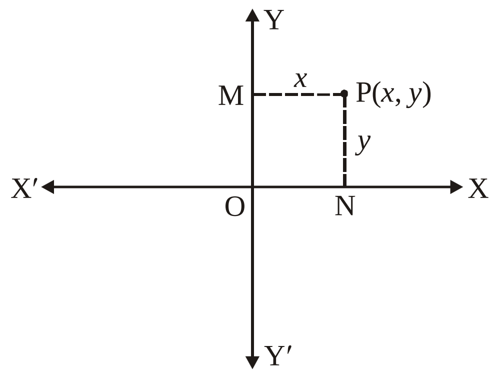
One of Descartes most enduring legacies was his development of Cartesian geometry, the algebraic system taught in schools today - essentially he invented graphs and graph paper. He also created exponential notation, indicated by numbers written in what is now referred to as superscript (x^2).

Introduction

Let X'OX and Y'OY are two mutually perpendicular lines in the plane. Then X'OX is called x-axis and Y'OY is called y-axis and two lines taken together is called co-ordinate axis. Point O is called as origin.

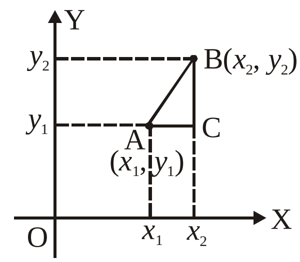
Let P be any point in the plane and if we draw PM and PN parallel to OX and OY respectively then PM is called x-coordinate or abscissa of point P and PN is called y-coordinate or ordinate of point P.

Let PM = ON = x and PN = OM = y then position of point P in the plane of the axis of the co-ordinates is represented by ordered pair (x, y).



❑ Distance Between Two Points

If A and B are two points whose coordinates are (x_1, y_1) and (x_2, y_2) respectively.



Then AB = $\sqrt{AC^2 + BC^2}$

□ AB = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

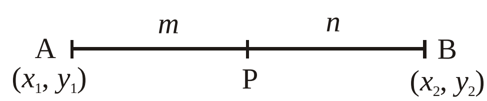
Note: Distance of any point $P(x, y)$ from origin

OP = $\sqrt{x^2 + y^2}$

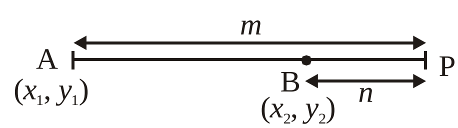
❑ Section Formula

Let A (x_1, y_1) and B (x_2, y_2) are two points and a point P lies on the line joining these two points.

(i) If point P divides join of two points A and B in the ratio of $m : n$ internally then co-ordinate of point P will be $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$



(ii) If point P divides join of points A and B in the ratio of $m : n$ externally then co-ordinate of point P will be $\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$



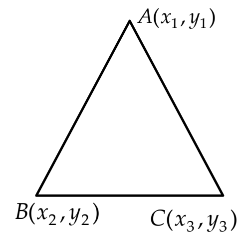
NOTE:

(i) Hence co-ordinates of mid point of line joining points (x_1, y_1) and (x_2, y_2) will be $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

(ii) Generally for convenience we take the ratio k : 1 (instead of $m : n$). If value of k is positive then it is internal division otherwise it is an external division.

❑ Area of Triangle

Let ABC is a triangle whose vertices are A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3) then area of triangle ABC is



$$\Delta = \frac{1}{2} \,|\, x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2) \,|$$

□ $$\Delta = \frac{1}{2} \left\| \begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{matrix} \right\|$$

PRACTICE EXERCISE

1. If two vertices of an equilateral triangle be (0, 0) $(3, \sqrt{3})$, find the third vertex.
2. If the coordinates of two points *A* an *B* are (3, 4) and (5, –2) respectively. Find the coordinates of any point *P* if *PA* = *PB* and Area of Δ*PAB* = 10.
3. Prove that the points (–2, –1), (1, 0), (4, 3) and (1, 2) are the vertices of a parallelogram. Is it a rectangle?
4. If the coordinates of the mid-points of the sides of a triangle are (1, 2) (0, –1) and (2, –1). Find the coordinates of its vertices.
5. If the segment joining the points *A*(*a*, *b*) and *B* (*c*, *d*) student an angle θ at the origin, prove that

$$\cos\theta = \frac{ac + bd}{\sqrt{(a^2 + b^2)\,(c^2 + d^2)}}.$$

Answers

1. $(3, -\sqrt{3})$ **2.** $\left|\frac{x + y - 2}{7}\right|$ **4.** *A*(1, –4), *B*(3, 2) and *C*(–1, 2)

3.1 Locus

When a point moves in accordance with certain geometrical conditions, the point traces out a path. This path is called locus of point.

❑ Method to Find The Locus of a Point

(i) Assume the co-ordinates of the point (*h*, *k*) whose locus is to be found.

(ii) Write the given conditions as equations in terms of known quantities involving *h*, *k*.

(iii) Some times some unknown quantities are included as parameters.

(iv) Eliminate the variables (parameters) so that eliminated contains only *h*, *k* and known quantities.

(v) Replace *h* by *x* and *k* by *y* in the eliminated.

The resulting equation will give the equation of locus of point.

ILLUSTRATIONS

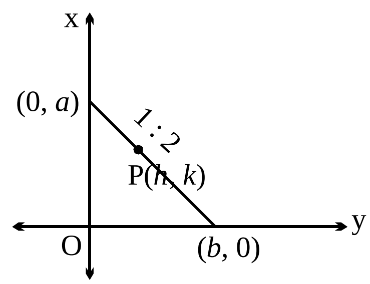
***Illustration* 1**

The ends of a rod of length l move on two mutually perpendicular lines. Find the locus of the point on the rod, which divides it in the ratio 2 : 1.

Solution

Suppose the two perpendicular lines are $x = 0$ and $y = 0$ and intercepts a and b are cut respectively on the two lines. Then the two points on these lines are $(0, a)$ and $(b, 0)$. The point P has coordinates given by

$$h = \frac{b + 2.0}{2+1},\ k = \frac{2.a + 1.0}{2+1} \quad □ \quad b = 3h \quad \text{and} \quad a = \frac{3k}{2}$$



Also $\quad l^2 = a^2 + b^2$

$$l^2 = \left(\frac{3k}{2}\right)^2 + (3h)^2$$

Thus the required locus is

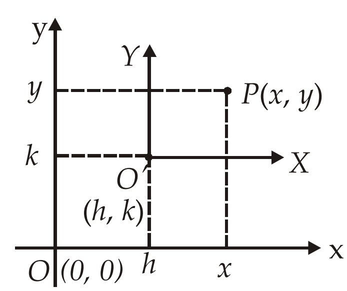
$$x^2 + \frac{y^2}{4} = \frac{l^2}{9}$$

3.2 Transformation of Axes

❑ **Shifting of Origin**

Let origin O be shifted to new position O'(h, k) without changing the direction of the axes. Let P(x, y) be any point in the plane. Then co-ordinate of point P(x, y) in the new plane are obtained by formulae.

$x = X + h$ and $y = Y + k$



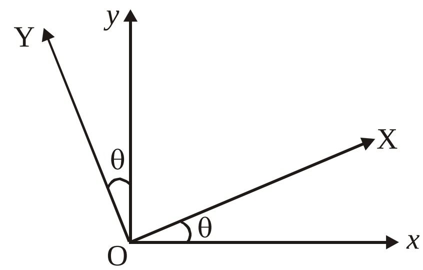
Where P(x, y) is referred to system Oxy whereas P(X, Y) is referred to system O′XY.

Hence $\quad X = x – h \quad$ and $\quad Y = y – k$

❑ **Rotation of Axes**

Let Ox and Oy are old axes and OX, OY are the new axes obtained by rotating Ox and Oy respectively through an angle □ about the origin in anticlockwise direction.

Let (x, y) and (X, Y) are the co-ordinate of any point P w.r.t. old axes and new axes respectively.



Then $\quad x = X\cos\theta - Y\sin\theta$

$\qquad\quad y = X\sin\theta + Y\cos\theta$

Also

$\qquad\quad X = x\cos\theta + y\sin\theta$

$\qquad\quad Y = y\cos\theta - x\sin\theta$

PRACTICE EXERCISE

6. Find the locus of a point, so that the join of (–5, 1) and (3, 2) subtends a right angle at the moving point.
7. Find the locus of the point of intersection of the lines $x \cos\alpha + y \cos\alpha = a$ and $x \sin\alpha - y \cos\alpha = b$, where α is a variable.
8. At what point the origin be shifted, if the coordinates of a point (4, 5) becomes (–3, 9) ?
9. Shift the origin to a suitable point so that the equation $y^2 + 4x + 8x - 2 = 0$ will not contain term in y and the constant term. Then the new origin is ______ .
10. If the axes are shifted to the point (1, –2) without rotation, what does the equation $2x^2 + y^2 - 4x + 4y = 0$ become?

Answers

6. $x^2 + y^2 + 2x - 3y - 13 = 0$ **7.** $x^2 + y^2 = a^2 + b^2$ **8.** (7, –4) **9.** $\left(\frac{3}{4}, -2\right)$

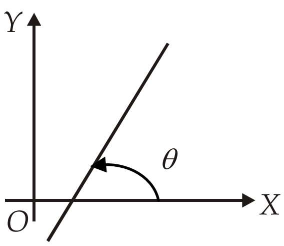
10. $5X^2 - 20Y^2 = 5a^2$ or $X^2 - 4Y^2 = a^2$

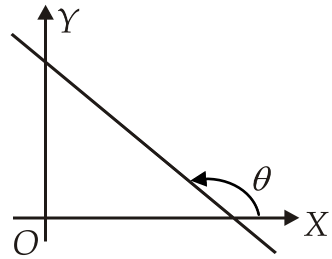
3.3 Straight Line

A straight line is a curve, every point on the line segment joining any two points on it lies on it.Any first degree equation in x, y which is of the form $ax + by + c = 0$ always represents a straight line (where a, b, c are constants and at least one of a and b is non-zero).

❑ **Slope**

(i) If a line makes angle θ with positive direction of x-axis in anticlockwise direction, then slope of line $m = \tan\theta$ (where $0 \le \theta < \pi$)





(ii) Slope of a line passing through two points (x_1, y_1) and (x_2, y_2) is $m = \frac{y_2 - y_1}{x_2 - x_1}$

❑ **Intercept of A Straight Line on The Axis**

If a line cuts the x-axis and y-axis at A and B respectively then OA and OB (with proper sign) are called the intercepts of the line on x-axis and y-axis respectively (where O is the origin)

If length of intercept is given (say 3) it means that intercept is either 3 or – 3.

If intercept of a line is k units then length of intercept will be $| k |$

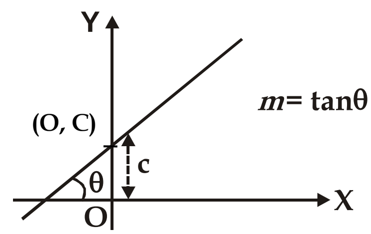
3.4 Standard Equations of Straight Line

❑ **Slope- Intercept form**

$y = mx + c$

where m = slope of line

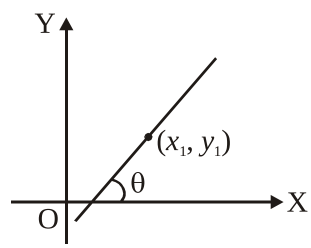
c = intercept of line on y-axis



(ii) **Slope-point form**

If a line passes through a point (x_1, y_1) whose slope is m (m = tan □) then equation of line is

$y - y_1 = m (x - x_1)$

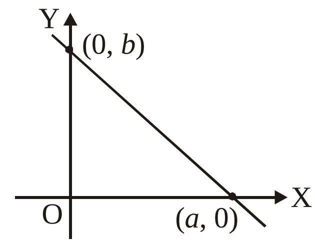


(iii) **Two point form**

Equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

(iv) **Intercept form**

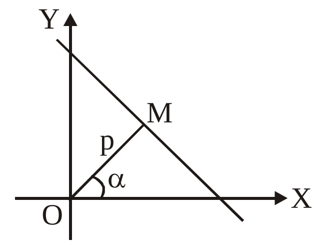


$$\frac{x}{a}+\frac{y}{b}=1$$

where a = intercept of line on x-axis

b = intercept of line on y-axis

(v) **Normal form**

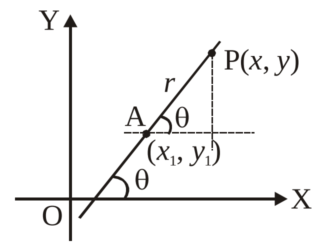


$x \cos \alpha + y \sin \alpha = p$

where p = length of perpendicular from origin to the line

α = angle which the perpendicular to the line from origin makes with x-axis.

(vi) **Parametric form**



Let a line passes through a point A(x_1, y_1) and makes angle θ with positive direction of x-axis. Let P(x, y) be any point on the line at a distance r from the point A(x_1, y_1) then parametric form of the line is used to find out the co-ordinate of point P(x, y).

Let AP = r

Hence from figure, $x - x_1 = r \cos \theta$

$y - y_1 = r \sin \theta$

$$\Rightarrow \quad \frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$$

The above equation is called as parametric equation of the given line.

NOTE: Co-ordinate of any point P on the line will be of the form $(x_1 + r\cos\theta,\ y_1 + r\sin\theta)$ where $|r|$ is the distance of point P from the fixed point (x_1, y_1).

❑ **Reduction of The General Equation of Line to Different Standard Forms**

(i) **Slope-Intercept form**

To reduce the equation Ax + By + C = 0 to the form $y = mx + c$

Given equation is Ax + By + C = 0 $\Rightarrow$ $y = \frac{-A}{B}x - \frac{C}{B}$

which is of the form $y = mx + c$

where $m=\frac{-A}{B}, c=-\frac{C}{B}$ (B $\neq$ 0)

Hence slope of line = $-\dfrac{A}{B}$

y-intercept of line = $-\dfrac{C}{B}$

(ii) **Intercept form**

To reduce the equation Ax + By + C = 0 to the form $\dfrac{x}{a}+\dfrac{y}{b}=1$

This reduction is possible only when C □ 0

Since the given equation is

Ax + By = – C

□ $\dfrac{x}{-C/A}+\dfrac{y}{-C/B}=1$

which is of the form $\dfrac{x}{a}+\dfrac{y}{b}=1$

& intercept on x-axis = $-\dfrac{C}{A}$

intercept on y-axis = $-\dfrac{C}{B}$

NOTE:

To find intercept of a line on x-axis, put y = 0 and value of x gives the intercept on x-axis.

Similarly intercept on y-axis can be obtained by putting x = 0 and solving for y.

(iii) **Normal form**

To reduce the equation Ax + By + C = 0 to the form x cos □ + y sin □ = p.

Given equation is Ax + By + C = 0 or Ax + By = – C

Case-I: when – C > 0, then normal form is

$$\frac{A}{\sqrt{A^2+B^2}}x+\frac{B}{\sqrt{A^2+B^2}}y=-\frac{C}{\sqrt{A^2+B^2}}$$

where cos □ = $\dfrac{A}{\sqrt{A^2+B^2}}$, sin □ = $\dfrac{B}{\sqrt{A^2+B^2}}$, $p=-\dfrac{C}{\sqrt{A^2+B^2}}$

Case-II: when – C < 0, then write the equation as – Ax – By = C

$$\frac{-A}{\sqrt{A^2+B^2}}x+\frac{-B}{\sqrt{A^2+B^2}}y=\frac{C}{\sqrt{A^2+B^2}}$$

where cos □ = $\dfrac{-A}{\sqrt{A^2+B^2}}$, sin □ = $\dfrac{-B}{\sqrt{A^2+B^2}}$, $p=\dfrac{C}{\sqrt{A^2+B^2}}$

NOTE:

In normal form, x cos □ + y sin □ = p, p is taken as positive.

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***Illustration* 2**

Find the equation of the straight line which makes an angles of 15° with the positive direction of x-axis and which cuts an intercept of length 4 on the negative direction of y-axis.

Solution

Here m = slope of the line = tan 15° $= \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3}$

and c = intercept on y-axis = – 4

Therefore, equation of the required line will be

$$y = \left(2-\sqrt{3}\right)x-4 \quad \text{or} \quad \left(2-\sqrt{3}\right)x-y-4=0$$

***Illustration* 3**

Find the equation of the straight line which passes through the point (3, 2) and whose gradient is $\frac{3}{4}$. Find the co-ordinates of the points on the lines that are 5 units away from the point (3, 2).

Solution

Let the required line make an angle □ with the positive direction x-axis and pass through P (3, 2).

As tan □ = $\frac{3}{4}$

□ cos □ = $\frac{4}{5}$ and sin □ = $\frac{3}{5}$

Now equation of the required line will be

$$\frac{x-3}{\frac{4}{5}} = \frac{y-2}{\frac{3}{5}} \quad \left[by\ formula\ \frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta}\right]$$

or 3 (x – 3) = 4 (y – 2) or 3x – 4y – 1 = 0 … (i)

Now the co-ordinates of the points which are at a distance of 5 units from P are (3 ± 5 cos □, 2 ± 5 sin □) or (3 ± 4), (2 ± 3)

or (7, 5) and (– 1, – 1)

***Illustration* 4**

Reduce the line 2x – 3y + 5 = 0, in slope intercept, intercept and normal form.

Solution

Slope Intercept Form

$y = \frac{2x}{3} + \frac{5}{3}, \ \tan\theta = m = \frac{2}{3}, \ c = \frac{5}{3}$

Intercept Form

$$\frac{x}{\left(-\frac{5}{2}\right)} + \frac{y}{\left(\frac{5}{3}\right)} = 1 \quad \text{hence} \quad a = -\frac{5}{2} \ , \quad b = \frac{5}{3}$$

Normal Form

$$-\frac{2x}{\sqrt{13}} + \frac{3y}{\sqrt{13}} = \frac{5}{\sqrt{13}}$$

$$\sin\alpha = \frac{3}{\sqrt{13}}, \cos\alpha = \frac{-2}{\sqrt{13}}, p = \frac{5}{\sqrt{13}}$$

Practice Exercise

11. Find the equation of the line which passes through the point (3, 4) and the sum of its intercept on the axes is 14.

12 A straight line is drawn through the point $P(2, 3)$ and is inclined at an angle 30° with the x-axis. Find the coordinates of two point on it at a distance 4 from P on either side of P.

13. The line joining two points $A(2, 0)$, $B(3, 1)$ is rotated about A in anti-clockwise direction through an angle of 15°. Find the equation of the line in the new position. If B goes to C in the new position, what will be the coordinates of C?

14. Find the distance of point (2, 3) from the line $2x - 3y + 9 = 0$ measured along a line $x - y + 1 = 0$.

15. Reduce the lines $3x - 4y + 4 = 0$ and $4x - 3y + 12 = 0$ to the normal form and hence determine which line is nearer to the origin.

Answers

11. $x + y = 7$ and $4x + 3y = 24$ **12.** $(2 \pm 2\sqrt{3}, 2 \pm 2)$ **13.** $\left(2 + \frac{1}{\sqrt{2}}, \frac{\sqrt{6}}{2}\right)$

14. $4\sqrt{2}$ **15.** $3x - 4y + 4 = 0$ is nearer

3.5 Position of Two Points With Respect To a Given Line

Let the given line be $ax + by + c = 0$ and $A(x_1, y_1)$, $B(x_2, y_2)$ be two points. If quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same signs, then both the points P and Q lie on the same side of the line $ax + by + c = 0$. If the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have opposite signs, then they lie on the opposite side of the line.

Note : Position of point P (x_1, y_1) w.r.t the line $ax + by + c = 0$ can be found by making c positive. Thus if $ax_1 + by_1 + c$ is > 0, =0 or < 0 then the point lies outside, on or inside the line respectively.

ILLUSTRATIONS

***Illustration* 5**

Find the range of □ in the interval (0, □) such that the point (3, 5) and (sin □, cos □) lie on the same side of the line $x + y – 1 = 0$.

Solution

$3 + 5 – 1 = 7 > 0$

□ $\sin\left(\frac{\pi}{4}+\theta\right) > \frac{1}{\sqrt{2}}$

□ $\frac{\pi}{4} < \frac{\pi}{4}+\theta < \frac{3\pi}{4}$

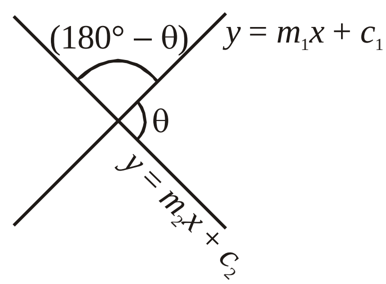
□ 0 < □ < $\frac{\pi}{2}$

3.6 Angle Between Two Straight Lines

If □ is the acute angle between two lines, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

where m_1 and m_2 are the slopes of the two lines and are finite.



Note:

(i) If two lines are parallel then $m_1 = m_2$

(ii) Any line parallel to the line $ax + by + c = 0$ is of the form $ax + by + k = 0$

(iii) If two lines are perpendicular to each other then $m_1 m_2 = – 1$

(vi) Any line perpendicular to $ax + by + c$ is of the form $bx – ay + k = 0$

(v) If one of the two lines are perpendicular to x-axis, then the slope of that line is infinite.

Let m_1 = □ then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{1 - \frac{m_2}{m_1}}{\frac{1}{m_1} + m_2}\right| = \left|\frac{1}{m_2}\right|$

□ □ = | 90 – □ | where tan □ = m_2

ILLUSTRATIONS

***Illustration* 6**

Find the equations of the straight lines passing through the point (2, 3) and inclined at $\frac{\pi}{4}$ radians to the line $2x + 3y = 5$.

Solution

Let the line $2x + 3y = 5$ makes angle θ with positive x-axis. Then $\tan\theta = -\frac{2}{3}$

Slope of the required lines are

$$\tan\left(\theta+\frac{\pi}{4}\right)=\frac{\tan\theta+\tan\frac{\pi}{4}}{1-\tan\theta+\tan\frac{\pi}{4}}=\frac{-\frac{2}{3}+1}{1-\left(-\frac{2}{3}\right)}=\frac{1}{5}$$

and $\tan\left(\theta-\frac{\pi}{4}\right)=-5$

$\therefore$ the equations of the required lines are

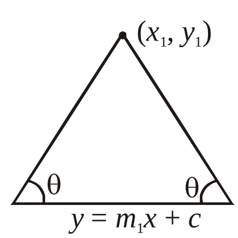
$y - 3 = \frac{1}{5}(x - 2)$ i.e., $x - 5y + 13 = 0$

$y - 3 = -5(x - 2)$ i.e., $5x + y - 13 = 0$

3.7 Equation of a Line Passing Through a Given Point and Equally Inclined to a Given

Line

Let the straight lines passes through point (x_1, y_1) and makes equal angles with the given line $y = m_1x + c$.



Let the angle is θ.

Let slope of the required line is m.

then $\tan\theta = \pm\left(\frac{m_1 - m}{1 + m_1 m}\right)$ … (i)

Equation (i) gives two values of m, hence we will get two lines.

Let the values of m are m_A and m_B

then $m_A = \frac{m + \tan\theta}{1 - m\tan\theta}$

& $m_B = \frac{m - \tan\theta}{1 + m\tan\theta}$

Hence equations of lines are

$y - y_1 = m_A(x - x_1)$ & $y - y_1 = m_B(x - x_1)$

ILLUSTRATIONS

***Illustration* 7**

Find the equation to the sides of an isosceles right-angled triangle, the equation of whose hypotenuse is $3x + 4y = 4$ and the opposite vertex is the point (2, 2).

Solution

The problem can be restated as:

Find the equations to the straight lines passing through the given point (2, 2) and making equal angles of 45° with the given straight line $3x + 4y - 4 = 0$. Slope of the line $3x + 4y - 4 = 0$ is $m_1 = -\frac{3}{4}$.

□ $\tan 45° = \pm\left(\frac{m - m_1}{1 + m_1 m_2}\right)$ i.e., $1 = \pm\left(\frac{m + \frac{3}{4}}{1 - \frac{3}{4}m}\right)$

$m_A = \frac{1}{7}$ and $m_B = -7$

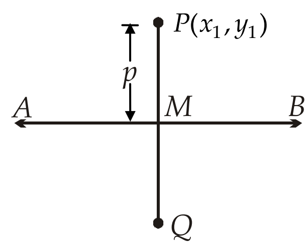
hence the required equations of the two lines are

$y - 2 = m_A(x - 2)$ and $y - 2 = m_B(x - 2)$

□ $7y - x - 12 = 0$ and $7x + y = 16$

3.8 Length of Perpendicular From a Point to a Line

Let length of perpendicular from a point $P(x_1, y_1)$ to a line $ax + by + c = 0$ is p then $p = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$



- **Foot of Normal or Perpendicular**

  The foot of perpendicular or normal from a point $P(x_1, y_1)$ to a line $ax + by + c = 0$ is

  $$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -\frac{(ax_1 + by_1 + c)}{a^2 + b^2}$$

- **Image of a Point w.r.t. a Line**

  The image of a point $P(x_1, y_1)$ w.r.t. a line $ax + by + c = 0$ is

  $$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -\frac{2(ax_1 + by_1 + c)}{a^2 + b^2}$$

❑ Distance Between two Parallel Lines

Distance between two parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $\left|\dfrac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right|$.

ILLUSTRATIONS

***Illustration* 8**

Three lines $x + 2y + 3 = 0$, $x + 2y - 7 = 0$ and $2x - y - 4 = 0$ form 3 sides of two squares. Find the equation of remaining sides of these squares.

Solution

Distance between the two parallel lines is $\dfrac{|7+3|}{\sqrt{5}} = 2\sqrt{5}$. The equations of sides A and C are of the form $2x - y + k = 0$.

Since distance between sides A and B = distance between sides B and C hence

$$\frac{|k-(-4)|}{\sqrt{5}} = 2\sqrt{5}$$

$\Rightarrow \quad \dfrac{k+4}{\sqrt{5}} = \pm 2\sqrt{5} \quad \Rightarrow k = 6, -14$

Hence the fourth sides of the two squares are

(i) $2x - y + 6 = 0$, (ii) $2x - y - 14 = 0$

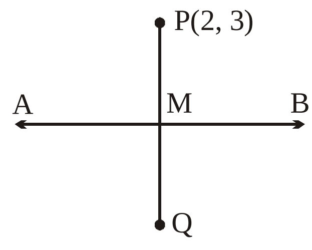
***Illustration* 9**

Find the foot of the perpendicular, drawn from the point (2, 3) to the line $3x - 4y + 5 = 0$. Also, find the image of (2, 3) in the given line.

Solution

Let AB $\equiv 3x - 4y + 5 = 0$

P $\equiv$ (2, 3) and PM $\perp$ AB



Foot of normal from P (2, 3) to $3x - 4y + 5 = 0$ is given by

$$\frac{x-2}{3} = \frac{y-3}{-4} = \frac{-[3\times2-4\times3+5]}{9+16}$$

$\Rightarrow \quad x = \dfrac{53}{25}, y = \dfrac{71}{21}$

Image of P (2, 3) on $3x - 4y + 5 = 0$ is given by

$$\frac{x-2}{3}=\frac{y-3}{-4}=-2\,\frac{[3\times2-4\times3+5]}{9+16}$$

$$\Rightarrow \quad x=\frac{56}{25}, y=\frac{67}{25}.$$

PRACTICE EXERCISE

16. Find the value of k if the straight line $2x - 3y + 4 + k(6x - y + 12) = 0$ is perpendicular to the line $7x + 5y - 4 = 0$.
17. A line passing through the points $(a, 2a)$ and $(-2, 3)$ is perpendicular to the line $4x + 3y + 5 = 0$; find the value of a.
18. Two sides of a square lie on the lines $x + y = 1$ and $x + y + 2 = 0$. What is its area?
19. Find the equations of the two straight lines through (7, 9) and making an angle of 60° with the line $x-\sqrt{3}y-2\sqrt{3}=0$.
20. A vertex of an equilateral triangle (2, 3) and opposite side is $x + y = 2$. Find the equations of other sides.

Answers

16. $k=-\frac{29}{37}$ **17.** 18/5 **18.** $\frac{9}{2}$ **19.** $x = 7$ and $x+\sqrt{3}\,y=7+9\sqrt{3}$

20. $(2+\sqrt{3})x - y = 1 + 2\sqrt{3}$ and $(2-\sqrt{3})x - y = 1 - 2\sqrt{3}$

3.9 Bisectors of The Angles Between Two Given Lines

Locus of a point, which moves such that its distance from two intersecting lines remains same, is called as angle bisector.

Let the two intersecting lines are

$$a_1x + b_1y + c_1 = 0 \qquad \ldots \text{(i)}$$

$$a_2x + b_2y + c_2 = 0 \qquad \ldots \text{(ii)}$$

Then equation of angle bisector of the angles between the lines (i) & (ii) will be

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\left(\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}\right) \qquad \ldots \text{(iii)}$$

Above equation gives two angle bisectors.

Note

(i) Angle bisectors of two lines are always mutually perpendicular.

(ii) If the two given lines are not perpendicular i.e., $a_1a_2 + b_1b_2 \neq 0$ then one of the bisector will be bisector of acute angle and other will be that of obtuse angle.

- **The Bisectors of The Acute and Obuse Angle**

  Take one of the lines and let its slope is m_1 and take one of the bisectors and let its slope be m_2. Then find $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

(i) If tan □ < 1 then the bisectors taken is the bisector of acute angle and the other will be the bisector of the obtuse angle.

(ii) If tan □ > 1 then the bisectors taken is the bisector of obuse angle and the other will be the bisector of the acute angle.

- **Equation of The Bisector of The Angle Containing Origin**

  First make the constants c_1 and c_2 in both the equations positive then equation $\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$ gives the equation of bisector of the angle containing origin.

NOTE

(i) If $a_1a_2 + b_1b_2 < 0$ then the origin lies in acute angle.

(ii) If $a_1a_2 + b_1b_2 > 0$ then the origin lies in obtuse angle.

Hence we can find the equation of bisector of acute angle and obtuse angle directly.

- **Equation of the Bisector of the Angle Which Contains a Given Point**

  Let the given point is (x_1, y_1)

(i) If $a_1x_1 + b_1y_1 + c_1$ and $a_2x_1 + b_2y_1 + c_2$ are of same sign then equation of bisector of angle containing point (x_1, y_1) is $\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$

(ii) If $a_1x_1 + b_1y_1 + c_1$ and $a_2x_1 + b_2y_1 + c_2$ are of opposite sign then equation of angle containing point (x_1, y_1) is $\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = -\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$

ILLUSTRATIONS

***Illustration* 10**

For the straight lines $4x + 3y - 6 = 0$ and $5x + 12y + 9 = 0$, find the equation of the

(i) Bisector of the obtuse angle between them

(ii) Bisector of the acute angle between them

(iii) Bisector of the angle which contains (0, 0)

(iv) Bisector of the angle which contains (1, 2)

Solution

Equations of bisectors of the angles between the given lines are

$$\frac{4x+3y-6}{\sqrt{4^2+3^2}} = \pm\frac{5x+12y+9}{\sqrt{5^2+12^2}}$$

$\Rightarrow$ $9x - 7y - 41 = 0$ and $7x + 9y - 3 = 0$

If θ is the acute angle between the line $4x + 3y - 6 = 0$ and the bisector $9x - 7y - 41 = 0$, then

$$\tan\theta = \left|\frac{-\frac{4}{3}-\frac{9}{7}}{1+\left(\frac{-4}{3}\right)\frac{9}{7}}\right| = \frac{11}{3} > 1$$

Hence

(i) The bisector of the obtuse angle is $9x - 7y - 41 = 0$

(ii) The bisector of the acute angle is $7x + 9y - 3 = 0$

(iii) The bisector of the angle containing the origin

$$\frac{-4x-3y+6}{\sqrt{(-4)^2+(-3)^2}} = \frac{5x+12y+9}{\sqrt{5^2+12^2}} \quad \Rightarrow \quad 7x + 9y - 3 = 0$$

(iv) For the point (1, 2), $4x + 3y - 6 = 4 \times 1 + 3 \times 2 - 6 > 0$

$5x + 12y + 9 = 12 \times 2 + 9 > 0$

Hence equation of the bisector of the angle containing the point (1, 2) is $\frac{4x+3y-6}{5} = \frac{5x+12y+9}{13}$

$\Rightarrow$ $9x - 7y - 41 = 0$

Alternative:

Making C_1 and C_2 positive in the given equations, we get $-4x - 3y + 6 = 0$ and $5x + 12y + 9 = 0$

Since $a_1a_2 + b_1b_2 = -20 - 36 = -56 < 0$,

So the origin will lie in the acute angle. Hence bisector of the acute angle is given by

$$\frac{4x+3y-6}{\sqrt{4^2+3^2}} = \frac{5x+12y+0}{\sqrt{5^2+12^2}} \qquad \text{i.e.,} \quad 9x - 7y - 41 = 0$$

Similarly bisector of obtuse angle is $7x + 9y - 3 = 0$

PRACTICE EXERCISE

21. Find the equations of the bisectors of the angles between the straight lines $3x - 4y + 7 = 0$ and $12x - 5y - 8 = 0$

22. Find the equation of the obtuse angle bisector of lines $12x - 5y + 7 = 0$ and $3y - 4x - 1 = 0$.

23. Find the bisector of the acute angle between the lines $3x + 4y - 11 = 0$ and $12x - 5y - 2 = 0$.

24. For the straight lines $4x + 3y - 6 = 0$ and $5x + 12y + 9 = 0$ find the equation of the bisector of the angle which contains the origin.

25. Find the equations of two straight lines which pass through the point (4, 5) and make equal angles with two straight lines given by the equations $3x - 4y - 7 = 0$ and $12x - 5y + 6 = 0$.

Answers

21. $99x - 77y + 51 = 0$ **22.** $4x + 7y + 11 = 0$ **23.** $11x + 3y - 17 = 0$

24. $7x + 9y - 3 = 0$ **25.** $7x + 9y - 73 = 0$

3.10 Families of Lines

If $L_1 = 0$ and $L_2 = 0$ are two given lines then family of lines through the point of intersection of the two given lines is $L_1 + \lambda L_2 = 0$.

- **One Parameter Family of Straight Line**

If a linear expression L_1 contains an unknown coefficient, then the line $L_1 = 0$ can not be a fixed line. Rather it represents a family of straight lines known as one parameter family of straight lines, e.g. family of lines parallel to the x-axis i.e., $y = C$ and family of straight lines passing through the origin i.e., $y = mx$.

Each member of the family passes through a fixed point. We have two methods to find the fixed point.

Method I:

If a family of straight lines can be written as $L_1 + \lambda L_2 = 0$, where L_1, L_2 are two fixed lines and λ is a parameter then each member of it will pass through a fixed point given by point of intersection of $L_1 = 0$ and $L_2 = 0$.

Method II:

Let the family of straight lines be of the form $ax + by + c = 0$ where a, b, c are variable parameters satisfying the condition $al + bm + cn = 0$, where l, m, n are given and $n \neq 0$.

Rewriting the condition as $a\left(\frac{l}{n}\right) + b\left(\frac{m}{n}\right) + c = 0$, and comparing with the given family of straight lines, we find that each member of it passes through the fixed point $\left(\frac{l}{n}, \frac{m}{n}\right)$.

Note: If $L_1 = 0$ and $L_2 = 0$ are parallel lines, they will meet at infinity.

ILLUSTRATIONS

***Illustration* 11**

Find the equation of the straight line passing through the point (2, 0) and through the point of intersection of the lines $x + 2y = 3$ and $2x – 3y = 4$.

Solution

Equation of any straight line passing through the intersection of the lines

$x + 2y = 3$ and $2x – 3y – 4 = 0$ is

□$(x + 2y – 3) + (2x – 3y – 4) = 4$ … (i)

Since it passes through the point (2, 0)

□ □$(2 + 0 – 3) + (4 – 0 – 4) = 0$

i.e., □ = 0

Now substituting this value of □ in (i) we get

$2x – 3y – 4 = 0$ as the equation of the required line

***Illustration* 12**

If the algebraic sum of perpendiculars from n given points on a variable straight line is zero then prove that the variable straight line passes through a fixed point.

Solution

Let n given points be (x_i, y_i) where $i = 1, 2, \ldots n$ and the variable straight line is $ax + by + c = 0$. Given that

$$\sum_{i=1}^{n}\left(\frac{ax_i + by_i + c}{\sqrt{a^2 + b^2}}\right) = 0$$

□ $a\sum x_i + b\sum y_i + cn = 0$

□ $a\frac{\sum x_i}{n} + b\frac{\sum y_i}{n} + c = 0$

Hence the variable straight line always passes through the fixed point $\left(\frac{\sum x_i}{n}, \frac{\sum y_i}{n}\right)$.

***Illustration* 13**

Prove that each member of the family of straight lines (3 sin □ + 4 cos □) x + (2 sin □ – 7 cos □) y + (sin □ + 2 cos □) = 0

(□ is a parameter) passes through a fixed point.

Solution

The given family of straight lines can be rewritten as

$(3x + 2y + 1)$ sin □ + $(4x – 7y + 2)$ cos □ = 0

or $(4x – 7y + 2)$ + tan □ $(3x + 2y + 1) = 0$

which is of the form $L_1 + \lambda L_2 = 0$

Hence each member of it will pass through a fixed point

which is the intersection of $4x - 7y + 2 = 0$ and $3x + 2y + 1 = 0$ i.e., $\left(\frac{-11}{29}, \frac{2}{29}\right)$.

3.11 Concurrency of Straight Lines

The condition for 3 lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$ to be concurrent is

(i) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

(ii) There exists 3 constants l, m, n (not all zero at the same time) such that $lL_1 + mL_2 + nL_3 = 0$, where $l + m + n = 0$, where $L_1 = 0$, $L_2 = 0$ and $L_3 = 0$ are the three given straight lines.

(iii) The three lines are concurrent if any one of the lines passes through the point of intersection of the other two lines.

3.12 Centroid, Incentre, Ex-Centre and Orthocentre

- **Centroid**

  The point of intersection of the medians of a triangle is called its centroid. It divides the medians in the raito 2 : 1. If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of a triangle, then the coordinates of its centroid are

  $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$

- **Incentre**

  The point of intersection of the internal bisectors of the angles of a triangle is called its in-centre. If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of a triangle ABC such that $BC = a$, $CA = b$ and $AB = c$, then the coordinates of its incentre are

  $\left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c}\right)$.

  If the equations of the sides of the triangle are given, then we find the bisectors of internal angles and then their point of intersection to determine the in-centre.

- **Circum-Centre**

  The point of intersection of the perpendicular bisectors of the sides of a triangle is called its circum-centre. It is equidistant from the vertices of a triangle.

  If O is the circumcentre of a triangle ABC, then $OA = OB = OC$ and OA is called the circum-radius.

To find the circum-centre of ΔABC, we use the relation $OA = OB = OC$. This gives two simultaneous linear equations and their solution provides the coordinate of circumcentre..

- **Ortho-Centre**

The point of intersection of the altitudes of a triangle is called its orthocenter.

To determine the orthocenter, first we find equations of lines passing through vertices and perpendicular to the opposite sides. Solving any two of these three equations we get the coordinates of orthocenter.

Note : The circumcentre O, centroid G and orthocenter O' of a triangle ABC are collinear such that G divides $O'O$ in the ratio 2 : 1 i.e., $O'G : OG = 2 : 1$.

ILLUSTRATIONS

***Illustration* 14**

If the lines $ax + y + 1 = 0$, $x + by + 1 = 0$ and $x + y + c = 0$ (a, b and c being distinct and different from 1) are concurrent, then prove that $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=1$.

Solution

Since the given lines are concurrent $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$

Operating $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$\begin{vmatrix} a & 1-a & 1-a \\ 1 & b-1 & 0 \\ 1 & 0 & c-1 \end{vmatrix} = 0$$

$\Rightarrow$ $a(b-1)(c-1) - (b-1)(1-a) - (c-1)(1-a) = 0$

$\Rightarrow$ $\frac{a}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=0$

$\Rightarrow$ $\left(\frac{a}{1-a}+1\right)+\frac{1}{1-b}+\frac{1}{1-c}=1$

$\Rightarrow$ $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=1$.

PRACTICE EXERCISE

26. Find the centroid, incentre circum-centre and orthocenter of the triangle whose sides have the equations $3x - 4y = 0$, $12y + 5x = 0$ and $y - 15 = 0$.
27. Find the circumcentre of the triangle whose sides are $3x - y + 3 = 0$, $3x + 4y + 3 = 0$ and $x + 3y + 11 = 0$.

28. Find the area of the triangle formed by the lines $y = m_1x + c_1$, $y = m_2x + c_2$ and $x = 0$.

29. Find the orthocenter of the triangle whose vertices are $(at_1 t_2, a (t_1 + t_2))$, $(at_2 t_3, a (t_2 + t_3)$ and $(at_1 t_3, a (t_1 + t_3))$.

30. If $3x - 4y = 0$, $4x + 3y - 8 = 0$ and $24x - 7y - 12 = 0$ are sides *AB*, *BC* and *CA* respectively of a triangle *ABC*, then find the internal bisector of $\angle ABC$.

Answers

26. **27.** (3, –3) **28.** $\frac{1}{2}\frac{(c_1-c_2)^2}{m_1-m_2}$ in magnitude **29.** $(-a, a(t_1 + t_2 + t_3 + t_1 t_2 t_3))$

30. $x + 7y - 8 = 0$

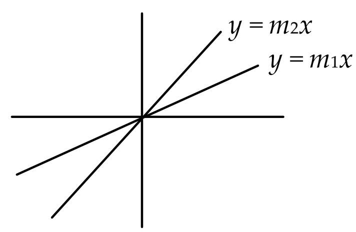
1.13 Pair of Straight Line

A rational integral algebraic equation in two variables *x* and *y* is said to be a homogeneous equation of second degree, if the sum of indices of *x* and *y* in each term is 2.

Thus the equation $ax^2 + 2hxy + by^2 = 0$ represents pair of straight line passing through origin.

Proof: We can write $ax^2 + 2hxy + by^2 = 0$

as $b\left(\frac{y}{x}\right)^2 + 2h\left(\frac{y}{x}\right) + a = 0$ …(i)



which is an quadratic equation in $\frac{y}{x}$

Thus $\frac{y}{x} = \frac{-2h \pm \sqrt{4h^2 - 4ab}}{2b}$

$\Rightarrow \quad y = \left(\frac{-h \pm \sqrt{h^2 - ab}}{b}\right)x$

Thus the equations are of the form $y = m_1x$ and $y = m_2x$ representing straight lines passing through origin.

1. The value of m_1 and m_2 are real and distinct if $h^2 > ab$
2. The value of m_1 and m_2 are real and coincident if $h^2 = ab$
3. The value of m_1 and m_2 are imaginary if $h^2 < ab$

❑ **Angle between lines**

Angle between pair of lines given by $ax^2 + 2hxy + by^2 = 0$. Let 'θ' be the angle between the lines, then from equation

$m_1 + m_2 = \frac{-2h}{b}$ …(a)

$m_1 m_2 = \frac{a}{b}$ …(b)

We know, $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

$\Rightarrow \quad \tan\theta = \left|\dfrac{2\sqrt{h^2 - ab}}{a+b}\right|$

Note:

(i) If the lines are parallel (coincident) i.e. θ = 0°, then $h^2 = ab$

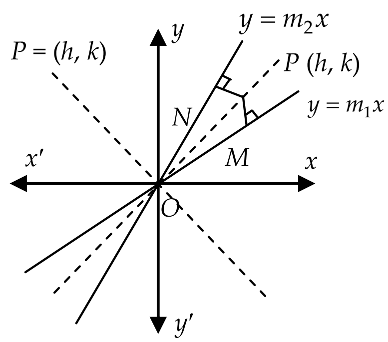
(ii) If the lines are perpendicular i.e. θ = 90°, then $a + b = 0$.

❑ Bisector of Angles Between the Lines

Let $y - m_1, x = 0$ and $y - m_2x = 0$ be the lines represented by $ax^2 + 2hxy + by^2 = 0$, then $m_1 + m_2 = -\dfrac{2h}{b}$ and $m_1 m_2 = \dfrac{a}{b}$.

Since bisector of the angles between two lines are the locus of a point which is equidistant from the two given lines. Then,

Let $P\,(h, k)$ be a point on a bisector of the angle between the given lines. Then, $PM = PN$.



$$\Rightarrow \quad \left|\frac{k - m_1 h}{\sqrt{1+m_1^2}}\right| = \left|\frac{k - m_2 h}{\sqrt{1+m_2^2}}\right|$$

$$\frac{k - m_1 h}{\sqrt{1+m_1^2}} = \pm \frac{k - m_2 h}{\sqrt{1+m_2^2}}$$

$$\frac{y - m_1 x}{\sqrt{1+m_1^2}} = \pm \frac{y - m_2 x}{\sqrt{1+m_1^2}}$$

The joint equation of the two bisector is

$$\left(\frac{y - m_1 x}{\sqrt{1+m_1^2}} + \frac{y - m_2}{\sqrt{1+m_2^2}}\right)\left(\frac{y - m_1 x}{\sqrt{1+m_1^2}} - \frac{y - m_2 x}{\sqrt{1+m_2^2}}\right) = 0$$

$\Rightarrow \quad (m_1 + m_2)(x^2 - y^2) = 2xy\,(1 - m_1 m_2)$

$\Rightarrow \quad \dfrac{x^2 - y^2}{a - b} = \dfrac{xy}{h}$ or $hx^2 - hy^2 = (a - b)xy$

NOTE:

Co-efficient of x^2 + coefficient of $y^2 = 0$. Thus the bisectors of the angle between the lines are perpendicular to each other.

1.14 General Equation of Second Degree

An equation of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ where a, b, c, f, g, h are constants is said to be general equation of second degree in x and y.

- **Condition for General Equation to Represent pair of Straight Lines**

  The necessary and sufficient condition for $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent pair of straight lines is that

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

- **Point of Intersection of Pair of Straight Lines Given by General Equation**

  The point of intersection of the lines represented by the equation

  $U: \quad ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ …(i)

  is given by

$$\frac{\delta U}{\delta x} = 0 \qquad \text{…(ii)}$$

$$\frac{\delta U}{\delta y} = 0 \qquad \text{…(iii)}$$

  where the roots of $\frac{\delta U}{\delta x} = 0$ gives x co-ordinates of the point of intersection and the roots of $\frac{\delta U}{\delta y} = 0$ gives y co-ordinates of the point of intersection of the lines.

  i.e. $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$

- **Angle Between Lines**

  Let $y = m_1x + c_1$ and $y = m_2x + c_2$ be the lines represented by general equation of second degree.

  Then $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = (y - m_1x - c_1)(y - m_2x - c_2) = 0$

  Comparing co-efficients, we get

$$\frac{m_1m_2}{a} = \frac{m_1 + m_2}{-2h} = \frac{1}{b}$$

$$\Rightarrow \quad m_1 + m_2 = \frac{-2h}{b} \text{ and } m_1m_2 = \frac{a}{b}$$

  Let θ be the angle between the lines.

  Thus $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$

$$= \left| \frac{2\sqrt{h^2 - ab}}{a+b} \right|$$

NOTE:

(i) The lines are parallel if $h^2 = ab$

(ii) The lines are perpendicular if $a + b = 0$

- **Equation of Bisector of Angles Between the Lines**

  Let $P(x_1, y_1)$ be the point of intersection of the lines and (x, y) be the moving point of the shifting of origin

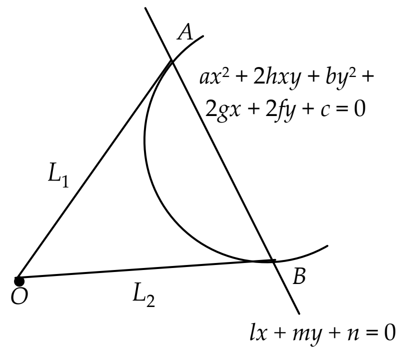
  Then equation of bisector is given by

$$\frac{(x-x_1)^2 - (y-y_1)^2}{a-b} = \frac{(x-x_1)\,(y-y_1)}{h}$$

1.15 HOMOGENIZATION OF GENERAL EQUATION

Let the curve be $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ …(i)

and the line be $lx + my + n = 0$ … (ii)



Then the lines joining origin to the point of intersection of curve and line is obtained by homogenizing the equation of curve.

As $lx + my + n = 0 \Rightarrow \frac{lx+my}{-n} = 1$ …(iii)

Equation (i) can be written as

$$ax^2 + 2hxy + by^2 + 2gx\,.\,1 + 2\,fy\,.\,1 + c\,.\,1^2 = 0$$

Thus

$$ax^2 + 2hxy + by^2 + 2gx\left(\frac{lx+my}{-n}\right) + 2fy\left(\frac{lx+my}{-n}\right) + c\left(\frac{lx+my}{-n}\right)^2 = 0$$

represents the lines from origin to point of intersection of curve and the given lines.

ILLUSTRATIONS

***Illustration* 15**

Prove that the equations to the straight lines passing through the origin and making an angle □ with the straight line $y + x = 0$ are given by $x^2 + 2xy \sec 2□ + y^2 = 0$.

Solution

Let $y = mx$ be the equation of the straight line passing through the origin and making angle □ with $y + x = 0$. Now, slope of line $y + x = 0$ is -1.

□ $\tan$ □ $= \pm \left(\frac{m-1}{1-m}\right)$ or $\tan^2$ □ $= \frac{(m+1)^2}{(m-1)^2}$ [squaring]

or $m^2(1 - \tan^2$ □$) + 2m(1 + \tan^2$ □$) + (1 - \tan^2$ □$) = 0$

This is a quadratic equation in m, therefore there will be two values of x and hence two straight lines which make an angle □ with $y + x = 0$. Let the two values of m be m_1 and m_2, then

$$m_1 + m_2 = -\frac{2\left(1+\tan^2\alpha\right)}{1-\tan^2\alpha}$$ $= -2 \sec 2$□ and $m_1 m_2 = \frac{1-\tan^2\alpha}{1-\tan^2\alpha} = 1$

Now the join equation of the straight line passing through the origin and making an angle □ with $y + x = 0$ is

$(y - m_1x)(y - m_2x) = 0$

or $y^2 - xy(m_1 + m_2) + m_1m_2x^2 = 0$

or $y^2 + 2xy \sec 2$□ $+ x^2 = 0$

***Illustration* 16**

Discuss the nature of the lines represented by the equation

(i) $2x^2 - 5xy + 2y^2 + 4x - 5y + 2 = 0$

(ii) $2x^2 + 2xy + y^2 + 6x + 2y + 5 = 0$

(iii) $x^2 + 2xy + y^2 + 12 = 0$

Solution

It is easy to see that in each case □ = 0.

In (i) $h^2 - ab > 0$, hence (i) represents a pair of interesting lines.

In (ii) $h^2 - ab < 0$, thus (ii) represents a pair of imaginary lines.

In (iii) $h^2 - ab = 0$ and $f^2 - bc < 0$.

Thus a pair of imaginary lines.

PRACTICE EXERCISE

31. Show that the equation $3x^2 + 7xy + 2y^2 + 5x + 5y + 2 = 0$ represents a pair of straight lines.

32. Prove that the lines $a^2x^2 + 2h(a + b)xy + b^2y^2 = 0$ are equally inclined to the lines $ax^2 + 2hxy + by^2 = 0$

33. Prove that the equation $6x^2 + 13xy + 6y^2 + 8x + 7y + 2 = 0$ represents a pair of straight lines intersecting at an angle $\tan^{-1}(5/12)$. Also, find the point of intersection of the lines.

34. Show that the difference of the tangents of the angles which the lines x^2 ($\sec^2\theta - \sin^2\theta$) – $2xy \tan \theta + y^2 \sin^2\theta = 0$ make with x-axis is 2.

35. Prove that the angle between the lines joiniong the origin to the points of intersection of the line $y = 3x + 2$ with the curve $x^2 + 2xy + 3y^2 + 4x + 8y = 11$ is $\tan^{-1}\left(\frac{2\sqrt{2}}{3}\right)$.

Answers

33. $\left(\frac{1}{5}, -\frac{4}{5}\right)$ **35.** $\theta = \tan^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

*****

MISCELLANEOUS PROBLEMS

OBJECTIVE TYPE

Example 1

The orthocentre of the triangle formed by the lines $2x^2 + 3xy - 2y^2 - 9x + 7y - 5 = 0$, $4x + 5y - 3 = 0$ lies at

(a) $\left(\frac{3}{5}, \frac{11}{5}\right)$ (b) $\left(\frac{6}{5}, \frac{11}{5}\right)$ (c) $\left(\frac{5}{6}, \frac{11}{5}\right)$ (d) None of these

Solution

The pair of straight lines $2x^2 + 3xy - 2y^2 - 9x + 7y - 5 = 0$ are perpendicular to each other so orthocentre is point of intersection of these lines

□ **Ans. (a)**

Example 2

If the line $y =$ □3 x cuts the curve $x^3 + y^3 + 3xy + 5x^2 + 3y^3 + 4x + 5y - 1 = 0$ at the points A, B, C then OA . OB . OC is

(a) $\frac{4}{13}\left(3\sqrt{3}-1\right)$ (b) $3\sqrt{3}+1$ (c) $\frac{2}{\sqrt{3}}+7$ (d) None of these

Solution

The line $y = \sqrt{3}x$ passes through the origin. Therefore it can be written as $\frac{x-0}{\frac{1}{2}} = \frac{y-0}{\frac{\sqrt{3}}{2}} = r$

where r is the distance of any point (x, y) on $y = \sqrt{3}x$ from (0, 0)

i.e., $\left(\frac{r}{2}, \frac{\sqrt{3}r}{2}\right)$ will always lie on the line $y = \sqrt{3}x$ at a distance r from (0, 0)

Since it cuts the curve

$$x^3 + y^3 + 3xy + 5x^2 + 3y^2 + 4x + 5y - 1 = 0$$

We have

$$\frac{r^3}{8} + \frac{3\sqrt{3}r^3}{8} + \frac{3\sqrt{3}r^2}{4} + \frac{5r^2}{4} + \frac{9}{4}r^2 + 2r + \frac{5\sqrt{3}r}{2} - 1 = 0$$

$$r^3\frac{\left(1+3\sqrt{3}\right)}{8} + r^2(\ldots) + r(\ldots) - 1 = 0$$

This is a cubic in r which shows that cuts the curve at 3 points.

□ OA . OB . OC = $\frac{1}{1+3\sqrt{3}} \,.\, 8 = \frac{4}{13}\left(3\sqrt{3}-1\right)$

∴ **Ans. (a)**

Example 3

The straight lines of the family $x(a + b) + y(a - b) = 2a$ (a and b being parameters) are

(a) Not concurrent (b) Concurrent at (1, – 1)

(c) Concurrent at (1, 1) (d) None of these

Solution

The given equation can be written as

$$a\,(x + y - 2) + b(x - y) = 0$$

or $(x + y - 2) + \frac{b}{a}\,(x - y) = 0$

This is a family of lines concurrent at point of intersection of $x - y = 0$ and $x + y - 2 = 0$.

On solving these two equations we get (1, 1)

∴ **Ans. (c)**

Example 4

Through the point P(□□□□), where □□ > 0, the straight line is drawn so as to form with coordinate axes a triangle of area s. If $ab > 0$, then least value of s is

(a) 2□□ (b) $\frac{1}{2}$ □□ (c) □□ (d) None of these

Solution

Given P □ (□, □)

Given line is $\frac{x}{a}+\frac{y}{b}=1$ … (i)

If line (i) cuts x and y axes at A and B respectively, then A □ $(a, 0)$ and B □ $(0, b)$.

Also the area of □OAB = s

$$\left(\frac{1}{2}\right) ab = s \quad □ \quad ab = 2s$$

Since line (i) passes through P (□, □)

$$\frac{\alpha}{a}+\frac{\beta}{b}=1 \quad □ \quad \frac{\alpha}{a}+\frac{a\beta}{2s}=1 \quad □ \quad a^2□ - 2as + 2□s = 0$$

Since □ is real, $4s^2 - 8□□s$ □ 0 □ s □ 2□□

Hence the least value of s = 2□□

∴ **Ans. (a)**

Example 5

The equation of the diagonal, through the origin, of the quadrilateral formed by the lines $x = 0$, $y = 0$, $x + y = 1$ and $6x + y = 3$ is given by

(a) $3x - y = 0$ (b) $3x - 2y = 0$ (c) $x - y = 0$ (d) $3x - 4y = 0$

Solution

(0, 0) is the one end of the diagonal.

The intersection of the other two lines, $x + y = 1$ and $6x + y = 3$ is $\left(\frac{2}{5}, \frac{3}{5}\right)$ which is the other end of the diagonal. The equation of the line that passes through these two points is $3x - 2y = 0$

$\therefore$ **Ans. (b)**

Example 6

The equation of the straight line equally inclined to axes and equidistant from the points (1, – 2) and (3, 4) is

(a) $x + y + 1 = 0$ (b) $x + y + 2 = 0$ (c) $x - y - 2 = 0$ (d) $x - y - 1 = 0$

Solution

Middle point of the line joining points (1, – 2) and (3, 4) is (2, 1) which lie on line $x - y - 1 = 0$, which is equally inclined to the axes and is at equal distance from the given points.

$\therefore$ **Ans. (d)**

Example 7

Let PQR be a right angled isosceles triangle, right angled at P(2, 1). If the equation of the line QR is $2x + y = 3$, then the equation representing the pair of lines PQ and PR is

(a) $3x^2 - 3y^2 + 8xy + 20x + 10y + 25 = 0$ (b) $3x^2 - 3y^2 + 8xy - 20x - 10y + 25 = 0$

(c) $3x^2 - 3y^2 + 8xy + 10x + 15y + 20 = 0$ (d) $3x^2 - 3y^2 - 8xy - 10x - 15y - 25 = 0$

Solution

If m is the slope of side PQ or PR, then

$\frac{m-(-2)}{1+(-2)m} = \pm \tan\frac{\pi}{4}$, Q slope of QR is – 2 and □PQR = □PRQ = 45°

□ $m = 3, -1/3$

□ Equations of side PQ and PR are

$3x - y - 5 = 0$ and $x + 3y - 5 = 0$

Their combined equation is

$(3x - y - 5)(x + 3y - 5) = 0$

i.e., $3x^2 - 3y^2 + 8xy - 20x - 10y + 25 = 0$

∴ **Ans. (b)**

Example 8

Equation of the bisector of the acute angle between the lines $3x - 4y + 7 = 0$ and $12x + 5y - 2 = 0$ is

(a) $21x + 77y - 101 = 0$ (b) $21x - 77y + 101 = 0$

(c) $11x - 3y + 9 = 0$ (d) $11x + 3y - 9 = 0$

Solution

The given equations can be rewritten as

$3x - 4y + 7 = 0$ and $-12x - 5y + 2 = 0$

(making constant terms positive)

Since $a_1a_2 + b_1b_2 = -36 + 20 < 0$

□ positive sign in equations of bisectors gives the bisector of acute angle. Hence acute angle bisector is

$$\frac{3x-4y+7}{\sqrt{(3^2+4^2)}} = \frac{-12x-5y+2}{\sqrt{(12^2+5^2)}}$$

or $11x - 3y + 9 = 0$,

∴ **Ans. (c)**

Example 9

Mixed term xy is to be removed from the general equation of second degree $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$, one should rotate the axes through an angle □ given by tan 2□ equal to

(a) $\frac{(a-b)}{2h}$ (b) $\frac{2h}{(a+b)}$ (c) $\frac{(a+b)}{2h}$ (d) $\frac{2h}{(a-b)}$

Solution

Let $(x' . y')$, be the coordinates on new axes, then, put

$x = x' \cos □ - y' \sin □$,

$y = x' \sin □ + y' \cos □$, in the equation

Then coefficient of $x'y'$ in transformed equation = 0, so,

$2(b - a) \sin □ \cos □ + 2h \cos 2□ = 0$

or $\tan 2□ = \frac{2h}{(a-b)}$

∴ **Ans. (d)**

Example 10

If one of the diagonals of a square is along the line $x = 2y$ and one of its vertices is (3, 0), then its sides through this vertex are given by the equations.

(a) $y - 3x + 9 = 0,\ 3y + x - 3 = 0$ (b) $y + 3x + 9 = 0,\ 3y + x - 3 = 0$

(c) $y - 3x + 9 = 0,\ 3y - x + 3 = 0$ (d) $y - 3x + 3 = 0,\ 3y + x + 9 = 0$

Solution

Diagonal of the square is along

$x = 2y$ … (i)

The point (3, 0) does not lie on (i)

Let the side through this vertex be

$y - 0 = m(x - 3)$

Angle between side (ii) and diagonal (i) is 45°

□ $\tan^{-1} \dfrac{m - \frac{1}{2}}{1 + m\left(\frac{1}{2}\right)} = \pm 45°$ □ $m = 3, -$

□ From (ii), the required sides are

$y - 3x + 9 = 0$ and $3y + x - 3 = 0$

∴ **Ans. (a)**

Example 11

The equation $x^2y^2 - 2xy^2 - 3y^2 - 4x^2y + 8xy + 12y = 0$ represents

(a) A pair of straight lines (b) A pair of straight line and a circle

(c) A pair of straight lines and a parabola (d) A set of four lines forming a square

Solution

$x^2y^2 - 2xy^2 - 3y^2 - 4x^2y + 8xy + 12y = y^2(x^2 - 2x - 3) - 4y(x^2 - 2x - 3) = 0$

□ $y(y - 4)(x - 3)(x + 1) = 0$ □ $y = 0,\ y = 4,\ x = 3,\ x = -1$

Hence the equation represents four straight lines which evidently form a square.

∴ **Ans. (d)**

Example 12

Let $2x - 3y = 0$ be a given line and P (sin □, 0) and Q (0, cos □) be the two points. Then P and Q lie on the same side of the given line, if □ lies in the

(a) 1st quadrant (b) 2nd quadrant (c) 3rd quadrant (d) 4th quadrant

Solution

P and Q lie on the same side if 2 sin □ and – 3 cos □ have the same signs i.e., sin □ and cos □ have opposite signs which is true for the 2nd and 4th quadrant.

∴ **Ans. (b, 4)**

Example 13

Consider the equation $y - y_1 = m(x - x_1)$. If m and x_1 are fixed and different lines are drawn for different values of y_1, then

(a) The line will pass through a fixed point
(b) There will be a set of parallel lines
(c) All the lines intersect the line $x = x_1$
(d) All the lines will be parallel to the line $y = y_1$

Solution

For a fixed value of m, the given lines form a set of parallel lines all with the slope m_1 □ answer (b) is correct, while (a) is incorrect. However this set will not be parallel to the line $y = y_1$, whose slope is zero, unless $m = 0$. And (d) is therefore excluded. If, furthermore, x_1 is fixed, the above set of lines will all intersect the line $x = x_1$. Hence answer (c) is also correct.

Ans. (b, c)

Example 14

The area of triangle is 5. Two of its vertices are (2, 1) and (3, – 2), the third vertex is lying on $y = x + 3$. The co-ordinates of the third vertex can be

(a) $\left(\frac{-3}{2}, \frac{3}{2}\right)$ (b) $\left(\frac{3}{2}, \frac{-3}{2}\right)$ (c) $\left(\frac{7}{2}, \frac{13}{2}\right)$ (d) $\left(\frac{-1}{4}, \frac{11}{4}\right)$

Solution

As the third vertex lies on the line $y = x = 3$, its co-ordiantes are of the form $(x, x + 3)$.

The area of triangle is $\frac{1}{2}\begin{vmatrix} x & x+3 & 1 \\ 2 & 1 & 1 \\ 3 & -2 & 1 \end{vmatrix} = \frac{1}{2}|4x-4| = |2x-2|$

According to given condition □ $2x - 2 = \pm 5$ □ $x = -\frac{3}{2}, \frac{7}{2}$

Hence the coordinates of third vertex can be $\left(\frac{-3}{2}, \frac{3}{2}\right), \left(\frac{7}{2}, \frac{13}{2}\right)$

∴ **Ans. (a, c)**

Example 15

The medias AD and BE of a triangle ABC with vertices A(0, b), B(0, 0) and C(a, 0) are perpendicular to each other if

(a) $b =$ □$2a$ (b) $a =$ □$2b$ (c) $b = -$□$2a$ (d) $a = -$□$2b$

Solution

The coordinates of the mid-point D of BC are $\left(\frac{a}{2}, 0\right)$ and the coordinates of the mid-point E of CA are $\left(\frac{a}{2}, \frac{b}{2}\right)$.

If AD and BE are mutually perpendicular, then slope of AD × slope of BE = – 1

$$\Rightarrow \quad \left(\frac{0-b}{\frac{a}{2}-0}\right)\left(\frac{\frac{b}{2}-0}{\frac{a}{2}-0}\right) = -1 \quad \Rightarrow \quad a = \pm\sqrt{2}b.$$

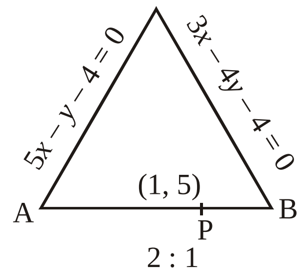
∴ **Ans. (b, d)**

SUBJECTIVE TYPE

Example 1

A line intersects the straight lines $5x - y - 4 = 0$ and $3x - 4y - 4 = 0$ at A and B respectively. If a point P (1, 5) on the line AB is such that AP : PB = 2 : 1 (internally), find the point A.

Solution



Let AP = $2r$.

Then PB = r

Let the slope of AB be $\tan\theta$.

The equation of the line AB is

$$\frac{x-1}{\cos\theta} = \frac{y-5}{\sin\theta} = r$$

So that

A $\equiv (1 + 2r\cos\theta, 5 + 2r\sin\theta)$

and B $\equiv (1 - r\cos\theta, 5 - r\sin\theta)$

or A $\equiv (1 - 2r\cos\theta, 5 - 2r\sin\theta)$

and B $\equiv (1 + r\cos\theta, 5 + r\sin\theta)$

The point 'A' lies on the line

$5x - y - 4 = 0$, So that

$5(1 + 2r\cos\theta) - (5 + 2r\sin\theta) - 4 = 0$ … (i)

The point 'B' lies on the line $3x - 4y - 4 = 0$

So that

$3(1 - r\cos\theta) + 4(5 - r\sin\theta) - 4 = 0$ … (ii)

Let $r\cos\theta = \alpha$

and $r\sin\theta = \beta$

$\Rightarrow$ $10\alpha - 2\beta - 4 = 0$

and $-3\alpha - 4\beta + 19 = 0$

On solving for α and β, we get

$$\alpha = \frac{27}{23}, \beta = \frac{89}{23}$$

Hence the point A is $\left(\frac{77}{83}, \frac{293}{23}\right)$

Similarly when A $\equiv (1 - 2r\cos\theta, 5 - 2r\sin\theta)$

and B = $(1 + r\cos\theta, 5 + r\sin\theta)$, we get

$-10\alpha + 2\beta - 4 = 0$

and $3\alpha + 4\beta + 19 = 0$

$\Rightarrow$ $\alpha = -\frac{27}{23}, \beta = -\frac{89}{23}$

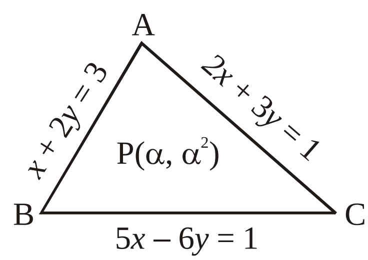
Hence the point A is $\left(\frac{77}{23}, \frac{293}{23}\right)$

Example 2

Find □, if (□, □^{2}) lies inside the triangle having sides along the lines $2x + 3y = 1$, $x + 2y - 3 = 0$ and $6y = 5x - 1$.

Solution

The vertices of the triangle ABC formed by the given lines are A □ (– 7, 5), B □□ $\left(\frac{5}{4}, \frac{7}{8}\right)$, C □ $\left(\frac{1}{3}, \frac{1}{9}\right)$



Sign of A w.r.t. BC is – ve

If P lies in-side the □ABC, then sign of P will be same as sign of A w.r.t. the line BC

□ 5□ – 6□^{2} – 1 < 0 … (i)

Similarly, 2□ + 3□^{2} – 1 > 0 … (ii)

and, □ + 2□^{2} – 1 < 0 … (iii)

Solving, (i), (ii) and (iii) for □ and then taking intersection we get

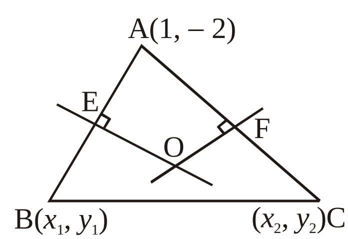
□□□ $\left(-\frac{3}{2}, -1\right) \cup \left(\frac{1}{2}, 1\right)$.

Example 3

The equations of the perpendicular bisectors of the sides AB and AC of a triangle ABC are respectively $x - y + 5 = 0$ and $x + 2y = 0$. If the coordinates of A are (1, – 2), find the equation of BC.

Solution

Let OE and OF be the perpendicular bisectors of AB and AC respectively.



Let B be (x_1, y_1) and C be (x_2, y_2). E is the mid-point of AB and F is the midpoint of AC.

E □□ $\left(\frac{x_1 + 1}{2}, \frac{y_1 - 2}{2}\right)$, F □ $\left(\frac{x_2 + 1}{2}, \frac{y_2 - 2}{2}\right)$

Since E and F lie on OE and OF

respectively, $x_1 - y_1 + 13 = 0$ … (i)

and $x_2 + 2y_2 - 3 = 0$ … (ii)

Also, slope of AB = – 1 and that of AC = 2, so that

$x_1 + y_1 + 1 = 0$ … (iii)

and $2x^2 - y^2 - 4 = 0$ … (iv)

Solving these equations, we get the co-ordinates of B and C as (– 7, 6) and

□ Equation of BC is $14x + 23y - 40 = 0$.

Example 4

Find the points nearest and farthest on the curve $5x^2 + 5y^2 + 6xy - 8 = 0$ from origin. Also determine the equations of the lines through the origin on which the points are occurring.

Solution

Any line through the origin is $\frac{x-0}{\cos\theta} = \frac{y-0}{\sin\theta} = r$.

Any point on this line is (r cos □, + r sin □).

If this point lies on the given curve, then

$5r^2 + 6r^2$ sin □ cos □ = 8

□ $r^2 = \frac{8}{5 + 3\sin 2\theta}$

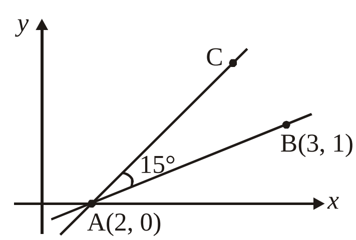
□ $r_{max.} = 2$ $\left(for\ \ \theta = \frac{3\pi}{4}\right)$ and $r_{min.} = 1$ $\left(for\ \ \theta = \frac{\pi}{4}\right)$

The required points are $\left(-\sqrt{2}, \sqrt{2}\right)$ and $\left(\sqrt{2}, \sqrt{2}\right)$. The lines are $y = -x$ and $y = x$.

Example 5

A line joining two points A (2, 0) and B (3, 1) is rotated about A in the anticlockwise direction through an angle of 15°. Find the equation of the line in the new position. If B goes to C, in the new position, what will be the coordinates of C.

Solution



Slope of the line AB is $m = 1$

□ tan □ = 1 □ □ = 45°

AB = □2 □ AC = □2

Slope of AC = tan (45° + 15°)

= tan (60°)

[because angle between AB and AC = 15°]

A is (2, 0) hence equation of line AC $\frac{x-2}{\cos 60°} = \frac{y-0}{\sin 60°} = r$

Here C is given by the formula

($2 + r\cos 60°$, $r\sin 60°$) where r = □2

C = (2 + □2 cos 60°, 0 + □2 sin 60°),

Hence point C is $\left(2 + \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right)$

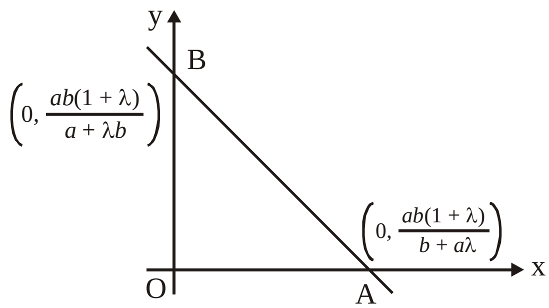
□ Equation of the line AC is y = □3(x – 2).

Example 6

A variable line through the point of intersection of the lines $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b}+\frac{y}{a}=1$ meets the coordinate axes in A and B. Show that the locus of the midpoint of AB is the curve: $2xy\,(a+b)=ab(x+y)$.

Solution

Let (h, k) be the mid point of the variable line AB.



The equation of the variable line AB is

$$(bx + ay - ab) + \square(ax + by - ab) = 0$$

Coordinates of A are $\left(\frac{ab(1+\lambda)}{b+\lambda a}, 0\right)$

Coordinates of B are $\left(0, \frac{ab(1+\lambda)}{a+\lambda b}\right)$

Mid point of AB is $\left(\frac{ab(1+\lambda)}{2(b+\lambda a)}, \frac{ab(1+\lambda)}{2(a+\lambda b)}\right)$

□ $h=\frac{ab(1+\lambda)}{2(b+\lambda a)};\ k=\frac{ab(1+\lambda)}{2(a+\lambda b)}$

□ $\frac{1}{2h}=\frac{b+\lambda a}{ab(1+\lambda)};\ \frac{1}{2k}=\frac{a+\lambda b}{ab(1+\lambda)}$

□ $\frac{1}{2h}+\frac{1}{2k}=\frac{a+b}{ab}$

□ $(h + k)\,ab = 2hk\,(a + b)$

Hence the locus of the mid-point of AB is

$$(x + y)\,ab = 2xy\,(a + b).$$

Example 7

Prove that the straight lines joining the origin to the points of intersection of the straight line $hx + ky = 2hk$ and the curve $(x - k)^2 + (y - h)^2 = c^2$ are at right angles if $h^2 + k^2 = c^2$.

Solution

Making the equation of the curve homogenous with the help of the line, we get

$$x^2 + y^2 - 2(kx + hy)\left(\frac{hx+ky}{2hk}\right) + (h^2 + k^2 - c^2)\left(\frac{hx+ky}{2hk}\right)^2 = 0$$

or $4h^2k^2x^2 + 4h^2k^2y^2 - 4hk^2x\,(hx + kx) - 4h^2ky\,(hx + ky) + (h^2 + k^2 - c^2)\,(h^2x^2 + k^2y^2 + 2hxy) = 0$

This is the equation of the pair of lines joining the origin to the points of intersection of the given line and curve. They will be at right angles if coefficient of x^2 + coefficient of $y^2 = 0$

$(h^2 + k^2)\,(h^2 + k^2 - c^2) = 0$

Since $[h^2 + k^2$ □ $0]$ □ $h^2 + k^2 = c^2$.

Example 8

The sides of a rhombus ABCD are parallel to the lines $x - y + 2 = 0$ and $7x - y - 3 = 0$. The diagonals intersect at (2, 3). The vertex A is on the y-axis. Find the possible position of A on the y-axis.

Solution

Let A be $(0, y_1)$

The sides of the rhombus are parallel to the lines $x - y + 2 = 0$ and $7x - y - 3 = 0$ and hence the diagonals of the rhombus are parallel to the bisectors of the angles between these lines. The bisectors have their slopes equal to the slopes of the diagonals.

The bisectors are $\frac{x - y + 2}{\sqrt{2}} = \pm \frac{7x - y - 3}{\sqrt{50}}$

i.e., $5x - 5y + 10 = 7x - y - 3$ i.e., $2x + 4y - 13 = 0$

or, $5x - 5y + 10 = -7x + y + 3$ i.e., $12x - 6y + 7 = 0$

The slope of a diagonal joining $(0, y_1)$ and (2, 3) is $\frac{3 - x_1}{2 - 0}$ and this is $= -\frac{1}{2}$ or 2

$3 - y_1 = -1$ or $3 - y_1 = 4$

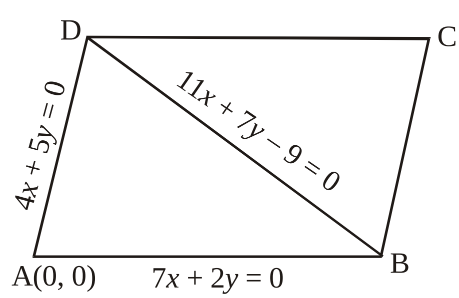
$y_1 = 4$ or $y_1 = -1$

The possible positions of A are (0, 4) or (0, – 1)

Example 9

The consecutive sides of a parallelogram are $4x + 5y = 0$ and $7x + 2y = 0$. If the equation of one diagonal be $11x + 7y = 9$, find the equation of the other diagonal.

Solution



A (0, 0) does not lie on $11x + 7y = 9$ and hence $11x + 7y = 9$ represents the diagonal BD.

Now $11x + 7y - 9 = 0$ can be written as $7x + 2y + 1\,(4x + 5y - 9) = 0$ and again $4x + 5y + 1\,(7x + 2y - 9) = 0$. But $7x + 2y + 1\,(4x + 5y - 9) = 0$ represents a straight line passing through the point of intersection of $7x + 2y = 0$ and $4x + 5y - 9 = 0$.

It follows that B is a point common to $11x + 7y - 9 = 0$, $7x + 2y = 0$ and $4x + 5y - 9 = 0$.

Also $4x + 5y - 9 = 0$ is clearly parallel to $4x + 5y = 0$. Hence $4x + 5y - 9 = 0$ represents the equation of BC. Similarly $7x + 2y - 9 = 0$ represents the equation of DC.

□ the equation of AC is of the form $4x + 5y - 9 + k(7x + 2y - 9) = 0$. Since AC passes through the origin, we have, $-9 - 9k = 0$, □ $k = -1$. Substituting the value of k, AC is given by $4x + 5y - 9 - (7x + 2y - 9) = 0$.

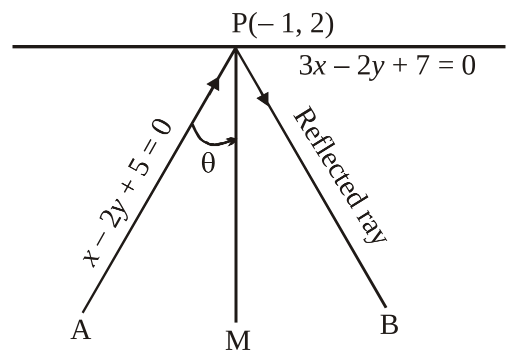
i.e. $-3x + 3y = 0$; i.e., $y = x$

Note: This example can also be done by finding the coordinates of the points B and D.

Example 10

A ray of light is sent along the line $x - 2y + 5 = 0$. Upon reaching the line $3x - 2y + 7 = 0$, the ray is reflected from it. Find the equation of the line containing the reflected ray.

Solution



The reflected ray PB passes through (– 1, 2)

Let its slope be m, slope of PM = – (PM is perpendicular to $3x - 2y + 7 = 0$)

□ $$\tan\theta = \frac{\frac{1}{2} + \frac{2}{3}}{1 - \frac{1}{2}\cdot\frac{2}{3}} = +\frac{7}{4}$$

□MPB = □, and this is the angle between PM whose slope is 2/3 and PB whose slope is m

□ $$\frac{-\frac{2}{3} - m}{1 - \frac{2}{3}m} = +\frac{7}{4}$$ □□ $8 + 12m = -21 + 14m$

□ $m = \frac{29}{2}$

□ equation to PB, the reflected ray is

$y - 2 = \frac{29}{2}(x + 1)$

i.e., $2y - 4 = 29x + 29$

i.e., $29x - 2y + 33 = 0$

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Exercise - I

OBJECTIVE TYPE QUESTIONS

Single Choice Questions

1. The locus of the mid point of the portion intercepted between the axes by the line $x \cos \alpha + y \sin \alpha = p$, where p is constant is

   (a) $x^2 + y^2 = 4p^2$ (b) $\frac{1}{x^2}+\frac{1}{y^2}=\frac{4}{p^2}$ (c) $x^2+y^2=\frac{4}{p^2}$ (d) $\frac{1}{x^2}+\frac{1}{y^2}=\frac{2}{p^2}$

2. The equation of a line which passes through $(a \cos^3 \theta, a \sin^3 \theta)$ and perpendicular to the line $x \sec\theta + y \operatorname{cosec} \theta = a$ is

   (a) $x \cos \theta + y \sin \theta = 2 a \cos 2\theta$ (b)$x \sin \theta - y \cos \theta = 2a \sin 2\theta$

   (c) $x \sin \theta + y \cos \theta = 2a \cos 2\theta$ (d)none of these

3. If a straight line L perpendicular to the line $5x - y = 1$ such that the area of the Δ formed by the line L and the coordinate axes is 5, then the equation of the line L is

   (a) $x + 5y + 5 = 0$ (b) $x + 5y \pm \sqrt{2} = 0$

   (c) $x + 5y \pm \sqrt{5} = 0$ (d) $x + 5y \pm 5\sqrt{2} = 0$

4. The equations of the lines on which the perpendicular from the origin make 30° angle with x–axis and which form a triangle of area $50/\sqrt{3}$ with axes, are

   (a) $x+\sqrt{3}y \pm 10 = 0$ (b) $\sqrt{3}x+y \pm 10 = 0$ (c) $x \pm \sqrt{3}y - 10 = 0$ (d) None of these

5. The line $3x + 2y = 24$ meets y-axis at A and x-axis at B. The perpendicular bisector of AB meets the line through (0, –1) parallel to x-axis at C. Then area of the triangle ABC is

   (a) 182 sq. units (b) 91 sq. units (c) 48 sq. units (d) None of these

6. If one vertex of an equilateral triangle is at (2, –1) and the base is $x + y - 2 = 0$, then the length of each side is

   (a) $\sqrt{\frac{3}{2}}$ (b) $\sqrt{\frac{2}{3}}$ (c) $\frac{2}{3}$ (d) $\frac{3}{2}$

7. The line joining the origin to the points of intersection of $2x^2 + 3xy - 4x + 1 = 0$, $3x + y = 1$ are given by

   (a) $x^2 - y^2 - 5xy = 0$ (b) $c^2 - y^2 + 5xy = 0$ (c) $x^2 + y^2 - 5xy = 0$ (d) $x^2 + y^2 + 5xy = 0$

8. The angle between the lines joining the origin to the points of intersection of the straight line $y = 3x + 2$ with the curve $x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$ is

   (a) $\tan^{-1}\frac{2\sqrt{2}}{3}$ (b) $\tan^{-1}\frac{\sqrt{2}}{3}$ (c) $\tan^{-1}\frac{1}{\sqrt{3}}$ (d) None of these

9. If the lines $x^2 + 2xy - 35y^2 - 4x + 44y - 12 = 0$ and $5x + ky - 8 = 0$ are concurrent then k is

(a) 2 (b) 1 (c) 3 (d) 4

10. Equation of the line passing through the point of intersection of lines $3x + 11y - 7 = 0$ and $5x + 7y + 1 = 0$ and is perpendicular to the line $7x - 5y + 5 = 0$ is

(a) $7x - 5y + 5 = 0$ (b) $11x - 3y + 1 - 0$ (c) $5x + 7y + 1 = 0$ (d) None of these

11. If coordinates of the points A, B, C, D are (13, 7) (–5, 2), (7, 3) and (3, 7) respectively. Let AB and CD meet at P then $PA : PB$ is equal to

(a) 6 : 7 (b) – 6 : 7 (c) 7 : 6 (d) None of these

12. If a ray traveling along the line $x = 1$ gets reflected from the line $x + y = 1$ then the equation of the line along which the reflected ray travels is

(a) $x - y = 1$ (b) $y = 0$ (c) $x = 0$ (d) None of these

13. The straight lines $L_1 \equiv 4x - 3y + 2 = 0$, $L_2 \equiv 3x + 4y - 4 = 0$ and $L_3 \equiv x - 7y + 6 = 0$

(a) form a right angled triangle(b)form a right angled and isosceles triangle

(c) are concurrent (d)none of these

14. The pair of straight lines joining the origin to the common points of $x^2 + y^2 = a^2$ and $y = mx + c$ are perpendicular to each other, if

(a) $2c^2 = a^2 (1 + m^2)$ (b) $c^2 - a^2 = m^2$ (c) $2c^2 (1 + m^2) = a^2$ (d) $c^2 + a^2 = m^2$

15. The three lines represented by $y^3 - 4x^2y = 0$ form a triangle which is

(a) isosceles (b) equilateral (c) right angled (d) None of these

Multiple choice questions MORE THAN ONE correct option

1. The area of a triangle is 5. Two of its vertices are (2, 1) and (3, –2). The third vertex lies on $y = x + 3$. The co-ordinates of the third vertex can be

(a) (– 3/2, 3/2) (b) (3/4, –3/2) (c) (7/2, 13/2) (d) (–1/4, 11/4)

2. If the lines $x - 2y - 6 = 0$, $3x + y - 4 = 0$ and $\lambda x + 4y + \lambda^2 = 0$ are concurrent, then

(a) $\lambda = 2$ (b) $\lambda = -3$ (c) $\lambda = 4$ (d) $\lambda = -4$

3. A line passing through the point (2, 2) and the axes enclose an area λ. The intercept on the axes made by the line are given by the two roots of

(a) $x^2 - 2|\lambda| x + |\lambda| = 0$ (b) $x^2 + |\lambda| x + 2|\lambda| = 0$ (c) $x^2 - |\lambda| x + 2|\lambda| = 0$ (d) None of these

4. Angles made with the x-axis by two lines drawn through the point (1, 2) cutting the line $x + y = 4$ at a distance $\sqrt{6}/3$ from the point (1, 2) are

(a) $\frac{\pi}{12}$ and $\frac{5\pi}{12}$ (b) $-\frac{7\pi}{12}$ and $-\frac{11\pi}{12}$ (c) $\frac{\pi}{8}$ and $\frac{3\pi}{8}$ (d) None of these

5. The point P (1, 1) is translated parallel to $2x = y$ in the first quadrant through a unit distance. The co-ordinates of the new position of P are

(a) $\left(1 \pm \frac{2}{\sqrt{5}}, 1 \pm \frac{1}{\sqrt{5}}\right)$ (b) $\left(1 \pm \frac{1}{\sqrt{5}}, 1 \pm \frac{2}{\sqrt{5}}\right)$ (c) $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ (d) $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

6. The point (4, 1) undergoes the following two successive transformations

(i) reflection about the line $y = x$ (ii) rotation through a distance 2 units along the positive x-axis

(a) (4, 3) (b) (3, 4) (c) (1, 4) (d) (7/2, 7/2)

7. A square with each side equal to a lies above the x-axis and has one vertex at the origin. One of the sides passing through the origin makes an angle $\alpha(0 < a < \pi/4)$ with the positive direction of the x-axis. Equation of a diagonal of the square is

(a) $y(\cos\alpha - \sin\alpha) = x(\sin\alpha + \cos\alpha)$ (b) $y(\sin\alpha + \cos\alpha) + x(\cos\alpha - \sin\alpha) = a$

(c) $x(\cos\alpha - \sin\alpha) = y(\cos\alpha + \sin\alpha)$ (d) $x(\cos\alpha - \sin\alpha) - y(\cos\alpha + \sin\alpha) = a$

8. The coordinates of the feet of the perpendiculars from the vertices of a triangle on the opposite sides are (20, 25), (8, 16) and (8, 9). The coordinates of a vertex of the triangle ae

(a) (5, 10) (b) (50, –5) (c) (15, 30) (d) (10, 15)

9. If the angle between the two lines represented by

$2x^2 + 5xy = 3y^2 + 6x + 7y + 4 = 0$ is $\tan^{-1}(m)$, then m is equal to

(a) 1/5 (b) –1 (c) –2/3 (d) None of these

10. The combined equation of three sides of a triangle is $(x^2 - y^2)(2x + 3y - 6) = 0$. If $(-2, a)$ is an interior point and $(b, 1)$ is an exterior point of the triangle then

(a) $2 < a < \frac{10}{3}$ (b) $-2 < a < \frac{10}{3}$ (c) $-1 < b < \frac{9}{2}$ (d) $-1 < b < 1$

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Exercise - II

ASSERTION & REASON , COMPREHENSION & MATCHING TYPE

Assertion and Reason Type Questions

In the following question, a statement of Assertion (A) is given which is followed by a corresponding statement of reason (R). Mark the correct answer out of the following options/codes.

(a) If both (A) and (R) are true and (R) is the correct explanation of (A).

(b) If both (A) and (R) are true but (R) is not correct explanation of (A).

(c) If (A) is true but (R) is false.

(d) If (A) is false but (R) is true.

1. A: The equation of the line through (2, 3) and parallel to the line $y \ \underset{x\to 0}{\text{Lt}} \frac{e^{xt}-1}{t}+5$ is $x - y + 1 = 0$.

   R: From the limit value, $y = x + 5$ which is a straight line with slope 1, therefore equation of the line through (2, 3) having slope 1 is $x - y + 1 = 0$

2. A: The incentre of the triangle formed by the line $x\cos\frac{\pi}{9}+y\sin\frac{\pi}{9}=\pi,\ \ x\cos\frac{8\pi}{9}+y\sin\frac{8\pi}{9}=\pi,$ $x\cos\frac{13\pi}{9}+y\sin\frac{13\pi}{9}=\pi$ is (0, 0)

   R: The point (0, 0) is equidistant from the three vertices of the triangle formed by the lines $x\cos\frac{\pi}{9}+y\sin\frac{\pi}{9}=\pi,\ \ x\cos\frac{8\pi}{9}+y\sin\frac{8\pi}{9}=\pi,\ \ x\cos\frac{13\pi}{9}+y\sin\frac{13\pi}{9}=\pi$.

3. A: The line $2x - y + 3 = 0$ is perpendicular to the line $x + 2y + 5 = 0$ and first line passes through (1, 2).

   R: Product of the slopes of the lines is equal to –1

4. A: The point (3, 2) and (1, 4) lie on opposite side of the $3x - 2y - 1 = 0$

   R: The algebraic perpendicular distance from the given point to the line have opposite sign

5. A: The equation $2x^2 - 3xy - 2y^2 + 5x - 5y + 3 = 0$ represents a pair of perpendicular straight lines.

   R: A pair of lines given by $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ are perpendicular if $(a + b) = 0$.

Passage Based Questions

Passage – I

$L_1 : 3x + 4y + 8 = 0$

$L_2 : 2x + 7y - 1 = 0$

1. If L_1, L_2 represent the sides AB and AC of the isosceles triangle ABC with $AB = AC = 2$ then the coordinates of

   (a) B are (28/5, – 11/5) (b) B are (28/5, 1/5)

   (c) C are $\left(\frac{14+4\sqrt{53}}{\sqrt{53}}, \frac{-4-\sqrt{5}}{\sqrt{5}}\right)$ (d) C are $\left(\frac{14+4\sqrt{53}}{\sqrt{53}}, \frac{4-\sqrt{5}}{\sqrt{5}}\right)$

2. Equation of the line through B parallel to AC is

   (a) $2x + 7y + 21 = 0$ (b) $10x + 35y = 63$ (c) $10x + 35y + 21 = 0$ (d) $2x + 7y = 63$

3. If D is the mid-point of BC and E is the mid-point of CA then DE is equal to

   (a) 1/4 (b) ½ (c) 1 (d) 2

Passage –II

Suppose two equations representing a pair of straight lines have identical portion of quadratic terms, then the pair of line represented by one equation will be parallel to the pair of the represented by the second equation. So if each of the following equation

$$ax^2 + 2hxy + by^2 + 2gx - 2fy + c = 0 \qquad \ldots(1)$$

and $ax^2 + 2hxy + by^2 - 2gx - 2fy + c = 0$ …(2)

represents a pair of straight lines then these four enclose a parallelogram

1. The value of k so that the equation $6x^2 + 11xy + 10y^2 + 31y + k = 0$ may represent a pair of straight lines is

   (a) – 15 (b) – 16 (c) 15 (d) 16

2. If the slope of one of the lines represented by the equations $ax^2 + 2hxy + by^2 = 0$ be square of the other then $\frac{a+b}{h} + \frac{8^2}{ab}$ is equal to

   (a) 9 (b) 8 (c) 7 (d) 6

3. The lines $(lx + my)^2 - 3(mx - ly)^2 = 0$ and $lx + my + n = 0$ form an

   (a) isosceles triangle (b) scalene triangle (c) equilateral triangle (d) None of these

Matching Type Questions

1. **Column I** **Column II**

(A) The equation of a line which passes through the point (–2, 3) and makes an angle of 30° with the positive

(P) $x + y + 1 = 0$

(B) A line passes through a point (–3, 2) and has a slope $-\sqrt{3}$. The equation of the line in symmetric form is

(Q) $4x + 3x + 12 = 0$

(C) The area of triangle formed by the co-ordinate axes and a line is 6 square units and the length of hypotenuse is 5 units. The equation of the line is

(R) $2y + 2\sqrt{3}x + 6(\sqrt{3} - 4) = 0$

(D) The equation of a line which passes through (1, –2) and cuts off equal intercepts on the axis is

(S) $x - \sqrt{3}y + (2 + 3\sqrt{3}) = 0$

(a) A-S, B-R, C-P, D-Q (b) A-Q, B-R, C-P, D-S
(c) A-P, B-R, C-Q, D-S (d) A-S, B-R, C-Q, D-P

2. **Column I** **Column II**

(A) If the equation $12x^2 - 10xy + 2y^2 + 11x$ $5y + c = 0$ represents a pair of straight lines and θ be the angle between them, then |tan θ| is equal to

(P) –2

(B) If the straight line $\frac{x}{a} + \frac{y}{b} = 1$ intersects the curve $\frac{x^2}{b^2} + \frac{y^2}{a^2}$ at the points *P* and *Q* and the chord *PQ* subtends angle θ at the origin then tan $\frac{\theta}{2}$ is equal to

(Q) 2

(C) If the lines $x^2 + 4xy - 2y^2 + 4x + 2fy + c^2 = 0$ intersect on the *x*-axis then *f* is equal to

(R) 4

(D) In the equation given in (iii) the value of *c* is equal to

(S) 1

(T) $\frac{1}{7}$

(a) A-T, B-S, C-R, D-P,Q (b) A-Q, B-R, S, T, C-T, D-S, T
(c) A-P,Q B-R, C-Q, D-S,T (d) A-R,T B-S, C-Q, D-P, R

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Exercise - III

SUBJECTIVE TYPE QUESTIONS

1. (p, q) is a point such that p and q are integers, $p \geq 50$ and the equation $px^2 + qx + 1 = 0$ has real roots. Square of the least distance of the point from the origin is.

2. The locus of the mid-point of the portion intercepted between the axes by the line $x \cos a + y \sin \alpha = p$ passes through the point $(p + 1, p - 1)$, then $71\,(4p^4 - 20p^2 + 99)$ is equal to.

3. If the point (3, 4) lies on the locus of the point of intersection of the lines $x \cos \alpha + y \sin \alpha = a$ and $x \sin \alpha - y \cos \alpha = b$, ($\alpha$ is a variable), the point (a, b) lies on the line $3x - 4y = 0$ then $9a^4 + 16b^4 + 34$ is equal to.

4. Through the point $P(3, -5)$, a line is drawn inclined at 45° with the positive direction of x-axis. It meets the line $x + y - 6 = 0$ at the point Q, O being the origin, then $9(PQ)^2 + 14(OP)^2 + 10(OQ)^2$ is equal to.

5. If the circumcentre of the triangle whose vertices are (0, 2), (3, 5) and (5, 8) is (h, k) then $4(h^2 + k^2)$ is equal to ______ .

6. The points (1, 3) and (5, 1) are two opposite vertices of a rectangle. The other two vertices lie on the line $y = 2x + c$. Find c and the remaining vertices.

7. Consider the family of lines, $5x + 3y \square 2 + K_1\,(3x \square y \square 4) = 0$ and $x \square y + 1 + K_2(2x \square y \square 2) = 0$. Find the equation of the line belonging to both the families without determining their vertices.

8. If the equation, $ax^2 + 2\,hxy + by^2 + 2\,gx + 2\,fy + c = 0$ represent a pair of straight lines, prove that the equation to the third pair of straight lines passing through the points where these meet the axes is, $ax^2 \square 2\,hxy + by^2 + 2\,gx + 2\,fy + c + \frac{4fg}{c}xy = 0$

9. A triangle has two sides $y = m_1x$ and $y = m_2x$ where m_1 and m_2 are the roots of the equation $b\alpha^2 + 2h\alpha + a = 0$. If (a, b) be the orthocentre of the triangle, then find the equation of the third side in terms of a, b and h.

10. If the origin line in the obtuse angle between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ prove that $(a_1a_2 + b_1b_2)\,c_1c_2 > 0$.

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Exercise - IV

IIT – JEE PROBLEMS

A. Fill in the blanks

1. $y = 10^x$ is the reflection of $y = \log_{10}x$ in the line whose equation is ______ .
2. The set of lines $ax + by + c = 0$, where $3a + 2b + 4c = 0$ is concurrent at the point ______ .
3. If a, b and c are in A.P., then the straight line $ax + by + c = 0$ will always pass through a fixed point whose coordinates are (______).
4. The sides AB, BC and CA of a triangle ABC have 3, 4 and 5 interior points respectively on them. The number of triangles that can be constructed these interior points as vertices is ______ .
5. The set of all real numbers a such that $a^2 + 2a + 3$ and $a^2 + 3a + 8$ are the sides of a triangle is ______ .
6. The orthocenter of the triangle formed by the lines $x + y = 1$, $2x + 3y = 6$ and $4x - y + 4 = 0$ lies in quadrant number ______ .
7. Let the algebraic sum of the perpendicular distance from the points (2, 0), (0, 2) and (1, 1) to a variable straight line be zero, then the line passes through a fixed point whose coordinates are ______ .
8. The vertices of a triangle are $A(-1, -7)$, $B(5, 1)$ and $C(1, 4)$. The equation of the bisector of the angle $\angle ABC$ is ______ .

B. True / False

9. The straight line $5x + 4y = 0$ passes through the point of intersection of the straight lines $x + 2y - 10 = 0$ and $2x + y + 5 = 0$.
10. No tangent can be drawn from the point (5/2, 1) to the circumcircle of the triangle with vertices $(1, \sqrt{3})$, $(1, -\sqrt{3})$, $(3, \sqrt{3})$.
11. If $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$, then the two triangles with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) and (a_1, b_1), (a_2, b_2), (a_3, b_3) must be congruent.
12. The lines $2x + 3y + 19 = 0$ and $9x + 6y - 17 = 0$ cut the coordinates axes in concyclic points.

C. Multiple Choice Questions with ONE correct answer

13. Given the four lines with the equations

    $x + 2y - 3 = 0, 3x + 4y - 7 = 0,$

$2x + 3y - 4 = 0$, $4x + 5y - 6 = 0$

then

(a) they are all concurrent (b)they are the sides of a quadrilateral

(c) None of these

14. The point (4, 1) undergoes the following three transformations successively

(i) Reflection about the line $y = x$

(ii) Transformation through a distance 2 units along the positive direction of x-axis

(iii) Rotation through an angle $\frac{\pi}{4}$ about the origin in the counter clockwise direction

Then the final position of the point is given by the coordinates

(a) $\left(\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$ (b) $\left(-\sqrt{2}, 7\sqrt{2}\right)$ (c) $\left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$ (d) $\left(\sqrt{2}, 7\sqrt{2}\right)$

15. The straight lines $x + y = 0$, $3x + y - 4 = 0$, $x + 3y - 4 = 0$ form a triangle which is

(a) isosceles (b) equilateral (c) right angled (d) None of these

16. The point $\left(0, \frac{8}{3}\right)$, $(1, 3)$ and (82, 30) are vertices of

(a) an obtuse angled triangle (b)an acute angled triangle

(c) a right angled triangle (d)None of these

17. If $P = (1, 0)$, $Q = (-1, 0)$ and $R = (2, 0)$ are three given points, then locus of the points satisfying the relation $SQ^2 + SR^2 = 2SP^2$, is

(a) a straight line parallel to x-axis (b) a circle passing through the origin

(c) a circle with the centre at the origin (d) a straight line parallel to y-axis

18. Line L has intercepts a and b on the coordinate axes. When the axes are rotated through a given angle, keeping the origin fixed, the same line L has intercepts p and q, then

(a) $a^2 + b^2 = p^2 + q^2$ (b) $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2} + \frac{1}{q^2}$ (c) $a^2 + p^2 = b^2 + q^2$ (d) $\frac{1}{a^2} + \frac{1}{p^2} = \frac{1}{b^2} + \frac{1}{q^2}$

19. If the sum of the distance of a point from two perpendicular lines in a plane is 1, then its locus is

(a) square (b) circle

(c) straight line (d) two intersecting lines

20. The diagonals of a parallelogram $PQRS$ are along the lines $x + 3y = 4$ and $6x - 2y = 7$. Then $PQRS$ must be a

(a) rectangle (b) square (c) cyclic quadrilateral (d) rhombus

21. If $P(1, 2)$, $Q(4, 6)$, $R(5, 7)$ and $S(a, b)$ are the vertices of a parallelogram $PQRS$, then

(a) $a = 2, b = 4$ (b) $a = 3, b = 4$ (c) $a = 2, b = 3$ (d) $a = 3, b = 5$

22. If the vertices P, Q, R of a triangle PQR are rational points, which of the following points of the triangle PQR is/(are) always rational point(s)

(a) centroid (b) incentre (c) circumcentre (d) orthocenter

23. Let $A_0, A_1, A_2, A_3, A_4, A_5$ be a regular hexagon inscribed in a circle of unit radius. Then the product of the lengths of the line segments A_0, A_1, $A_0 A_2$ and $A_0 A_4$ is

(a) $\frac{3}{4}$ (b) $3\sqrt{3}$ (c) 3 (d) $\frac{3\sqrt{3}}{2}$

24. Let PS be the median of the triangle with vertices $P(2, 2)$, $Q(6, -1)$ and $R(7, 3)$. The equation of the line passing through $(1, -1)$ and parallel to PS is

(a) $2x - 9y - 7 = 0$ (b) $2x - 9y - 11 = 0$ (c) $2x + 9y - 11 = 0$ (d) $2x + 9y + 7 = 0$

25. The incentre of the triangle with vertices $(1, \sqrt{3})$, $(0, 0)$ and $(2, 0)$ is

(a) $\left(1, \frac{\sqrt{3}}{2}\right)$ (b) $\left(\frac{2}{3}, \frac{1}{\sqrt{3}}\right)$ (c) $\left(\frac{2}{3}, \frac{\sqrt{3}}{2}\right)$ (d) $\left(1, \frac{1}{\sqrt{3}}\right)$

26. The number of integer values of m, for which the x-coordinate of the point of intersection of the lines $3x + 4y = 9$ and $y = mx + 1$ is also an integer, is

(a) 2 (b) 0 (c) 4 (d) 1

27. Area of the parallelogram formed by the lines $y = mx$, $y = mx + 1$, $y = nx$ and $y = nx + 1$ equals

(a) $\frac{|m+n|}{(m-n)^2}$ (b) $\frac{2}{|m+n|}$ (c) $\frac{1}{|m+n|}$ (d) $\frac{1}{|m-n|}$

28. Let $P = (-1, 0)$, $Q = (0, 0)$ and $R = (3, 3\sqrt{3})$ be three points. Then the equations of the bisector of the angle PQR is

(a) $\frac{\sqrt{3}}{2} x + y = 0$ (b) $x + \sqrt{3}y = 0$ (c) $\sqrt{3}x + y = 0$ (d) $x + \frac{\sqrt{3}}{2} y = 0$

29. A straight line through the origin O meets the parallel lines $4x + 2y = 9$ and $2x + y + 6 = 0$ at points P and Q respectively. Then the point O divides the segment PQ in the ratio

(a) 1 : 2 (b) 3 : 4 (c) 2 : 1 (d) 4 : 3

D. Multiple Choice Questions with ONE or MORE THAN ONE correct answer

30. Three lines $px + qy + r = 0$, $qx + ry + p = 0$ and $rx + py + q = 0$ are concurrent, if

(a) $p + q + r = 0$ (b) $p^2 - q^2 + r^2 = pr + rq$

(c) $p^3 + q^3 + r^3 = 3pqr$ (d) None of these

31. All points lying inside the triangle formed by the points (1, 3), (5, 0) and (–1, 2) satisfy

(a) $3x + 2y \geq 0$ (b) $2x + y - 13 \geq 0$ (c) $2x - 3y - 12 \leq 0$ (d) $-2x + y \geq 0$

32. Let L_1 be a straight line passing through the origin and I_2 be the straight line $x + y = 1$. If the intercepts made by the circle $x^2 + y^2 - x + 3y = 0$ on L_1 and L_2 are equal, then which of the following equation can represent L_1 ?

(a) $x + y = 0$ (b) $x - y = 0$ (c) $x + 7y = 0$ (d) $x - 7y = 0$

E. Subjective Questions

33. A straight line L is perpendicular to the line $5x - y = 1$. The area of the triangle formed by the line L and the coordinate axes is 5. Find the equation of the line L.

34. The points (1, 3) and (5, 1) are two opposite vertices of a rectangle. The other two vertices lie on the line $y = 2x + c$. Find c and the remaining vertices.

35. The ends A, B of a straight line segment of constant length c slide upon the fixed rectangular axes OX, OY respectively. If the rectangle $OAPB$ be completed, then show that the locus of the foot of the perpendicular drawn from P to AB is $x^{2/3} + y^{2/3} = c^{2/3}$

36. The coordinates of A, B, C are (6, 3), (–3, 5), (4, –2) respectively and P is any point (x, y). Show that the ratio of the areas of the triangles $\Delta\, PBC$ and $\Delta\, ABC$ is $\left|\frac{x+y-2}{7}\right|$.

37. The vertices of a triangle are

$[at_1 t_2, a(t_1 + t_2)]$, $[at_2t_3, a(t_2 + t_3)]$, $[at_3 t_1, a(t_3 + t_1)$. Find the orthocenter of the triangle.

38. Two equal sides of an isosceles triangle are given by the equations $7x - y + 3 = 0$ and $x + y - 3 = 0$ and its third side passes through the point (1, –10). Determine the equation of the third side.

39. Two sides of a rhombus $ABCD$ are parallel to the lines $y = x + 2$ and $y = 7x + 3$. If the diagonals of the rhombus intersect at the point (1, 2) and the vertex A is on the y-axis, find possible coordinates of A.

40. One of the diameter of the circle circumscribing the rectangle $ABCD$ $4y = x + 7$. If A and B are the points (–3, 4) and (5, 4) respectively, then find the area of rectangle.

41. The equations of the perpendicular bisectors of the sides AB and AC of a triangle ABC are $x - y + 5 = 0$ and $x + 2y = 0$, respectively. If the points A is (1, –2), find the equation of the line BC.

42. Lines $L_1 \equiv ax = by + c = 0$ and $L_2 \equiv lx + my + n = 0$ intersect at the point P and make an angle θ with each other. Find the equation of a line L different from L_2 which passes through P and makes the same angle θ with L_1.

43. Let ABC be a triangle with $AB = AC$. If D is mid-point of BC, the foot of the perpendicular drawn from D to AC and F the mid-point of DE. Prove that AF is perpendicular to BE.

44. A line cuts the x-axis at $A(7, 0)$ and the y-axis at $B(0, -5)$. A variable line PQ is drawn perpendicular to AB cutting the x-axis in P and the y-axis in Q. If AQ and BP intersect at R find the locus of R.

45. Straight lines $3x + 4y = 5$ and $4x - 3y = 15$ intersect t the point A. Points B and C are chosen on these two lines such that $AB = AC$. Determine the possible equations of the line BC passing through the point (1, 2).

46. Find the equation of the line passing through the point (2, 3) and making intercept of length 3 units between the lines $y + 2x = 2$ and $y + 2x = 5$.

47. Determine all values of α for which the point (α, α^2) lies inside the triangles formed by the lines $2x + 3y - 1 = 0$ $x + 2y - 3 = 0$, $5x - 6y - 1 = 0$.

48. A line through $A(-5, -4)$ meets the line $x + 3y + 2 = 0$, $2x + y + 4 = 0$ and $x - y - 5 = 0$ at the points B, C and D respectively. If $(15/AB)^2 + (10/AC)^2 = (6/AD)^2$, find the equation of the line.

49. A rectangle $PQRS$ has its side PQ parallel to the line $y = mx$ and vertices PQ and S on the lines $y = a$, $x = b$ and $x = -b$, respectively. Find the locus of the vertex R.

50. A variable straight line of slope 4 intersects the hyperbola $xy = 1$ at two points. Find the locus of the point which divides the line segment between these two pointing the ratio 1 : 2.

51. Using co-ordinate geometry, prove that the three altitudes of any triangle are concurrent.

52. For points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ of the co-ordinate plane, a new distance $d(P, Q)$ is defined by $d(P, Q) = |x_1 - x_2| + |y_1 - y_2|$. Let $O = (0, 0)$ and $A = (3, 2)$. Prove that the set of points in the first quadrant which are equidistant (with respect to the new distance) from O and A consists of the union of a line segment of finite length and an infinite ray. Sketch this set in a labeled diagram.

53. A straight line L through the origin meets the line $x + y = 1$ and $x + y = 3$ at P and Q respectively. Through P and Q two straight lines L_1 and L_2 are drawn, parallel to $2x - y = 5$ and $3x + y = 5$ respectively. Lines L_1 and L_2 intersect at R, show that the locus of R as L varies, is a straight line.

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Answers

Exercise - I

Only One Option is correct

1.	(b)	2.	(d)	3.	(b)	4.	(c)	5.	(b)
6.	(a)	7.	(a)	8.	(a)	9.	(a)	10.	(c)
11.	(d)	12.	(b)	13.	(c)	14.	(c)	15.	(d)

More Than One Choice Correct

1.	(a, c)	2.	(a, d)	3.	(c)	4.	(a, b)	5.	(b)
6.	(b)	7.	(a, b)	8.	(a, b, c)	9.	(a)	10.	(a, d)

Exercise - II

Assertion and Reason

1.	(a)	2.	(c)	3.	(d)	4.	(a)	5.	(d)

Passage Based Questions

Passage – I

1.	(a, c)	2.	(c)	3.	(c)

Passage –II

1.	(a)	2.	(d)	3.	(c)

Matching Type Questions

1.	(d)	2.	(a)

Exercise - III

Subjective Type

1.	2725	2.	6461	3.	3634	4.	1264	5.	1850

6. $c =$ □ 4, $B(2, 0)$, $D(4, 4)$ 7. $5x$ □ $2y$ □ $7 = 0$

9. $(a + b)(ax + by) = ab(a + b - 2h)$

Exercise - IV

IIT-JEE Level Problem

Section - A

1. $y = x$ 2. $\left(\frac{3}{4}, \frac{1}{2}\right)$ 3. $(1, -2)$ 4. 205 5. $a > 5$

6. $\left(\frac{41}{7}, \frac{22}{7}\right)$ 7 $(1, 1)$ 8. $7y = x + 2$

Section - B

9. True 10. True 11. False 12. True

Section – C

13. (c) 14. (c) 15. (a) 16. (d) 17. (d)
18. (b) 19. (a) 20. (d) 21. (c) 22. (a)
23. (c) 24. (d) 25. (d) 26. (a) 27. (d)
28. (c) 29. (b)

Section - D

30. (a, c) 31. (a, c) 32. (b, c)

Section - E

33. $x + 5y = \pm 5\sqrt{2}$

34. $c = -4$, (4, 4), (4, 0)

37. $(-a, a(t_1 + t_2 + t_3 + t_1t_2t_3))$

38. $x - 3y - 31 = 0$ and $3x + y + 7 = 0$

39. $\left(0, \frac{5}{2}\right), (0, 0)$

40. 32 sq. untis

41. $14x + 23y - 40 = 0$

42. $2(al + bm)(ax + by + c) - (a^2 + b^2)(ln + my + n) = 0$

44. $x^2 + y^2 - 7x + 5y = 0$

45. $x - 7y + 13 = 0$ and $7x + y + 9 = 0$

46. $x = 2$ and $3x + 4y = 18$

47. $-\frac{3}{2} < \alpha < -1 \cup \frac{1}{2} < \alpha < 1$

48. $2x + 3y + 22 = 0$

49. $(m^2 - 1)x - my + b(m^2 + 1) + am = 0$

50. $(2y - x)(2x - y) = -4$

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