RELATIONS

1.2.1 Definition

Let A and B be two non-empty sets, then every subset of $A \times B$ defines a relation from A to B and every relation from A to B is a subset of $A \times B$.

Let $R \subseteq A \times B$ and $(a, b) \in R$. Then we say that a is related to b by the relation R and write it as aRb. If $(a, b) \in R$, we write it as aRb.

Example: Let $A = \{1, 2, 5, 8, 9\}$, $B = \{1, 3\}$ we set a relation from A to B as: $a\ R\ b$ iff $a \leq b$, $a \in A$, $b \in B$. Then $R = \{(1, 1)\}, (1, 3), (2, 3)\} \subset A \times B$

(1) **Total number of relations :** Let A and B be two non-empty finite sets consisting of m and n elements respectively. Then $A \times B$ consists of mn ordered pairs. So, total number of subset of $A \times B$ is 2^{mn}. Since each subset of $A \times B$ defines relation from A to B, so total number of relations from A to B is 2^{mn}. Among these 2^{mn} relations the void relation ϕ and the universal relation $A \times B$ are trivial relations from A to B.

(2) **Domain and range of a relation :** Let R be a relation from a set A to a set B. Then the set of all first components or coordinates of the ordered pairs belonging to R is called the domain of R, while the set of all second components or coordinates of the ordered pairs in R is called the range of R.

Thus, Dom $(R) = \{a : (a, b) \in R\}$ and Range $(R) = \{b : (a, b) \in R\}$.

It is evident from the definition that the domain of a relation from A to B is a subset of A and its range is a subset of B.

(3) **Relation on a set :** Let A be a non-void set. Then, a relation from A to itself *i.e.* a subset of $A \times A$ is called a relation on set A.

Example: 1 Let $A = \{1, 2, 3\}$. The total number of distinct relations that can be defined over A is

(a) 2^9 (b) 6 (c) 8 (d) None of these

Solution: (a) $n(A \times A) = n(A).n(A) = 3^2 = 9$

So, the total number of subsets of $A \times A$ is 2^9 and *a* subset of $A \times A$ is a relation over the set A.

Example: 2 Let $X = \{1, 2, 3, 4, 5\}$ and $Y = \{1, 3, 5, 7, 9\}$. Which of the following is/are relations from X to Y

(a) $R_1 = \{(x, y) \mid y = 2 + x, x \in X, y \in Y\}$ (b) $R_2 = \{(1, 1), (2, 1), (3, 3), (4, 3), (5, 5)\}$

(c) $R_3 = \{(1, 1), (1, 3)(3, 5), (3, 7), (5, 7)\}$ (d) $R_4 = \{(1, 3), (2, 5), (2, 4), (7, 9)\}$

Solution: (a,b,c) R_4 is not a relation from X to Y, because $(7, 9) \in R_4$ but $(7, 9) \notin X \times Y$.

Example: 3 Given two finite sets A and B such that $n(A) = 2$, $n(B) = 3$. Then total number of relations from A to B is

(a) 4 (b) 8 (c) 64 (d) None of these

Solution: (c) Here $n(A \times B) = 2 \times 3 = 6$

Since every subset of $A \times B$ defines a relation from A to B, number of relation from A to B is equal to number of subsets of $A \times B = 2^6 = 64$, which is given in (c).

Example: 4 The relation R defined on the set of natural numbers as $\{(a, b) : a$ differs from b by 3$\}$, is given by

(a) $\{(1, 4, (2, 5), (3, 6),.....\}$ (b) $\{(4, 1), (5, 2), (6, 3),.....\}$

(c) $\{(1, 3), (2, 6), (3, 9),..\}$ (d) None of these

Solution: (b) $R = \{(a, b) : a, b \in N, a - b = 3\} = \{((n+3), n) : n \in N\} = \{(4, 1), (5, 2), (6, 3).....\}$

1.2.2 Inverse Relation

Let A, B be two sets and let R be a relation from a set A to a set B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$

Clearly $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$. Also, Dom ($R$) = Range (R^{-1}) and Range (R) = Dom (R^{-1})

Example : Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$ and $R = \{(a, 1), (a, 3), (b, 3), (c, 3)\}$.

Then, (i) $R^{-1} = \{(1, a), (3, a), (3, b), (3, c)\}$

(ii) Dom (R) = $\{a, b, c\}$ = Range (R^{-1})

(iii) Range (R) = $\{1, 3\}$ = Dom (R^{-1})

Example: 5 Let $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$. A relation $R : A \to B$ is defined by $R = \{(1, 3), (1, 5), (2, 1)\}$. Then R^{-1} is defined by

(a) $\{(1,2), (3,1), (1,3), (1,5)\}$ (b) $\{(1, 2), (3, 1), (2, 1)\}$ (c)

Solution: (c) $(x, y) \in R \Leftrightarrow (y, x) \in R^{-1}$, $\therefore$ $R^{-1} = \{(3, 1), (5, 1), (1, 2)\}$.

Example: 6 The relation R is defined on the set of natural numbers as $\{(a, b) : a = 2b\}$. Then R^{-1} is given by

(a) $\{(2, 1), (4, 2), (6, 3).....\}$ (b) $\{(1, 2), (2, 4), (3, 6)....\}$ (c)

Solution: (b) $R = \{(2, 1), (4, 2), (6, 3),......\}$ So, $R^{-1} = \{(1, 2), (2, 4), (3, 6),.....\}$.

1.2.3 Types of Relations

(1) **Reflexive relation :** A relation R on a set A is said to be reflexive if every element of A is related to itself.

Thus, R is reflexive $\Leftrightarrow (a, a) \in R$ for all $a \in A$.

A relation R on a set A is not reflexive if there exists an element $a \in A$ such that $(a, a) \notin R$.

Example: Let $A = \{1, 2, 3\}$ and $R = \{(1, 1); (1, 3)\}$

Then R is not reflexive since $3 \in A$ but $(3, 3) \notin R$

Note : ❑ The identity relation on a non-void set A is always reflexive relation on A. However, a reflexive relation on A is not necessarily the identity relation on A.

- ❑ The universal relation on a non-void set A is reflexive.

(2) **Symmetric relation :** A relation R on a set A is said to be a symmetric relation *iff*

$$(a, b) \in R \Rightarrow (b, a) \in R \text{ for all } a, b \in A$$

i.e. $aRb \Rightarrow bRa$ for all $a, b \in A$.

it should be noted that R is symmetric iff $R^{-1} = R$

Note : ❑ The identity and the universal relations on a non-void set are symmetric relations.

- ❑ A relation R on a set A is not a symmetric relation if there are at least two elements $a, b \in A$ such that $(a, b) \in R$ but $(b, a) \notin R$.
- ❑ A reflexive relation on a set A is not necessarily symmetric.

(3) **Anti-symmetric relation :** Let A be any set. A relation R on set A is said to be an anti-symmetric relation *iff* $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ for all $a, b \in A$.

Thus, if a ≠ b then a may be related to b or b may be related to a, but never both.

Example: Let N be the set of natural numbers. A relation $R \subseteq N \times N$ is defined by xRy iff x divides y(*i.e.*, x/y).

Then $xRy,\ yRx \Rightarrow x$ divides y, y divides $x \Rightarrow x = y$

Note : ❑ The identity relation on a set A is an anti-symmetric relation.

- ❑ The universal relation on a set A containing at least two elements is not anti-symmetric, because if $a \neq b$ are in A, then a is related to b and b is related to a under the universal relation will imply that $a = b$ but $a \neq b$.
- ❑ The set $\{(a, a) : a \in A\} = D$ is called the diagonal line of $A \times A$. Then "the relation R in A is antisymmetric iff $R \cap R^{-1} \subseteq D$".

(4) **Transitive relation :** Let A be any set. A relation R on set A is said to be a transitive relation *iff*

$(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$ *i.e.*, aRb and $bRc \Rightarrow aRc$ for all $a, b, c \in A$.

In other words, if a is related to b, b is related to c, then a is related to c.

Transitivity fails only when there exists a, b, c such that $a\ R\ b$, $b\ R\!\!\!\!/\ c$ but $a\ R\ c$.

Example: Consider the set $A = \{1, 2, 3\}$ and the relations

$R_1 = \{(1, 2), (1, 3)\}$; $R_2 = \{(1, 2)\}$; $R_3 = \{(1, 1)\}$; $R_4 = \{(1, 2), (2, 1), (1, 1)\}$

Then R_1, R_2, R_3 are transitive while R_4 is not transitive since in $R_4, (2, 1) \in R_4; (1, 2) \in R_4$ but $(2, 2) \notin R_4$.

Note : ❑ The identity and the universal relations on a non-void sets are transitive.

- ❑ The relation 'is congruent to' on the set T of all triangles in a plane is a transitive relation.

(5) **Identity relation :** Let A be a set. Then the relation $I_A = \{(a, a) : a \in A\}$ on A is called the identity relation on A.

In other words, a relation I_A on A is called the identity relation if every element of A is related to itself only. Every identity relation will be reflexive, symmetric and transitive.

Example: On the set = {1, 2, 3}, $R = \{(1, 1), (2, 2), (3, 3)\}$ is the identity relation on A .

Note : ❑ It is interesting to note that every identity relation is reflexive but every reflexive relation need not be an identity relation.

Also, identity relation is reflexive, symmetric and transitive.

(6) **Equivalence relation :** A relation R on a set A is said to be an equivalence relation on A *iff*

(i) It is reflexive *i.e.* $(a, a) \in R$ for all $a \in A$

(ii) It is symmetric *i.e.* $(a, b) \in R \Rightarrow (b, a) \in R$, for all $a, b \in A$

(iii) It is transitive *i.e.* $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$.

Note : ❑ **Congruence modulo (m) :** Let m be an arbitrary but fixed integer. Two integers a and b are said to be congruence modulo m if $a - b$ is divisible by m and we write $a \equiv b$ (mod m).

Thus $a \equiv b$ (mod m) $\Leftrightarrow a - b$ is divisible by m. For example, $18 \equiv 3$ (mod 5) because 18 – 3 = 15 which is divisible by 5. Similarly, $3 \equiv 13$ (mod 2) because 3 – 13 = –10 which is divisible by 2. But 25 ≠ 2 (mod 4) because 4 is not a divisor of 25 – 3 = 22.

The relation "Congruence modulo *m*" is an equivalence relation.

Important Tips

- *If R and S are two equivalence relations on a set A , then $R \cap S$ is also an equivalence relation on A.*
- *The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.*
- *The inverse of an equivalence relation is an equivalence relation.*

1.2.4 Equivalence Classes of an Equivalence Relation

Let R be equivalence relation in $A(\neq \phi)$. Let $a \in A$. Then the equivalence class of a, denoted by $[a]$ or $\{\bar{a}\}$ is defined as the set of all those points of A which are related to a under the relation R. Thus $[a] = \{x \in A : x\,R\,a\}$.

It is easy to see that

(1) $b \in [a] \Rightarrow a \in [b]$ (2) $b \in [a] \Rightarrow [a] = [b]$ (3) Two equivalence classes are either disjoint or identical.

As an example we consider a very important equivalence relation $x \equiv y (\text{mod } n)$ iff n divides $(x - y), n$ is a fixed positive integer. Consider $n = 5$. Then

$[0] = \{x : x \equiv 0 \text{ (mod 5)}\} = \{5p : p \in Z\} = \{0, \pm 5, \pm 10, \pm 15, ...\}$

$[1] = \{x : x \equiv 1 (\text{mod } 5)\} = \{x : x - 1 = 5k, k \in Z\} = \{5k + 1 : k \in Z\} = \{1, 6, 11, ..., -4, -9, ...\}$.

One can easily see that there are only 5 distinct equivalence classes *viz*. [0], [1], [2], [3] and [4], when $n = 5$.

Example: 7 Given the relation $R = \{(1, 2), (2, 3)\}$ on the set $A = \{1, 2, 3\}$, the minimum number of ordered pairs which when added to R make it an equivalence relation is

(a) 5 (b) 6 (c) 7 (d) 8

Solution: (c) R is reflexive if it contains (1, 1), (2, 2), (3, 3)

$\Theta\ (1, 2) \in R, (2, 3) \in R$

$\therefore$ R is symmetric if $(2, 1), (3, 2) \in R$. Now, $R = \{(1, 1), (2, 2), (3, 3), (2, 1), (3, 2), (2, 3), (1, 2)\}$

R will be transitive if $(3, 1); (1, 3) \in R$. Thus, R becomes an equivalence relation by adding (1, 1) (2, 2) (3, 3) (2, 1) (3,2) (1, 3) (3, 1). Hence, the total number of ordered pairs is 7.

Example: 8 The relation $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$ on set $A = \{1, 2, 3\}$ is

(a) Reflexive but not symmetric (b)Reflexive but not transitive

(c) Symmetric and Transitive (d)Neither symmetric nor transitive

Solution: (a) Since $(1, 1); (2, 2); (3, 3) \in R$ therefore R is reflexive. $(1, 2) \in R$ but $(2, 1) \notin R$, therefore R is not symmetric. It can be easily seen that R is transitive.

Example: 9 Let R be the relation on the set R of all real numbers defined by $a R b$ iff $|a - b| \le 1$. Then R is]

(a) Reflexive and Symmetric (b)Symmetric only

(c)Transitive only (d)Anti-symmetric only

Solution: (a) $|a - a| = 0 < 1 \ \therefore aRa \ \forall \ a \in R$

$\therefore$ R is reflexive, Again $a R b \Rightarrow |a - b| \le 1 \Rightarrow |b - a| \le 1 \Rightarrow bRa$

$\therefore$ R is symmetric, Again $1R\frac{1}{2}$ and $\frac{1}{2}R1$ but $\frac{1}{2} \ne 1$

$\therefore$ R is not anti-symmetric

Further, $1 R 2$ and $2 R 3$ but $1 \not R 3$

$[\Theta |1 - 3| = 2 > 1]$

$\therefore$ R is not transitive.

Example: 10 The relation "less than" in the set of natural numbers is

(a) Only symmetric (b) Only transitive

(c) Only reflexive (d) Equivalence relation

Solution: (b) Since $x < y, y < z \Rightarrow x < z \ \forall \ x, y, z \in N$

$\therefore$ $xRy, yRz \Rightarrow xRz$, $\therefore$ Relation is transitive , $\therefore$ $x < y$ does not give $y < x$, $\therefore$ Relation is not symmetric.

Since $x < x$ does not hold, hence relation is not reflexive.

Example: 11 With reference to a universal set, the inclusion of a subset in another, is relation, which is

(a) Symmetric only (b) Equivalence relation (c)Reflexive only (d) None of these

Solution: (d) Since $A \subseteq A$ $\therefore$ relation '$\subseteq$' is reflexive.

Since $A \subseteq B, \; B \subseteq C \Rightarrow A \subseteq C$

$\therefore$ relation '$\subseteq$' is transitive.

But $A \subseteq B \not\Rightarrow B \subseteq A$, $\therefore$ Relation is not symmetric.

Example: 12 Let $A = \{2, 4, 6, 8\}$. A relation R on A is defined by $R = \{(2,4), (4,2), (4,6), (6,4)\}$. Then R is

(a) Anti-symmetric (b) Reflexive (c) Symmetric (d) Transitive

Solution: (c) Given $A = \{2, 4, 6, 8\}$

$R = \{(2, 4)(4, 2)\ (4, 6)\ (6, 4)\}$

$(a, b) \in R \Rightarrow (b, a) \in R$ and also $R^{-1} = R$. Hence R is symmetric.

Example: 13 Let $P = \{(x, y) \mid x^2 + y^2 = 1, x, y \in R\}$. Then P is

(a) Reflexive (b) Symmetric (c) Transitive (d) Anti-symmetric

Solution: (b) Obviously, the relation is not reflexive and transitive but it is symmetric, because $x^2 + y^2 = 1 \Rightarrow y^2 + x^2 = 1$.

Example: 14 Let R be a relation on the set N of natural numbers defined by $nRm \Leftrightarrow n$ is a factor of m (*i.e.*, $n|m$). Then R is

(a) Reflexive and symmetric (b) Transitive and symmetric

(c) Equivalence (d) Reflexive, transitive but not symmetric

Solution: (d) Since $n \mid n$ for all $n \in N$, therefore R is reflexive. Since $2 \mid 6$ but $6 \nmid 2$, therefore R is not symmetric.

Let $n\,R\,m$ and $m\,R\,p \Rightarrow n|m$ and $m|p \Rightarrow n|p \Rightarrow nRp$. So R is transitive.

Example: 15 Let R be an equivalence relation on a finite set A having n elements. Then the number of ordered pairs in R is

(a) Less than n (b)Greater than or equal to n

(c)Less than or equal to n (d)None of these

Solution: (b) Since R is an equivalence relation on set A, therefore $(a, a) \in R$ for all $a \in A$. Hence, R has at least n ordered pairs.

Example: 16 Let N denote the set of all natural numbers and R be the relation on $N \times N$ defined by $(a, b)\ R\ (c, d)$ if $ad(b + c) = bc(a + d)$, then R is

(a) Symmetric only (b) Reflexive only

(c) Transitive only (d) An equivalence relation

Solution: (d) For $(a, b), (c, d) \in N \times N$

$(a, b) R (c, d) \Rightarrow ad(b + c) = bc(a + d)$

Reflexive: Since $ab(b+a) = ba(a+b) \forall a,b \in N$,

$\therefore (a,b)R(a,b)$, $\therefore$ R is reflexive.

Symmetric: For $(a,b),(c,d) \in N \times N$, let $(a,b)R(c,d)$

$\therefore ad(b+c) = bc(a+d) \Rightarrow bc(a+d) = ad(b+c) \Rightarrow cb(d+a) = da(c+b) \Rightarrow (c,d)R(a,b)$

$\therefore R$ is symmetric

Transitive: For $(a,b),(c,d),(e,f) \in N \times N$, Let $(a,b)R(c,d),(c,d)R(e,f)$

$\therefore ad(b+c) = bc(a+d)$, $cf(d+e) = de(c+f)$

$\Rightarrow adb + adc = bca + bcd$ .....(i) and $cfd + cfe = dec + def$ ....(ii)

(i) × ef + (ii) × ab gives, $adbef + adcef + cfdab + cfeab = bcaef + bcdef + decab + defab$

$\Rightarrow adcf(b+e) = bcde(a+f) \Rightarrow af(b+e) = be(a+f) \Rightarrow (a,b)R(e,f)$. $\therefore$ R is transitive. Hence R is an equivalence relation.

Example: 17 For real numbers x and y, we write $x\,Ry \Leftrightarrow x - y + \sqrt{2}$ is an irrational number. Then the relation R is

(a) Reflexive (b) Symmetric (c) Transitive (d) None of these

Solution: (a) For any $x \in R$, we have $x - x + \sqrt{2} = \sqrt{2}$ an irrational number.

$\Rightarrow xRx$ for all x. So, R is reflexive.

R is not symmetric, because $\sqrt{2}R1$ but $1 \not R \sqrt{2}$, R is not transitive also because $\sqrt{2}\,R\,1$ and $1R2\sqrt{2}$ but $\sqrt{2} \not R 2\sqrt{2}$.

Example: 18 Let X be a family of sets and R be a relation on X defined by 'A is disjoint from B'. Then R is

(a) Reflexive (b) Symmetric (c) Anti-symmetric (d) Transitive

Solution: (b) Clearly, the relation is symmetric but it is neither reflexive nor transitive.

Example: 19 Let R and S be two non-void relations on a set A. Which of the following statements is false

(a) R and S are transitive $\Rightarrow R \cup S$ is transitive

(b) R and S are transitive $\Rightarrow R \cap S$ is transitive

(c) R and S are symmetric $\Rightarrow R \cup S$ is symmetric

(d) R and S are reflexive $\Rightarrow R \cap S$ is reflexive

Solution: (a) Let $A = \{1,2,3\}$ and $R = \{(1, 1), (1, 2)\}$, $S = \{(2, 2)\ (2, 3)\}$ be transitive relations on A.

Then $R \cup S = \{(1, 1); (1, 2); (2, 2); (2, 3)\}$

Obviously, $R \cup S$ is not transitive. Since $(1, 2) \in R \cup S$ and $(2,3) \in R \cup S$ but $(1, 3) \notin R \cup S$.

Example: 20 The solution set of $8x \equiv 6 (\text{mod } 14), x \in Z$, are

(a) $[8] \cup [6]$ (b) $[8] \cup [14]$ (c) $[6] \cup [13]$ (d) $[8] \cup [6] \cup [13]$

Solution: (c) $8x - 6 = 14P (P \in Z) \Rightarrow x = \frac{1}{8}[14P + 6], x \in Z$

$\Rightarrow x = \frac{1}{4}(7P + 3) \Rightarrow x = 6, 13, 20, 27, 34, 41, 48, ...$

$\therefore$ Solution set = {6, 20, 34, 48,.....} $\cup$ {13, 27, 41, ...} = [6] $\cup$ [13].

Where [6], [13] are equivalence classes of 6 and 13 respectively.

1.2.5 Composition of Relations

Let *R* and *S* be two relations from sets *A* to *B* and *B* to *C* respectively. Then we can define a relation *SoR* from *A* to *C* such that $(a, c) \in SoR \Leftrightarrow \exists\, b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.

This relation is called the composition of *R* and *S*.

For example, if *A* = {1, 2, 3}, *B* = {*a*, *b*, *c*, *d*}, *C* = {*p*, *q*, *r*, *s*} be three sets such that *R* = {(1, *a*), (2, *c*), (1, *c*), (2, *d*)} is a relation from *A* to *B* and *S* = {(*a*, *s*), (*b*, *q*), (*c*, *r*)} is a relation from *B* to *C*. Then *SoR* is a relation from *A* to *C* given by *SoR* = {(1, *s*) (2, *r*) (1, *r*)}

In this case *RoS* does not exist.

In general $RoS \neq SoR$. Also $(SoR)^{-1} = R^{-1}oS^{-1}$.

Example: 21 If *R* is a relation from a set *A* to a set *B* and *S* is a relation from *B* to a set *C*, then the relation *SoR*

(a) Is from *A* to *C* (b) Is from *C* to *A* (c) Does not exist (d) None of these

Solution: (a) It is obvious.

Example: 22 If $R \subset A \times B$ and $S \subset B \times C$ be two relations, then $(SoR)^{-1} =$

(a) $S^{-1}oR^{-1}$ (b) $R^{-1}oS^{-1}$ (c) SoR (d) RoS

Solution: (b) It is obvious.

Example: 23 If *R* be a relation < from *A* = {1,2, 3, 4} to *B* = {1, 3, 5} *i.e.*, $(a, b) \in R \Leftrightarrow a < b$, then RoR^{-1} is

(a) {(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)}

(b) {(3, 1) (5, 1), (3, 2), (5, 2), (5, 3), (5, 4)}

(c) {(3, 3), (3, 5), (5, 3), (5, 5)}

(d) {(3, 3) (3, 4), (4, 5)}

Solution: (c) We have, *R* = {(1, 3); (1, 5); (2, 3); (2, 5); (3, 5); (4, 5)}

$R^{-1} =$ {(3, 1), (5, 1), (3, 2), (5, 2); (5, 3); (5, 4)}

Hence RoR^{-1} = {(3, 3); (3, 5); (5, 3); (5, 5)}

Example: 24 Let a relation *R* be defined by *R* = {(4, 5); (1, 4); (4, 6); (7, 6); (3, 7)} then $R^{-1}oR$ is

(a) {(1, 1), (4, 4), (4, 7), (7, 4), (7, 7), (3, 3)} (b) {(1, 1), (4, 4), (7, 7), (3, 3)}

(c) {(1, 5), (1, 6), (3, 6)} (d) None of these

Solution: (a) We first find R^{-1}, we have $R^{-1} = \{(5,4);(4,1);(6,4);(6,7);(7,3)\}$ we now obtain the elements of $R^{-1}oR$ we first pick the element of R and then of R^{-1}. Since $(4,5) \in R$ and $(5,4) \in R^{-1}$, we have $(4,4) \in R^{-1}oR$

Similarly, $(1,4) \in R, (4,1) \in R^{-1} \Rightarrow (1,1) \in R^{-1}oR$

$(4,6) \in R, (6,4) \in R^{-1} \Rightarrow (4,4) \in R^{-1}oR,$ $\quad (4,6) \in R, (6,7) \in R^{-1} \Rightarrow (4,7) \in R^{-1}oR$

$(7,6) \in R, (6,4) \in R^{-1} \Rightarrow (7,4) \in R^{-1}oR,$ $\quad (7,6) \in R, (6,7) \in R^{-1} \Rightarrow (7,7) \in R^{-1}oR$

$(3,7) \in R, (7,3) \in R^{-1} \Rightarrow (3,3) \in R^{-1}oR,$

Hence $R^{-1}oR =$ {(1, 1); (4, 4); (4, 7); (7, 4), (7, 7); (3, 3)}.

1.2.6 Axiomatic Definitions of the Set of Natural Numbers (Peano's Axioms)

The set N of natural numbers (N = {1, 2, 3, 4......}) is a set satisfying the following axioms (known as peano's axioms)

(1) N is not empty.

(2) There exist an injective (one-one) map $S: N \to N$ given by $S(n) = n^+$, where n^+ is the immediate successor of n in N *i.e.*, $n + 1 = n^+$.

(3) The successor mapping S is not surjective (onto).

(4) If $M \subseteq N$ such that,

(i) M contains an element which is not the successor of any element in N, and

(ii) $m \in M \Rightarrow m^+ \in M$, then $M = N$

This is called the axiom of induction. We denote the unique element which is not the successor of any element is 1. Also, we get $1^+ = 2, 2^+ = 3$.

Note : ❑ Addition in N is defined as,

$$n + 1 = n^+$$

$$n + m^+ = (n + m)^+$$

❑ Multiplication in N is defined by,

$$n.1 = n$$

$$n.m^+ = n.m + n$$

ASSIGNMENT

Basic Level

1. A relation from P to Q is

(a) A universal set of $P \times Q$ (b) $P \times Q$

(c) An equivalent set of $P \times Q$ (d) A subset of $P \times Q$

2. Let R be a relation from a set A to set B, then

(a) $R = A \cup B$ (b) $R = A \cap B$ (c) $R \subseteq A \times B$ (d) $R \subseteq B \times A$

3. Let $A = \{a, b, c\}$ and $B = \{1, 2\}$. Consider a relation R defined from set A to set B. Then R is equal to set

(a) A (b) B (c) $A \times B$ (d) $B \times A$

4. Let $n(A) = n$. Then the number of all relations on A is

(a) 2^n (b) $2^{(n)!}$ (c) 2^{n^2} (d) None of these

5. If R is a relation from a finite set A having m elements to a finite set B having n elements, then the number of relations from A to B is

(a) 2^{mn} (b) $2^{mn}-1$ (c) $2mn$ (d) m^n

6. Let R be a reflexive relation on a finite set A having n-elements, and let there be m ordered pairs in R. Then

(a) $m \geq n$ (b) $m \leq n$ (c) $m = n$ (d) None of these

7. The relation R defined on the set $A = \{1, 2, 3, 4, 5\}$ by $R = \{(x, y) : |x^2 - y^2| < 16\}$ is given by

(a) $\{(1, 1), (2, 1), (3, 1), (4, 1), (2, 3)\}$ (b) $\{(2, 2), (3, 2), (4, 2), (2, 4)\}$

(c) $\{(3, 3), (3, 4), (5, 4), (4, 3), (3, 1)\}$ (d) None of these

8. A relation R is defined from $\{2, 3, 4, 5\}$ to $\{3, 6, 7, 10\}$ by; $xRy \Leftrightarrow x$ is relatively prime to y. Then domain of R is

(a) $\{2, 3, 5\}$ (b) $\{3, 5\}$ (c) $\{2, 3, 4\}$ (d) $\{2, 3, 4, 5\}$

9. Let R be a relation on N defined by $x + 2y = 8$. The domain of R is

(a) $\{2, 4, 8\}$ (b) $\{2, 4, 6, 8\}$ (c) $\{2, 4, 6\}$ (d) $\{1, 2, 3, 4\}$

10. If $R = \{(x, y) \mid x, y \in Z, x^2 + y^2 \leq 4\}$ is a relation in Z, then domain of R is

(a) $\{0, 1, 2\}$ (b) $\{0, -1, -2\}$ (c) $\{-2, -1, 0, 1, 2\}$ (d) None of these

11. If $A = \{1, 2, 3\}$, $B = \{1, 4, 6, 9\}$ and R is a relation from A to B defined by 'x is greater than y'. The range of R is

(a) $\{1, 4, 6, 9\}$ (b) $\{4, 6, 9\}$ (c) $\{1\}$ (d) None of these

12. R is a relation from $\{11, 12, 13\}$ to $\{8, 10, 12\}$ defined by $y = x - 3$. Then R^{-1} is

(a) $\{(8, 11), (10, 13)\}$ (b) $\{(11, 18), (13, 10)\}$ (c) $\{(10, 13), (8, 11)\}$ (d) None of these

13. Let A = {1, 2, 3}, B = {1, 3, 5}. If relation R from A to B is given by R ={(1, 3), (2, 5), (3, 3)}. Then R^{-1} is

(a) {(3, 3), (3, 1), (5, 2)} (b){(1, 3), (2, 5), (3, 3)} (c){(1, 3), (5, 2)} (d) None of these

14. Let R be a reflexive relation on a set A and I be the identity relation on A. Then

(a) $R \subset I$ (b) $I \subset R$ (c) $R = I$ (d) None of these

15. Let A = {1, 2, 3, 4} and R be a relation in A given by R = {(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (3, 1), (1, 3)}. Then R is

(a) Reflexive (b) Symmetric (c) Transitive (d)An equivalence relation

16. An integer m is said to be related to another integer n if m is a multiple of n. Then the relation is

(a) Reflexive and symmetric (b) Reflexive and transitive

(c) Symmetric and transitive (d) Equivalence relation

17. The relation R defined in N as $aRb \Leftrightarrow b$ is divisible by a is

(a) Reflexive but not symmetric (b) Symmetric but not transitive

(c) Symmetric and transitive (d) None of these

18. Let R be a relation on a set A such that $R = R^{-1}$, then R is

(a) Reflexive (b) Symmetric (c) Transitive (d) None of these

19. Let R = {(a, a)} be a relation on a set A. Then R is

(a) Symmetric (b) Antisymmetric

(c) Symmetric and antisymmetric (d) Neither symmetric nor anti-symmetric

20. The relation "is subset of" on the power set $P(A)$ of a set A is

(a) Symmetric (b) Anti-symmetric (c) Equivalency relation (d)None of these

21. The relation R defined on a set A is antisymmetric if $(a,b) \in R \Rightarrow (b,a) \in R$ for

(a) Every (a, b) $\in R$ (b) No $(a,b) \in R$ (c) No $(a,b), a \neq b, \in R$ (d) None of these

22. In the set A = {1, 2, 3, 4, 5}, a relation R is defined by R = {(x, y)| x, $y \in A$ and $x < y$}. Then R is

(a) Reflexive (b) Symmetric (c) Transitive (d) None of these

23. Let A be the non-void set of the children in a family. The relation 'x is a brother of y' on A is

(a) Reflexive (b) Symmetric (c) Transitive (d) None of these

24. Let A = {1, 2, 3, 4} and let R= {(2, 2), (3, 3), (4, 4), (1, 2)} be a relation on A. Then R is

(a) Reflexive (b) Symmetric (c) Transitive (d) None of these

25. The void relation on a set A is

(a) Reflexive (b)Symmetric and transitive

(c)Reflexive and symmetric (d)Reflexive and transitive

26. Let R_1 be a relation defined by $R_1 = \{(a, b) \mid a \geq b, a, b \in R\}$. Then R_1 is

(a) An equivalence relation on *R* (b)Reflexive, transitive but not symmetric

(c) Symmetric, Transitive but not reflexive (d)Neither transitive not reflexive but symmetric

27. Let *A* = {*p*, *q*, *r*}. Which of the following is an equivalence relation on *A*

(a) R_1 = {(*p*, *q*), (*q*, *r*), (*p*, *r*), (*p*, *p*)} (b) R_2 = {(*r*, *q*), (*r*, *p*), (*r*, *r*), (*q*, *q*)}

(c) R_3 = {(*p*, *p*), (*q*, *q*), (*r*, *r*), (*p*, *q*)} (d) None of these

28. Which one of the following relations on *R* is an equivalence relation

(a) $aR_1b \Leftrightarrow |a| = |b|$ (b) $aR_2b \Leftrightarrow a \geq b$ (c) $aR_3b \Leftrightarrow a \text{ divides } b$ (d) $aR_4b \Leftrightarrow a < b$

29. If *R* is an equivalence relation on a set *A*, then R^{-1} is

(a) Reflexive only (b) Symmetric but not transitive

(c) Equivalence (d) None of these

30. *R* is a relation over the set of real numbers and it is given by $nm \geq 0$. Then *R* is

(a) Symmetric and transitive (b) Reflexive and symmetric

(c) A partial order relation (d) An equivalence relation

31. In order that a relation *R* defined on a non-empty set *A* is an equivalence relation, it is sufficient, if *R*

(a) Is reflextive (b) Is symmetric

(c) Is transitive (d) Possesses all the above three properties

32. The relation "congruence modulo *m*" is

(a) Reflexive only (b) Transitive only

(c) Symmetric only (d) An equivalence relation

33. Solution set of $x \equiv 3$ (mod 7), $x \in Z$, is given by

(a) {3} (b) $\{7p - 3 : p \in Z\}$ (c) $\{7p + 3 : p \in Z\}$ (d) None of these

34. Let *R* and *S* be two equivalence relations on a set *A*. Then

(a) $R \cup S$ is an equivalence relation on *A* (b) $R \cap S$ is an equivalence relation on *A*

(c) $R - S$ is an equivalence relation on *A* (d) None of these

35. Let *R* and *S* be two relations on a set *A*. Then

(a) *R* and *S* are transitive, then $R \cup S$ is also transitive

(b) *R* and *S* are transitive, then $R \cap S$ is also transitive

(c) *R* and *S* are reflexive, then $R \cap S$ is also reflexive

(d) *R* and *S* are symmetric then $R \cup S$ is also symmetric

36. Let *R* = {(1, 3), (2, 2), (3, 2)} and *S* = {(2, 1), (3, 2), (2, 3)} be two relations on set *A* = {1, 2, 3}.

Then RoS =

(a) {(1, 3), (2, 2), (3, 2), (2, 1), (2, 3)} (b) {(3, 2), (1, 3)}

(c) {(2, 3), (3, 2), (2, 2)} (d) {(2, 3), (3, 2)}

37. In problem 36, $RoS^{1} =$

(a) {(2, 2), (3, 2) (b) {(1, 2), (2, 2), (3, 2)}

(c) {(1, 2), (2, 2)} (d) {(1, 2), (2, 2), (3, 2), (2, 3)}

Advance Level

38. Let R be a relation on the set N be defined by $\{(x, y) | x, y \in N, 2x + y = 41\}$. Then R is

(a) Reflexive (b) Symmetric (c) Transitive (d) None of these

39. Let L denote the set of all straight lines in a plane. Let a relation R be defined by $\alpha R \beta \Leftrightarrow \alpha \perp \beta, \alpha, \beta \in L$. Then R is

(a) Reflexive (b) Symmetric (c) Transitive (d) None of these

40. Let T be the set of all triangles in the Euclidean plane, and let a relation R be defined on T by aRb iff $a \approx b, a, b \in T$. Then R is

(a) Reflexive but not transitive (b) Transitive but not symmetric

(c) Equivalence (d) None of these

41. Two points P and Q in a plane are related if $OP = OQ$, where O is a fixed point. This relation is

(a) Partial order relation (b) Equivalence relation

(c) Reflexive but not symmetric (d) Reflexive but not transitive

42. Let r be a relation over the set $N \times N$ and it is defined by $(a, b) r (c, d) \Rightarrow a + d = b + c$. Then r is

(a) Reflexive only (b) Symmetric only (c) Transitive only (d)An equivalence relation

43. Let L be the set of all straight lines in the Euclidean plane. Two lines l_1 and l_2 are said to be related by the relation R iff l_1 is parallel to l_2. Then the relation R is

(a) Reflexive (b) Symmetric (c) Transitive (d) Equivalence

44. Let n be a fixed positive integer. Define a relation R on the set Z of integers by, $aRb \Leftrightarrow n | a - b|$. Then R is

(a) Reflexive (b) Symmetric (c) Transitive (d) Equivalence

***

ANSWER

Basic & Advance Level

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
d	c	c	c	a	a	d	d	c	c	c	a	a	b	a,b	b	a	b	c	b
21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
c	c	b,c	c	b	b	d	a	c	d	d	d	c	b	b,c,d	c	b	d	b	c
41	42	43	44																
b	d	a,b,c,d	a,b,c,d																

FUNCTIONS

If A and B are two non-empty sets, then a rule f which associated to each $x \in A$, a unique number $y \in B$, is called a function from A to B and we write, $f : A \to B$.

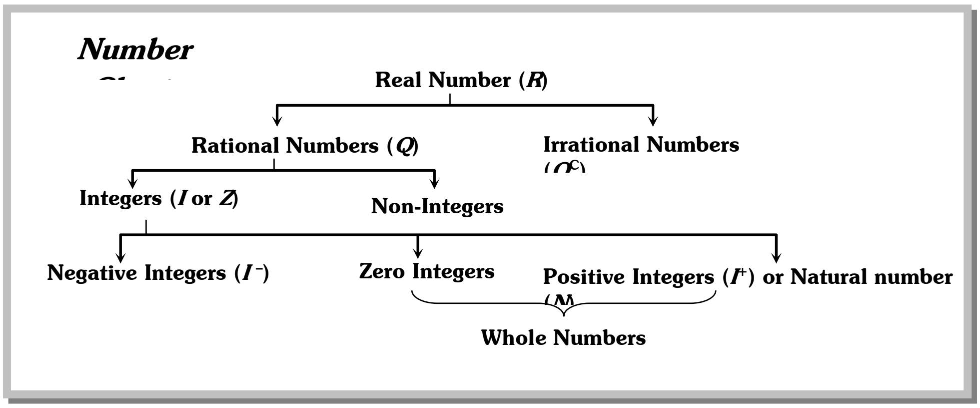
2.1.1 Some Important Definitions

(1) **Real numbers :** Real numbers are those which are either rational or irrational. The set of real numbers is denoted by R.

(i) **Rational numbers :** All numbers of the form p/q where p and q are integers and $q \neq 0$, are called rational numbers and their set is denoted by Q. *e.g.* $\frac{2}{3}, -\frac{5}{2}, 4 \left(\text{as } 4 = \frac{4}{1} \right)$ are rational numbers.

(ii) **Irrational numbers :** Those are numbers which can not be expressed in form of p/q are called irrational numbers and their set is denoted by Q^c (*i.e.*, complementary set of Q) *e.g.* $\sqrt{2}, 1 - \sqrt{3}, \pi$ are irrational numbers.

(iii) **Integers :** The numbers …….– 3, – 2, – 1, 0, 1, 2, 3, …….. are called integers. The set of integers is denoted by I or Z. Thus, I or Z ={…….,– 3, – 2, –1, 0, 1, 2, 3,……}



Note :
- ❒ Set of positive integers I^+ = {1, 2, 3, ...}
- ❒ Set of negative integers I^- = {– 1, – 2, – 3, ……}.
- ❒ Set of non negative integers ={0, 1, 2, 3, ..}
- ❒ Set of non positive integers = {0, – 1, – 2, – 3,…..}
- ❒ Positive real numbers: $R^+ = (0, \infty)$
- ❒ Negative real numbers: $R^- = (-\infty, 0)$
- ❒ R_0 : all real numbers except 0 (Zero)
- ❒ Imaginary numbers: $C = \{i, \omega, ...\}$
- ❒ Even numbers: $E = \{0, 2, 4, 6, ...\}$
- ❒ Odd numbers: $0 = \{1, 3, 5, 7, ...\}$
- ❒ Prime numbers : The natural numbers greater than 1 which is divisible by 1 and itself only, called prime numbers.
- ❒ In rational numbers the digits are repeated after decimal

- 0 (zero) is a rational number
- In irrational numbers, digits are not repeated after decimal
- π and e are called special irrational quantities
- ∞ is neither a rational number nor an irrational number

(2) Related quantities : When two quantities are such that the change in one is accompanied by the change in other, *i.e.*, if the value of one quantity depends upon the other, then they are called related quantities. *e.g.* the area of a circle $(A = \pi r^2)$ depends upon its radius (r) as soon as the radius of the circle increases (or decreases), its area also increases (or decreases). In the given example, A and r are related quantities.

(3) **Variable:** A variable is a symbol which can assume any value out of a given set of values. The quantities, like height, weight, time, temperature, profit, sales etc, are examples of variables. The variables are usually denoted by x, y, z, u, v, w, t etc. There are two types of variables mainly:

(i) **Independent variable :** A variable which can take any arbitrary value, is called independent variable.

(ii) **Dependent variable :** A variable whose value depends upon the independent variable is called dependent variable. *e.g.* $y = x^2$, if $x = 2$ then $y = 4 \Rightarrow$ so value of y depends on x. y is dependent and x is independent variable here.

(4) **Constant :** A constant is a symbol which does not change its value, *i.e.*, retains the same value throughout a set of mathematical operation. These are generally denoted by a, b, c etc. There are two types of constant.

(i) **Absolute constant :** A constant which remains the same throughout a set of mathematical operation is known as absolute constant. All numerical numbers are absolute constants, *i.e.* $2, \sqrt{3}, \pi$ etc. are absolute constants.

(ii) **Arbitrary constant :** A constant which remains same in a particular operation, but changes with the change of reference, is called arbitrary constant *e.g.* $y = mx + c$ represents a line. Here m and c are constants, but they are different for different lines. Therefore, m and c are arbitrary constants.

(5) **Absolute value :** The absolute value of a number x, denoted by $|x|$, is a number that satisfies the conditions

$|x| = \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases}$. We also define $|x|$ as follows, $|x|$ = maximum $\{x, -x\}$ or $|x| = \sqrt{x^2}$

The properties of absolute value are

The inequality $|x| \le a$ means $-a \le x \le a$ (ii) The inequality $|x| \ge a$ means $x \ge a$ or $x \le -a$

(iii) $|x \pm y| \le |x| + |y|$ and $|x \pm y| \ge |x| - |y|$ (iv) $|xy| = |x|\,|y|$

(v) $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}, y \ne 0$

(6) **Greatest integer:** Let $x \in R$. Then $[x]$ denotes the greatest integer less than or equal to x; *e.g.* $[1.34]=1$, $[-4.57]=-5$, $[0.69]=0$ etc.

(7) **Fractional part :** We know that $x \geq [x]$. The difference between the number 'x' and it's integral value '$[x]$' is called the fractional part of x and is symbolically denoted as $\{x\}$. Thus, $\{x\} = x - [x]$

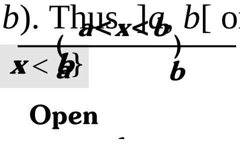
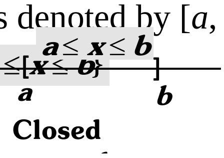
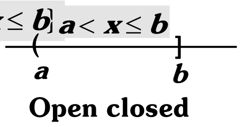
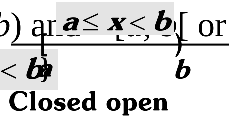
e.g., if $x = 4.92$ then $[x] = 4$ and $\{x\} = 0.92$.

Note : ❒ Fractional part of any number is always non-negative and less than one.

2.1.2 Intervals

If a variable x assumes any real value between two given numbers, say a and b ($a<b$) as its value, then x is called a continuous variable. The set of real numbers which lie between two specific numbers, is called the interval.

There are four types of interval:

(1)	**Open interval :** Let a and b be two real numbers such that $a<b$, then the set of all real numbers lying strictly between a and b is called an open interval and is denoted by $]a, b[$ or (a, b). Thus, $]a, b[$ or $(a, b) = \{x \in R: a < x < b\}$ 	(2)	**Closed interval :** Let a and b be two real numbers such that $a<b$, then the set of all real numbers lying between a and b including a and b is called a closed interval and is denoted by $[a, b]$. Thus, $[a, b] = \{x \in R: a \leq x \leq b\}$ 
(3)	**Open-Closed interval :** It is denoted by $]a, b]$ or $(a, b]$ and $]a, b]$ or $(a, b] = \{x \in R: a < x \leq b\}$ 	(4)	**Closed-Open interval :** It is denoted by $[a, b[$ or $[a, b)$ and $[a, b[$ or $[a, b) = \{x \in R: a \leq x < b\}$ 

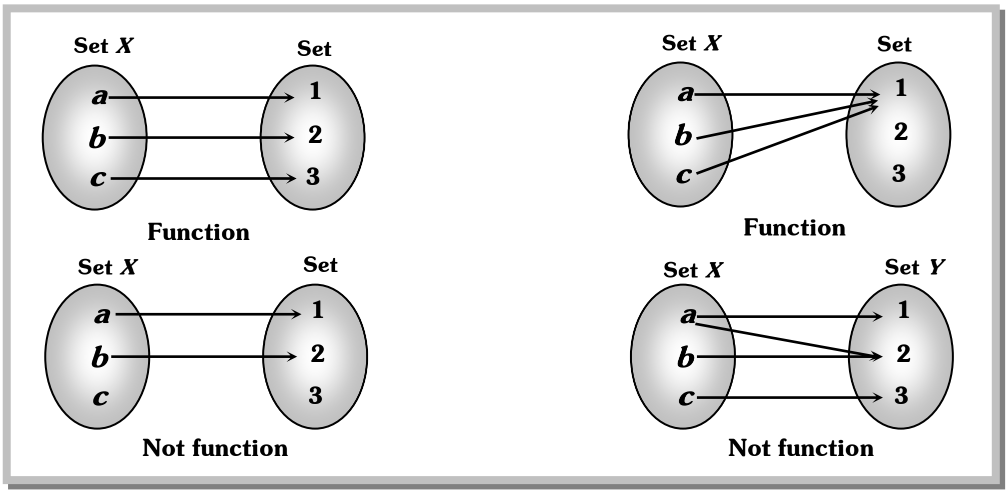
2.1.3 Definition of Function

(1) Function can be easily defined with the help of the concept of mapping. Let X and Y be any two non-empty sets. "A function from X to Y is a rule or correspondence that assigns to each element of set X, one and only one element of set Y". Let the correspondence be 'f' then mathematically we write $f: X \to Y$ where $y = f(x), x \in X$ and $y \in Y$. We say that 'y' is the image of 'x' under f (or x is the pre image of y).

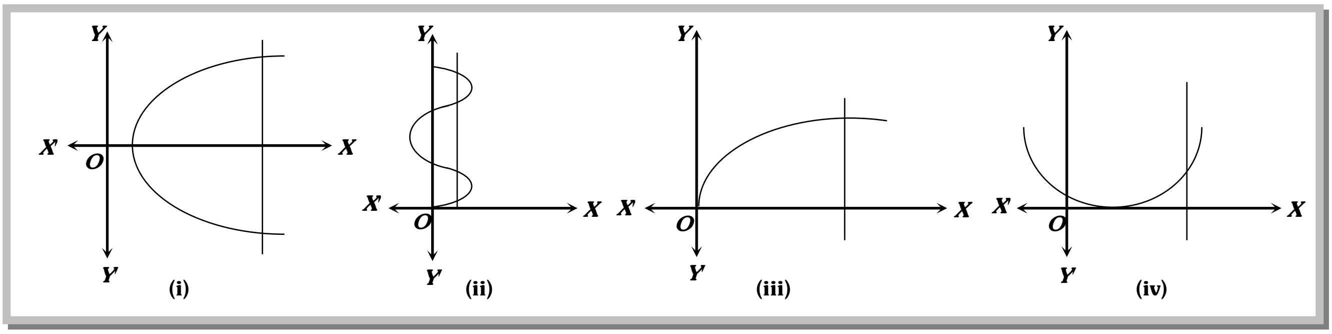
Two things should always be kept in mind:

(i) A mapping $f: X \to Y$ is said to be a function if each element in the set *X* has it's image in set *Y*. It is also possible that there are few elements in set *Y* which are not the images of any element in set *X*.

(ii) Every element in set *X* should have one and only one image. That means it is impossible to have more than one image for a specific element in set *X*. Functions can not be multi-valued (A mapping that is multi-valued is called a relation from *X* and *Y*) *e.g.*



(2) ***Testing for a function by vertical line test :*** *A relation* $f: A \to B$ *is a function or not it can be checked by a graph of the relation. If it is possible to draw a vertical line which cuts the given curve at more than one point then the given relation is not a function and when this vertical line means line parallel to* Y*-axis cuts the curve at only one point then it is a function. Figure (iii) and (iv) represents a function.*



(3) **Number of functions :** Let *X* and *Y* be two finite sets having *m* and *n* elements respectively. Then each element of set *X* can be associated to any one of *n* elements of set *Y*. So, total number of functions from set *X* to set *Y* is n^m.

(4) ***Value of the function :*** *If* $y = f(x)$ *is a function then to find its values at some value of* x*, say* $x = a$, *we directly substitute* x = a *in its given rule* $f(x)$ *and it is denoted by* $f(a)$.

e.g. If $f(x) = x^2 + 1$, then $f(1) = 1^2 + 1 = 2$, $f(2) = 2^2 + 1 = 5$, $f(0) = 0^2 + 1 = 1$ *etc.*

Example: 1 *If* A *contains 10 elements then total number of functions defined from* A *to* A *is*

(a) 10 (b) 2^{10} (c) 10^{10} (d) $2^{10}-1$

Solution: (c) According to formula, total number of functions = n^n

Here, n = 10. So, total number of functions = 10^{10}.

Example: 2 If $f(x)=\frac{x-|x|}{|x|}$, then $f(-1)=$

(a) 1 (b) – 2 (c) 0 (d) 2

Solution: (b) $f(-1)=\frac{-1-|-1|}{|-1|}=\frac{-1-1}{1}=-2$.

Example: 3 If $f(y)=\log y$, then $f(y)+f\left(\frac{1}{y}\right)$ is equal to

(a) 2 (b) 1 (c) 0 (d) – 1

Solution: (c) Given $f(y)=\log y \Rightarrow f(1/y)=\log(1/y)$, then $f(y)+f\left(\frac{1}{y}\right)=\log y+\log(1/y)=\log 1=0$.

Example: 4 If $f(x)=\log\left[\frac{1+x}{1-x}\right]$, then $f\left[\frac{2x}{1+x^2}\right]$ is equal to

(a) $[f(x)]^2$ (b) $[f(x)]^3$ (c) $2f(x)$ (d) $3f(x)$

Solution: (c) $f(x)=\log\left(\frac{1+x}{1-x}\right)$

$$\therefore f\left(\frac{2x}{1+x^2}\right)=\log\left[\frac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}\right]=\log\left[\frac{x^2+1+2x}{x^2+1-2x}\right]=\log\left[\frac{1+x}{1-x}\right]^2=2\log\left[\frac{1+x}{1-x}\right]=2f(x)$$

Example: 5 If $f(x)=\cos[\pi^2]x+\cos[-\pi^2]x$, then

(a) $f\left(\frac{\pi}{4}\right)=2$ (b) $f(-\pi)=2$ (c) $f(\pi)=1$ (d) $f\left(\frac{\pi}{2}\right)=-1$

Solution: (d) $f(x)=\cos[\pi^2]x+\cos[-\pi^2]x$

$$f(x)=\cos(9x)+\cos(-10x)=\cos(9x)+\cos(10x)=2\cos\left(\frac{19x}{2}\right)\cos\left(\frac{x}{2}\right)$$

$$f\left(\frac{\pi}{2}\right)=2\cos\left(\frac{19\pi}{4}\right)\cos\left(\frac{\pi}{4}\right);\quad f\left(\frac{\pi}{2}\right)=2\times\frac{-1}{\sqrt{2}}\times\frac{1}{\sqrt{2}}=-1.$$

Example: 6 If $f:R\to R$ satisfies $f(x+y)=f(x)+f(y)$, for all $x,y\in R$ and $f(1)=7$, then $\sum_{r=1}^{n}f(r)$ is

(a) $\frac{7n}{2}$ (b) $\frac{7(n+1)}{2}$ (c) $7n(n+1)$ (d) $\frac{7n(n+1)}{2}$

Solution: (d) $f(x+y)=f(x)+f(y)$

put $x=1,y=0 \Rightarrow f(1)=f(1)+f(0)=7$

put $x=1,y=1 \Rightarrow f(2)=2.f(1)=2.7$; similarly $f(3)=3.7$ and so on

$$\therefore \sum_{r=1}^{n}f(r)=7(1+2+3+.....+n)=\frac{7n(n+1)}{2}.$$

Example: 7 If $f(x)=\frac{1}{\sqrt{x+2\sqrt{2x-4}}}+\frac{1}{\sqrt{x-2\sqrt{2x-4}}}$ for $x>2$, then $f(11)=$

(a) $\frac{7}{6}$ (b) $\frac{5}{6}$ (c) $\frac{6}{7}$ (d) $\frac{5}{7}$

Solution: (c) $f(x)=\frac{1}{\sqrt{x+2\sqrt{2x-4}}}+\frac{1}{\sqrt{x-2\sqrt{2x-4}}}$

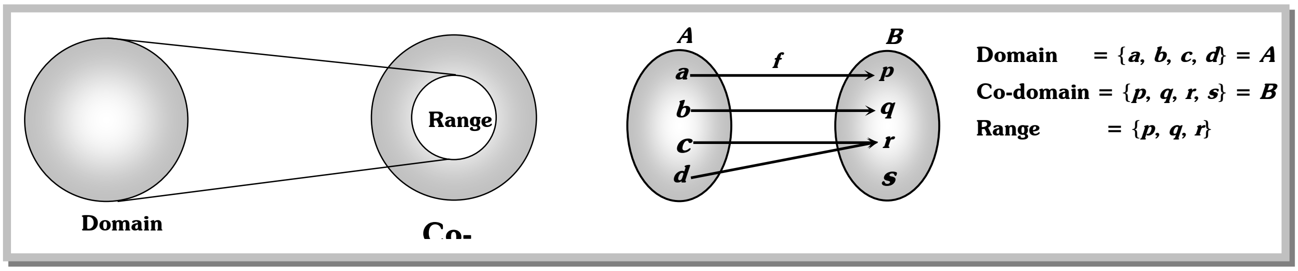
$$f(11)=\frac{1}{\sqrt{11+2\sqrt{18}}}+\frac{1}{\sqrt{11-2\sqrt{18}}}=\frac{1}{3+\sqrt{2}}+\frac{1}{3-\sqrt{2}}=\frac{3-\sqrt{2}}{7}+\frac{3+\sqrt{2}}{7}=\frac{6}{7}.$$

2.1.4 Domain, Co-domain and Range of Function

If a function f is defined from a set of A to set B then for $f: A \to B$ set A is called the domain of function f and set B is called the co-domain of function f. The set of all f-images of the elements of A is called the range of function f.

In other words, we can say Domain = All possible values of x for which $f(x)$ exists.

Range = For all values of x, all possible values of $f(x)$.



(1) Methods for finding domain and range of function

(i) Domain

Expression under even root (*i.e.*, square root, fourth root etc.) ≥ 0

Denominator $\neq 0$.

If domain of $y=f(x)$ and $y=g(x)$ are D_1 and D_2 respectively then the domain of $f(x)\pm g(x)$ or $f(x).g(x)$ is $D_1 \cap D_2$.

While domain of $\frac{f(x)}{g(x)}$ is $D_1 \cap D_2 - \{g(x)=0\}$.

Domain of $\left(\sqrt{f(x)}\right)=D_1 \cap \{x: f(x)\geq 0\}$

(ii) **Range :** Range of $y=f(x)$ is collection of all outputs $f(x)$ corresponding to each real number in the domain.

If domain $\in$ finite number of points $\Rightarrow$ range $\in$ set of corresponding $f(x)$ values.

If domain $\in R$ or R – [some finite points]. Then express x in terms of y. From this find y for x to be defined (*i.e.*, find the values of y for which x exists).

If domain $\in$ a finite interval, find the least and greatest value for range using monotonicity.

Important Tips

If f(x) is a given function of x and if a is in its domain of definition, then by f(a) it means the number obtained by replacing x by a in f(x) or the value assumed by f(x) when x = a.

Range is always a subset of co-domain.

Example: 8 Domain of the function $\frac{1}{\sqrt{x^2-1}}$ is

(a) $(-\infty,-1)\cup(1,\infty)$ (b) $(-\infty,-1]\cup(1,\infty)$ (c) $(-\infty,-1)\cup[1,\infty)$ (d) None of these

Solution: (a) For domain, $x^2-1>0 \Rightarrow (x-1)(x+1)>0$

$\Rightarrow x<-1$ or $x>1 \Rightarrow x\in(-\infty,-1)\cup(1,\infty)$.

Example: 9 The domain of the function $f(x)=\frac{1}{\sqrt{|x|-x}}$ is

(a) R^+ (b) R^- (c) R_0 (d) R

Solution: (b) For domain, $|x|-x>0 \Rightarrow |x|>x$. This is possible, only when $x\in R^-$.

Example: 10 Find the domain of definition of $f(x)=\frac{\log_2(x+3)}{x^2+3x+2}$

(a) $(-3,\infty)$ (b) $\{-1,-2\}$ (c) $(-3,\infty)-\{-1,-2\}$ (d) $(-\infty,\infty)$

Solution: (c) Here $f(x)=\frac{\log_2(x+3)}{x^2+3x+2}=\frac{\log_2(x+3)}{(x+1)(x+2)}$ exists if,

Numerator $x+3>0 \Rightarrow x>-3$ …... (i)

and denominator $(x+1)(x+2)\neq 0 \Rightarrow x\neq -1,-2$ …… (ii)

Thus, from (i) and (ii); we have domain of $f(x)$ is $(-3,\infty)-\{-1,-2\}$.

Example: 11 The domain of the function $f(x)=\sqrt{(2-2x-x^2)}$ is

(a) $-3\le x\le\sqrt{3}$ (b) $-1-\sqrt{3}\le x\le -1+\sqrt{3}$

(c) $-2\le x\le 2$ (d) None of these

Solution: (b) The quantity square root is positive, when $-1-\sqrt{3}\le x\le -1+\sqrt{3}$.

Example: 12 If the domain of function $f(x)=x^2-6x+7$ is $(-\infty,\infty)$, then the range of function is

(a) $(-\infty,\infty)$ (b) $[-2,\infty)$ (c) $(-2,3)$ (d) $(-\infty,-2)$

Solution: (b) $x^2-6x+7=(x-3)^2-2$ Obviously, minimum value is –2 and maximum ∞.

Example: 13 The domain of the function $f(x)=\sqrt{x-x^2}+\sqrt{4+x}+\sqrt{4-x}$ is

(a) $[-4,\infty)$ (b) $[-4,4]$ (c) $[0,4]$ (d) $[0,1]$

Solution: (d) $f(x)=\sqrt{x-x^2}+\sqrt{4+x}+\sqrt{4-x}$

clearly $f(x)$ is defined if

$4+x\geq 0 \Rightarrow x\geq -4$

$4-x\geq 0 \Rightarrow x\leq 4$

$x(1-x)\geq 0 \Rightarrow x\geq 0$ and $x\leq 1$

$\therefore$ Domain of $f=(-\infty,4]\cap[-4,\infty)\cap[0,1]=[0,1]$.

Example: 14 The domain of the function $\sqrt{\log(x^2-6x+6)}$ is

(a) $(-\infty,\infty)$ (b) $(-\infty,3-\sqrt{3})\cup(3+\sqrt{3},\infty)$

(c) $(-\infty,1]\cup[5,\infty)$ (d) $[0,\infty)$

Solution: (c) The function $f(x)=\sqrt{\log(x^2-6x+6)}$ is defined when $\log(x^2-6x+6)\geq 0$

$\Rightarrow x^2-6x+6\geq 1 \Rightarrow (x-5)(x-1)\geq 0$

This inequality hold if $x\leq 1$ or $x\geq 5$. Hence, the domain of the function will be $(-\infty,1]\cup[5,\infty)$.

Example: 15 The domain of definition of the function $y(x)$ given by $2^x+2^y=2$ is

(a) (0, 1] (b) [0, 1] (c) $(-\infty,0]$ (d) $(-\infty,1)$

Solution: (d) $2^y=2-2^x$

y is real if $2-2^x\geq 0 \Rightarrow 2>2^x \Rightarrow 1>x$

$x\in(-\infty,1)$

Example: 16 The domain of the function $f(x)=\sin^{-1}[\log_2(x/2)]$ is

(a) [1, 4] (b) [– 4, 1] (c) [– 1, 4] (d) None of these

Solution: (a) $f(x)=\sin^{-1}[\log_2(x/2)]$

Domain of $\sin^{-1}x$ is $x\in[-1,1]$

$\Rightarrow -1\leq \log_2(x/2)\leq 1 \Rightarrow \frac{1}{2}\leq\frac{x}{2}\leq 2 \Rightarrow 1\leq x\leq 4$

$\therefore x\in[1,4]$.

Example: 17 The domain of the derivative of the function $f(x)=\begin{cases}\tan^{-1}x & , |x|\leq 1\\ \frac{1}{2}(|x|-1) & , |x|>1\end{cases}$ is

(a) $R-\{0\}$ (b) $R-\{1\}$ (c) $R-\{-1\}$ (d) $R-\{-1,1\}$

Solution: (c) $f(x)=\begin{cases}\frac{1}{2}(-x-1), x<-1\\ \tan^{-1}x, -1\leq x\leq 1\\ \frac{1}{2}(x+1), x>1\end{cases} \Rightarrow f'(x)=\begin{cases}-\frac{1}{2}, & x<-1\\ \frac{1}{1+x^2}, -1<x<1\\ \frac{1}{2}, & x>1\end{cases}$

$f'(-1-0)=-\frac{1}{2};\ f'(-1+0)=\frac{1}{1+(-1+0)^2}=\frac{1}{2}$

$f'(1-0)=\frac{1}{1+(1-0)^2}=\frac{1}{2};\ f'(1+0)=\frac{1}{2}$

$\therefore f'(-1)$ does not exist. $\therefore$ domain of $f'(x)=R-\{-1\}$.

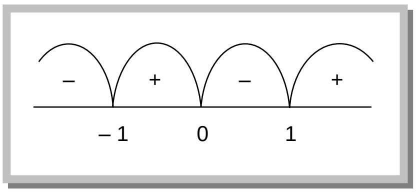
Example: 18 Domain of definition of the function $f(x)=\frac{3}{4-x^2}+\log_{10}(x^3-x)$, is

(a) (1, 2) (b) (–1, 0) ∪ (1, 2) (c) (1, 2) ∪ (2, ∞) (d) (–1, 0) ∪ (1, 2) ∪ (2, ∞)

Solution: (d) $f(x)=\frac{3}{4-x^2}+\log_{10}(x^3-x)$

So, $4-x^2\neq 0 \Rightarrow x\neq\pm\sqrt{4} \Rightarrow x\neq\pm 2$

and $x^3-x>0 \Rightarrow x(x^2-1)>0 \Rightarrow x>0, |x|>1$



$\therefore\ D=(-1,0)\cup(1,\infty)-\{2\}$

$D=(-1,0)\cup(1,2)\cup(2,\infty)$.

Example: 19 The domain of the function $f(x)=\log_{3+x}(x^2-1)$ is

(a) $(-3,-1)\cup(1,\infty)$ (b) $[-3,-1)\cup[1,\infty)$

(c) $(-3,-2)\cup(-2,-1)\cup(1,\infty)$ (d) $[-3,-2)\cup(-2,-1)\cup[1,\infty]$

Solution: (c) $f(x)$ is to be defined when $x^2-1>0$

$\Rightarrow x^2>1, \Rightarrow x<-1 \text{ or } x>1$ and $3+x>0$

$\therefore\ x>-3$ and $x\neq-2$

$\therefore\ D_f=(-3,-2)\cup(-2,-1)\cup(1,\infty)$.

Example: 20 Domain of definition of the function $f(x)=\sqrt{\sin^{-1}(2x)+\frac{\pi}{6}}$, for real value x, is

(a) $\left[-\frac{1}{4},\frac{1}{2}\right]$ (b) $\left[-\frac{1}{2},\frac{1}{2}\right]$ (c) $\left(-\frac{1}{2},\frac{1}{9}\right)$ (d) $\left[-\frac{1}{4},\frac{1}{4}\right]$

Solution: (a) $-\frac{\pi}{6}\leq\sin^{-1}(2x)\leq\frac{\pi}{2}\Rightarrow-\frac{1}{2}\leq 2x\leq 1 \Rightarrow x\in\left[-\frac{1}{4},\frac{1}{2}\right]$.

Example: 21 *The range of* $f(x)=\cos x-\sin x$, *is*

(a) $(-1,1)$ (b) $[-1,1)$ (c) $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ (d) $[-\sqrt{2},\sqrt{2}]$

Solution: (d) Let, $f(x)=\cos x-\sin x \Rightarrow f(x)=\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x\right) \Rightarrow f(x)=\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)$

Now since, $-1\leq\cos\left(x+\frac{\pi}{4}\right)\leq 1 \Rightarrow -\sqrt{2}\leq f(x)\leq\sqrt{2} \Rightarrow f(x)\in[-\sqrt{2},\sqrt{2}]$

Trick : Θ Maximum value of $\cos x-\sin x$ is $\sqrt{2}$ and minimum value of $\cos x-\sin x$ is $-\sqrt{2}$. Hence, range of $f(x)=[-\sqrt{2},\sqrt{2}]$.

Example: 22 *The range of* $\frac{1+x^2}{x^2}$ *is*

(a) $(0,1)$ (b) $(1,\infty)$ (c) [0, 1] (d) $[1,\infty)$

Solution: (b) Let $y=\frac{1+x^2}{x^2}$ $\Rightarrow x^2y=1+x^2 \Rightarrow x^2(y-1)=1 \Rightarrow x^2=\frac{1}{y-1}$

Now since, $x^2 > 0 \Rightarrow \frac{1}{y-1} > 0 \Rightarrow (y-1) > 0 \Rightarrow y > 1 \Rightarrow y \in (1, \infty)$

Trick : $y = \frac{1+x^2}{x^2} = 1 + \frac{1}{x^2}$. Now since, $\frac{1}{x^2}$ is always > 0 $\Rightarrow y > 1 \Rightarrow y \in (1, \infty)$.

Example: 23 For real values of *x*, range of the function $y = \frac{1}{2 - \sin 3x}$ is

(a) $\frac{1}{3} \le y \le 1$ (b) $-\frac{1}{3} \le y < 1$ (c) $-\frac{1}{3} > y > -1$ (d) $\frac{1}{3} > y > 1$

Solution: (a) $\because y = \frac{1}{2 - \sin 3x}$, $\therefore 2 - \sin 3x = \frac{1}{y} \Rightarrow \sin 3x = 2 - \frac{1}{y}$

Now since,

$-1 \le \sin 3x \le 1 \Rightarrow -1 \le 2 - \frac{1}{y} \le 1 \Rightarrow -3 \le -\frac{1}{y} \le -1 \Rightarrow 1 \le \frac{1}{y} \le 3 \Rightarrow \frac{1}{3} \le y \le 1$.

Example 24 If $f(x) = a\cos(bx + c) + d$, then range of $f(x)$ is

(a) $[d + a, d + 2a]$ (b) $[a - d, a + d]$ (c) $[d + a, a - d]$ (d) $[d - a, d + a]$

Solution: (d) $f(x) = a\cos(bx + c) + d$ ….. (i)

For minimum $\cos(bx + c) = -1$

from (i), $f(x) = -a + d = (d - a)$,

for maximum $\cos(bx + c) = 1$

from (i), $f(x) = a + d = (d + a)$

$\therefore$ Range of $f(x) = [d - a, d + a]$.

Example: 25 The range of the function $f(x) = \frac{x+2}{|x+2|}$ is

(a) {0, 1} (b) {– 1, 1} (c) *R* (d) $R - \{-2\}$

Solution: (b) $f(x) = \frac{x+2}{|x+2|} = \begin{cases} -1, & x < -2 \\ 1, & x > -2 \end{cases}$

$\therefore$ Range of $f(x)$ is $\{-1, 1\}$.

Example: 26 The range of $f(x) = \sec\left(\frac{\pi}{4}\cos^2 x\right), -\infty < x < \infty$ is

(a) $[1, \sqrt{2}]$ (b) $[1, \infty)$ (c) $[-\sqrt{2}, -1] \cup [1, \sqrt{2}]$ (d) $(-\infty, -1] \cup [1, \infty)$

Solution: (a) $f(x) = \sec\left(\frac{\pi}{4}\cos^2 x\right)$

We know that, $0 \le \cos^2 x \le 1$ at $\cos x = 0$, $f(x) = 1$ and at $\cos x = 1$, $f(x) = \sqrt{2}$

$\therefore 1 \le x \le \sqrt{2} \Rightarrow x \in [1, \sqrt{2}]$.

Example: 27 Range of the function $f(x) = \frac{x^2 + x + 2}{x^2 + x + 1}; x \in R$ is

(a) $(1, \infty)$ (b) $(1, 11/7)$ (c) $(1, 7/3]$ (d) $(1, 7/5]$

Solution: (c) $f(x) = 1 + \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} \Rightarrow$ Range $= (1, 7/3]$.

2.1.5 ALGEBRA OF FUNCTIONS

Let $f(x)$ and $g(x)$ be two real and single-valued functions, with domains X_f, X_g and ranges Y_f and Y_g respectively. Let $X = X_f \cap X_g \neq \phi$. Then, the following operations are defined.

(1) **Scalar multiplication of a function :** $(c\,f)(x) = c\,f(x)$, where *c* is a scalar. The new function $c\,f(x)$ has the domain X_f.

(2) **Addition/subtraction of functions :** $(f \pm g)(x) = f(x) \pm g(x)$. The new function has the domain *X*.

(3) **Multiplication of functions :** $(fg)(x) = (g\,f)(x) = f(x)g(x)$. The product function has the domain *X*.

(4) **Division of functions :**

(i) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$. The new function has the domain *X*, except for the values of *x* for which $g(x) = 0$.

(ii) $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)}$. The new function has the domain *X*, except for the values of *x* for which $f(x) = 0$.

(5) **Equal functions :** Two function *f* and *g* are said to be equal functions, if and only if

(i) Domain of *f* = domain of *g*

(ii) Co-domain of *f* = co-domain of *g*

(iii) $f(x) = g(x) \forall x \in$ their common domain

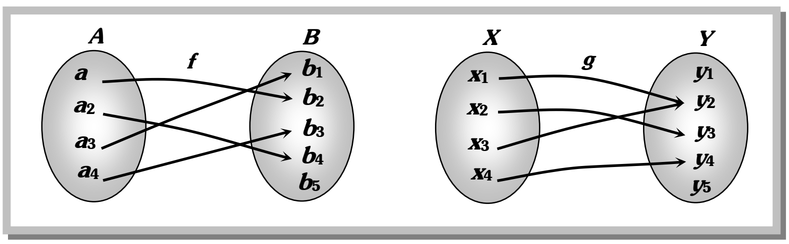
(6) **Real valued function :** If *R*, be the set of real numbers and *A*, *B* are subsets of *R*, then the function $f: A \rightarrow B$ is called a real function or real –valued function.

2.1.6 KINDS OF FUNCTION

(1) **One-one function (injection) :** A function $f: A \rightarrow B$ is said to be a one-one function or an injection, if different elements of *A* have different images in *B*. Thus, $f: A \rightarrow B$ is one-one.

$\Leftrightarrow a \neq b \Rightarrow f(a) \neq f(b)$ for all $a, b \in A$ $\Leftrightarrow f(a) = f(b) \Rightarrow a = b$ for all $a, b \in A$.

e.g. Let $f: A \rightarrow B$ and $g: X \rightarrow Y$ be two functions represented by the following diagrams.



Clearly, $f: A \to B$ is a one-one function. But $g: X \to Y$ is not one-one function because two distinct elements x_1 and x_3 have the same image under function *g*.

(i) **Method to check the injectivity of a function**

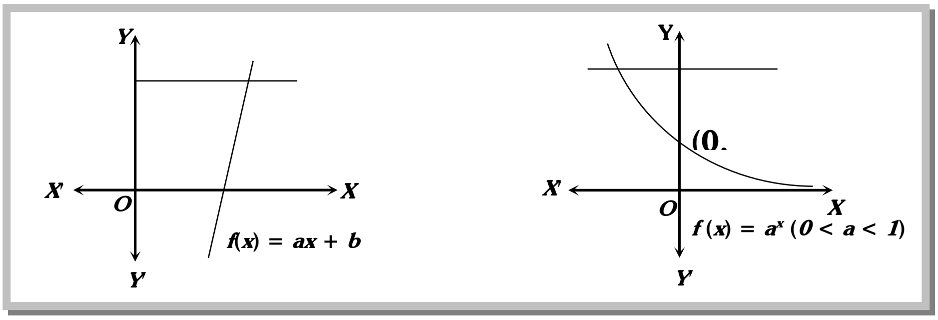
Step I : Take two arbitrary elements *x*, *y* (say) in the domain of *f*.

Step II : Put $f(x) = f(y)$.

Step III : Solve $f(x) = f(y)$. If $f(x) = f(y)$ gives *x* = *y* only, then $f: A \to B$ is a one-one function (or an injection). Otherwise not.

Note : ❒ If function is given in the form of ordered pairs and if two ordered pairs do not have same second element then function is one-one.

❒ If the graph of the function $y = f(x)$ is given and each line parallel to *x*-axis cuts the given curve at maximum one point then function is one-one. *e.g.*

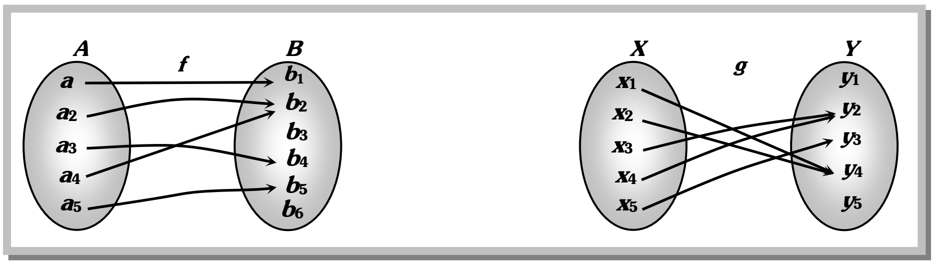


(ii) **Number of one-one functions (injections) :** If *A* and *B* are finite sets having *m* and *n* elements respectively, then number of one-one functions from *A* to *B* $= \begin{cases} {}^nP_m, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}$

(2) **Many-one function :** A function $f: A \to B$ is said to be a many-one function if two or more elements of set *A* have the same image in *B*.

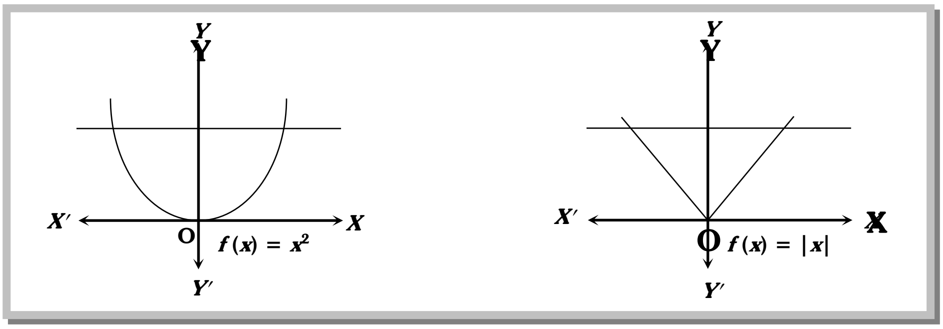
Thus, $f: A \to B$ is a many-one function if there exist $x, y \in A$ such that $x \neq y$ but $f(x) = f(y)$.

In other words, $f: A \to B$ is a many-one function if it is not a one-one function.



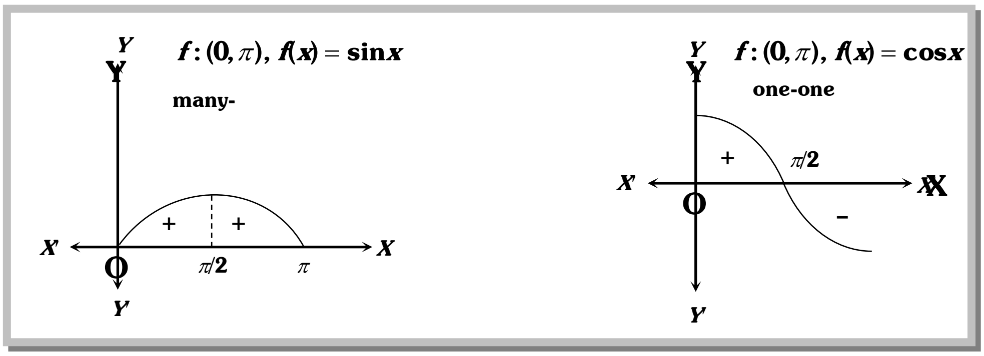
Note : ❒ If function is given in the form of set of ordered pairs and the second element of atleast two ordered pairs are same then function is many-one.

❒ If the graph of $y = f(x)$ is given and the line parallel to *x*-axis cuts the curve at more than one point then function is many-one.



If the domain of the function is in one quadrant then the trigonometrical functions are always one-one.

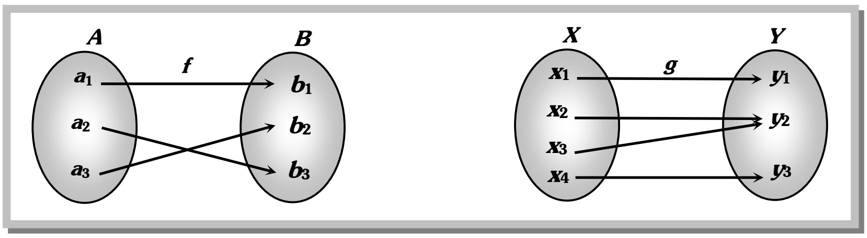
If trigonometrical function changes its sign in two consecutive quadrants then it is one-one but if it does not change the sign then it is many-one.



❒ In three consecutive quadrants trigonometrical functions are always many-one.

(3) **Onto function (surjection) :** A function $f : A \to B$ is onto if each element of *B* has its pre-image in *A*. Therefore, if $f^{-1}(y) \in A, \forall y \in B$ then function is onto. In other words, Range of *f* = Co-domain of *f*.

e.g. The following arrow-diagram shows onto function.

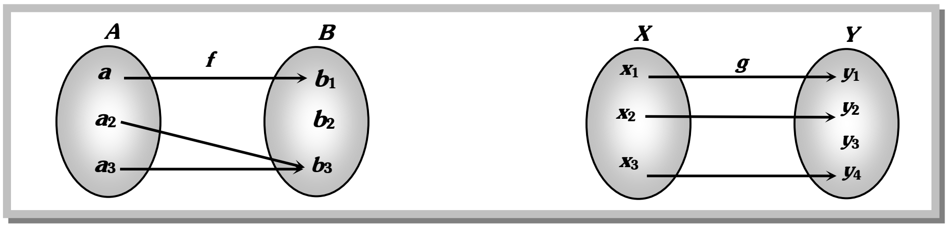


(i) **Number of onto function (surjection) :** If *A* and *B* are two sets having *m* and *n* elements respectively such that $1 \le n \le m$, then number of onto functions from *A* to *B* is $\sum_{r=1}^{n} (-1)^{n-r}\, {}^nC_r\, r^m$.

(4) **Into function :** *A* function $f : A \to B$ is an into function if there exists an element in *B* having no pre-image in *A*.

In other words, $f : A \to B$ is an into function if it is not an onto function.

e.g. The following arrow-diagram shows into function.



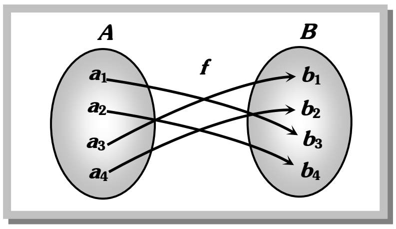
(i) **Method to find onto or into function**

(a) If range = co-domain, then $f(x)$ is onto and if range is a proper subset of the co-domain, then $f(x)$ is into.

(b) Solve $f(x) = y$ by taking x as a function of y *i.e.*, $g(y)$ (say).

(c) Now if $g(y)$ is defined for each $y \in$ co-domain and $g(y) \in$ domain for $y \in$ co-domain, then $f(x)$ is onto and if any one of the above requirements is not fulfilled, then $f(x)$ is into.

(5) **One-one onto function (bijection) :** A function $f: A \to B$ is a bijection if it is one-one as well as onto.



In other words, a function $f: A \to B$ is a bijection if

(i) It is one-one i.e., $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in A$.

(ii) It is onto i.e., for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.

Clearly, f is a bijection since it is both injective as well as surjective.

Number of one-one onto function (bijection) : If A and B are finite sets and $f: A \to B$ is a bijection, then A and B have the same number of elements. If A has n elements, then the number of bijection from A to B is the total number of arrangements of n items taken all at a time *i.e.* $n!$.

(6) **Algebraic functions :** Functions consisting of finite number of terms involving powers and roots of the independent variable and the four fundamental operations +, –, × and ÷ are called algebraic functions.

e.g., (i) $x^{\frac{3}{2}} + 5x$ (ii) $\frac{\sqrt{x+1}}{x-1}, x \neq 1$ (iii) $3x^4 - 5x + 7$

The algebraic functions can be classified as follows:

(i) **Polynomial or integral function :** It is a function of the form $a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$,

where $a_0 \neq 0$ and $a_0, a_1, ..., a_n$ are constants and $n \in N$ is called a polynomial function of degree n

e.g. $f(x) = x^3 - 2x^2 + x + 3$ is a polynomial function.

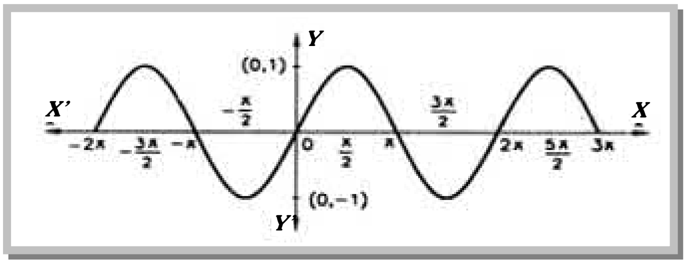
Note : ❐ The polynomial of first degree is called a linear function and polynomial of zero degree is called a constant function.

(ii) **Rational function :** The quotient of two polynomial functions is called the rational function. *e.g.* $f(x) = \frac{x^2 - 1}{2x^3 + x^2 + 1}$ is a rational function.

(iii) **Irrational function :** An algebraic function which is not rational is called an irrational function. *e.g.* $f(x) = x + \sqrt{x} + 6,\ g(x) = \dfrac{x^3 - \sqrt{x}}{1 + x^{1/4}}$ are irrational functions.

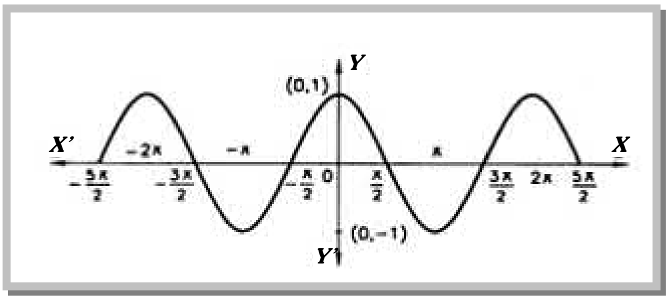
(7) **Transcendental function :** A function which is not algebraic is called a transcendental function. *e.g.*, trigonometric; inverse trigonometric , exponential and logarithmic functions are all transcendental functions.

(i) **Trigonometric functions :** A function is said to be a trigonometric function if it involves circular functions (sine, cosine, tangent, cotangent, secant, cosecant) of variable angles.

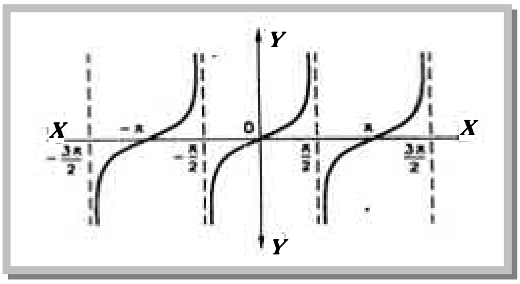


(a) **Sine function :** The function that associates to each real numbers *x* to $\sin x$ is called the sine function. Here *x* is the radian measure of the angle. The domain of the sine function is *R* and the range is [–1, 1].

(b) **Cosine function:** The function that associates to each real number *x* to $\cos x$ is called the cosine function. Here *x* is the radian measure of the angle. The domain of the cosine function is *R* and the range is [– 1, 1].

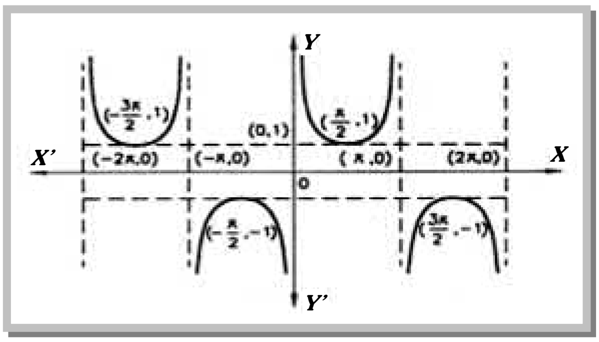


(c) **Tangent function :** The function that associates a real number *x* to tan *x* is called the tangent function.



Clearly, the tangent function is not defined at odd multiples of $\dfrac{\pi}{2}$ *i.e.,* $\pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}$ etc. So, the domain of the tangent function is $R - \{(2n+1)\dfrac{\pi}{2} \mid n \in I\}$. Since it takes every value between $-\infty$ and ∞. So, the range is *R*. Graph of $f(x) = \tan x$ is shown in figure.

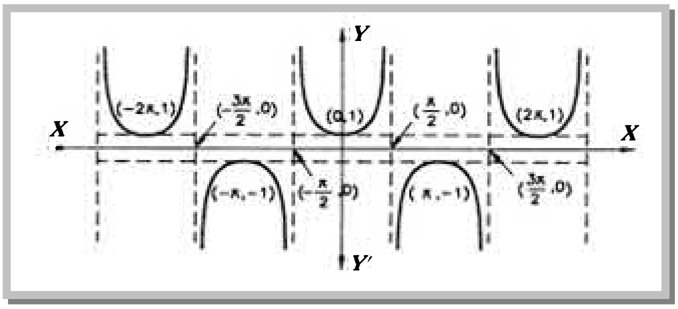
(d) **Cosecant function :** The function that associates a real number *x* to cosec*x* is called the cosecant function.



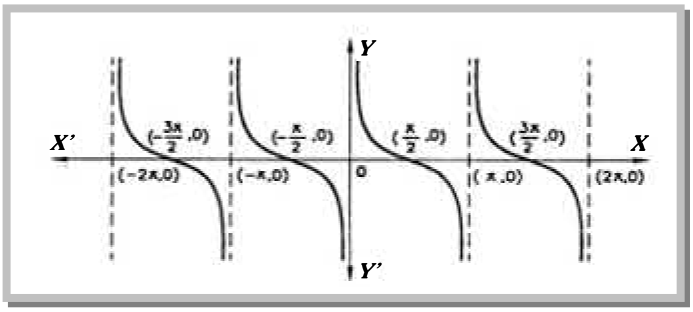
Clearly, cosec *x* is not defined at $x = n\pi, n \in I$. *i.e.,* $0, \pm\pi, \pm 2\pi, \pm 3\pi$ etc. So, its domain is $R - \{n\pi \mid n \in I\}$. Since $\text{cosec}\, x \geq 1$ or $\text{cosec}\, x \leq -1$. Therefore, range is $(-\infty, -1] \cup [1, \infty)$. Graph of $f(x) = \text{cosec}\, x$ is shown in figure.

(e) **Secant function :** The function that associates a real number *x* to sec *x* is called the secant function.

Clearly, sec *x* is not defined at odd multiples of $\frac{\pi}{2}$ *i.e.*, $(2\pi+1)\frac{\pi}{2}$, where $n \in I$. So, its domain is $R - \{(2n+1)\frac{\pi}{2} \mid n \in I\}$. Also, $|\sec x| \geq 1$, therefore its range is $(-\infty, -1] \cup [1, \infty)$. Graph of $f(x) = \sec x$ is shown in figure.



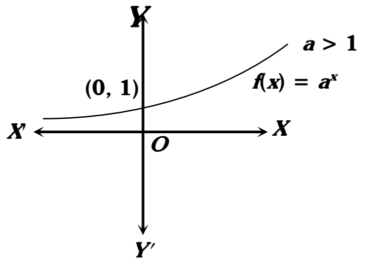
(f) **Cotangent function :** The function that associates a real number *x* to cot *x* is called the cotangent function. Clearly, cot *x* is not defined at $x = n\pi, n \in I$ i.e., at $n = 0, \pm\pi, \pm 2\pi$ etc. So, domain of $\cot x$ is $R - \{n\pi \mid n \in I\}$. The range of $f(x) = \cot x$ is *R* as is evident from its graph in figure.



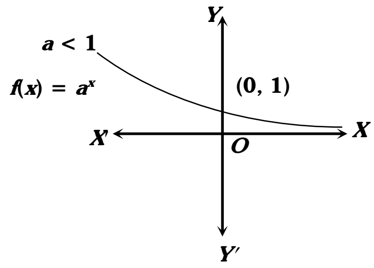
(ii) **Inverse trigonometric functions**

Function	**Domain**	**Range**	**Definition of the function**
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$	$y = \sin^{-1} x \Leftrightarrow x = \sin y$
$\cos^{-1} x$	[–1, 1]	[0, π]	$y = \cos^{-1} x \Leftrightarrow x = \cos y$
$\tan^{-1} x$	(–∞, ∞) or *R*	(–π/2, π/2)	$y = \tan^{-1} x \Leftrightarrow x = \tan y$
$\cot^{-1} x$	(–∞, ∞) or *R*	(0, π)	$y = \cot^{-1} x \Leftrightarrow x = \cot y$
$\text{cosec}^{-1} x$	*R* – (–1, 1)	$[-\pi/2, \pi/2] - \{0\}$	$y = \text{cosec}^{-1} x \Leftrightarrow x = \text{cosec}\, y$
$\sec^{-1} x$	*R* – (–1, 1)	$[0, \pi] - [\pi/2]$	$y = \sec^{-1} x \Leftrightarrow x = \sec y$

(iii) **Exponential function :** Let $a \neq 1$ be a positive real number. Then $f : R \to (0, \infty)$ defined by $f(x) = a^x$ is called exponential function. Its domain is *R* and range is $(0, \infty)$.

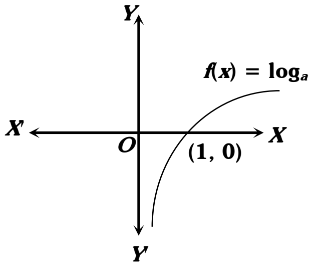


graph of $f(x) = a^x$, when $a > 1$

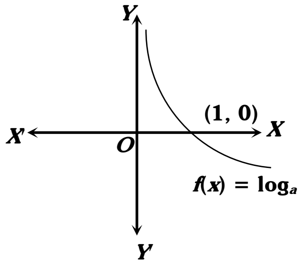


graph of $f(x) = a^x$, when $a < 1$

(iv) **Logarithmic function :** Let $a \neq 1$ be a positive real number. Then $f:(0,\infty) \to R$ defined by $f(x) = \log_a x$ is called logarithmic function. Its domain is $(0,\infty)$ and range is *R*.



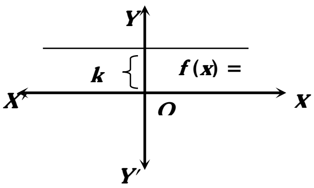
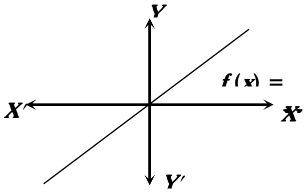
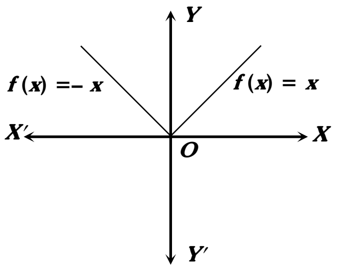
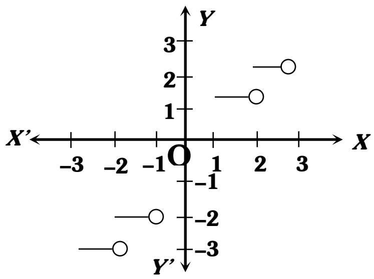
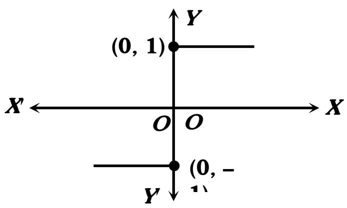
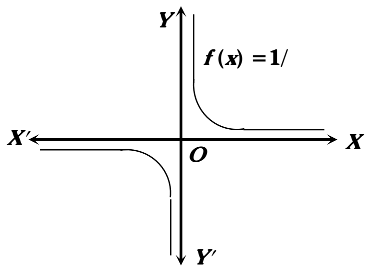
graph of $f(x) = \log_a x$, when $a > 1$



graph of $f(x) = \log_a x$, when $a <$

(8) **Explicit and implicit functions :** A function is said to be explicit if it can be expressed directly in terms of the independent variable. If the function can not be expressed directly in terms of the independent variable or variables, then the function is said to be implicit. *e.g.* $y = \sin^{-1} x + \log x$ is explicit function, while $x^2 + y^2 = xy$ and $x^3 y^2 = (a - x)^2 (b - y)^2$ are implicit functions.

(9) **Constant function :** Let *k* be a fixed real number. Then a function *f*(*x*) given by $f(x) = k$ for all $x \in R$ is called a constant function. The domain of the constant function $f(x) = k$ is the complete set of real numbers and the range of *f* is the singleton set {*k*}. The graph of a constant function is a straight line parallel to *x*-axis as shown in figure and it is above or below the *x*-axis according as *k* is positive or negative. If *k* = 0, then the straight line coincides with *x*-axis.	(10) **Identity function :** The function defined by $f(x) = x$ for all $x \in R$, is called the identity function on *R*. Clearly, the domain and range of the identity function is *R*. The graph of the identity function is a straight line passing through the origin and inclined at an angle of 45° with positive direction of *x*-axis.

	
(11) **Modulus function :** The function defined by $f(x) = \|x\| = \begin{cases} x, \text{ when } x \geq 0 \\ -x, \text{ when } x < 0 \end{cases}$ is called the modulus function. The domain of the modulus function is the set *R* of all real numbers and the range is the set of all non-negative real numbers. 	(12) **Greatest integer function:** Let $f(x) = [x]$, where [*x*] denotes the greatest integer less than or equal to *x*. The domain is *R* and the range is *I*. *e.g*. [1.1] = 1, [2.2] = 2, [– 0.9] = –1, [– 2.1] = – 3 etc. The function *f* defined by $f(x) = [x]$ for all $x \in R$, is called the greatest integer function. 
(13) **Signum function :** The function defined by $f(x) = \begin{cases} \frac{\|x\|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ or $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$ is called the signum function. The domain is *R* and the range is the set {–1, 0, 1}. 	(14) **Reciprocal function:** The function that associates each non-zero real number x to be reciprocal $\frac{1}{x}$ is called the reciprocal function. The domain and range of the reciprocal function are both equal to $R - \{0\}$ *i.e.*, the set of all non-zero real numbers. The graph is as shown. 

Domain and Range of Some Standard Functions

Function	Domain	Range
Polynomial function	R	R
Identity function x	R	R
Constant function K	R	$\{K\}$
Reciprocal function $\frac{1}{x}$	R_0	R_0
$x^2, \lvert x \rvert$	R	$R^+ \cup \{0\}$
$x^3, x\lvert x \rvert$	R	R
Signum function	R	$\{-1, 0, 1\}$
$x + \lvert x \rvert$	R	$R^+ \cup \{0\}$
$x - \lvert x \rvert$	R	$R^- \cup \{0\}$
$[x]$	R	I
$x - [x]$	R	$[0, 1)$
$\sqrt{x}$	$[0, \infty)$	R
a^x	R	R^+
$\log x$	R^+	R
$\sin x$	R	$[-1, 1]$
$\cos x$	R	$[-1, 1]$
$\tan x$	$R - \left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots\ldots\right\}$	R
$\cot x$	$R - \{0, \pm\pi, \pm 2\pi, \ldots\ldots\ldots\}$	R
$\sec x$	$R - \left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots\ldots\ldots\right\}$	$R - (-1, 1)$
$\operatorname{cosec} x$	$R - \{0, \pm\pi, \pm 2\pi, \ldots\ldots\ldots\}$	$R - (-1, 1)$
$\sin^{-1} x$	$[-1, 1]$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	R	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1} x$	R	$(0, \pi)$
$\sec^{-1} x$	$R - (-1, 1)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$\operatorname{cosec}^{-1} x$	$R - (-1, 1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

Important Tips

Any function, which is entirely increasing or decreasing in the whole of a domain, is one-one.

Any continuous function f(x), which has at least one local maximum or local minimum, is many-one.

If any line parallel to the x-axis cuts the graph of the function at most at one point, then the function is one-one and if there exists a line which is parallel to the x-axis and cuts the graph of the function in at least two points, then the function is many-one.

Any polynomial function $f : R \to R$ is onto if degree of f is odd and into if degree of f is even.

An into function can be made onto by redefining the co-domain as the range of the original function.

Example: 28 Function $f : N \to N, f(x) = 2x + 3$ is

(a) One-one onto (b) One-one into (c) Many-one onto (d) Many –one into

Solution: (b) f is one-one because $f(x_1) = f(x_2) \Rightarrow 2x_1 + 3 = 2x_2 + 3 \Rightarrow x_1 = x_2$

Further $f^{-1}(x) = \frac{x-3}{2} \notin N$(domain) when x = 1, 2, 3 etc.

$\therefore$ f is into which shows that f is one-one into.

Example: 29 The function $f : R \to R$ defined by $f(x) = (x-1)(x-2)(x-3)$ is

(a) One-one but not onto (b)Onto but not one-one

(c) Both one-one and onto (d)Neither one-one nor onto

Solution: (b) We have $f(x) = (x-1)(x-2)(x-3) \Rightarrow f(1) = f(2) = f(3) = 0 \Rightarrow f(x)$ is not one-one

For each $y \in R$, there exists $x \in R$ such that $f(x) = y$. Therefore f is onto.

Hence, $f : R \to R$ is onto but not one-one.

Example: 30*Find number of surjection from* A *to* B *where* $A = \{1, 2, 3, 4\}, B = \{a, b\}$

(a) 13 (b) 14 (c) 15 (d) 16

Solution: (b) Number of surjection from A to B $= \sum_{r=1}^{2} (-1)^{2-r}\, {}^2C_r (r)^4$

$= (-1)^{2-1}\, {}^2C_1 (1)^4 + (-1)^{2-2}\, {}^2C_2 (2)^4 = -2 + 16 = 14$

Therefore, number of surjection from A to B = 14.

Trick : Total number of functions from A to B is 2^4 of which two function $f(x) = a$ for all $x \in A$ and

$g(x) = b$ for all $x \in A$ are not surjective. Thus, total number of surjection from A to B

$= 2^4 - 2 = 14$.

Example: 31 If $A = \{a, b, c\}$, then total number of one-one onto functions which can be defined from A to A is

(a) 3 (b) 4 (c) 9 (d) 6

Solution: (d) Total number of one-one onto functions = $3!$

Example: 32 If $f:R\to R$, then $f(x)=|x|$ is

(a) One-one but not onto (b) Onto but not one-one

(c) One-one and onto (d) None of these

Solution: (d) $f(-1)=f(1)=1$ $\therefore$ function is many-one function.

Obviously, *f* is not onto so *f* is neither one-one nor onto.

Example: 33 Let $f:R\to R$ be a function defined by $f(x)=\frac{x-m}{x-n}$, where $m\neq n$. Then

(a) *f* is one-one onto (b) *f* is one-one into (c) *f* is many one onto (d) *f* is many one into

Solution: (b) For any $x,y\in R$, we have

$$f(x)=f(y)\Rightarrow\frac{x-m}{x-n}=\frac{y-m}{y-n}\Rightarrow x=y$$

$\therefore f$ is one-one

Let $\alpha\in R$ such that $f(x)=\alpha\Rightarrow\frac{x-m}{x-n}=\alpha\;\Rightarrow\;x=\frac{m-n\alpha}{1-\alpha}$

Clearly $x\notin R$ for $\alpha=1$. So, *f* is not onto.

Example: 34 The function $f:R\to R$ defined by $f(x)=e^x$ is

(a) Onto (b) Many-one (c) One-one and into (d) Many one and onto

Solution: (c) Function $f:R\to R$ is defined by $f(x)=e^x$. Let $x_1,x_2\in R$ and $f(x_1)=f(x_2)$ or $e^{x_1}=e^{x_2}$ or $x_1=x_2$. Therefore *f* is one-one. Let $f(x)=e^x=y$. Taking log on both sides, we get $x=\log y$. We know that negative real numbers have no pre-image or the function is not onto and zero is not the image of any real number. Therefore function *f* is into.

Example: 35 A function *f* from the set of natural numbers to integers defined by

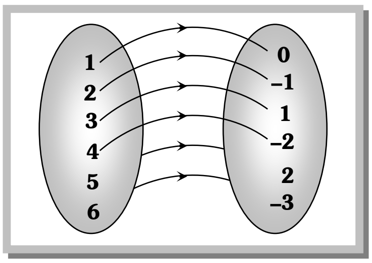
$$f(n)=\begin{cases}\frac{n-1}{2}, & \text{when } n \text{ is odd}\\ -\frac{n}{2}, & \text{when } n \text{ is even}\end{cases}$$, is

(a) One-one but not onto (b)Onto but not one-one

(c) One-one and onto both (d)Neither one-one nor onto

Solution: (c) $f:N\to I$

$f(1)=0, f(2)=-1, f(3)=1, f(4)=-2, f(5)=2$ and $f(6)=-3$ so on.



In this type of function every element of set *A* has unique image in set *B* and there is no element left in set *B*. Hence *f* is one-one and onto function.

2.1.7 Even and Odd function

(1) **Even function :** If we put (–*x*) in place of *x* in the given function and if $f(-x) = f(x)$, $\forall x \in$ domain then function *f*(*x*) is called even function. *e.g.* $f(x) = e^x + e^{-x}$, $f(x) = x^2$, $f(x) = x\sin x$, $f(x) = \cos x$, $f(x) = x^2\cos x$ all are even function.

(2) **Odd function :** If we put (–*x*) in place of *x* in the given function and if $f(-x) = -f(x)$, $\forall x \in$ domain then *f*(*x*) is called odd function. *e.g.* $f(x) = e^x - e^{-x}$, $f(x) = \sin x$, $f(x) = x^3$, $f(x) = x\cos x$, $f(x) = x^2\sin x$ all are odd function.

Important Tips

The graph of even function is always symmetric with respect to y-axis.

The graph of odd function is always symmetric with respect to origin.

The product of two even functions is an even function.

The sum and difference of two even functions is an even function.

The sum and difference of two odd functions is an odd function.

The product of two odd functions is an even function.

The product of an even and an odd function is an odd function

It is not essential that every function is even or odd. It is possible to have some functions which are neither even nor odd function. e.g. $f(x) = x^2 + x^3$, $f(x) = \log_e x$, $f(x) = e^x$.

The sum of even and odd function is neither even nor odd function.

Zero function f(x) = 0 is the only function which is even and odd both.

Example: 36 Which of the following is an even function

(a) $x\left(\frac{a^x - 1}{a^x + 1}\right)$ (b) $\tan x$ (c) $\frac{a^x - a^{-x}}{2}$ (d) $\frac{a^x + 1}{a^x - 1}$

Solution: (a) We have : $f(x) = x\left(\frac{a^x - 1}{a^x + 1}\right)$

$$f(-x) = -x\left(\frac{a^{-x} - 1}{a^{-x} + 1}\right) = -x\left(\frac{\frac{1}{a^x} - 1}{\frac{1}{a^x} + 1}\right) = -x\left(\frac{1 - a^x}{1 + a^x}\right) = x\left(\frac{a^x - 1}{a^x + 1}\right) = f(x)$$

So, $f(x)$ is an even function.

Example: 37 *Let* $f(x) = \sqrt{x^4 + 15}$, *then the graph of the function* $y = f(x)$ *is symmetrical about*

(a) The *x*-axis (b) The *y*-axis (c) The origin (d) The line $x = y$

Solution: (b) $f(x)=\sqrt{x^4+15} \Rightarrow f(-x)=\sqrt{(-x)^4+15}=\sqrt{x^4+15}=f(x)$

$\Rightarrow f(-x)=f(x) \Rightarrow f(x)$ is an even function $\Rightarrow f(x)$ is symmetric about y-axis.

Example: 38 The function $f(x)=\log(x+\sqrt{x^2+1})$ is

(a) An even function (b) An odd function (c) Periodic function (d) None of these

Solution: (b) $f(x)=\log(x+\sqrt{x^2+1})$ and $f(-x)=-\log(x+\sqrt{x^2+1})=-f(x)$, so $f(x)$ is an odd function.

Example: 39 Which of the following is an even function

(a) $f(x)=\frac{a^x+1}{a^x-1}$ (b) $f(x)=x\left(\frac{a^x-1}{a^x+1}\right)$ (c) $f(x)=\frac{a^x-a^{-x}}{a^x+a^{-x}}$ (d) $f(x)=\sin x$

Solution: (b) In option (a), $f(-x)=\frac{a^{-x}+1}{a^{-x}-1}=\frac{1+a^x}{1-a^x}=-\frac{a^x+1}{a^x-1}=-f(x)$ So, It is an odd function.

In option (b), $f(-x)=(-x)\frac{a^{-x}-1}{a^{-x}+1}=-x\frac{(1-a^x)}{1+a^x}=x\frac{(a^x-1)}{(a^x+1)}=f(x)$ So, It is an even function.

In option (c), $f(-x)=\frac{a^{-x}-a^x}{a^{-x}+a^x}=-f(x)$ So, It is an odd function.

In option (d), $f(-x)=\sin(-x)=-\sin x=-f(x)$ So, It is an odd function.

Example: 40 The function $f(x)=\sin\left(\log(x+\sqrt{x^2+1})\right)$ is

(a) Even function (b) Odd function (c) Neither even nor odd (d) Periodic function

Solution: (b) $f(x)=\sin\left(\log(x+\sqrt{1+x^2})\right)$

$\Rightarrow f(-x)=\sin[\log(-x+\sqrt{1+x^2})] \Rightarrow f(-x)=\sin\log\left((\sqrt{1+x^2}-x)\frac{(\sqrt{1+x^2}+x)}{(\sqrt{1+x^2}+x)}\right)$

$\Rightarrow f(-x)=\sin\log\left[\frac{1}{(x+\sqrt{1+x^2})}\right] \Rightarrow f(-x)=\sin\left[\log(x+\sqrt{1+x^2})^{-1}\right]$

$\Rightarrow f(-x)=\sin\left[-\log(x+\sqrt{1+x^2})\right] \Rightarrow f(-x)=-\sin\left[\log(x+\sqrt{1+x^2})\right] \Rightarrow f(-x)=-f(x)$

$\therefore f(x)$ is odd function.

2.1.8 Periodic Function

A function is said to be periodic function if its each value is repeated after a definite interval. So a function $f(x)$ will be periodic if a positive real number T exist such that, $f(x+T)=f(x), \forall x \in$ domain. Here the least positive value of T is called the period of the function. Clearly $f(x)=f(x+T)=f(x+2T)=f(x+3T)=......$ *e.g.* $\sin x, \cos x, \tan x$ are periodic functions with period $2\pi, 2\pi$ and π respectively.

Some standard results on periodic functions

Functions	**Periods**
(1) $\sin^n x, \cos^n x, \sec^n x, \operatorname{cosec}^n x$	$\begin{cases} \pi; \text{ if } n \text{ is even} \\ 2\pi; \text{ if } n \text{ is odd or fraction} \end{cases}$
(2) $\tan^n x, \cot^n x$	$\pi; n$ is even or odd.
(3) $\|\sin x\|, \|\cos x\|, \|\tan x\|, \|\cot x\|, \|\sec x\|, \|\operatorname{cosec} x\|$	π
(4) $x - [x]$	1
(5) Algebraic functions e.g., $\sqrt{x}, x^2, x^3 + 5, \dots \text{etc}$	Period does not exist

Important Tips

If f(x) is periodic with period T, then c.f(x) is periodic with period T, f(x + c) is periodic with period T and f(x) ± c is periodic with period T. where c is any constant.

If a function f(x) has a period T, then the function f(ax+b) will have a period $\frac{T}{|a|}$.

If f(x) is periodic with period T then $\frac{1}{f(x)}$ is also periodic with same period T.

If f(x) is periodic with period T, $\sqrt{f(x)}$ is also periodic with same period T.

If f(x) is periodic with period T, then a f(x) + b, where $a, b \in R (a \neq 0)$ is also a periodic function with period T.

If $f_1(x)$, $f_2(x)$, $f_3(x)$ are periodic functions with periods T_1, T_2, T_3 respectively then; we have

$h(x) = af_1(x) \pm bf_2(x) \pm cf_3(x)$, has period as,

$$= \begin{cases} \text{L.C.M of } \{T_1, T_2, T_3\}; & \text{if } h(x) \text{ is not an even function} \\ \frac{1}{2} \text{L.C.M. of } \{T_1, T_2, T_3\}; & \text{if } h(x) \text{ is an even function} \end{cases}$$

Example: 41 The period of the function $f(x) = 2\cos\frac{1}{3}(x - \pi)$ is

(a) 6π (b) 4π (c) 2π (d) π

Solution: (a) $f(x) = 2\cos\frac{1}{3}(x - \pi) = 2\cos\left(\frac{x}{3} - \frac{\pi}{3}\right)$

Now, since $\cos x$ has period $2\pi \Rightarrow \cos\left(\frac{x}{3} - \frac{\pi}{3}\right)$ has period $\frac{2\pi}{\frac{1}{3}} = 6\pi$

$\Rightarrow 2\cos\left(\frac{x}{3} - \frac{\pi}{3}\right)$ has period $= 6\pi$.

Example: 42 The function $f(x)=\sin\frac{\pi x}{2}+2\cos\frac{\pi x}{3}-\tan\frac{\pi x}{4}$ is periodic with period

(a) 6 (b) 3 (c) 4 (d) 12

Solution: (d) $\because$ $\sin x$ has period $=2\pi \Rightarrow$ $\sin\frac{\pi x}{2}$ has period $=\frac{2\pi}{\frac{\pi}{2}}=4$

$\because$ $\cos x$ has period $=2\pi \Rightarrow$ $\cos\frac{\pi x}{3}$ has period $=\frac{2\pi}{\frac{\pi}{3}}=6 \Rightarrow 2\cos\frac{\pi x}{3}$ has period = 6

$\because$ $\tan x$ has period $=\pi$ $\Rightarrow$ $\tan\frac{\pi x}{4}$ has period $=\frac{\pi}{\frac{\pi}{4}}=4$.

L.C.M. of 4, 6 and 4 =12, period of $f(x)$ = 12.

Example: 43 The period of $|\sin 2x|$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) π (d) 2π

Solution: (b) Here $|\sin 2x|=\sqrt{\sin^2 2x}=\sqrt{\frac{(1-\cos 4x)}{2}}$

Period of $\cos 4x$ is $\frac{\pi}{2}$. Hence, period of $|\sin 2x|$ will be $\frac{\pi}{2}$

Trick : $\because$ $\sin x$ has period $=2\pi \Rightarrow$ $\sin 2x$ has period $=\frac{2\pi}{2}=\pi$

Now, if $f(x)$ has period p then $|f(x)|$ has period $\frac{p}{2} \Rightarrow$ $|\sin 2x|$ has period $=\frac{\pi}{2}$.

Example: 44 If $f(x)$ is an odd periodic function with period 2, then $f(4)$ equals

(a) 0 (b) 2 (c) 4 (d) – 4

Solution: (a) Given, $f(x)$ is an odd periodic function. We can take $\sin x$, which is odd and periodic.

Now since, $\sin x$ has period = 2 and $f(x)$ has period = 2.

So, $f(x)=\sin(\pi x) \Rightarrow$ $f(4)=\sin(4\pi)=0$.

Example: 45 The period of the function $f(x)=\sin^2 x$ is

(a) $\frac{\pi}{2}$ (b) π (c) 2π (d) None of these

Solution: (b) $\sin^2 x=\frac{1-\cos 2x}{2} \Rightarrow$ Period $=\frac{2\pi}{2}=\pi$.

Example: 46 The period of $f(x)=x-[x]$, if it is periodic, is

(a) $f(x)$ is not periodic (b) $\frac{1}{2}$ (c) 1 (d) 2

Solution: (c) Let $f(x)$ be periodic with period *T*. Then,

$f(x+T)=f(x)$ for all $x\in R$ $\Rightarrow$ $x+T-[x+T]=x-[x]$ for all $x\in R$ $\Rightarrow$ $x+T-x=[x+T]-[x]$

$\Rightarrow$ $[x+T]-[x]=T$ for all $x\in R$ $\Rightarrow$ $T=1,2,3,4,........$

The smallest value of *T* satisfying,

$f(x+T)=f(x)$ for all $x\in R$ is 1.

Hence $f(x)=x-[x]$ has period 1.

Example: 47 The period of $f(x)=\sin\left(\frac{\pi x}{n-1}\right)+\cos\left(\frac{\pi x}{n}\right),\quad n\in Z,\quad n>2$ is

(a) $2\pi n(n-1)$ (b) $4n(n-1)$ (c) $2n(n-1)$ (d) None of these

Solution: (c) $f(x)=\sin\left(\frac{\pi x}{n-1}\right)+\cos\left(\frac{\pi x}{n}\right)$

Period of $\sin\left(\frac{\pi x}{n-1}\right)=\frac{2\pi}{\left(\frac{\pi}{n-1}\right)}=2(n-1)$ and period of $\cos\left(\frac{\pi x}{n}\right)=\frac{2\pi}{\left(\frac{\pi}{n}\right)}=2n$

Hence period of $f(x)$ is LCM of $2n$ and $2(n-1)\Rightarrow 2n(n-1)$.

Example: 48 If a,b be two fixed positive integers such that $f(a+x)=b+[b^3+1-3b^2f(x)+3b\{f(x)\}^2-\{f(x)\}^3]^{\frac{1}{3}}$ for all real x, then $f(x)$ is a periodic function with period

(a) a (b) $2a$ (c) b (d) $2b$

Solution: (b) $f(a+x)=b+(1+\{b-f(x)\}^3)^{1/3}\Rightarrow f(a+x)-b=\{1-\{f(x)-b\}^3\}^{1/3}$

$\Rightarrow \phi(a+x)=\{1-\{\phi(x)\}^3\}^{1/3}\quad [\phi(x)=f(x)-b]\Rightarrow \phi(x+2a)=\{1-\{\phi(x+a)\}^3\}^{1/3}=\phi(x)$

$\Rightarrow f(x+2a)-b=f(x)-b\Rightarrow f(x+2a)=f(x)$

$\therefore$ $f(x)$ is periodic with period $2a$.

2.1.9 Composite Function

If $f:A\to B$ and $g:B\to C$ are two function then the composite function of *f* and *g*,

$gof\,A\to C$ will be defined as $gof(x)=g[f(x)],\forall x\in A$

(1) **Properties of composition of function :**

(i) *f* is even, *g* is even $\Rightarrow$ fog even function.

(ii) *f* is odd, *g* is odd $\Rightarrow$ fog is odd function.

(iii) *f* is even, *g* is odd $\Rightarrow$ fog is even function.

(iv) *f* is odd, *g* is even $\Rightarrow$ fog is even function.

(v) Composite of functions is not commutative *i.e.*, $fog\neq gof$

(vi) Composite of functions is associative *i.e.*, $(fog)oh=fo(goh)$

(vii) If $f:A\to B$ is bijection and $g:B\to A$ is inverse of *f*. Then $fog=I_B$ and $gof=I_A$.

where, I_A and I_B are identity functions on the sets *A* and *B* respectively.

(viii) If $f:A\to B$ and $g:B\to C$ are two bijections, then $gof:A\to C$ is bijection and $(gof)^{-1}=(f^{-1}og^{-1})$.

(ix) $fog\neq gof$ but if , $fog=gof$ then either $f^{-1}=g$ or $g^{-1}=f$ also, $(fog)(x)=(gof)(x)=(x)$.

Important Tips

gof(x) is simply the g-image of f(x), where f(x) is f-image of elements $x \in A$.

Function gof will exist only when range of f is the subset of domain of g.

fog does not exist if range of g is not a subset of domain of f.

fog and gof may not be always defined.

If both f and g are one-one, then fog and gof are also one-one.

If both f and g are onto, then gof is onto.

Example: 49 If $f: R \to R, f(x) = 2x - 1$ and $g: R \to R,\ g(x) = x^2$ then $(gof)(x)$ equals

(a) $2x^2 - 1$ (b) $(2x-1)^2$ (c) $4x^2 - 2x + 1$ (d) $x^2 + 2x - 1$

Solution: (b) $gof(x) = g\{f(x)\} = g(2x-1) = (2x-1)^2$.

Example: 50 If $f: R \to R, f(x) = (x+1)^2$ and $g: R \to R, g(x) = x^2 + 1$, then $(fog)(-3)$ is equal to

(a) 121 (b) 144 (c) 112 (d) 11

Solution: (a) $fog(x) = f\{g(x)\} = f(x^2+1) = (x^2+1+1)^2 = (x^2+2)^2 \Rightarrow fog(-3) = (9+2)^2 = 121$.

Example: 51 $f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cos\left(x + \frac{\pi}{3}\right)$ and $g\left(\frac{5}{4}\right) = 1$, then $(gof)(x)$ is equal to

(a) 1 (b) –1 (c) 2 (d) – 2

Solution: (a) $f(x) = \sin^2 x + \sin^2(x + \pi/3) + \cos x \cos(x + \pi/3) = \frac{1 - \cos 2x}{2} + \frac{1 - \cos(2x + 2\pi/3)}{2} + \frac{1}{2}\{2\cos x \cos(x + \pi/3)\}$

$= \frac{1}{2}[1 - \cos 2x + 1 - \cos(2x + 2\pi/3) + \cos(2x + \pi/3) + \cos \pi/3]$

$= \frac{1}{2}\left[\frac{5}{2} - \{\cos 2x + \cos\left(2x + \frac{2\pi}{3}\right)\} + \cos\left(2x + \frac{\pi}{3}\right)\right] = \frac{1}{2}\left[\frac{5}{2} - 2\cos\left(2x + \frac{\pi}{3}\right)\cos\frac{\pi}{3} + \cos\left(2x + \frac{\pi}{3}\right)\right] = 5/4$ for all x.

$\therefore\ gof(x) = g(f(x)) = g(5/4) = 1$ [$\because g(5/4) = 1$ (given)]

Hence, $gof(x) = 1$, for all x.

Example: 52 If $g(x) = x^2 + x - 2$ and $\frac{1}{2}(gof)(x) = 2x^2 - 5x + 2$, then $f(x)$ is equal to

(a) $2x - 3$ (b) $2x + 3$ (c) $2x^2 + 3x + 1$ (d) $2x^2 - 3x - 1$

Solution: (a) $g(x) = x^2 + x - 2 \Rightarrow (gof)(x) = g[f(x)] = [f(x)]^2 + f(x) - 2$

Given, $\frac{1}{2}(gof)(x) = 2x^2 - 5x + 2$ $\therefore$ $\frac{1}{2}[f(x)]^2 + \frac{1}{2}f(x) - 1 = 2x^2 - 5x + 2$

$\Rightarrow [f(x)]^2 + f(x) = 4x^2 - 10x + 6 \Rightarrow f(x)[f(x) + 1] = (2x-3)[(2x-3)+1] \Rightarrow f(x) = 2x - 3$.

Example: 53 If $f(y) = \frac{y}{\sqrt{1-y^2}}, g(y) = \frac{y}{\sqrt{1+y^2}}$, then $(fog)(y)$ is equal to

(a) $\frac{y}{\sqrt{1-y^2}}$ (b) $\frac{y}{\sqrt{1+y^2}}$ (c) y (d) $\frac{1-y^2}{\sqrt{1+y^2}}$

Solution: (c) $f[g(y)] = \frac{y/\sqrt{1+y^2}}{\sqrt{1 - \left(\frac{y}{\sqrt{1+y^2}}\right)^2}} = \frac{y}{\sqrt{1+y^2}} \times \frac{\sqrt{1+y^2}}{\sqrt{1+y^2-y^2}} = y$

Example: 54 If $f(x)=\frac{2x-3}{x-2}$, then $[f\{f(x)\}]$ equals

(a) x (b) $-x$ (c) $\frac{x}{2}$ (d) $-\frac{1}{x}$

Solution: (a) $f[f(x)]=\frac{2\left(\frac{2x-3}{x-2}\right)-3}{\left(\frac{2x-3}{x-2}\right)-2}=x$

Example: 55 Suppose that $g(x)=1+\sqrt{x}$ and $f(g(x))=3+2\sqrt{x}+x$, then $f(x)$ is

(a) $1+2x^2$ (b) $2+x^2$ (c) $1+x$ (d) $2+x$

Solution: (b) $g(x)=1+\sqrt{x}$ and $f(g(x))=3+2\sqrt{x}+x$ ….. (i)

$\Rightarrow f(1+\sqrt{x})=3+2\sqrt{x}+x$

Put $1+\sqrt{x}=y \Rightarrow x=(y-1)^2$

then, $f(y)=3+2(y-1)+(y-1)^2=2+y^2$

therefore, $f(x)=2+x^2$.

Example: 56 Let $g(x)=1+x-[x]$ and $f(x)=\begin{cases}-1, & x<0\\ 0, & x=0\\ 1, & x>0\end{cases}$, then for all x, $f(g(x))$ is equal to

(a) x (b) 1 (c) $f(x)$ (d) $g(x)$

Solution: (b) Here $g(x)=1+n-n=1, x=n\in Z$

$1+n+k-n=1+k,\ x=n+k$ (where $n\in Z,\ 0<k<1$)

Now $f(g(x))=\begin{cases}-1, & g(x)<0\\ 0, & g(x)=0\\ 1, & g(x)>0\end{cases}$

Clearly, $g(x)>0$ for all x. So, $f(g(x))=1$ for all x.

Example: 57 If $f(x)=\frac{2x+1}{3x-2}$, then $(fof)(2)$ is equal to

(a) 1 (b) 3 (c) 4 (d) 2

Solution: (d) Here $f(2)=\frac{5}{4}$

Hence $(fof)(2)=f(f(2))=f\left(\frac{5}{4}\right)=\frac{2\times\frac{5}{4}+1}{3\times\frac{5}{4}-2}=2$.

Example: 58 If $f:R\to R$ and $g:R\to R$ are given by $f(x)=|x|$ and $g(x)=[x]$ for each $x\in R$, then $\{x\in R: g(f(x))\le f(g(x))\}=$

(a) $Z\cup(-\infty,0)$ (b) $(-\infty,0)$ (c) Z (d) R

Solution: (d) $g(f(x))\le f(g(x)) \Rightarrow g(|x|)\le f[x] \Rightarrow [|x|]\le|[x]|$. This is true for $x\in R$

2.1.10 Inverse Function

If $f: A \to B$ be a one-one onto (bijection) function, then the mapping $f^{-1}: B \to A$ which associates each element $b \in B$ with element $a \in A$, such that $f(a) = b$, is called the inverse function of the function $f: A \to B$

$f^{-1}: B \to A,\ f^{-1}(b) = a \Rightarrow f(a) = b$

In terms of ordered pairs inverse function is defined as $f^{-1} = (b, a)$ if $(a, b) \in f$.

Note : ❒For the existence of inverse function, it should be one-one and onto.

Important Tips

Inverse of a bijection is also a bijection function.

Inverse of a bijection is unique.

$(f^{-1})^{-1} = f$

If f and g are two bijections such that (gof) exists then $(gof)^{-1} = f^{-1}og^{-1}$.

If f : A→ B is a bijection then f^{-1}: B → A is an inverse function of f. $f^{-1}of = I_A$ and $fof^{-1} = I_B$.

Here I_A,is an identity function on set A, and I_B, is an identity function on set B.

Example: 59 If $f: R \to R$ is given by $f(x) = 3x - 5$, then $f^{-1}(x)$

(a) Is given by $\frac{1}{3x-5}$
(b) Is given by $\frac{x+5}{3}$
(c) Does not exist because *f* is not one-one
(d) Does not exist because *f* is not onto

Solution: (b) Clearly, $f: R \to R$ is a one-one onto function. So, it is invertible.

Let $f(x) = y$. then, $3x - 5 = y \Rightarrow x = \frac{y+5}{3} \Rightarrow f^{-1}(y) = \frac{y+5}{3}$. Hence, $f^{-1}(x) = \frac{x+5}{3}$.

Example: 60 Let $f: R \to R$ be defined by $f(x) = 3x - 4$, then $f^{-1}(x)$ is

(a) $3x + 4$
(b) $\frac{1}{3}x - 4$
(c) $\frac{1}{3}(x+4)$
(d) $\frac{1}{3}(x-4)$

Solution: (c) $f(x) = 3x - 4 = y \Rightarrow y = 3x - 4 \Rightarrow x = \frac{y+4}{3} \Rightarrow f^{-1}(y) = \frac{y+4}{3} \Rightarrow f^{-1}(x) = \frac{x+4}{3}$.

Example: 61 If the function $f: R \to R$ be such that $f(x) = x - [x]$, where $[y]$ denotes the greatest integer less than or equal to *y*, then $f^{-1}(x)$ is

(a) $\frac{1}{x-[x]}$
(b) $[x] - x$
(c) Not defined
(d) None of these

Solution: (c) $f(x) = x - [x]$ Since, for $x = 0 \Rightarrow f(x) = 0$

For $x = 1 \Rightarrow f(x) = 0$.

For every integer value of x, $f(x) = 0$

$\Rightarrow$ $f(x)$ is not one-one $\Rightarrow$ So $f^{-1}(x)$ is not defined.

Example: 62 If $f:[1,\infty)\to[1,\infty)$ is defined as $f(x)=2^{x(x-1)}$ then $f^{-1}(x)$ is equal to

(a) $\left(\frac{1}{2}\right)^{x(x-1)}$ (b) $\frac{1}{2}\left(1+\sqrt{1+4\log_2 x}\right)$

(c) $\frac{1}{2}\left(1-\sqrt{1+4\log_2 x}\right)$ (d) Not defined

Solution: (b) Given $f(x)=2^{x(x-1)}\Rightarrow x(x-1)=\log_2 f(x)$

$\Rightarrow x^2-x-\log_2 f(x)=0\Rightarrow x=\frac{1\pm\sqrt{1+4\log_2 f(x)}}{2}$

Only $x=\frac{1+\sqrt{1+4\log_2 f(x)}}{2}$ lies in the domain

$\therefore\quad f^{-1}(x)=\frac{1}{2}[1+\sqrt{1+4\log_2 x}]$

Example: 63 Which of the following function is invertible

(a) $f(x)=2^x$ (b) $f(x)=x^3-x$ (c) $f(x)=x^2$ (d) None of these

Solution: (a) A function is invertible if it is one-one and onto.

Example: 64 If $f(x)=x^2+1$, then $f^{-1}(17)$ and $f^{-1}(-3)$ will be

(a) 4, 1 (b) 4, 0 (c) 3, 2 (d) None of these

Solution: (d) Let $y=x^2+1\quad\Rightarrow\quad x=\pm\sqrt{y-1}$

$\Rightarrow f^{-1}(y)=\pm\sqrt{y-1}\Rightarrow f^{-1}(x)=\pm\sqrt{x-1}$

$\Rightarrow f^{-1}(17)=\pm\sqrt{17-1}=\pm 4$

and $f^{-1}(-3)=\pm\sqrt{-3-1}=\pm\sqrt{-4}$, which is not possible.

ASSIGNMENT

Basic Level

1. If $f(x)=\frac{1-x}{1+x}$, then $f[f(\cos 2\theta)]$ equal to

(a) $\tan 2\theta$ (b) $\sec 2\theta$ (c) $\cos 2\theta$ (d) $\cot 2\theta$

2. If $f(x)=\frac{\cos^2 x+\sin^4 x}{\sin^2 x+\cos^4 x}$ for $x\in R$, then $f(2002)=$

(a)1 (b) 2 (c) 3 (d) 4

3. If $\phi(x)=a^x$, then $\{\phi(p)\}^3$ is equal to

(a) $\phi(3p)$ (b) $3\phi(p)$ (c) $6\phi(p)$ (d) $2\phi(p)$

4. If $f(x)=\cos(\log x)$, then $f(x)f(y)-\frac{1}{2}[f\left(\frac{x}{y}\right)+f(xy)]=$

(a) $\frac{1}{2}$ (b) 2 (c) 0 (d) 1

5. If $f(\theta)=\tan\theta$, then $\frac{f(\theta)-f(\phi)}{1+f(\theta)f(\phi)}$ is equal to

(a) $f(\theta-\phi)$ (b) $f(\phi-\theta)$ (c) $f(\theta+\phi)$ (d) None of these

6. If $f(x)=2x\sqrt{1-x^2}$, then $f\left(\sin\frac{x}{2}\right)$ equals

(a) $\sin 2x$ (b) $\sin x$ (c) $2\sin x$ (d) $2\sin\frac{x}{2}$

7. If $f(x)=\frac{x}{x-1}$, then $\frac{f(a)}{f(a+1)}$ is equal to

(a) $f(-a)$ (b) $f\left(\frac{1}{a}\right)$ (c) $f(a^2)$ (d) $f\left(\frac{-a}{a-1}\right)$

8. If $f(x)=\begin{cases}2x-3 & , x\geq 2\\ x & , x<2\end{cases}$, then $f(1)$ is equal to

(a) $2f(2)$ (b) $f(2)$ (c) $-f(2)$ (d) $\frac{1}{2}f(2)$

9. If $f(x)=x^2-x^{-2}$, then $f\left(\frac{1}{x}\right)$ is equal to

(a) $f(x)$ (b) $-f(x)$ (c) $\frac{1}{f(x)}$ (d) $[f(x)]^2$

10. If $f(x)=4x^3+3x^2+3x+4$, then $x^3 f\left(\frac{1}{x}\right)$ is

(a) $f(-x)$ (b) $\frac{1}{f(x)}$ (c) $\left[f\left(\frac{1}{x}\right)\right]^2$ (d) $f(x)$

11. The equivalent function of $\log x^2$ is

(a) $2\log x$ (b) $2\log|x|$ (c) $|\log x^2|$ (d) $(\log x)^2$

Advance Level

If $f(x)=\cos[\pi]x+\cos[\pi x]$, where $[y]$ is the greatest integer function of y then $f\left(\frac{\pi}{2}\right)$ is equal to

(a) $\cos 3$ (b) 0 (c) $\cos 4$ (d) None of these

Let $f(x)=\begin{cases}1+|x| & , x<-1\\ [x] & , x\geq -1\end{cases}$, where [.] denotes the greatest integer function. Then $f\{f(-2.3)\}$ is equal to

(a) 4 (b) 2 (c)–3 (d) 3

If $f(x_1)+f(x_2)=f\left(\frac{x_1+x_2}{1+x_1x_2}\right), x_1x_2\in(-1,1)$, then $f(x)$ is equal to

(a) $\log\left(\frac{1-x}{1+x}\right)$ (b) $\tan^{-1}\left(\frac{1-x}{1+x}\right)$ (c) $\log\left(\frac{2x}{1-x^2}\right)$ (d) $\tan^{-1}\left(\frac{1+x}{1-x}\right)$

If $f(x)=\frac{|x|}{x}, x\neq 0$, then the value of function

(a) 1 (b) 0 (c)–1 (d) Does not exists

If a function $g(x)$ is defined in $[-1,1]$ and two vertices of an equilateral triangle are (0,0) and (x, $g(x)$) and its area is $\frac{\sqrt{3}}{4}$, then $g(x)$ equals

(a) $\sqrt{1+x^2}$ (b) $-\sqrt{1+x^2}$ (c) $\sqrt{1-x^2}$ (d) None of these

If $f(x)=\frac{2^x+2^{-x}}{2}$, then $f(x+y).\,f(x-y)$ is equal to

(a) $\frac{1}{2}[f(x+y)+f(x-y)]$ (b) $\frac{1}{2}[f(2x)+f(2y)]$ (c) $\frac{1}{2}[f(x+y).\,f(x-y)]$ (d) None of these

$f(1)=1$ and $f(n+1)=2f(n)+1$ if $n\geq 1$, then $f(n)$ is

(a) 2^{n+1} (b) 2^n (c) 2^n-1 (d) $2^{n-1}-1$

If $2f(x)-3f(1/x)=x^2$, $x\neq 0$, then $f(2)$ is equal to

(a) 5/2 (b) –7/4 (c)–1 (d) None of these

If $f(x)=|x-1|$, then correct statement is

(a) $f(x^2)=[f(x)]^2$ (b) $f(|x|)=|f(x)|$ (c) $f(x+y)=f(x)+f(y)$ (d) None of these

DOMAIN OF FUNCTION

Basic Level

The domain of the function $f(x)=\sqrt{\log_{0.5}x}$ is

(a) (0, 1] (b) (0, ∞) (c)(0.5, ∞) (d) [1, ∞)

The domain of definition of the real function $f(x)=\sqrt{\log_{12}x^2}$ of the real variable x is

(a) $x>0$ (b) $|x|\geq 1$ (c) $|x|\geq 4$ (d) $x\geq 4$

The natural domain of the real valued function defined by $f(x)=\sqrt{x^2-1}+\sqrt{x^2+1}$

(a) $1<x<\infty$ (b) $-\infty<x<\infty$ (c) $-\infty<x<-1$ (d) $(-\infty,\infty)-(-1,1)$

The domain of the function $y=\sqrt{\frac{1}{x}-1}$ is,

(a) $x \le 1$ (b) $0 \le x \le 1$ (c) $0 \le x < 1$ (d) $0 < x \le 1$

Domain of $f(x) = \log|\log x|$ is

(a) $(0, \infty)$ (b) $(1, \infty)$ (c) $(0,1) \cup (1, \infty)$ (d) $(-\infty, 1)$

Domain of function $f(x) = \left[\log_{10}\left(\frac{5x - x^2}{4}\right)\right]^{1/2}$ is

(a) $-\infty < x < \infty$ (b) $1 \le x \le 4$ (c) $4 \le x \le 16$ (d) $-1 \le x \le 1$

Domain of the function $\sin^{-1}\left[\log_2\left(\frac{x^2}{2}\right)\right]$ is

(a) $[1, 2]$ (b) $[-1, 2]$ (c) $[-2, 2] - (-1, 1)$ (d) $[-2, 2] - \{ \}$

The domain of the function $f(x) = \frac{\sqrt{4 - x^2}}{\sin^{-1}(2 - x)}$ is

(a) [0, 2] (b) [0, 2) (c)[1, 2) (d) [1, 2]

The domain of the function $f(x) = \log(\sqrt{x - 4} + \sqrt{6 - x})$ is

(a) $[4, \infty)$ (b) $(-\infty, 6]$ (c) $[4, 6]$ (d) None of these

Advance Level

The largest set of real values of x for which $f(x) = \sqrt{(x + 2)(5 - x)} - \frac{1}{\sqrt{x^2 - 4}}$ is a real function

(a) [1, 2] $\cup (2, 5]$ (b) $(2, 5]$ (c) $[3, 4]$ (d) None of these

The domain of the function $f(x) = \frac{1}{\log_{10}(1 - x)} + \sqrt{x + 2}$ is

(a) $]-3, -2.5[\cup]-2.5, -2[$ (b) $[-2, 0[\cup]0, 1[$

(c) $]0, 1[$ (d) None of these

The domain of the function $f(x) = \log_e(x - [x])$, where [.] denotes the greatest integer function, is

(a) *R* (b) *R*–*Z* (c) $(0, +\infty)$ (d) None of these

The domain of the function $f(x) = \frac{\sin^{-1}(3 - x)}{\ln(|x| - 2)}$ is

(a) $[2, 4]$ (b) (2, 3) ∪ (3, 4] (c)[2, ∞) (d) $(-\infty, -3) \cup [2, \infty)$

Domain of the function $f(x) = \sin^{-1}(1 + 3x + 2x^2)$ is

(a) $(-\infty, \infty)$ (b) $(-1, 1)$ (c) $\left[\frac{-3}{2}, 0\right]$ (d) $\left(-\infty, \frac{-1}{2}\right) \cup (2, \infty)$

Domain of the function $\sin \ln\left(\frac{\sqrt{4 - x^2}}{1 - x}\right)$

(a) $[-2, 1]$ (b) $(-2, 1)$ (c) $[-2, 1)$ (d) $(-2, 1]$

Domain of the function $f(x) = \sqrt{\log_{0.5}(3x - 8) - \log_{0.5}(x^2 + 4)}$ is

(a) $\left(\frac{8}{3}, \infty\right)$ (b) $\left(-\infty, \frac{8}{3}\right)$ (c) $(-\infty, \infty)$ (d) $(0, \infty)$

The domain of $f(x) = \frac{1}{\sqrt{|\cos x| + \cos x}}$ is

(a) $[-2n\pi, 2n\pi]$ (b) $(2n\pi, \overline{2n+1}\pi)$ (c) $\left(\frac{(4n+1)\pi}{2}, \frac{(4n+3)\pi}{2}\right)$ (d) $\left(\frac{(4n-1)\pi}{2}, \frac{(4n+1)\pi}{2}\right)$

The domain of $f(x) = \sin^{-1}\left(\frac{1+x^2}{2x}\right) + \sqrt{1-x^2}$ is

(a) {1} (b) $(-1,1)$ (c){1,–1} (d) None of these

The domain of the function $f(x) = \sqrt{\log\left(\frac{1}{|\sin x|}\right)}$ is

(a) $R - \{-\pi, \pi\}$ (b) $R - \{n\pi \mid n \in Z\}$ (c) $R - \{2n\pi \mid n \in Z\}$ (d) $(-\infty, \infty)$

The domain of the function $f(x) = {}^{16-x}C_{2x-1} + {}^{20-3x}P_{4x-5}$, where the symbols have their usual meanings, is the set

(a) {2, 3} (b) {2, 3, 4} (c){1, 2, 3, 4} (d) {1, 2, 3, 4, 5}

Domain of the function $f(x) = \sin^{-1}\{1+e^x\}^{-1}$ is

(a) (– ∞, ∞) (b) $[-1, 0]$ (c) $[0, 1]$ (d) [–1, 1]

If *n* is an integer then domain of the function $\sqrt{\sin 2x}$ is

(a) $[n\pi - \frac{\pi}{2}, n\pi]$ (b) $\left[n\pi, n\pi + \frac{\pi}{2}\right]$ (c) $[(2n-1)\pi, 2n\pi]$ (d) $[2n\pi, (2n+1)\pi]$

RANGE OF FUNCTION

Basic Level

If $A\{-2, -1, 0, 1, 2\}$ and $f: A \to Z, f(x) = x^2 + 1,$ then the range of f is

(a) {0, 1, 2, 5} (b) {1, 2, 5} (c){–5, –2, 1, 2, 3} (d) *A*

The range of the function $f: [0,1] \to R, f(x) = x^3 - x^2 + 4x + 2\sin^{-1} x$ is

(a) $[-\pi - 2, 0]$ (b) [2, 3] (c)[0, 4+ π] (d) $[0, 2+\pi]$

The range of $f(x) = \cos(x/3)$ is

(a) $[-1/3, 1/3]$ (b) $[-3, 3]$ (c) $[1/3, -1/3]$ (d) [– 1, 1]

Range of $f(x) = \frac{x^2 + 34x - 71}{x^2 + 2x - 7}$ is

(a) $[5, 9]$ (b) $(-\infty, 5] \cup [9, \infty)$ (c) $(5, 9)$ (d) None of these

Range of the function $f(x) = \frac{x^2 - x + 1}{x^2 + x + 1}$

(a) *R* (b) [3, ∞) (c) $\left[\frac{1}{3}, 3\right]$ (d) None of these

Advance Level

The range of the function $f(x) = \cos[x]$, where $\frac{-\pi}{2} < x < \frac{\pi}{2}$ is

(a) {– 1, 1, 0} (b) $\{\cos 1, 1, \cos 2\}$ (c) $\{\cos 1, -\cos 1, 1\}$ (d) None of these

The range of the function $f(x) = |x-1| + |x-2|, -1 \le x \le 3$ is

(a) [1, 3] (b) [1, 5] (c) [3, 5] (d) None of these

Let $f(x) = (1+b^2)x^2 + 2bx + 1$ and $m(b)$ the minimum value of $f(x)$ for a given *b*. As *b* varies, the range of $m(b)$ is

(a) $[0, 1]$ (b) $\left(0, \frac{1}{2}\right]$ (c) $\left[\frac{1}{2}, 1\right]$ (d) $(0, 1]$

KINDS OF FUNCTION

Basic Level

Which of the following functions defined from *R* to *R* is onto

(a) $f(x) = |x|$ (b) $f(x) = e^{-x}$ (c) $f(x) = x^3$ (d) $f(x) = \sin x$

The number of bijective function from set *A* to itself when *A* contains 106 elements is

(a) 106 (b) $(106)^2$ (c) 106 ! (d) 2^{106}

If *A* contains 3 elements and *B* contains 4 elements, then the number of all one – one functions defined from *A* to *B* is

(a) 144 (b) 12 (c) 24 (d) 64

If $A = \{a, b\}$, then total number of functions which can be defined from *A* to *A* is

(a) 2 (b) 3 (c) 4 (d) 1

Function $f: R \to R, f(x) = x^3 + 7$ is

(a) One – one onto (b) One – one into (c) Many – one onto (d) Many – one into

Which of the following functions from *R* to *R* is into

(a) x^5 (b) $3x - 7$ (c) x^3 (d) $\sin x$

Function $f: R \to R, f(x) = x^2$ is

(a) One – one but not onto (b) Onto but not one- one (c) Neither one-one nor onto (d) One- one onto

If $A = R - \{3\}, B = R - \{1\}$ and $f: A \to B$, $f(x) = \frac{x-2}{x-3}$, then *f* is

(a) One-one (b) Onto (c) One-one onto (d) Many-one into

Advance Level

Let $f(x) = \frac{x^2 - 4}{x^2 + 4}$ for $|x| > 2$, then the function $f: (-\infty, -2] \cup [2, \infty) \to (-1, 1)$ is

(a) One-one into (b) One-one onto (c)Many one into (d) Many one onto

Let the function $f: R \to R$ be defined by $f(x) = 2x + \sin x, x \in R$ Then f is

(a) One-to-one and onto (b) One-to-one but not onto

(c) Onto but not one-to-one (d) Neither one-to-one nor onto

function $f: R \to R, f(x) = x|x|$ is

(a) One – one but not onto (b) Onto but not one – one

(c) One – one onto (d) Neither one – one nor onto

If for two function f and g; gof is a bijection, then correct statement is

(a) Both g and f must be bijections (b) g must be a bijection

(c) f must be a bijection (d) Neither of them may be a bijection

If $f:[0,\infty) \to [0,\infty)$ and $f(x) = \frac{x}{1+x}$, then f is

(a) One – one and onto (b)One – one but not onto

(c)Onto but not one – one (d) Neither one – one nor onto

The number of all onto functions which can be defined from $A = \{1, 2, 3, ..., n\}$, $n \geq 2$ to $B = \{a, b\}$ is

(a) ${}^nP_2$ (b) $2^n - 2$ (c) $2^n - 1$ (d) None of these

If $1 + 2x$ is a function having $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ as domain and $(-\infty, \infty)$ as co- domain, then it is

(a) Onto but not one- one (b) One – one but not onto

(c)One – one and onto (d) Neither one – one nor onto

If $A = \{x \mid -1 \leq x \leq 1\} = B$ and $f: A \to B$; $f(x) = \sin \pi x$, then f is

(a) One – one (b) Onto (c)One – one onto (d) Many one into

If the real-valued function $f(x) = px + \sin x$ is a bijective function then the set of possible valued of $p \in R$ is

(a) $R - \{0\}$ (b) R (c) $(0, +\infty)$ (d) None of these

EVEN / ODD FUNCTIONS

Basic Level

The function $f(x) = x \cos x$ is

(a) Even function (b) Odd function (c) Neither even nor odd (d) Periodic function

A function whose graph is symmetrical about the y-axis is given by

(a) $f(x) = \log_e(x + \sqrt{x^2 + 1})$ (b) $f(x+y) = f(x) + f(y)$ for all $x, y \in R$

(c) $f(x) = \cos x + \sin x$ (d) None of these

Let $f(x+y) = f(x) + f(y)$ for all $x, y \in R$. Then

(a) $f(x)$ is an even function (b) $f(x)$ is an odd function (c) $f(0) = 0$ (d) $f(n) = nf(1), n \in N$

If $f(x)$ is an odd function then

(a) $\frac{f(-x) + f(x)}{2}$ is an even function (b) $[|f(x)| + 1]$ is even, where $[x]$ = the greatest integer $\leq x$

(c) $\frac{f(x)-f(-x)}{2}$ is neither even nor odd (d)None of these

Advance Level

If $f(x)$ and $g(x)$ are two functions of *x* such that $f(x)+g(x)=e^x$ and $f(x)-g(x)=e^{-x}$ then

(a) $f(x)$ is an odd function (b) $g(x)$ is an odd function

(c) $f(x)$ is an even function (d) $g(x)$ is an even function

If $f(x)=\begin{cases} x^2\sin\frac{\pi x}{2}, & |x|<1 \\ x|x|, & |x|\geq 1 \end{cases}$ then $f(x)$ is

(a) An even function (b) An odd function (c)A periodic function (d) None of these

Which of the following is an even function? Here [.] denotes the greatest integer function and *f* is any function

(a) $[x]-x$ (b) $f(x)-f(-x)$ (c) $e^{3-2x}.\tan^2 x$ (d) $f(x)+f(-x)$

PERIODIC FUNCTION

Basic Level

The period of |cos *x*| is

(a) 2π (b) π (c) $\frac{\pi}{2}$ (d) $\frac{3\pi}{2}$

The period of the function $\sin\left(\frac{\pi x}{2}\right)+\cos\left(\frac{\pi x}{2}\right)$ is

(a) 4 (b) 6 (c)12 (d) 24

If $f(x)$ is a periodic function of the period *T*, then $f(ax+b)$ where $a>0$, is a periodic function of the period

(a) *T*/*b* (b) *aT* (c)*bT* (d) *T*/*a*

The period of the function $f(x)=\sin\left(\frac{2x+3}{6\pi}\right)$ is

(a) 2π (b) 6π (c) $6\pi^2$ (d) None of these

The period of the function $f(x)=3\sin\frac{\pi x}{3}+4\cos\frac{\pi x}{4}$ is

(a) 6 (b) 24 (c)8 (d) 2π

The period of the function $f(x)=|\sin x|+|\cos x|$ is

(a) π (b) $\pi/2$ (c) 2π (d) None of these

Advance Level

Let $f(x)=\cos 3x+\sin\sqrt{3}x$. Then $f(x)$ is

(a) A periodic function of period 2π (b) A periodic function of period $\sqrt{3}\pi$

(c) Not a periodic function (d) None of these

$f(x) = \cos\sqrt{x}$, correct statement is

(a) $f(x)$ is periodic & its period $= \sqrt{2}\pi$ (b) $f(x)$ is periodic & its period $= 4\pi^2$

(c) $f(x)$ is periodic & its period $= \sqrt{\pi}$ (d) $f(x)$ is not periodic

COMPOSITE FUNCTIONS

Basic Level

If $f: R \to R, f(x) = \sin x; g: R \to R, g(x) = x^2$, then $(fog)(x)$ equals to

(a) $\sin x^2$ (b) $\sin^2 x$ (c) $\sin x + x^2$ (d) $\sin\frac{x}{x^2}$

If $f(x) = (a - x^n)^{1/n}$, where $a > 0$ and n is a positive integer, then $f[f(x)] =$

(a) x^3 (b) x^2 (c) x (d) None of these

If $f(x) = \frac{x}{\sqrt{1+x^2}}$, then $fofof(x)$ is equal to

(a) $\frac{x}{\sqrt{1+3x^2}}$ (b) $\frac{x}{\sqrt{1+2x^2}}$ (c) $\frac{x}{\sqrt{1+x^2}}$ (d) None of these

Let *f* and *g* be functions defined by $f(x) = \frac{x}{x+1}, g(x) = \frac{x}{1-x}$, then (*fog*) (*x*) is

(a) $\frac{1}{x}$ (b) $\frac{1}{x-1}$ (c) $x - 1$ (d) *x*

If $f(x) = ax + b$ and $g(x) = cx + d$, then $f(g(x)) = g(f(x))$ is equivalent to

(a) $f(a) = g(c)$ (b) $f(b) = g(b)$ (c) $f(d) = g(b)$ (d) $f(c) = g(a)$

Advance Level

If $f(x) = \sqrt{|x-1|}$ and $g(x) = \sin x$, then $(fog)(x)$ is equal to

(a) $\sin\sqrt{|x-1|}$ (b) $|\sin x/2 - \cos x/2|$ (c) $|\sin x - \cos x|$ (d) None of these

If *f* and *g* are two real valued function defined by $f(x) = e^x$ and $g(x) = 3x - 2$, then $(fog)^{-1}(x)$ is equal to

(a) $\log(3x-2)$ (b) $\frac{2+\log x}{3}$ (c) $\log\left(\frac{x+2}{3}\right)$ (d) None of these

If $f(x) = \frac{1}{1-x}, x \neq 0, 1$, then the graph of the function $y = f\{f(f(x))\}, x > 1$, is

(a) A circle (b) An ellipse

(c) A straight line (d) A pair of straight lines

If $f(x)$ is defined on [0, 1] by the rule $f(x) = \begin{cases} x, \text{if } x \text{ is rational} \\ 1-x, \text{if } x \text{ is irrational} \end{cases}$. Then for all $x \in [0,1], f(f(x))$ is

(a) Constant (b) $1 + x$ (c) x (d) None of these

INVERSE FUNCTION

Basic Level

$f: R \to R$ is a function defined by $f(x) = 10x - 7$. If $g = f^{-1}$, then $g(x) =$

(a) $\frac{1}{10x-7}$ (b) $\frac{1}{10x+7}$ (c) $\frac{x+7}{10}$ (d) $\frac{x-7}{10}$

If $y = f(x) = \frac{x+2}{x-1}$, then $x =$

(a) $f(y)$ (b) $2f(y)$ (c) $\frac{1}{f(y)}$ (d) None of these

Inverse of the function $y = 2x - 3$ is

(a) $\frac{x+3}{2}$ (b) $\frac{x-3}{2}$ (c) $\frac{1}{2x-3}$ (d) None of these

Advance Level

The value of α for which the function $f(x) = 1 + \alpha x, \alpha \neq 0$ is inverse of itself will be

(a) – 2 (b) – 1 (c) 1 (d) 2

If $f: [1, +\infty) \to [2, +\infty)$ is given by $f(x) = x + \frac{1}{x}$ then f^{-1} equals

(a) $\frac{x+\sqrt{x^2-4}}{2}$ (b) $\frac{x}{1+x^2}$ (c) $\frac{x-\sqrt{x^2-4}}{2}$ (d) $1+\sqrt{x^2-4}$

The inverse of the function $f(x) = \frac{10^x - 10^{-x}}{10^x + 10^{-x}}$ is

(a) $\log_{10}(2-x)$ (b) $\frac{1}{2}\log_{10}\left(\frac{1+x}{1-x}\right)$ (c) $\frac{1}{2}\log_{10}(2x-1)$ (d) $\frac{1}{4}\log\left(\frac{2x}{2-x}\right)$

The inverse of the function $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} + 2$ is given by

(a) $\log_e\left(\frac{x-2}{x-1}\right)^{\frac{1}{2}}$ (b) $\log_e\left(\frac{x-1}{3-x}\right)^{\frac{1}{2}}$ (c) $\log_e\left(\frac{x}{2-x}\right)^{\frac{1}{2}}$ (d) $\log_e\left(\frac{x-1}{x+1}\right)^{-2}$

***

Answer

ASSIGNMENT

Basic & Advance Level

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
c	a	a	c	a	b	c	b	b	d	b	c	d	a	d	c	b	c	b	d
21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
a	b	d	d	c	b	c	c	c	b	b	b	b	c	b	a	d	c	b	a
41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
a	b	b	c	d	b	c	b	b	d	c	c	c	c	a	d	c	c	c	a
61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
c	a	b	b	b	b	d	b	d	b,c, d	a, b	b, c	b	d	b	a	d	c	b	b
81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98		
c	d	a	c	a	d	c	b	b	c	c	c	a	a	b	a	b	b		