SEQUENCE & SERIES

Leonardo Fibonacci

Leonardo Fibonacci, a 13th century Pisan mathematician/astronomer, formulated a beautifully simple progression of numbers. The series starts with 1. The next number is also 1. Adding these two numbers produces 2. Adding the preceeding produces 3. 2 + 3 = 5. 5 + 3 = 8. 8 + 5 = 13, and so on.

The ratio between each succeeding pair of numbers (1 : 1 . 618) describes the ratio of the "Golden Section", a famous classical proportion, as well as the growth rate of many organisms, as seen in the seedhead of the sunflower, the end view of a pine cone, growth rings of clam shells, etc. The Fibonacci series of numbers is found over and over in plant growth. Looking at the seedhead of a sunflower, one will see that the seeds are arranged in 55 clockwise spirals, and 89 or 34 counterclockwise spirals, numbers adjacent to each other in the series. Likewise, in a daisy, pine cone and other natural forms, their structure is based on a combination of numbers from the Fibonacci series.

Although Fibonacci formulated his series in the 13th century, the relationship between living things and geometry was observed by pre-historic people. Their observations, spirals, and other forms, were carved in bone or painted on cave walls at least 20,000 years B.C.

Introduction

A sequence is a function whose domain is the set of natural numbers. If the range of a sequence is a subset of a real number (complex number), then it is called a real sequence (complex sequence). If $t_n = f(n)$, then the sequence is written as $\{t_1, t_2, t_3, \ldots\}$.

2.1 Arithmetic Progression (A.P.)

An arithmetic progression (A.P.) is a progression in which the difference of any term and its preceding term is constant through out. And this constant is called the common difference of the A.P.

Or, we can say that an arithmetic progression (A.P.) is a sequence which increases or decreases by a fixed number (common difference).

If $a_1, a_2, a_3,$ .... are in A.P. then, common difference, $d = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \ldots\ldots.. = a_k - a_{k-1}$, where $k \in N$

NOTE:

(*i*) If a is the first term and d is the common difference, then the A.P. can be written as,
$a, a + d, a + 2d, a + 3d, \ldots\ldots.$

(*ii*) nth term of this A.P. $a_n = a + (n - 1)\, d$

(*iii*) Sum of n terms of this A.P. $S_n = \frac{n}{2}\{2a + (n-1)\,d\}$.

If the series is having n terms and the last term (nth term) is l then, $S_n = \frac{n}{2}(a + l)$

❑ Properties of A.P.

(*i*) If $a_1, a_2, a_3, \ldots\ldots\ldots$ are in A.P. then
$a_1 \pm k, a_2 \pm k, a_3 \pm k, a_4 \pm k, \ldots\ldots\ldots$ are also in A.P. with the same common difference

(*ii*) If $a_1, a_2, a_3, \ldots\ldots\ldots$ are in A.P. then,
$a_1k, a_2k, a_3k, \ldots\ldots\ldots.$ and $a_1/k, a_2/k, a_3/k, \ldots\ldots$ are also in A.P. ($k \neq 0$) with common difference kd or d/k respectively.

(*iii*) If $a_1, a_2, a_3, \ldots\ldots..$ are in A.P. and $b_1, b_2, b_3, \ldots\ldots..$ are also in A.P. then,
$a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3, \ldots\ldots\ldots$ are also in A.P.

(*iv*) If $a_1, a_2, a_3, \ldots\ldots..$ are in A.P. then, $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots\ldots.. = a_k + a_{n+1-k}$

(*v*) $a_r = \frac{a_{r-k} + a_{r+k}}{2}\ \forall\, r$, $0 \leq k \leq n - r$

(*vi*) If the nth term of a sequence is a linear expression in n, $a_n = an + b$, then the sequence is A.P. with common difference 'a'.

(*vii*) If the sum of its n terms is of the type, $S_n = an^2 + bn$, then the sequence is an A.P. with common difference '$2a$'.

❑ Selection of Terms

(*a*) To select odd number of terms

$a - d, a, a + d$ (three terms)

$a - 2d, a - d, a, a + d, a + 2d$ (five terms)

$a - 3d, a - 2d, a - d, a_1a + d, a + 2d, a + 3d$ (seven terms)

(*b*) To select even number of terms

$a - d, a + d$ (two terms)

$a - 3d, a - d, a + d, a + 3d$ (four terms)

Arithmetic mean

(a) Arithmetic mean of n numbers

Let a_1, a_2, a_3 ...., a_n be the given 'n' numbers and 'A' be there arithmetic mean, then

$$A = \frac{a_1 + a_2 + a_3 + \ldots\ldots\ldots + a_n}{n}$$

(*b*) Insertion of single Arithmetic mean between 'a' and 'b'

Let 'A' be the arithmetic mean of a and b, then a, A, b form an A.P.

Thus $A - a = b - A$

$$A = \left(\frac{a+b}{2}\right)$$

(*c*) Inserting 'n' arithmetic means between 'a' and 'b'.

Let A_1, A_2, A_3, ..., A_n be the AM's.

then a, A_1, A_2, ..., A_n, b will be in A.P.

Now, $b = a + (n + 1)\ d$

$d = \frac{b-a}{n+1}$ (common difference)

□ $A_1 = a + d = a + \left(\frac{b-a}{n+1}\right)$

$A_2 = a + 2d = a + 2\left(\frac{b-a}{n+1}\right)$

$A_3 = a + 3d = a + 3\left(\frac{b-a}{n+1}\right)$

.. ..

.. ..

so kth term of the A.M. is

□ $A_k = a + k\left(\frac{b-a}{n+1}\right)$

ILLUSTRATIONS

***Illustration* 1**

The interior angles of a polygon are in arithmetic progression. The smallest angle is 120° and the common difference is 5°. Find the number of sides of the polygon.

Solution

Let the number of sides of the polygon be n.

The sum of the interior angles of the polygon = $(n - 2)$ □ = $(n - 2)$. 180°

Also the first term of the A.P. = a = 120°. The common difference = d = 5°

□ sum of all the interior angles is

$$= \frac{n}{2}\left[2\times120°+(n-1)5°\right]$$

□ $\frac{n}{2}$ □ (240° + (n – 1) 5°) = (n – 2) 180°

□ n (48 + n – 1) = (n – 2)72

□ $n^2 - 25n + 144 = 0$,

□ n = 9 or 16

For n = 16, the largest angle = 120° + 15 × 5° = 195°.

This is not possible as an interior angle of a polygon cannot be greater than 180°.

Hence n = 9.

***Illustration* 2**

If $a_1, a_2, \ldots\ldots\ldots\ldots a_n$ are in A.P. ($a_i > 0$ for all i),

Show that $\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\ldots\ldots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$.

Solution

L.H.S. = $\frac{\sqrt{a_1}-\sqrt{a_2}}{a_1-a_2}+\frac{\sqrt{a_2}-\sqrt{a_3}}{a_2-a_3}+\ldots\ldots+\frac{\sqrt{a_{n-1}}-\sqrt{a_n}}{a_{n-1}-a_n}$

If d is the common difference, then

L.H.S. $= \frac{\sqrt{a_1}-\sqrt{a_2}}{-d}+\frac{\sqrt{a_2}-\sqrt{a_3}}{-d}+\ldots\ldots+\frac{\sqrt{a_{n-1}}-\sqrt{a_n}}{-d}$

□ L.H.S. = $-\frac{1}{d}\left[\sqrt{a_1}-\sqrt{a_2}+\sqrt{a_2}-\sqrt{a_3}+\ldots\ldots+\sqrt{a_{n-1}}-\sqrt{a_n}\right]$

$$= -\frac{1}{d}\frac{(a_1-a_n)}{\sqrt{a_1}+\sqrt{a_n}}=\frac{(a_n-a_1)}{d}\cdot\frac{1}{\sqrt{a_1}+\sqrt{a_n}}$$

$$= \frac{a_1+(n-1)d-a_1}{d}\cdot\frac{1}{\sqrt{a_1}+\sqrt{a_n}}$$

$$= \frac{n-1}{\sqrt{a_1}+\sqrt{a_n}} = \text{R.H.S.}$$

***Illustration* 3**

Four different integers form an increasing A.P. One of these numbers is equal to the sum of the sum of the squares of the other three numbers. Find the numbers.

Solution

Let the numbers be

$a - d, a, a + d, a + 2d$

where, $a, d \in I, d > 0$

according the hypothesis

$$(a-d)^2 + a^2 + (a+d)^2 = a + 2d$$

i.e., $2d^2 - 2d + 3a^2 - a = 0$

$$\therefore \quad d = \frac{1}{2}\left[1 \pm \sqrt{(1+2a-6a^2)}\right]$$

Since, d is positive integer,

$$1 + 2a - 6a^2 > 0$$

$$\therefore \quad a^2 - \frac{a}{3} - \frac{1}{6} < 0$$

$$\therefore \quad \left(a - \frac{1-\sqrt{7}}{6}\right)\left(a - \frac{1+\sqrt{7}}{6}\right) < 0$$

$$\therefore \quad \left(\frac{1-\sqrt{7}}{6}\right) < a < \left(\frac{1+\sqrt{7}}{6}\right)$$

Since, a is an integer,

$$a = 0$$

then, $d = \frac{1}{2}[1 \pm 1] = 1$ or 0. Since, $d > 0 \quad \therefore d = 1$

Hence, the numbers are – 1, 0, 1, 2

PRACTICE EXERCISE

1. If $a_1, a_2, a_3, \ldots,$ be an A.P. of non-zero terms, prove that

$$\frac{1}{a_1 a_n} + \frac{1}{a_2 a_{n-1}} + \frac{1}{a_3 a_{n-2}} + \ldots + \frac{1}{a_n a_1} = \frac{2}{a_1 + a_n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right).$$

2. Find the sum of first 24 terms of the A.P. $a_1, a_2, a_3, \ldots,$ if it is known that $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$.

3. Solve $1 + 6 + 11 + 16 + \ldots + x = 148$.

4. The ratio of the sum of n terms of two A.P.'s is $(7n + 1) : (4n + 27)$. Find the ratio of their mth terms.

5. For what values of the parameter a are there values of x such that $5^{1+x} + 5^{1-x}$, $a/2$, $25^x + 25^{-x}$ are three consecutive terms of an A.P.?

Answers

2. 900 **3.** 36 **4.** $(14m - 6) : (8m + 23)$ **5.** $a \geq 12$

2.2 Geometric Progression

A progression is called geometric progression if ratio of any term and its preceding term is constant. This constant is called common ratio of that series.

A sequence $a_1, a_1, a_3, \ldots a_n$ is called a geometric progression if $\frac{a_n}{a_{n-1}}$ = constant

Note:

(*a*) If t_1, t_2, t_3, ... are in G.P. then the common ratio is

$$r = \frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_4}{t_3} = \ldots\ldots\ldots = \frac{t_k}{t_{k-1}}$$

(*b*) If a is the first term and r is the common ratio then G.P. can be written as,

$a, ar, ar^2, \ldots\ldots\ldots\ldots, ar^{n-1}, \ldots\ldots\ldots$ (a □ 0)

(i) nth term of this G.P.

$$t_n = ar^{n-1}$$

(ii) nth term from end.

$t'_n = \frac{l}{r^{n-1}}$ where l is the last term.

(*iii*) sum of the first n term of this G.P. is

$$S = a + ar + ar^2 + \ldots\ldots\ldots\ldots + ar^{n-1}$$

$S = \frac{a(r^n - 1)}{(r-1)} = \frac{ar^n - a}{r-1}$ if r □ 1

$= an$, if $r = 1$

(iv) Sum of infinite G.P. when $|r| < 1$

$S = a + ar = ar^2 + \ldots\ldots\ldots$ □

$S = \frac{a}{1-r}$, $|r| < 1$

❑ **Properties of G.P.**

(*a*) If $a_1, a_2, a_3, \ldots\ldots\ldots a_n$ are in G.P. then, $a_1k, a_2k, a_3k, \ldots\ldots\ldots$ and $\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}$, ... are also in G.P. with common ratio 'r'

(*b*) If $a_1, a_2, \ldots\ldots\ldots a_n$ are in G.P. and $b_1, b_2, \ldots\ldots\ldots b_n$ are in G.P. then

$a_1b_1, a_2b_2, a_3b_3, \ldots\ldots\ldots a_nb_n$ is also in G.P. and same way, $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots\ldots\ldots\ldots, \frac{a_n}{b_n}$

(*c*) If $a_1, a_2, a_3, \ldots\ldots\ldots$ are in G.P. ($a_i > 0$) then $\log a_1, \log a_2, \log a_3, \ldots\ldots$ are in A.P.

(*d*) If $a_1, a_2, a_3, \ldots\ldots\ldots a_n$ are in G.P. then $a_1^k, a_2^k, .a_3^k, a_4^k, \ldots\ldots\ldots\ldots$ is also in G.P.

(*e*) If $a_1, a_2, a_3, ... a_n$ are in G.P. then $a_1 a_n = a_2 a_{n-1} = a_3 a_{n-2} = ... = a_k \, . \, a_{n+1-k}$, $0 < k$ □ n & k □ N.

(*f*) If $a_1, a_2, a_3, ..., a_n$ are in G.P. then $a_r = \sqrt{a_{r-k} . a_{r+k}}$, 0 □ k □ r,

or we can say each term is the G.M. of its two equidistant terms.

❑ Selection of Terms

(*a*) When number of terms are odd,

$\frac{a}{r}, a, ar$ (three terms)

$\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$ (five terms)

$\frac{a}{r^3}, \frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2, ar^3$ (seven terms)

(*b*) when number of terms are even:

$\frac{a}{r^3}, \frac{a}{r}, ar, ar^2$ (four terms)

$\frac{a}{r^5}, \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, ar^5$ (six terms)

❑ Geometric Means

(a) Geometric mean of n numbers $t_1, t_2, t_3, \ldots t_n$ is

$G = (t_1 . t_2 . t_3 . \ldots . t_n)^{1/n}$

(b) Insertion of single geometric mean between a and b

Let 'G' be the geometric mean of a and b

Thus a, G, b form on G.P.

$$\Rightarrow \frac{G}{a} = \frac{b}{G}$$

or $G^2 = ab$

$$\Rightarrow G = \sqrt{ab}$$

(c) Insertion of n geometric means between a and b

Let $G_1, G_2, G_3 \ldots .. G_n$ be n geometric means between any two numbers a and b. Then

$a, G_1, G_2, \ldots G_n, b$ form *a G.P.*

Thus $b = a \, . \, r^{n+1}$, where 'r' is the common ratio of the G.P.

$$\Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$\therefore \quad G_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

$G_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$

M M

$G_k = ar^k = a\left(\frac{b}{a}\right)^{\frac{k}{n+1}}$

ILLUSTRATIONS

***Illustration* 4**

If the mth, nth, and pth terms of an A.P. and G.P. be equal and be respectively x, y, z, then prove that

$x^{y-z} \cdot y^{z-x} \cdot z^{x-y} = 1$ or $x^y y^z z^x = x^z \cdot y^x \cdot z^y$

Solution

Let 'a' be the first term and d be the common difference of the A.P.

 $x = a + (m - 1) d, y = a + (n - 1) d, z = a + (p - 1) d.$

Let A be the first term and R be the common ratio of the G.P.

 $x = AR^{m-1}, y = AR^{n-1}, z = AR^{p-1}$

 $x^{y-z} \cdot y^{z-x} \cdot z^{x-y} = (AR^{m-1})^{y-z} \cdot (AR^{n-1})^{z-x} \cdot (AR^{p-1})^{x-y}$

$= A^\circ \cdot (R^{m-1})^{(n-p)d} (R^{n-1})^{(p-m)d} (R^{p-1})^{(m-n)d}$

$= R^\circ = 1$

 $x^y y^z z^x = x^z \cdot yx \cdot z^y$

PRACTICE EXERCISE

6. The third term of a G.P. is 4. Find the product of its first five terms.
7. If a, b, c are respectively the pth, qth and rth terms of a G.P., show that $(q - r) \log a + (r - p) \log b + (p - a) \log c = 0$.
8. The sum of first three terms of a G.P. is 16 and the sum of the next three terms is 128. Find the sum of n terms of the G.P.
9. If $x = \left(a + \frac{a}{r} + \frac{a}{r^2} + \ldots\infty\right)$, $y = \left(b - \frac{b}{r} + \frac{b}{r^2} - \ldots\infty\right)$ and $z = \left(c + \frac{c}{r^2} + \frac{c}{r^4} + \ldots\infty\right)$, prove that $\frac{xy}{z} = \frac{ab}{c}$.
10. Let x be the arithmetic mean and y, z be two geometric means between any two positive numbers. Then, prove that $\frac{y^3 + z^3}{xyz} = 2$

Answers

6. 4^5 **8.** $\frac{16}{7}(2^n - 1)$

2.3 Harmonic Progression (H.P.)

A sequence is said to be harmonic progression (H.P.) if the reciprocal of its terms are in Arithmetic progression (A.P.)

If $a_1, a_2, a_3, ..., a_n$ are in H.P. then

$$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, ..., \frac{1}{a_n} \text{ are in A.P.}$$

nth term of this A.P. from start.

$$t_n = \frac{1}{t'_n}$$

where t'_n = nth term of A.P. $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}$ ....

$$= \frac{1}{a_1} + (n-1)\left(\frac{1}{a_2} - \frac{1}{a_1}\right)$$

$$\therefore \quad t_n = \frac{1}{\frac{1}{a_1} + (n-1)\left(\frac{1}{a_2} - \frac{1}{a_1}\right)}$$

NOTE:

(*i*) If a, b, c are in H.P. then,

$1/a, 1/b, 1/c$ are in A.P.

$$\therefore \frac{2}{b} = \frac{1}{a} + \frac{1}{c} = \frac{a+c}{ac}$$

$$b = \frac{2ac}{a+c}$$

(*ii*) There is no formula for the sum of *a* H.P.

(*iii*) No term of H.P. can be zero.

❑ Harmonic Mean

(*i*) H.M. of a and b is

$$H = \frac{2ab}{a+b}$$

(*ii*) n H.Ms. inserted between a and b are given by

Let $H_1, H_2, H_3, ..., H_n$ are the HMs, then

$a, H_1, H_2, H_3, ..., H_n, b$ are in H.P.

$\therefore$ $1/a, 1/H_1, 1/H_2, ..., 1/H_n, 1/b$ are in A.P.

□ $\frac{1}{H_m} = \frac{1}{a} + (m)\frac{\left(\frac{1}{b} - \frac{1}{a}\right)}{(n+1)}$

$\therefore \quad H_m = \dfrac{1}{\frac{1}{a} + (m)\frac{\left(\frac{1}{b} - \frac{1}{a}\right)}{(n+1)}}$

- **Properties of Arithmetic, Geometric & Harmonic Means**

  Let A, G and H be arithmetic, Geometric and Harmonic means of any two numbers 'a' and 'b'. $a, b \in R^+$. Then

  $A = \frac{a+b}{2}, G = \sqrt{ab}$ and $H = \frac{2ab}{a+b}$

(i) $A \geq G \geq H$. The equality hold if and only if $a = b$.

(ii) A, G, H form a G.P. i.e. $G^2 = AH$

(iii) The equation having 'a' and 'b' as its roots is $x^2 - 2Ax + G^2 = 0$

ILLUSTRATIONS

Illustration **5**

If H_1, H_2, H_3,.........., H_n are n-harmonic means lying between a and b then show that $\frac{H_1 + a}{H_1 - a} + \frac{H_n + b}{H_n - b} = 2n$.

Solution

Given that

$a, H_1, H_2, H_3, \ldots\ldots\ldots, H_n, b$ are in H.P.

Then, $\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3} \ldots\ldots\ldots \frac{1}{H_n}, \frac{1}{b}$ .... are in A.P.

Let d be its common difference.

□ $\frac{1}{b} = \frac{1}{a}$ □ + $(n + 2 - 1)\, d$ $\left\{\because \frac{1}{b} \text{ is } (n+2)^{th} \text{ term}\right\}$

thus $d = \frac{a-b}{ab(n+1)}$

$\frac{1}{H_1} = \frac{1}{a} + d = \frac{1}{a} + \frac{a-b}{ab(n+1)}$

or, $\frac{1}{H_1} = \frac{bn+a}{ab(n+1)}$

or, $\frac{a}{H_1} = \frac{bn+a}{b(n+1)}$

Using componendo & dividendo rule, we get

$$\frac{H_1 + a}{H_1 - a} = \frac{2bn + (a+b)}{b-a} \qquad \ldots \text{(i)}$$

Again, $\dfrac{1}{H_n} = \dfrac{1}{b} - d = \dfrac{1}{b} - \dfrac{a-b}{ab(n+1)}$

or, $\dfrac{1}{H_n} = \dfrac{an+b}{ab(n+1)}$

or, $\dfrac{b}{H_n} = \dfrac{an+b}{a(n+1)}$

or, $\dfrac{H_n + b}{H_n - b} = \dfrac{2an + (a+b)}{a-b} \qquad \ldots \text{(ii)}$

From (i) and (ii), we get

$$\frac{H_1 + a}{H_1 - a} + \frac{H_n + b}{H_n - b} = \frac{2(b-a).n}{(b-a)} = 2n$$

2.4 Arithmetico-Geometric Series

A series is said to be A.G.P. if its terms are formed by multiplying corresponding terms of an A.P. and a G.P.

i.e. $S = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\ldots\ldots$

or, $S = 3 + 5x + 7x^2 + 9x^3 + \ldots\ldots\ldots\ldots$

(*a*) Sum of *n* terms of A.G.P. (Arithematico Geometric Progression):

$$S_n = a + (a+d)\,r + (a+2d)\,r^2 + \ldots\ldots\ldots\ldots + (a + (n-1)\,d)\,r^{n-1} \qquad \ldots (i)$$

$$\therefore \quad rS_n = 0 + ar + (a+d)\,r^2 + \ldots\ldots + \ldots\ldots + (a + (n-1)d)r^n \qquad \ldots (ii)$$

Subtracting (*ii*) from (*i*),

$$(1-r)S_n = a + dr + dr^2 + \ldots\ldots + dr^{n-1} - (a + (n-1)d)r^n$$

$$= a + dr\,\frac{(1-r^{n-1})}{(1-r)} - (a + (n-1)d)r^n$$

$$\therefore \quad S_n = \frac{a}{(1-r)} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{(a+(n-1)d)r^n}{(1-r)}$$

(*b*) Sum of infinite A.G.P.

$$S_\infty = a + (a+d)\,r + (a+2d)\,r^2 + (a+3d)\,r^3 + \ldots\ldots\ldots \infty$$

$$\therefore \quad rS_\infty = 0 + ar + (a+d)\,r^2 + (a+2d)\,r^3 + \ldots\ldots\ldots\ldots \infty$$

$$(1-r)\,S_\infty = a + dr + dr^2 + dr^3 + \ldots\ldots\ldots \infty$$

$$= a + \frac{dr}{1-r}$$

$$S_{\square} = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

NOTE:

(*i*) $\square n = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

(ii) $\square n^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)\,(2n+1)}{6}$

(iii) $\square n^3 = 1^2 + 2^2 + 3^2 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

ILLUSTRATIONS

***Illustration* 6**

Find the sum to infinity of the series

$1 - 3x + 5x^2 - 7x^3 + \ldots, \infty$ when $|x| < 1$

Solution

Clearly, the given series is an arithmetico-geometric series whose corresponding G.P. is $1, -x, x^2, -x^3, \ldots$

The common ratio of this G.P. is $-x$. Let S_∞ denote the sum of the given infinite series. Then,

$S_\infty = 1 - 3x + 5x^2 - 7x^3 + \ldots \infty$

$\Rightarrow$ $(-x)\, S_\infty = -x + 3x^2 - 5x^3 + \ldots \infty$

Subtracting (ii) from (i), we get

$(1 + x)\, S_\infty = 1 + [-2x + 2x^2 - 2x^3 + \ldots\infty]$

$$= 1 + \left[\frac{-2x}{1-(-x)}\right]$$

$$= 1 - \frac{2x}{1+x} = \frac{1-x}{1+x}$$

$$\Rightarrow \quad S_\infty = \frac{1-x}{(1+x)^2}.$$

***Illustration* 7**

Find the sum of the following series to *n* terms

$$\frac{1^3}{1} + \frac{1^3 + 2^3}{1+3} + \frac{1^3 + 2^3 + 3^3}{1+3+5} + \ldots$$

Solution

Let T_n be the nth term of the given series. Then,

$$T_n = \frac{1^3 + 2^3 + 3^3 + ... + n^3}{1 + 3 + 5 + ... + (2n-1)} = \frac{\left\{\frac{n(n+1)}{2}\right\}^2}{\frac{n}{2}[2 + (n-1)2]}$$

$$= \frac{(n+1)^2}{4} = \frac{1}{4}(n^2 + 2n + 1)$$

Let S_n denote the sum of n terms of the given series. Then,

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} \frac{1}{4}(k^2 + 2k + 1) = \frac{1}{4}\left[\sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1\right]$$

$$= \frac{1}{4}\left[\frac{n(n+1)(2n+1)}{6} + 2\left(\frac{n(n+1)}{2}\right) + n\right]$$

$$= \frac{n}{24}(2n^2 + 9n + 13).$$

PRACTICE EXERCISE

11. Find the nth terms of the series $1 + \frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + ...$
12. Find the sum to n terms of the series $1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + ...$
13. Sum to infinity of the following series $1^2 + 5^2x + 9^2 x^2 + 13^2 x^3 + \ldots\infty$, where $|x| < 1$.
14. Find the sum to n terms of the series $1^2 + 3^2 + 5^2 + \ldots$ to n terms.
15. Sum to n terms the series $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \ldots$

Answers

11. $\frac{n}{3^{n-1}}$ **12.** $S_n = \frac{5}{4} + \frac{15}{16}\left(1 - \frac{1}{5^{n-1}}\right) - \frac{(3n-2)}{4(5^{n-1})}$ **13.** $\frac{1 + 22x + 9x^2}{(1-x)^3}$

14. $\frac{n}{3}(4n^2 - 1)$ **15.** $\frac{n(n+1)}{2}$

*****

MISCELLANEOUS PROBLEMS

OBJECTIVE TYPE

Example 1

If *a*, *b*, *c* are in harmonic progression, then the value of $\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{b}\right)$ is

(a) $\frac{2}{bc}-\frac{1}{b^2}$ (b) $\frac{1}{4}\left(\frac{3}{c^2}+\frac{2}{ca}-\frac{1}{a^2}\right)$ (c) $\frac{3}{b^2}-\frac{1}{ab}$ (d)None of these

Solution

As *a*, *b*, *c* are in H.P. 1/*a*, 1/*b*, 1/*c* are in A.P.

□ $\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b}$ □□ or $\frac{2}{b}=\frac{1}{a}+\frac{1}{c}$ … (i)

Therefore,

$$\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{b}\right)=\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{c}-\frac{2}{b}\right)\left(\frac{1}{c}+\frac{1}{b}-\frac{1}{c}\right) \quad \text{[using (i)]}$$

$$=\left(\frac{2}{c}-\frac{1}{b}\right)\left(\frac{1}{b}\right)=\frac{2}{bc}-\frac{1}{b^2}$$

Also $$\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{b}\right)=\left(\frac{1}{2a}+\frac{1}{2c}+\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{2a}-\frac{1}{2c}\right) \quad \text{[using (i)]}$$

$$=\left(\frac{3}{2c}-\frac{1}{2a}\right)\left(\frac{1}{2c}+\frac{1}{2a}\right)=\frac{1}{4}\left(\frac{3}{c^2}+\frac{2}{ca}-\frac{1}{a^2}\right)$$

Lastly $$\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{b}\right)=\left(\frac{1}{b}+\frac{2}{b}-\frac{1}{a}-\frac{1}{a}\right)\left(\frac{1}{b}-\frac{1}{a}+\frac{1}{a}\right)$$

$$=\left(\frac{3}{b}-\frac{2}{a}\right)\frac{1}{b}=\frac{3}{b^2}-\frac{2}{ab}$$

∴ **Ans. (a, b, c)**

Example 2

If $\sum_{r=1}^{n} r(r+1)(2r+3) = an^4 + bn^3 + cn^2 + dn + e$, then

(a) $a = 1/2$ (b) $b = 8/3$ (c) $c = 9/2$ (d) $e = 0$

Solution

We have

$$\sum_{r=1}^{n} r(r+1)(2r+3)=\sum_{r=1}^{n}\left(2r^3+5r^2+3r\right)$$

$$=2\sum_{r=1}^{n} r^3+5\sum_{r=1}^{n} r^2+3\sum_{r=1}^{n} r$$

$$=2\left[\frac{1}{4}n^2(n+1)^2\right]+5\left[\frac{1}{6}n(n+1)(2n+1)\right]+3\left[\frac{1}{2}n(n+1)\right]$$

$= \frac{1}{6}\ n\ (n + 1)\ [3n\ (n + 1) + 5\ (2n + 1) + 9] = \frac{1}{6}\ n\ (n + 1)\ (3n^2 + 13n + 14)$

$= \frac{1}{6}\ [3n^4 + 16n^3 + 27n^2 + 14n] = \frac{1}{2}n^4 + \frac{8}{3}n^3 + \frac{9}{2}n^2 + \frac{7}{3}n$

$\therefore$ **Ans. (a, b, c, d)**

Example 3

The sum to n terms of the series $1^2 + (1^2 + 3^2) + (1^2 + 3^2 + 5^2) + ...$ is

(a) $\frac{1}{6} n\ (n + 1)\ (2n^2 + 2n - 1)$ (b) $\frac{1}{3}\ (n^4 + 2n^2)$ (c) $\frac{1}{3}\ (n^3 + 3n^2 - n)$

(d) None of these

Solution

We have $t_r = 1^2 + 3^2 + 5^2 + ... + (2r - 1)^2$

$= \sum_{k=1}^{r}(2k-1)^2 = \sum_{k=1}^{r}\left(4k^2 - 4k + 1\right)$

$= 4\left\{\frac{1}{6} r\ (r+1)\ (2r+1)\right\} - \frac{4}{2} r\ (r+1) + r$

$= \frac{1}{6}\ r\ [4(r + 1)\ (2r + 1) - 12\ (r + 1) + 6]$

$= \frac{1}{6}\ r\ [8r^2 - 2] = \frac{1}{3}\ (4r^3 - r)$

Thus, sum of n terms of the given series

$\frac{4}{3}\sum_{r=1}^{n} r^3 - \frac{1}{3}\sum_{r=1}^{n} r = \frac{4}{3}.\frac{1}{4}n^2\ (n+1)^2 - \frac{1}{3}.\frac{1}{2}n(n+1)$

$= \frac{1}{6}\ n\ (n + 1)\ [2n\ (n + 1) - 1] = n\ (n + 1)\ (2n^2 + 2n - 1)$

$\therefore$ **Ans. (a)**

Example 4

The sum of n terms of the series $\frac{3}{1^2} + \frac{5}{1^2 + 2^2} + \frac{7}{1^2 + 2^2 + 3^2} + ...$ is

(a) $\frac{3n}{n+1}$ (b) $\frac{6n}{n+1}$ (c) $\frac{9n}{n+1}$ (d) $\frac{12n}{n+1}$

Solution

$T_r = \frac{\frac{3+(r-1).2}{r(r+1)(2r+1)}}{6} = \frac{6}{r(r+1)} = 6\left(\frac{1}{r} - \frac{1}{r+1}\right)$

$\sum_{r=1}^{n} T_r = 6.\left(\frac{1}{1} - \frac{1}{n+1}\right) = \frac{6n}{n+1}$ $\therefore$ **Ans. (b)**

SUBJECTIVE TYPE

Example 1

For what value of n, $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}\infty$ is (a □ b)

(i)A.M. between a and b (ii)G.M. between a and b (iii)H.M. between a and b

Solution

(i) A.M. between a and b is $\frac{a+b}{2}$

we have $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}=\frac{a+b}{2}$

□ $2a^{n+1} + 2b^{n+1} = (a + b)(a^n + b^n)$

□ $2a^{n+1} + 2b^{n+1} = a^{n+1} + ab^n + ba^n + b^{n+1}$

□ $a^{n+1} - ab^n + b^{n+1} - ba^n = 0$

□ $a^n (a - b) - b^n (a - b) = 0$

□ $(a - b)(a^n - b^n) = 0$

$a - b$ □ 0

□ $a^2 - b^n = 0$

□ $\left(\frac{a}{b}\right)^n = 1$

□ $\left(\frac{a}{b}\right)^n = \left(\frac{a}{b}\right)^0$

Hence, n = 0

(ii) Q G.M. between a and b is $\sqrt{ab}$

we have $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}=\sqrt{ab}$

□ $a^{n+1} + b^{n+1} = \sqrt{ab}\,(a^n + b^n)$

□ $a^{n+1} + b^{n+1} = a^{n+1/2} \,.\, b^{1/2} + a^{1/2} \,.b^{n+1/2}$

□ $a^{n+1} - a^{n+1/2} \,.\, b^{1/2} - a^{1/2} \,.\, b^{n+1/2} + b^{n+1} = 0$

□ $a^{n+1/2} (a^{1/2} - b^{1/2}) - b^{n+1/2} (a^{1/2} - b^{1/2}) = 0$

□ $a^{1/2} - b^{1/2}$ □ 0

□ $a^{n+1/2} - b^{n+1/2} = 0$

□ $\left(\frac{a}{b}\right)^{n+\frac{1}{2}} = 1 = \left(\frac{a}{b}\right)^0$

□ $n + \frac{1}{2} = 0$

Hence, $n = -\frac{1}{2}$

(iii) Because H.M. between a and b is $\frac{2ab}{a+b}$

We have $\frac{a^{n+1} + b^{n+1}}{a^n + b^n} = \frac{2ab}{a+b}$

$\Rightarrow \quad (a^{n+1} + b^{n+1})(a + b) = 2ab\,(a^n + b^n)$

$\Rightarrow \quad a^{n+2} + a^{n+1}b + b^{n+1}a + b^{n+2} = 2a^{n+1}b + 2ab^{n+1}$

$\Rightarrow \quad a^{n+2} - a^{n+1}b - ab^{n+1}(a - b) = 0$

$\Rightarrow \quad a^{n+1}(a - b) - b^{n+1}(a - b) = 0$

$\Rightarrow \quad (a - b)(a^{n+1} - b^{n+1}) = 0$

$a - b \neq 0$

$\Rightarrow \quad a^{n+1} - b^{n+1} = 0$

$\Rightarrow \quad a^{n+1} = b^{n+1}$

$\Rightarrow \quad \left(\frac{a}{b}\right)^{n+1} = 1 = \left(\frac{a}{b}\right)^0$

Hence, $n + 1 = 0$

Therefore, $n = -1$

Example 2

If a be one A.M. and G_1 and G_2 be two geometric means between b and c, prove that $G_1^3 + G_2^3 = 2abc$.

Solution

Given a = A.M. between b and c

$\Rightarrow \quad a = \frac{b+c}{2}$

$\Rightarrow \quad 2a = b + c$ ... (i)

G_1, G_2 are two G.M.'s between b and c then b, G_1, G_2, c are in G.P.

If r is the common ratio of this G.P. Then $G_1 = br$, $G_2 = br^2$, $c = br^3$

Now : $G_1^3 + G_2^3 = (br)^3 + (br^2)^3 = b^3r^3 + b^3r^6$

$= b^3\left(\frac{c}{b}\right) + b^3\left(\frac{c}{b}\right)^2 \quad (\because\ c = br^3)$

$= b^2c + bc^2 = bc\,(b + c)$

$= bc\,(2a) = 2abc$ {from (i)}

Hence, $G_1^3 + G_2^3 = 2abc$

Example 3

If A_1, A_2 be the A.M.'s, G_1, G_2 be the G.M.'s and H_1, H_2 be the H.M.'s between any two numbers then show that

$$\frac{G_1G_2}{H_1H_2} = \frac{A_1 + A_2}{H_1 + H_2}$$

Solution

Let the two numbers be *a* and *b*.

Then $A_1 + A_2$ = sum of A.M.'s = $\left(\frac{a+b}{2}\right)2 = a+b$ … (i)

and G_1G_2 = Product of G.M.'s = $\left(\sqrt{ab}\right)^2 = ab$ … (ii)

and = $\frac{1}{H_1} + \frac{1}{H_2}$ sum of reciprocals of H.M.'s = $2\left(\frac{1}{\left(\frac{2ab}{a+b}\right)}\right)$

$$= \frac{a+b}{ab}$$

$$= \frac{A_1 + A_2}{G_1G_2} \qquad \text{\{from (i) and (ii)\}}$$

□ $\frac{1}{H_1} + \frac{1}{H_2} = \frac{A_1 + A_2}{G_1G_2}$

□ $\frac{H_1 + H_2}{H_1H_2} = \frac{A_1 + A_2}{G_1G_2}$

Hence, $\frac{G_1G_2}{H_1H_2} = \frac{A_1 + A_2}{H_1 + H_2}$

Example 4

Find the sum of the series

$$\frac{1^3}{1} + \frac{1^3 + 2^3}{1+3} + \frac{1^3 + 2^3 + 3^3}{1+3+5} + ... + n \text{ terms}$$

Solution

Here *n*th term

$$T_n = \frac{1^3 + 2^3 + 3^3 + ... + n^3}{1+3+5+....+(2n-1)}$$

$$= \frac{\sum n^3}{\frac{n}{2}(1+2n-1)}$$

$$= \frac{\left[n\left(\frac{n+1}{2}\right)\right]^2}{n^2}$$

$= \frac{1}{4}(n+1)^2$

$= \frac{1}{4}(n^2 + 2n + 1)$

$= \frac{1}{4}n^2 + \frac{1}{2}n + \frac{1}{4}$

□ sum of n terms

$S_n = \sum T_n$

$= \frac{1}{4}\sum n^2 + \frac{1}{2}\sum n + \frac{1}{4}\sum 1$

$= \frac{1}{4} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{1}{2} \cdot \frac{n(n+1)}{2} + \frac{1}{4} \cdot n$

$= \frac{1}{24} n(n+1)(2n+1) + \frac{1}{4} n(n+1) + \frac{1}{4} n$

$= \frac{n}{24}\{2n^2 + 3n + 1 + 6n + 6 + 6\}$

Hence, $S_n = \frac{n}{24}(2n^2 + 9n + 13)$

Example 5

Find the sum of the series $1 + 2^2x + 3^2x^2 + 4^2x^3 + \ldots$ □, $|x| < 1$.

Solution

Here the numbers $1^2, 2^2, 3^2, 4^2, \ldots$ i.e., 1, 4, 9, 16, … are not in A.P. but 1, 4 – 1 = 3, 9 – 4 = 5, 16 – 9 = 7, are in A.P.

Let $S_{\square} = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \ldots$ □

$S_{\square} = 1 + 4x + 9x^2 + 16x^3 + \ldots$ □ … (i)

□ $xS_{\square} = 0 + x + 4x^2 + 9x^3 + \ldots$ □ … (ii)

Subtracting (ii) from (i), we get

$(1 - x) S_{\square} = 1 + 3x + 5x^2 + 7x^3 + \ldots$ □ … (iii)

Again $x(1 - x) S_{\square} = 0 + x + 3x^2 + 5x^3 + \ldots$ □ … (iv)

Subtracting (iv) from (iii), we get

$(1 - x)(1 - x) S_{\square} = 1 + 2x + 2x^2 + 2x^3 + \ldots$ □

$(1 - x)^2 S_{\square} = 1 + \frac{2x}{1-x}$

$= \frac{1+x}{1-x}$

Hence, $S_\infty = \frac{(1+x)}{(1-x)^3}$

Example 6

Find nth term and sum of n terms of the series, 1 + 3 + 7 + 15 + 31 + …

Solution

Let nth term and sum of n terms of the series be denoted by T_n and S_n respectively.

□ $S_n = 1 + 3 + 7 + 15 + 31 + \ldots + T_{n-1} + T_n$ … (i)

$S_n = 1 + 3 + 7 + 15 + \ldots + T_{n-1} + T_n$ … (ii)

Subtracting (ii) from (i), we get

$0 = 1 + 2 + 4 + 8 + 16 + \ldots + (T_n - T_{n-1}) - T_n$

□ $T_n = 1 + 2 + 4 + 8 + 16 + \ldots$ n terms

$= 1.\frac{(2^n - 1)}{2-1}$

Hence, $T_n = (2^n - 1)$

□ sum of n term

$S_n = \sum T_n$

$= \sum (2^n - 1)$

$= \sum 2^n - \sum 1$

$= (2 + 2^2 + 2^3 + \ldots + 2^n) - n$

Hence, $S_n = 2.\frac{(2^n - 1)}{2-1} - n$

$= 2(2^n - 1) - n$

Example 7

Three numbers are in G.P. whose sum is 70. If the extremes be each multiplied by 4 and the mean by 5, they will be in A.P. Find the numbers.

Solution

Let the three numbers in G.P. be $\frac{a}{r}$, a, ar

Given $\frac{a}{r} + a + ar = 70$ … (i)

and $\frac{4a}{r}$, $5a$, $4ar$ are in A.P.

□ $10a = \frac{4a}{r} + 4ar$

or $\frac{10a}{4} = \frac{a}{r} + ar$

or $\frac{5a}{2} = 70 - a$ {from (i)}

or $5a = 140 - 2a$

or $7a = 140$

□ $a = 20$

from (i), we get

$\frac{20}{r} + 20 + 20r = 70$

or $\frac{20}{r} + 20r = 50$

or $2 + 2r^2 = 5r$

or $2r^2 - 5r + 2 = 0$

or $(r - 2)(2r - 1) = 0$

□ $r = 2$ or $\frac{1}{2}$

Hence, the three numbers are 10, 20, 40 or 40, 20, 10

Example 8

If the sum of *m* terms of an A.P. is equal to the sum of either the next *n* terms or the next *p* terms then prove that

$$(m + n)\left(\frac{1}{m} - \frac{1}{p}\right) = (m + p)\left(\frac{1}{m} - \frac{1}{n}\right)$$

Solution

Let the A.P. be $a, a + d, a + 2d, ...$

Given $T_1 + T_2 + \ldots + T_m = T_{m+1} + \ldots + T_{m+n}$ … (i)

Adding $T_1 + T_2 + \ldots + T_m$ on both sides in (i), we get

$2(T_1 + T_2 + \ldots + T_m) = T_1 + T_2 + \ldots + T_{m+1} + \ldots + \ldots + T_{m+n}$

□ $2S_m = S_{m+n}$

□ $2 \cdot \frac{m}{2}\{2a + (m - 1)d\} = \frac{m+n}{2}\{2a + (m + n - 1)d\}$

Let $2a + (m - 1)d = x$

□ $mx = \frac{m+n}{2}\{x + nd\}$

□ $(m - n)x = (m + n)nd$ … (ii)

Again $T_1 + T_2 + \ldots + T_m = T_{m+2} + \ldots + T_{m+p}$

Similarly $(m - p)\, x = (m + p)\, pd$ … (iii)

Dividing (ii) by (iii), we get

$$\frac{m-n}{m-p} = \frac{(m+n)\,n}{(m+p)\,p}$$

□ $(m - n)(m + p)\,p = (m - p)(m + n)\,n$

Dividing both sides by *mnp*, we have

$$(m+p)\left(\frac{1}{n}-\frac{1}{m}\right) = (m+n)\left(\frac{1}{p}-\frac{1}{m}\right)$$

Hence, $(m+n)\left(\frac{1}{m}-\frac{1}{p}\right) = (m+p)\left(\frac{1}{m}-\frac{1}{n}\right)$

Example 9

Find the sum to *n* terms of the series, 5 + 55 + 555 + …

Solution

Let S_n = 5 + 55 + 555 + … *n* terms

= 5 {1 + 11 + 111 + … *n* terms}

= {9 + 99 + 999 + … *n* terms}

= {(10 – 1) + (100 – 1) + (1000 – 1) + … *n* terms}

$$= \frac{5}{9}\left[\frac{10\left(10^n-1\right)}{10-1} - n\right]$$

Hence, $S_n = \frac{50}{81}\left(10^n - 1\right) - \frac{5}{9}n$.

Example 10

If the *n*th term of a certain series is of the form $a + bn + c.\, 2^n$ where *a, b, c* are constants. The first three terms are 2, – 1, and – 3, then show that *a, c, b* are in A.P. Also find the sum of first *n* terms of the series.

Solution

$t_n = a + bn + c.2^n$

$t_1 = a + b + 2c = 2$ … (*i*)

$t_2 = a + 2b + 4c = -1$ … (*ii*)

$t_3 = a + 3b + 8c = -3$ … (*iii*)

On solving (*i*), (*ii*), (*iii*), we get

$a = 5,\ b = -4,\ c = 1/2$

Clearly $a + b = 2c$

□ *a*, *c*, *b* are in A.P.

Now, $t_n = 5 - 4n + 2^{n-1}$

∴ $t_1 = 5 - 4.1 + 1$

$t_2 = 5 - 4.2 + 2$

$t_3 = 5 - 4.3 + 2^2$

.. ..

.. ..

$t_n = 5 - 4.n + 2^{n-1}$

$t_1 + t_2 + t_3 + ... + t_n = 5n - 4.\frac{n(n+1)}{2} + \frac{2^n - 1}{2-1}$

$S = 2^n - 2n^2 + 3n - 1.$

*****

Exercise - I

OBJECTIVE TYPE QUESTIONS

1. Given a sequence of four numbers such that the first three are in G.P. and the last three are in A.P. with common difference 6. If the first and the fourth number are equal, then common ratio of the G.P. is

(a) $-\frac{1}{2}$ (b) 2 (c) 3 (d) –3

2. The sum of the series $\frac{7}{1^3.2^3}+\frac{19}{2^3.3^3}+\frac{37}{3^3.4^3}+....$ upto ∞ is

(a) 3 (b) 2 (c) 1 (d) 1.5

3. Let a, b, c be three distinct positive numbers which are in G.P. If $\log_c a$, $\log_b c$, $\log_a b$ are in A.P., then the common difference of the A.P. is

(a) 1/2 (b) 3/2 (c) 5/2 (d) 7/2

4. Let a, b, c be in A.P., and $(b - c)x^2 + (c - a)x + (a - b) = 0$ and $(a + c)x^2 + (b + c)x = 0$ have a common root, then

(a) a^2, b^2, c^2 are in G.P. (b) a^2, b^2, c^2 are in A.P. (c) a^2, c^2, b^2 are in A.P. (d) a^2, c^2, b^2 are in G.P.

5. If $\sqrt{b}+\sqrt{c}, \sqrt{c}+\sqrt{a}, \sqrt{a}+\sqrt{b}$ are in H.P., then a, b, c are in

(a) H.P. (b) G.P. (c) A.P. (d) None of these

6. If unit's digit of $1^2 + 2^2 + \ldots + n^2$ is 5, then unit's digit of $1^{10} + 2^{10} + \ldots + n^{10}$ is

(a) 1 (b) 3 (c) 5 (d) 7

7. Suppose $t_r = 1^2 + 2^2 + \ldots + r^2$. Let $t_1 + t_2 + \ldots + t_n = \frac{1}{12} n(n + 1)(n + 2)k$, then value of k is

(a) $n + 1$ (B) $2n + 1$ (c) $3n - 1$ (d) n

8. If $\cos(x - y)$, $\cos x$ and $\cos(x + y)$ are in H.P., then value of $\cos x \sec(y/2)$ is

(a) $\pm\sqrt{2}$ (b) $\pm\sqrt{3}$ (c) ± 2 (d) ± 1

9. The sum to infinity of the series

$S = 1 + \frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+....$ is

(a) $\frac{16}{35}$ (b) $\frac{11}{8}$ (c) $\frac{35}{16}$ (d) $\frac{17}{6}$

10. If the sum to n terms of an A.P. is $3n^2 + 5n$, while $T_m = 164$, then value of m is

(a) 25 (b) 26 (c) 27 (d) 28

11. If G_1 and G_2 are two geometric means and A is the arithmetic mean inserted between two position numbers a and b, then the value of $\frac{G_1^2}{G_2}+\frac{G_2^2}{G_1}$ is

(a) A (b) $2A$ (c) $A/2$ (d) $3A/2$

12. Suppose for each $n \in N$. $(1^2 - a_1) + (2^2 - a_2) + \ldots + (n^2 - a_n) = \frac{1}{3}\, n\,(n^2 - 1)$, then a_n equals

(a) n (b) $n - 1$ (c) $n + 1$ (d) $2n$

13. The n^{th} terms of the two series 3 + 10 + 17 + …. and 63 + 65 + 67 + …… are equal, then the value of n is

(a) 9 (b) 13 (c) 19 (d) none of these

14. If a, b, c are in H.P, then the value of $\frac{b+a}{b-a}+\frac{b+c}{b-c}$ is

(a) 1 (b) 2 (c) 3 (d) none of these

15. In any triangle ABC the angles A, B, C are in A.P, then the value of sin 2B is given by

(a) 1/2 (b) $\sqrt{3}/2$ (c) $1/\sqrt{2}$ (d) none of these

16. rth term of sequence $\frac{1}{1\cdot3\cdot5}+\frac{1}{3\cdot5\cdot7}+\frac{1}{5\cdot7\cdot9}+\ldots\ldots$ is given by

(a) $\frac{1}{r(r+2)(r+4)}$ (b) $\frac{1}{(2r+1)(2r+3)(2r+5)}$ (c) $\frac{1}{(2r-1)(2r+1)(2r+3)}$ (d) none of these

17. The sum of numbers in the n^{th} group of the following (1, 3), (5, 7, 9, 11), (13, 15, 17, 19, 21, 23) , ... is

(a) $\frac{n(n+1)(n+2)}{3}$ (b) $2n^3$ (c) $n^2(n+1)^2$ (d) $4n^3$

18. If a, b, c are three positive real numbers, then the minimum value of the expression $\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}$ is

(a) 1 (b) 2 (c) 3 (d) None of these

19. If $x_i > 0$, i = 1, 2, …., 50 and $x_1 + x_2 + \ldots + x_{50} = 50$, then the minimum value of $\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_{50}}$ equals to

(a) 50 (b) $(50)^2$ (c) $(50)^3$ (d) $(50)^4$

20. Let r^{th} term of a series be given by $T_r = \frac{r}{1-3r^2+r^4}$. Then $\lim_{n\to\infty}\sum_{r=1}^{n} T_r$ is

(a) 3/2 (b) 1/2 (c) –1/2 (d) –3/2

21. If $x_1, x_2, \ldots x_{20}$ are in H.P. then $x_1 x_2 + x_2 x_3 + \ldots + x_{19} x_{20}$ =

(a) $x_1 x_{20}$ (b) $19\, x_1 x_{20}$ (c) $20\, x_1 x_{20}$ (d) none of these

22. The first two terms of an H.P. are $\frac{2}{5}$ and $\frac{12}{23}$. The value of the largest term of the H.P. is

(a) $\frac{72}{73}$ (b) 6 (c) $\frac{1}{6}$ (d) none of these

23. The co-efficient of x^{n-2} in $(x-1)(x-2)(x-3)..........(x-n)$ is

(a) $\frac{n(n^2+1)(3n+1)}{24}$ (b) $\frac{n(n^2-1)(3n+2)}{24}$

(c) $\frac{n(n^2+1)(3n+4)}{24}$ (d) None of these

24. If $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+........\infty=\frac{\pi^4}{90}$ then $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+........\infty$ is equal to

(a) $\frac{\pi^4}{96}$ (b) $\frac{\pi^4}{45}$ (c) $\frac{\pi^4}{90}$ (d) $\frac{\pi^4}{46}$

25 If a, a_1, a_2…….a_{2n-1}, b are in A.P, a, b_1, b_2…….b_{2n-1}, b are in G.P. and a c_1, c_2…….c_{2n-1}, b are in H.P. where a, b are positive then the equation $a_nx^2 - b_nx + c_n = 0$ has its roots

(a) real and unequal (b) real and equal (c) imaginary (d) do not exist

26. If mth, nth and pth terms of an A.P. and G.P. are equal and are respectively x, y, z then

(a) $x^yy^zz^x = x^zy^xz^y$ (b) $(x - y)^x (y - z)^y = (z - x)^z$

(c) $(x - y)^z (y - z)^x = (z - x)^y$ (d) none of these

27. If the sum to n terms of an A.P. is cn(n –1), where c ≠ 0. The sum of the squares of these terms is

(a) $c^2n^2(n + 1)^2$ (b) $\frac{2}{3}$ c^2n (n –1) (2n –1)

(c) $\frac{2}{3}$ c^2n (n + 1) (2n + 1) (d) none of these

28. Let $\sum_{r=1}^{n} r^4 = f(n)$, then $\sum_{r=1}^{n}(2r-1)^4 =$

(a) f (2n) –16 f (n); ∀ n ∈ N (b) f (n) –16 f$\left(\frac{n-1}{2}\right)$, when n is odd

(c) f (n) –16 f $\left(\frac{n}{2}\right)$, when n is even (d) none of these

Multiple Choice Questions with MORE THAN ONE option correct

1. If the first and (2*n* – 1)th terms of an A.P., a G.P. and a H.P. are equal and their nth terms are *a*, *b* and *c* respectively, then

(a) $a = b = c$ (b) $a + c = b$ (c) $a \geq b \geq c$ (d) $ac = b^2$

2. If *a*, *b*, *c* be three unequal positive quantities in H.P. then

(a) $a^{100} + c^{100} > 2b^{100}$ (b) $a^3 + c^3 > 2b^3$ (c) $a^5 + c^5 > 2b^5$ (d) $a^2 + c^2 > 2b^2$

3. The sum of the products taken two at a time of the numbers 1, 2, 2^2 , 2^3,…, 2^{n-2}. 2^{n-1} is

(a) $\frac{1}{3}2^{2n}+\frac{2}{3}$ (b) $\frac{1}{3}2^{2n}-2^n+\frac{1}{3}$ (c) $\frac{1}{3}2^{2n}-\frac{1}{3}$ (d) $\frac{1}{3}2^{2n}-2^n+\frac{2}{3}$

4. If $<a_n>$ and $<b_n>$ be two sequence given by $a_n=(x)^{1/2^n}+(y)^{1/2^n}$ *and* $b_n=(x)^{1/2^n}-(y)^{1/2^n}$ for all $n \in N$ then $a_1\, a_2\, a_3 \ldots a_n$ is

(a) $\frac{x+y}{b_n}$ (b) $\frac{x-y}{b_n}$ (c) $\frac{x^2+y^2}{b_n}$ (d) $\frac{x^2-y^2}{b_n}$

5. If $\sum_{r=1}^{n} t_r = \frac{n(n+1)\,(n+2)\,(n+3)}{8}$, where t_r denotes the rth term of a series, then $\underset{n\to\infty}{\text{Lim}} \sum_{r=1}^{\infty}\frac{1}{t_r}$ is

(a) $\frac{1}{8}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) 1

6. Given that $0 < x < \pi/4$ and $\pi/4 < y < \pi/2$ and $\sum_{k=0}^{\infty}(-1)^k \tan^{2k} x = a, \sum_{k=0}^{\infty}(-1)^k \cot^{2k} y = b,$ then $\sum_{k=0}^{\infty}\tan_k^{2k} \cot_y^{2k}$ is

(a) $\frac{1}{a}+\frac{1}{b}-\frac{1}{ab}$ (b) $a + b - ab$ (c) $\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{ab}}$ (d) $\frac{ab}{a+b-1}$

7. If a, b, c are in H.P., then the value of $\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{b}\right)$ is

(a) $\frac{2}{bc}-\frac{1}{b^2}$ (b) $\frac{1}{4}\left(\frac{3}{c^2}+\frac{2}{ca}-\frac{1}{a^2}\right)$ (c) $\frac{3}{b^2}-\frac{2}{ab}$ (d) None of these

8. $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j}1 = \ldots.$

(a) $\frac{n(n+1)\,(n+2)}{6}$ (b) $\sum n^2$ (c) ${}^{n}C_3$ (d) ${}^{n+2}C_3$

9. If an A.P., $a_7 = 9$ if $a_1\, a_2\, a_7$ is least, the common difference is

(a) $\frac{13}{20}$ (b) $\frac{23}{20}$ (c) $\frac{33}{20}$ (d) $\frac{43}{20}$

10. If b_1, b_2, b_3 ($b_1 > 0$) are three successive terms of a G.P. with common ratio r, the value of r for which the inequality $b_3 > 4b_2 - 3b_1$ holds is given by

(a) $r > 3$ (b) $r < 1$ (c) $r = 3.5$ (d) $r = 5.2$

11. If the ratio of A.M. between two positive real numbers a and b to their H.M. is $m : n$; then $a : b$

(a) $\frac{\sqrt{(m-n)}+\sqrt{n}}{\sqrt{(m-n)}-\sqrt{n}}$ (b) $\frac{\sqrt{n}+\sqrt{(m-n)}}{\sqrt{n}-\sqrt{(m-n)}}$ (c) $\frac{\sqrt{m}+\sqrt{(m-n)}}{\sqrt{m}-\sqrt{(m-n)}}$ (d) $\frac{\sqrt{m}-\sqrt{(m-n)}}{\sqrt{m}+\sqrt{(m-n)}}$

*****

Exercise - II

ASSERTION & REASON , COMPREHENSION & MATCHING TYPE

Assertion & Reason Type

In the following question, a statement of Assertion (A) is given which is followed by a corresponding statement of reason (R). Mark the correct answer out of the following options/codes.

(a) If both (A) and (R) are true and (R) is the correct explanation of (A).

(b) If both (A) and (R) are true but (R) is not correct explanation of (A).

(c) If (A) is true but (R) is false.

(d) If both (A) and (R) are false.

(e) If (A) is false but (R) is true.

1. A: 4, 8, 16 are in G.P. and 12, 16, 24 are in H.P.
   R: If middle term is added in three consecutive terms of a G.P., resultant will be in H.P.
2. A: If sum of n terms of a series is $4n^2 + 2n + 1$, then the series is in A.P.
   R: Sum of n terms of an A.P. is always of the form $an^2 + bn$.
3. A: If sides of a quadrilateral be in A.P. then a circle can be inscribed in it.
   R: $ABCD$ will be a quadrilateral if $AB + CD = AD + BC$.
4. A: If a, b, c are three positive numbers in G.P. then $\frac{a+b+c}{3}\,\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} = \left(\sqrt[3]{abc}\right)^2$
   R: $(AM)\,(HM) = (GM)^2$ is true for positive numbers.
5. A: $a + b + c = 12$ ($a, b, c > 0$) then maximum value of $abc = 64$
   R: Maximum value occurs when $a = b = c$.

Passage Based Questions

Passage – I

The sum of three terms of a strictly increasing G.P. is αS and sum of the squares of these terms is S^2.

1. α^2 lies

   (a) (1/3, 2) (b) (1, 2) (c) (1/3, 3) (d) None of these

2. If $\alpha^2 = 2$, then value of r equals

   (a) $\frac{1}{2}\left(5-\sqrt{3}\right)$ (b) $\frac{1}{2}\left(3+\sqrt{5}\right)$ (c) $\frac{1}{2}\left(\sqrt{5}+\sqrt{3}\right)$ (d) $\frac{1}{3}\left(\sqrt{3}+\sqrt{5}\right)$

3. If common ratio $r = \sqrt{2}$, then α^2 equals

   (a) $\frac{1}{7}(2 \pm \sqrt{3})$ (b) $\frac{1}{4}(11 \pm \sqrt{7})$ (c) $\frac{1}{7}(11 \pm 6\sqrt{2})$ (d) $\frac{1}{5}(11 \pm \sqrt{7})$

Passage – II

Suppose x_1, x_2 be the roots of $ax^2 + bx + c = 0$ and x_3, x_4 be the roots of $px^2 + qx + r = 0$

1. If $x_1, x_2, \frac{1}{x_3}, \frac{1}{x_4}$ are in A.P., then $\frac{b^2-4ac}{q^2-4pr}$ equals

   (a) a^2/r^2 (b) b^2/q^2 (c) c^2/p^2 (d) a^2/p^2

2. If x_1, x_2, x_3, x_4 are in A.P., then common difference of this A.P. is

   (a) $\frac{1}{2}\left(\frac{b}{a}-\frac{q}{p}\right)$ (b) $\frac{1}{8}\left(\frac{b}{a}-\frac{q}{p}\right)$ (c) $\frac{1}{3}\left(\frac{c}{a}-\frac{q}{p}\right)$ (d) None of these

3. If a, b, c are in G.P. as well as x_1, x_2, x_3, x_4 = are in G.P., then p, q, r are in

   (a) A.P. (b) G.P. (c) H.P. (d) A.G.P.

Matching Type Questions

1. For $0 < \theta < \pi/4$, let $x = \sum_{n=0}^{\infty}(\sin\theta)^{2n}$, $y = \sum_{n=0}^{\infty}(\cos\theta)^{2n}$ Sum of the series

Column I	Column II
(A) $\sum_{n=0}^{\infty}\sin^{2n}\theta\cos^{2n}\theta$	(P) $\frac{xy^2}{xy^2-1}$
(B) $\sum_{n=0}^{\infty}\tan^{2n}\theta$	(Q) $\frac{y}{x-y}$
(C) $\sum_{n=0}^{\infty}\sin^{2n}\theta\cos^{4n}\theta$	(R) $\frac{xy}{xy-1}$
(D) $\sum_{n=0}^{\infty}\cos^{2n}\sin^{4n}\theta$	(S) $\frac{x^2y}{x^2y-1}$

(a) A-R, B-P, C-S, D-Q (b) A-S, B-P, C-Q, D-R

(c) A-R, B-Q, C-S, D-P (d) A-R, B-S, C-Q, D-P

2. Value of x when

Column I	Column II
(A) $5^2\, 5^4\, 5^6 \ldots 5^{2x} = (0.04)^{-28}$	(P) $3\log_3 5$
(B) $x^2 = (0.2)^{\log_{\sqrt{5}}}\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots\right)$	(Q) 4
(C) $x = (0.16)^{\log 2.5}\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots\right)$	(R) 2
(D) $3^{x-1} + 3^{x-2} + 3^{x-3} + \ldots$	(S) 7

(a) A-R, B-P, C-S, D-Q (b) A-S, B-P, C-Q, D-R

(c) A-R, B-Q, C-S, D-P (d) A-S, B-R, C-Q, D-P

*****

Exercise - III

SUBJECTIVE TYPE

1. (a) If $a_1, a_2, \ldots, a_n$ are in arithmetic progression, where $a_i > 0$ for all i show that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\ldots \frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\ldots+\frac{1}{\sqrt{a_{n-1}+\sqrt{a_n}}}=\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

   (b) Does there exist a geometric progression containing 27, 8 and 12 as three of its terms? If it exists how many such progression are possible?

2. Find the numbers a, b, c between 2 and 18 such that (i) their sum is 25 (ii) the numbers 2, a, b are consecutive terms of an A.P. and (iii) the numbers b, c 18 are consecutive terms of a G.P.

3. The sum of the squares of three distinct real numbers, which are in G.P., is S^2. If their sum is aS, show that

$$a^2 \in \left(\frac{1}{3}, 1\right) \cup (1, 3)$$

4. Let p be the first of n arithmetic means between two numbers and q the first of n harmonic means between the same numbers. Show that q does not lie between p and $\left(\frac{n+1}{n-1}\right)^2 p$.

5. If $S_1, S_2, S_3, \ldots, S_n$ are the sums of infinite geometric series, whose first terms are 1, 2, 3…, n and whose common ratio are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n+1}$ respectively, then find the values of $S_1^2 + S_2^2 + S_3^2 + \ldots + S_{2n-1}^2$

6. The real numbers x_1, x_2, x_3 satisfying the equation $x^3 - x^2 + \beta x + \gamma = 0$ are in A.P. Find the intervals in which β and γ lie.

7. The fourth power of the common difference of an arithmetic progression with integer entries is added to the product of any four consecutive terms of it. Prove that resulting sum is the square of an integer.

8. Let $a_1, a_2\ldots$ be positive real numbers in geometric progression. For each n, let A_n, G_n H_n be respectively, the arithmetic mean, geometric mean, and harmonic mean of $a_1, a_2, \ldots, a_n$. Find an expression for the geometric mean of $G_1, G_2, \ldots, G_n$ in terms of $A_1, A_2, \ldots, A_n$ $H_1, H_2, \ldots H_n$.

9. Let a, b be positive real numbers. If a, A_1, A_2, b are in arithmetic progression a, G_1, G_2, b are in geometric progression and a, H_1, H_2, b are in harmonic progression, show that $\frac{G_1G_2}{H_1H_2} = \frac{A_1+A_2}{H_1+H_2} = \frac{(2a+b)\,(a+2b)}{9ab}$

10. If a, b, c are in A.P., a^2, b^2, c^2 are in H.P., then prove that either $a = b = c$ or a, b, $-\frac{c}{2}$ form a G.P.

11. If a, b, c are positive real numbers, then prove that $\{(1 + a)(1 + b)(1 + c)\}^7 > 7^7 a^4 b^4 c^4$

12. Let $A_n = \left(\frac{3}{4}\right) - \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \ldots + (-1)^{n-1}\left(\frac{3}{4}\right)^n$ $B_n = 1 - A_n$. Find a least odd natural number n_0, so that $B_n > A_n$ $\forall$, $n \geq n_0$.

*****

Exercise - IV

IIT – JEE PROBLEMS

A. Fill in the blanks

1. The sum of integers from 1 to 100 that are divisible by 2 or 5 is ______ .
2. In a triangle ABC, if cot A, cot B, cot C are in A.P., then a, b, c are in ______ progression.
3. The sum of the first n terms of the series $1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 +$ _____ is $\frac{n(n+1)^2}{2}$, when n is even. When n is odd, the sum is _____ .
4. Let the harmonic mean and geometric mean of two positive numbers be in the ratio 4 : 5. Then the two numbers are in the ratio ______ .
5. Let x be the arithmetic mean and y, z be the two geometric means between any two positive numbers, then $\frac{y^3 + z^3}{xyz}$ = ______
7. Let p and q be roots of the equation $x^2 - 2x + A = 0$ and let r and s be the roots of the equation $x^2 - 18x + B = 0$. If $p < q < r < s$ are in arithmetic progression then A = ______ and B = ______ .

B. True / False

8. If three complex numbers are in A.P. Then they lie on a circle in the complex plane.

C. Multiple Choice Questions with ONE correct answer

9. The third term of a geometric progression is 4. The product of the first five terms is
   (a) 4^3 (b) 4^5 (c) 4^4 (d) None of these
10. If a, b, c are in G.P. then the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root if $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in
   (a) A.P. (b) G.P. (c) H.P. (d) None of these
11. If a, b, c, d and p are distinct real numbers such that $(a^2 + b^2 + c^2)\,p^2 - 2\,(ab + bc + cd)\,p + (b^2 + c^2 + d^2) \leq 0$, then a, b, c, d
   (a) are in A.P. (b) are in G.P. (c) are in H.P. (d) satisfy $ab = cd$
12. In a triangle the lengths of the two larger sides are 10 and 9, respectively. If the angles are in A.P., then the length of the third side can be
   (a) $\sqrt{91}$ (b) $3\sqrt{3}$ (c) 5 (d) None of these

13. Sum of the first n terms of the series $\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+....$ is equal to

(a) $2^n - n - 1$ (b) $1 - 2^{-n}$ (c) $n + 2^{-n} - 1$ (d) $2^n + 1$

14. The product of n positive numbers is unity, then their sum is

(a) a positive integer (b) divisible by n (c) equal to $n + \frac{1}{n}$ (d) never less than n

15. Let T_r be the rth term of an A.P., for r = 1, 2, 3…. if for some positive integers m, n we have $T_m = \frac{1}{n}$ and $T_n = \frac{1}{m}$, then T_{mn} equals

(a) $\frac{1}{mn}$ (b) $\frac{1}{m}+\frac{1}{n}$ (c) 1 (d) 0

16. If $x > 1, y > 1, z > 1$ are in G.P., then $\frac{1}{1+\ln x}, \frac{1}{1+\ln y}, \frac{1}{1+\ln z}$ are in

(a) A.P. (b) H.P. (c) G.P. (d)None of the above

17. If x_1, x_2, x_3 as well as y_1, y_2, y_3 ae in G.P. with the same common ratio, then the points (x_1, y_1), (x_2, y_2) and (x_3, y_3)

(a) lie on a straight line (b) lie on an ellipse
(c) lie on a circle (d) are vertices of a triangle

18. The harmonic mean of the roots of the equation $(5+\sqrt{2})x^2 - (4+\sqrt{5})x + 8 + 2\sqrt{5} = 0$ is

(a) 2 (b) 4 (c) 6 (d) 8

19. Let $a_1, a_2…., a_{10}$ be in A.P. and $h_1, h_2,…, h_{10}$ be in H.P. If $a_1 = h_1 = 2$ and $a_{10} = h_{10} = 3$, then a_4h_7 is

(a) 2 (b) 3 (c) 5 (d) 6

20. If a, b, c are positive real numbers such that $a + b + c + d = 2$, then $M = (a + b)(c + d)$ satisfies the relation

(a) $0 \le M \le 1$ (b) $1 \le M \le 2$ (c) $2 \le M \le 3$ (d) $3 \le M \le 4$

21. Consider an infinite geometric series with first term a and common ratio r. If its sum is 4 and the second term is 3/4, then

(a) $a = \frac{4}{7}, r = \frac{3}{7}$ (b) $a = 2, r = 3/8$ (c) $a = 3/2, r = 1/2$ (d) $a = 3, r = ¼$

22. Let α, β be the roots of $x^2 - x + p = 0$ and γ, δ be the roots of $x^2 - 4x + q = 0$. If $\alpha, \beta, \gamma, \delta$ are in G.P. , then the integer values of p and q respectively are

(a) –2, – 32 (b) –2, 3 (c) –6, 3 (d) –6, –32

23. If the sum of the first $2n$ terms of the A.P. 2, 5, 8,…., is equal to the sum of the first n terms of the A.P. 57, 59, 61,…, then n equals.

(a) 10 (b) 12 (c) 11 (d) 13

24. Let the positive number a, b, c, d be in A.P. Then abc, abd, acd, bcd are

(a) not in A.P./G.P./H.P (b)in A.P. (c) in G.P. (d) in H.P.

25. If $a_1, a_2…., a_n$ are positive real numbers whose product is a fixed number c, then the minimum value of $a_1 + a_2 +…+ a_{n-1} + 2a_n$ is

(a) $n(2c)^{1/n}$ (b) $(n+1)c^{1/n}$ (c) $2nc^{1/n}$ (d) $(n + 1)(2c)^{1/n}$

26. Suppose a, b, c are in A.P. and a^2, b^2, c^2 are in G.P. If $a < b < c$ and $a + b + c = \frac{3}{2}$, then the value of a is

(a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{2\sqrt{3}}$ (c) $\frac{1}{2}+\frac{1}{\sqrt{2}}$ (d) $\frac{1}{2}-\frac{1}{\sqrt{2}}$

27. If $\alpha \in \left(0, \frac{\pi}{2}\right)$, then $\sqrt{x^2+x}+\frac{\tan^2\alpha}{\sqrt{x^2+x}}$ is always greater than or equal to

(a) $2\tan\alpha$ (b) 1 (c) 2 (d) $\sec^2\alpha$

28. An infinite G.P. has first term 'x' and sum 5, then x belongs to

(a) $x < -10$ (b) $-10 < x < 0$ (c) $0 < x < 10$ (d) $x > 10$

29. □, □ are root of $f(x) = ax^2 + bx + c$, $a \equiv 0$ and $\Delta = b^2 - 4ac$. If $\alpha + \beta$, $\alpha^2 + \beta^2$ and $\alpha^3 + \beta^3$ are in G.P., then

(a) $\Delta \neq 0$ (b) $b\Delta = 0$ (c) $c\Delta = 0$ (d) $bc \neq 0$

D. Multiple Choice Questions with ONE or MORE THAN ONE correct answer

30. If the first and the $(2n - 1)$th term of an A.P., G.P. and H.P. are equal and their nth terms are a, b and c respectively, then

(a) $a = b = c$ (b) $a \geq b \geq c$ (c) $a + c = b$ (d) $ac - b^2 = 0$

31. For a positive integer n let

$a(n) = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...+\frac{1}{(2^n)-1}$, then

(a) $a(100) \leq 100$ (b) $a(100) > 100$ (c) $a(200) \leq 100$ (d) $a(200) > 100$

32. Let S_1, S_2… be squares such that for each $n \geq 1$ the length of a side of S_n equals the length of a diagonal of S_{n+1}. If the length of a side of S_1 is 10 cm, then for which of the following values of n is the area of S_n less than 1 sq. cm?

(a) 7 (b) 8 (c) 9 (d) 10

*****

Answers

Exercise - I

Only One Option is correct

1.	(a)	2.	(c)	3.	(b)	4.	(c)	5.	(c)
6.	(c)	7.	(a)	8.	(a)	9.	(c)	10.	(c)
11.	(b)	12.	(a)	13	(b)	14.	(b)	15.	(b)
16.	(c)	17.	(d)	18.	(d)	19.	(a)	20.	(c)
21.	(b)	22.	(b)	23.	(b)	24.	(a)	25.	(c)
26.	(a)	27.	(b)	28.	(a)				

More Than One Choice Correct

1.	(a, c, d)	2.	(a, b, c, d)	3.	(d)	4.	(b)	5.	(c)
6.	(c, d)	7.	(a, b, c)	8.	(a, d)	9.	(c)	10.	(a, b, c, d)
11.	(c, d)								

Exercise - II

Assertion and Reason

1.	(a)	2.	(d)	3.	(c)	4.	(c)	5.	(a)

Passage – I

1.	(d)	2.	(b)	3.	(c)

Passage – II

1.	(a)	2.	(b)	3.	(b)

Matching Type Questions

1.	(c)	2.	(d)

Exercise - III

Subjective Type

2. $a = 5, b = 8, c = 12$ 5. $\frac{1}{6}(2n)(2n+1)(4n+1) - 1$

6. $\beta \in [-\infty, 1/3]$ and $\gamma \in [-1/27, \infty)$

9. $G_m = (A_1A_2\ldots A_nH_1H_2\ldots H_n)^{1/2n}$ 12. 6

Exercise - IV

IIT-JEE Level Problem
Section - A

1. 3050 2. A.P. 3. $\frac{n^2(n+1)}{2}$ 4. 4 : 1 5.

2

7. $A = -3, B = 77$

Section - B

8. False

Section – C

9.	(b)	10.	(a)	11.	(b)	12.	(a)	13	(c)
14.	(d)	15.	(c)	16.	(b)	17.	(a)	18.	(b)
19.	(d)	20.	(a)	21.	(d)	22.	(a)	23.	(c)
24.	(d)	25.	(a)	26.	(d)	27.	(a)	28.	(c)
29.	(c)								

Section - D

30.	(a, b, d)	31.	(a, d)	32.	(b, c, d)

*****