# QUADRATIC EQUATIONS

## Omar Khayyam

He was famous during his lifetime as a mathematician, well known for inventing the method of solving equations by intersecting a parabola with a circle. Although his approach at achieving this had earlier been attempted by Menaechmus and others, Khayyám provided a generalization extending it to all cubics. In addition he discovered the binomial expansion, and authored criticisms of Euclid's theories of parallels which made their way to Europe, where they contributed to the eventual development of non-Euclidean geometry.

In 1070 he wrote his greatest work on algebra. In it he classified equations according to their degree, and gave rules for solving quadratic equations, which are very similar to the ones in use today, and a geometric method for solving cubic equations with real roots. He also wrote on the triangular array of binomial coefficients known as Pascal's triangle. In 1077, Omar wrote Sharh ma ashkala min musadarat kitab Uqlidis (Explanations of the Difficulties in the Postulates of Euclid). An important part of the book is concerned with Euclid's famous parallel postulate, which had also attracted the interest of Thabit ibn Qurra. Al-Haytham had previously attempted a demonstation of the postulate; Omar's attempt was a distinct advance. Omar Khayyám also had other notable work in geometry, specifically on the theory of proportions.

**IIT-JEE Syllabus**

Equation Quadratic equations with real coefficients, relations between roots and coefficients, formation of quadratic equations with given roots, symmetric functions of roots.

## Introduction

Equation of the form $ax^2 + bx + c = 0$ where $a, b, c \in R$ and $a \neq 0$ is called a quadratic equation, where $a, b, c$ are called coefficients of this equation.

- **Roots of Quadratic Equation**

  To find roots of $ax^2 + bx + c = 0$

  $$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad \text{as} \quad a \neq 0$$

  $$\Rightarrow \left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} = 0$$

□ $\left(x+\frac{b}{2a}\right)^2=\left(\frac{b^2-4ac}{4a^2}\right)$

□ $x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$

□ $x=\left(\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right)$ these are the roots of the given equation.

Let the roots are □ & □ then

□□□□ = $\left(\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right)$

(***a***) sum of the roots, □□□□□ $= -\frac{b}{a}$

(***b***) product of the roots, □□□□□ $= \frac{c}{a}$

(***c***) If □ & □ are the roots of $ax^2 + bx + c = 0$ then, $ax^2 + bx + c = a$ ($x$ – □) ($x$ – □)

**(d)** the equation whose roots are □ and □ is

($x$ – □) ($x$ – □) = 0

□ $x^2$ – (□□□□□) $x$ + □□ = 0

⇒ $x^2$ – (sum of roots) $x$ + (product of roots) = 0

❑ **Nature of Roots**

To find nature of roots of $ax^2 + bx + c = 0$, where $a, b, c$ □ $R$,

Find discriminant, $D = b^2 - 4ac$

(***a***) If $D < 0$, ($b^2 - 4ac < 0$), then the roots of the quadratic equation are non-real i.e. complex roots. The roots are complex conjugates of each other.

(***b***) If $D = 0$, ($b^2 - 4ac = 0$), then the roots are real and equal i.e. □□□□□ $= -\frac{b}{2a}$ .

(***c***) If $D > 0$, ($b^2 - 4ac > 0$), then the roots of quadratic equation are real and distinct.

($i$) If $D = b^2 - 4ac$ is a perfect square and $a, b, c$ are rational numbers then the roots are rational.

($ii$) If $D = b^2 - 4ac$ is not a perfect square and $a, b$ and $c$ are rational number then the roots are irrational conjugates of each other.

**(d)** If $a = 1$ and $b, c$ □□$N$ and the roots are rational numbers, then these roots must be integers.

(***e***) If $a + b + c = 0$ then 1 is a root of the equation and if $a, b, c$ are rational numbers then both the roots are rational.

**(f)** If a quadratic equation in $x$ has more than two roots, then it is an identity in $x$. (i.e. true for all value of $x$) and $a = b = c = 0$.

## ILLUSTRATIONS

***Illustration* 1**

Prove that the roots of the equation $(b - c)\, x^2 + 2(c - a)\, x + (a - b) = 0$ are always real.

***Solution***

Given $(b - c)x^2 + 2(c - a)x + (a - b) = 0$

Discriminant = $[2(c - a)]^2 - 4(b - c)\,(a - b)$

For real roots $D > 0$

□ To prove $4\,(c - a)^2 - 4(b - c)\,(a - b)$ □ 0

□ $(c - a)^2 - (b - c)\,(a - b)$ □ 0

□ $(c^2 + a^2 - 2ac) - (ab - ac - b^2 + bc)$ □ 0

□ $a^2 + c^2 - 2ac - ab + ac + b^2 - bc$ □ 0

□ $a^2 + b^2 + c^2 - ac - ab - bc$ □ 0

□ $2(a^2 + b^2 + c^2 - ac - ab - bc)$ □ 0

□ $(a^2 - 2ab + b^2) + (b^2 - 2bc + c^2) + (c^2 - 2ac + a^2)$ □ 0

□ $(a - b)^2 + (b - c)^2 + (c - a)^2$ □ 0, which is always true. Hence proved.

## 1.1 Condition For Common Root (s)

Consider two quadratic equations,

$$a_1x^2 + b_1x + c_1 = 0 \quad \text{and} \quad a_2x^2 + b_2x + c_2 = 0$$

**($a$) For two common roots**

In such a case, two equations should be identical for that, the ratio of coefficients must be same. $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

**($b$) For one common root**

Let □ be the common root of two equations. So □ should satisfy the two equations.

$$a_1□^2 + b_1□ + c_1 = 0 \quad \text{and} \quad a_2□^2 + b_2□ + c_2 = 0$$

Solving the two equations by Cramer's Rule,

$$\frac{\alpha^2}{(b_1c_2 - b_2c_1)} = \frac{\alpha}{(c_1a_2 - c_2a_1)} = \frac{1}{(a_1b_2 - a_2b_1)}$$

$$\square \quad \square^2 = \left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}\right) \quad \& \quad \square = \left(\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$$

$$(a_1c_2 - a_2c_1)^2 = (a_1b_2 - a_2b_1)\,(b_1c_2 - b_2c_1)$$

***Note:*** To find the common root between the two equations, remove the term containing $x^2$ term from both the equations and find the value of $x$, this is the common root.

## 1.2 Quadratic Expression

Expression of the form $ax^2 + bx + c$, where $a, b, c$ □ $R$ and $a$ □ 0 is called a quadratic expression (quadratic polynomial) in x. So in general quadratic expression is represented by

$f(x) = ax^2 + bx + c$ or $y = ax^2 + bx + c$

### ❑ Graph of Quadratic Expression

$y = ax^2 + bx + c$ or $f(x) = ax^2 + bx + c$

$$= a\left\{\left(x + \frac{b}{2a}\right)^2 + \left(\frac{4ac - b^2}{4a^2}\right)\right\}$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a}$$

$$\square \quad \left(y + \frac{D}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2$$

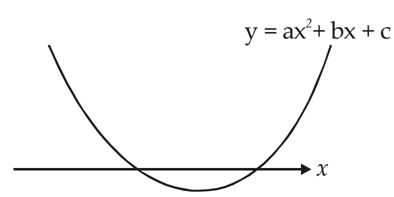
Let $x + \frac{b}{2a} = X$ and $y + \frac{D}{4a} = Y$

$\square \quad Y = aX^2$

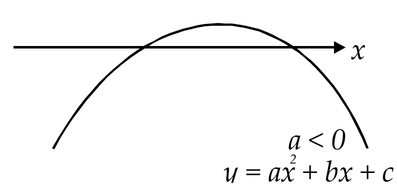
$\square \quad X^2 = \frac{Y}{a}$

So the shape of $y = f(x)$ is a parabola

**(a)** If $a > 0$, the parabola opens upwards.




**(b)** If $a < 0$, then the parabola opens downwards.



## ❑ Greatest And Least Value of $f(x) = ax^2 + bx + c$

$y = ax^2 + bx + c$

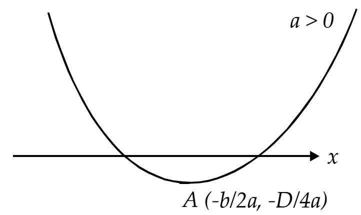
$$= a\left\{x^2 + \frac{b}{a}x + \frac{c}{a}\right\}$$

$$= a\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right\}$$

$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{(4ac - b^2)}{4a}$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a}$$

**Case I**




$a > 0$, then $f(x)$ (or $y$) is least when

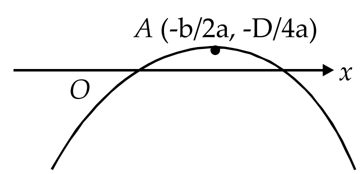
$x + \frac{b}{2a} = 0$ or $x = -\frac{b}{2a}$ then

$y = -\frac{D}{4a}$ is minimum value.

**Case II**

$a < 0$

$$y = a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a}$$




then $y$ will be maximum when, $x + \frac{b}{2a} = 0$

$\therefore$ $x = -\frac{b}{2a}$ and then maximum value of $y$ will be

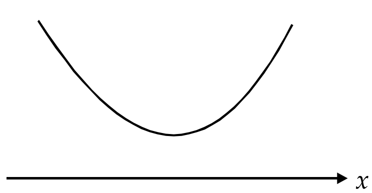
$y = -\frac{D}{4a}$, maximum value of y (or $f(x)$)

## ❑ Sign of Quadratic Expression

Let $f(x) = ax^2 + bx + c$ or $y = ax^2 + bx + c$

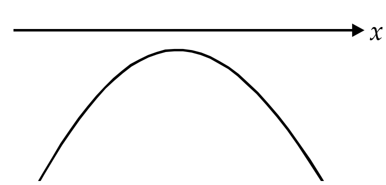
where $a, b, c \in R$ and $a \neq 0$, then following cases may occur,

*(i)* $a > 0$ & $D < 0$:



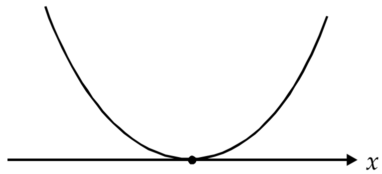

$\Rightarrow$ $f(x) > 0$ for all $x \in R$. *i.e.*, $f(x)$ will be always positive.

*(ii)* $a < 0$ & $D < 0$



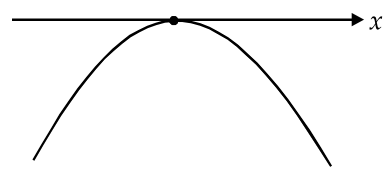

$\Rightarrow$ $f(x) < 0$ for all $x \in R$ i.e., $f(x)$ is negative for all values of $x$.

($iii$) $a > 0$ & $D = 0$,



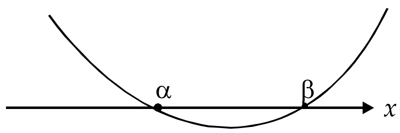

$\Rightarrow$ $f(x)$ □ 0 for all $x$ □ $R$.

($iv$) $a < 0$ & $D$




$\Rightarrow$ $f(x)$ □ 0 for all $x$ □ $R$.

($v$) $a > 0$ and $D > 0$




Let $f(x) = 0$ have two real roots □ and □,

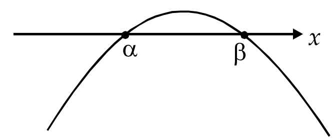
let □□□□□, then

$f(x) > 0$ for all

$x$ □ (– □□□□) □ (□□□□□□□)

and $f(x) < 0$ for all $x$ □ (□□□□)

($vi$) $a < 0$ and $D > 0$




Let $f(x) = 0$ have two real roots □ and □

(□□□□□) then $f(x) < 0$

for all $x$ □ (– □□□□) □ (□□□□□□□)

and $f(x) > 0$ for all $x$ □ (□□□□□

## PRACTICE EXERCISE

**1.** If $l$, $m$, $n$ are real, $l \neq m$, then the roots of the equation $(l - m)x^2 - 5(l + m)x - 2(l - m) = 0$ are

(a) real and equal (b) complex (c) real and unequal (d) None of these

**2.** If the roots of the equation $x^2 - bx + c = 0$ are two consecutive integers, then $b^2 - 4c$ equals

(a) –2 (b) 3 (c)2 (d) 1

**3.** The value of α for which the sum of the squares of the roots of the equation $x^2 - (\alpha - 2)x - \alpha - 1 = 0$ assumes the least value, is

(a) 1 (b) 0 (c) 3 (d) 2

**4.** If the equation $y = px + a\sqrt{1+p^2}$ is regarded as a quadratic in $p$, it will have equal root if, $x^2 + y^2$ is

(a) $-a^2$ (b) 0 (c) $a^2$ (d) None of these

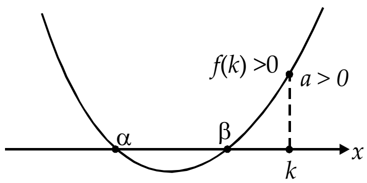
## Answers

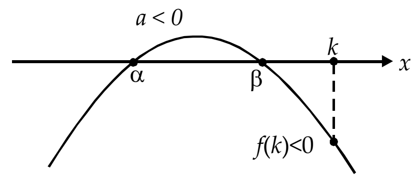
**1.** (c) **2.** (d) **3.** (a) **4.** (c)

## ❑ Location of Roots

Let $f(x) = ax^2 + bx + c$, where $a, b, c$ □ $R$ and $a$ □ $0$ and □□□□ be the roots of $f(x) = 0$

(*i*) Condition if both the roots are less than k:




$D \geq 0$ … (*i*)

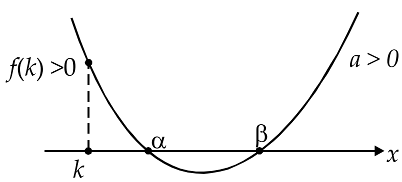
$a\,f(k) > 0$ … (*ii*)

$k > -\frac{b}{2a}$ …(iii)

(*ii*) If both the roots are greater than k.

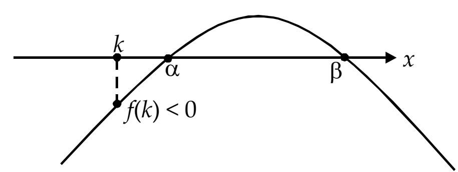
**Case I**

$a > 0$




**Case II**

$a < 0$




$D \geq 0$ … (*i*)

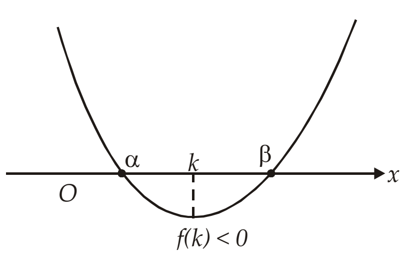
$a \,.\, f(k) > 0$ … (*ii*)

and $k < -\frac{b}{2a}$ …(iii)

(*iii*) If $k$ lies in between the roots

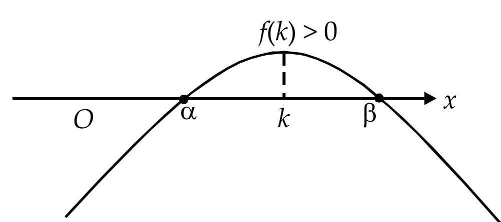
**Case I**

$a > 0$




**Case II**

$a < 0$



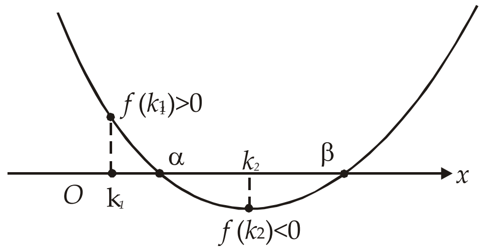

$D > 0$ … (*i*)

and $a\,f(k) < 0$ … (*ii*)

(*iv*) Let the smaller root lies in the interval $(k_1, k_2)$.

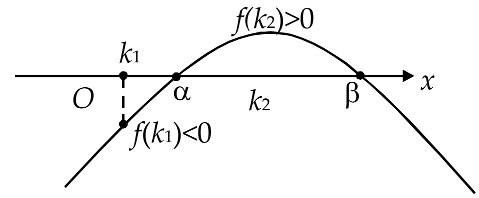
**Case I**

for $a > 0$,




**Case II**

$a < 0$




$D > 0$ ..(i)

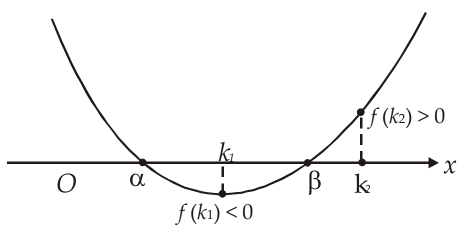
$f(k_1) \cdot f(k_2) < 0$ …(ii)

$k_1 < -\dfrac{b}{2a}$ …(iii)

(*v*) If the bigger root lies in the internal $(k_1, k_2)$

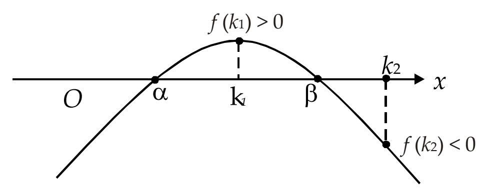
**Case I**

$a > 0$




**Case II**

$a < 0$




for $a < 0$

$D > 0$ ...(i)

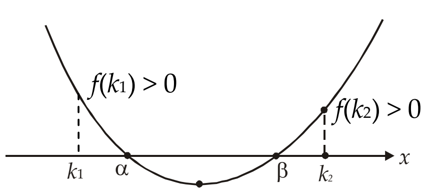
$f(k_1) \cdot f(k_2) < 0$ …(ii)

$k_2 > -\dfrac{b}{2a}$ …(iii)

(*vi*) If both the roots lie in the interval $(k_1, k_2)$

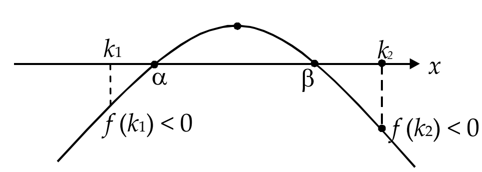
**Case I**

$a > 0$




**Case II**

$a < 0$



$D \geq 0$ ...(i)

$f(k_1) \cdot f(k_2) > 0$ …(ii)

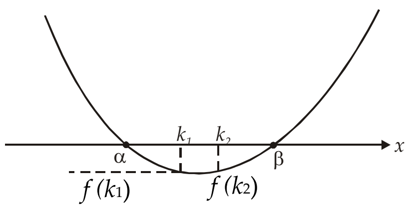
$k_1 < -\frac{b}{2a}$ …(iii)

$k_2 > -\frac{b}{2a}$ …(iv)

(*vii*) If $k_1$ & $k_2$ lie in interval (□□□□) where $k_1 < k_2$ and □□□□□.

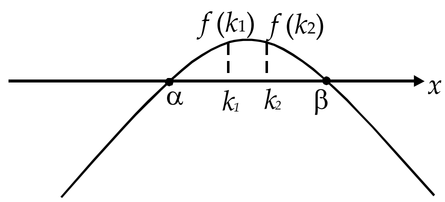
**Case I**

$a > 0$




**Case II**

$a < 0$




$D \geq 0$ ..(i)

$f(k_1) \cdot f(k_2) > 0$ …(ii)

## ❑ Theory of Polynomial Equation

(*i*) If $a_0, a_1, a_2, \ldots\ldots, a_n$ □ $R$ and $a_n$ □ 0 then $P(x) = 0$ has exactly $n$ roots (real or imaginary)

(*ii*) Imaginary roots always occur in conjugate pairs *i.e.* if $b$ □□0 and $a + ib$ is a root of $P(x) = 0$, then $a - ib$ is also a root.

(*iii*) A polynomial equation in $x$ of odd degree has at least one real root (moreover it has an odd number of real roots)

(*iv*) If $x_1, x_2, x_3, \ldots\ldots x_n$ are roots of $P(x) = 0$, then $P(x)$ can be written in the form,

$$P(x) = a_n (x - x_1)(x - x_2) \ldots\ldots\ldots (x - x_n)$$

(*v*) If $a$ is a root of $P(x) = 0$, then $(x - a)$ is a factor of $P(x)$ and vice-versa.

(*vi*) If $x_1, x_2 \ldots. x_n$ are the roots of $P(x) = 0$ *i.e.*,

$a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots. + a_1 x + a_0 = 0$, $a_n$ □ 0, then

$$\Sigma x_1 = -\frac{a_{n-1}}{a_n};$$

$$\Sigma x_1 x_2 = (-1)^2 \frac{a_{n-2}}{a_n};$$

$$\Sigma x_1 x_2 x_3 = (-1)^3 \frac{a_{n-3}}{a_n} = -\frac{a_{n-3}}{a_n}$$

$$x_1 x_2 x_3 \ldots\ldots x_n = (-1)^n \frac{a_0}{a_n}$$

(*vii*) If $P(x) = 0$ has roots □□□□□□□□□□□□□□□□□□□□□□□□$_n$ then the equation having roots

$f(\square_1), f(\square_2), f(\square_3), \ldots\ldots\ldots, f(\square_n)$ is given by $P(f^{-1}(x)) = 0$.

In particular if we have roots to be $\square_1 + k, \square_2 + k, \ldots\ldots, \square_n + k$, then the corresponding equation is $P(x - k) = 0$ and if we have roots to be $k\square_1, k\square_2, \ldots\ldots\ldots, k\square_n$, then the corresponding equation is $P(x/k) = 0$.

(*viii*) If equation, $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots + a_1 x + a_0 = 0$ is satisfied by more than $n$ number then $P(x)$ is identically zero or we can say it become an identity *i.e.* $a_n = a_{n-1} = a_{n-2} = \ldots\ldots = a_0 = 0$

(*ix*) If coefficients of $P(x)$ have '*m*' changes in sign, then $P(x) = 0$ have at most '*m*' positive roots and if $P(-x)$ have '*t*' changes in sign, then $P(x) = 0$ have at most '*t*' negative roots. Using this we can find maximum no. of real roots, of a polynomial equation.

(*x*) If $f(x)$ is a polynomial expression & $f(b) = f(a)$, then there exists atleast one root of $f'(x) = 0$ in $(a, b)$

(*xi*) If $f(a) f(b) < 0$, then there will be atleast one root in $(a, b)$ of $f(x) = 0$.

## ILLUSTRATIONS

***Illustration* 2**

If equation $(a - 1)x^2 + (a^2 + a - 2)x + (a^2 - 3a + 2) = 0$ has more than two roots then $a = \ldots\ldots\ldots\ldots$

***Solution***

$(a - 1)x^2 + (a + 2)(a - 1)x + (a - 2)(a - 1) = 0$

as it is an identity

$a - 1 = 0$ □ $a = 1$

$(a + 2)(a - 1) = 0$ □ $a = 1, -2$

$(a - 2)(a - 1) = 0$ □ $a = 1, 2$

So, $a = 1$

***Illustration* 3**

Find the maximum number of real roots of $x^6 - 3x^5 + 4x^3 + 3x^2 + 4 = 0$

Let $f(x) = x^6 - 3x^5 + 4x^3 + 3x^2 + 4$

Here $f$(x) has two changes in sign, so $f(x) = 0$ has at most two positive real roots.

$f(-x) = x^6 + 3x^5 - 4x^3 + 3x^2 + 4$, has two changes in sign so it has almost two negative real roots. Hence maximum number of real roots = 4.

***Illustration* 4**

The coefficient of $x$ in the quadratic equation $x^2 + px + q = 0$ was taken as 17 in place of 13, its roots were found to be – 2 and – 15. Find the roots of the original equation.

***Solution***

Here $q = (-2) \times (-15) = 30$, correct value of $p = 13$. Hence original equation is $x^2 + 13x + 30 = 0$ or $(x + 10)(x + 3) = 0$

□ Roots are – 10, – 3

***Illustration* 5**

Let $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ be two roots of the equation $(x + 1)^n + x^n + 1 = 0$, where □ and □ are the roots of the equation $x^2 + px + q = 0$. If □□□□ are also roots of the equation $x^{2n} + p^n x^n + q^n = 0$, then show that $n$ must be an even integer, when $p$ □ 0.

***Solution***

As $\frac{\alpha}{\beta}$ is a root of $(x + 1)^n + x^n + 1 = 0$

$$\left(\frac{\alpha}{\beta}+1\right)^n + \left(\frac{\alpha}{\beta}\right)^n + 1 = 0$$

or (□□□□□)$^n$ + □$^n$ + □$^n$ = 0 … (*i*)

But □□□□ are roots of $x^2 + px + q = 0$

□ □□□□□ = – p, □□ = q

□ from (i), $(-p)^n$ + □$^n$ + □$^n$ = 0 … (*ii*)

As □, □ are roots of $x^{2n} + p^n x^n + q^n = 0$, we get

□$^{2n}$ + $p^n$□$^n$ + $q^n$ = 0 and □$^{2n}$ + $p^n$□$^n$ + $q^n$ = 0

Subtracting, □$^{2n}$ – □$^{2n}$ + $p^n$(□$^n$ – □$^n$) = 0

or (□$^n$ + □$^n$) (□$^n$ – □$^n$) + $p^n$(□$^n$ – □$^n$) = 0

□ either □$^n$ + □$^n$ + $p^n$ = 0

or □$^n$ – □$^n$ = 0

Case (*i*) when □$^n$ + □$^n$ + $p^n$ = 0 … (*iii*)

From (*ii*) – (*iii*), $(-p)^n - p^n = 0$

□ $(-p)^n = p^n$

□ $n$ is an even integer.

**Case (ii)** when $\square^n - \square^n = 0$ *i.e.*, $\square^n = \square^n$ … (*iv*)

$(-p)^n + 2\square^n = 0$ or $\square^n = -\dfrac{(-p)^n}{2}$ … (*v*)

Also $\square\square = q$ □ $\square^n\square^n = q^n$ □ $\square^{2n} = q^n$ … (*vi*)

From (*v*) and (*vi*) $q^n = \left(\dfrac{-(-p)^n}{2}\right)^2 = \dfrac{(-p)^{2n}}{4} = \dfrac{p^{2n}}{4}$ … (*vii*)

Now, $\square^{2n} + p^n\square^n + q^n = 0$ □ $\left(-\dfrac{(-p)^n}{2}\right)^2 + p^n\left(-\dfrac{(-p)^n}{2}\right) + \dfrac{p^{2n}}{4} = 0$

□ $\dfrac{p^{2n}}{4} - (-1)^n\dfrac{p^{2n}}{2} + \dfrac{p^{2n}}{4} = 0$ □ $\dfrac{p^{2n}}{4} - (-1)^n\dfrac{p^{2n}}{2} = 0$

□ $1 - (-1)^n = 0$ [ $p$ □ 0]□ $n$ is an even integer

Thus from both the cases it follows that $n$ must be an even integer.

***Illustration* 6**

Find all the integral values of a for which the quadratic equation $(x - a)(x - 10) + 1 = 0$ has integral roots.

**Solution**

Here the equation is $x^2 - (10 + a)x + 10a + 1 = 0$… (*i*)

Since integral roots will always be rational it means $D$ should be a perfect square

Form (i) $D = a^2 - 20a + 96$

□ $D = (a - 10)^2 - 4$

□ $4 = (a - 10)^2 - D$

If $D$ is a perfect square it means we want difference of two perfect square as 4 which is possible only when $(a - 10)^2 = 4$ and $D = 0$

□ $(a - 10) = \pm 2$

□ $a = 12, 8$

**Alternative Method**

$(x - a)(x - 10) = -1$

Here $(x - a)$ and $(x - 10)$ are integers. So $(x - a)$ & $(x - 10)$ have to be opposite in sign and their magnitude will be unity.

So $(x - a) = 1$ then $(x - 10) = -1$

$\Rightarrow \quad a = 8$

and if $(x - a) = -1$ then $\quad (x - 10) = 1$

$\Rightarrow \quad a = 12$

***Illustration* 7**

Find the constant $m$ so that '1' lies between the roots of the quadratic equation

$(2m + 1)\, x^2 - mx + m - 2 = 0$

***Solution***

$(2m + 1)x^2 - mx + (m - 2) = 0$

or $\quad f(x) \equiv x^2 - \dfrac{m}{(2m+1)}x + \dfrac{(m-2)}{(2m+1)} = 0 \qquad$ (As $m \neq -1/2$)

The statement '1' lies between the roots $\Leftrightarrow$ $f(1) < 0$

$$\Rightarrow \quad 1 - \frac{m}{2m+1} + \frac{m-2}{2m+1} < 0$$

$$\frac{2m+1-m+m-2}{2m+1} < 0 \qquad \Rightarrow \quad -\frac{1}{2} < m < \frac{1}{2}$$

***Illustration* 8**

If $\alpha, \beta$ and $\gamma$ are the roots of the equation $x^3 + px^2 + qx + r = 0$, then form the equation whose roots are $\alpha - \dfrac{1}{\beta\gamma}$, $\beta - \dfrac{1}{\gamma\alpha}$ and $\gamma - \dfrac{1}{\alpha\beta}$ .

***Solution***

$$\alpha - \frac{1}{\beta\gamma} = \alpha - \frac{\alpha}{\alpha\beta\gamma} = \alpha + \frac{\alpha}{r} = \alpha\left(1 + \frac{1}{r}\right)$$

Thus the roots of the required equation are $\alpha\left(1+\dfrac{1}{r}\right), \beta\left(1+\dfrac{1}{r}\right)$ and $\gamma\left(1+\dfrac{1}{r}\right)$. Hence the required equation is obtained from the given equation by replacing $x$ by $x/\left(1+\dfrac{1}{r}\right) = \dfrac{rx}{r+1}$ .

Hence required equation is $\left(\dfrac{rx}{r+1}\right)^3 + p\left(\dfrac{rx}{r+1}\right)^2 + q\left(\dfrac{rx}{r+1}\right) + r = 0$

$\Rightarrow \quad r^2x^3 + pr\,(r + 1)\,x^2 + q\,(r + 1)^2\,x + (r + 1)^3 = 0.$

## PRACTICE EXERCISE

**6.** If the roots of the quadratic equation $x^2 - 2kx + k^2 + k - 5 = 0$ are less than 5, then $k$ lies in the interval

(a) (5, 6] (b) $(6, \infty)$ (c) $(-\infty, 4)$ (d) [4, 5]

**7.** Let $f(x) = x^2 + 2(m - 1)x + m + 5 = 0$, where $m$ is a real parameter, then

(a) $f(x) = 0$ has both roots positive, if $m \in (-5, -1)$ (b) $f(x) > 0$ for all $x$, if $m \in (-1, 4)$

(c) $f(x) > 0$ for all $x > 0$, if $m \in (-1, \infty)$ (d) All above

**8.** Let $f(x) = x^2 + ax + 8$ where $a \in R$. Let $f(x) = 0$ has roots $\alpha$ and $\beta$ then,

(a) If roots of $f(x) = 0$ are lying between 1 and 5 then $a$ must lie in the range $\left(-\frac{33}{5}, -\sqrt{32}\right)$

(b) If $f(x) > 0 \ \forall \ x \in R$ then $a \in \left(-\sqrt{32}, \sqrt{32}\right)$

(c) If only one root lies between 1 and 5, then $a \in \left(-9, \frac{33}{5}\right)$

(d) For any real values of $a$, zero can never lie between the roots of $f(x)$

**9.** The expression $\frac{ax^2 - 7x + 5}{5x^2 - 7x + a}$ will take all real values if $a$ lies in the interval $[-k, 2]$. The numerical quantity $k$ should be equal to

**10.** $a$, $b$, $c$ are positive integers. The equation $ax^2 + bx + c = 0$ has one root in $(-1, 0)$ and other root in (0, 1). If the least value of $a + c$ is $b + k$, then $k$ must be equal to

### Answers

**6.** (c) **7.** (d) **8.** (a, b, c, d) **9.** 12 **10.** 0

*****

# MISCELLANEOUS PROBLEM

## OBJECTIVE TYPE

**Example 1**

The value of $'a'$ for which the sum of the square of the roots of the equation $x^2 - (a - 2)x - a - 1 = 0$ assumes the least value is

(a) 0 (b) 1 (c) 2 (d) 3

**Solution**

Let $\alpha, \beta$ be the roots of the given equation. Then, $\alpha + \beta = a - 2$ and $\alpha\beta = -(a + 1)$.

Now, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (a - 2)^2 + 2(a + 1)$

$= a^2 - 2a + 6 = (a - 1)^2 + 5$

Clearly, $\alpha^2 + \beta^2 \geq 5$. So, the minimum value of $\alpha^2 + \beta^2$ is 5 which it attains at $a = 1$.

 **Ans. (b)**

**Example 2**

The real values of $'a'$ for which the quadratic equation $2x^2 - (a^3 + 8a - 1)x + a^2 - 4a = 0$ possesses roots of opposite signs are given by

(a) $a > 5$ (b) $0 < a < 4$ (c) $a > 0$ (d) $a > 7$

**Solution**

The roots of the given equation will be of opposite signs, if they are real and their product is negative, i.e. they are real and their product is negative, i.e.

Disc $\geq 0$ and product of roots $< 0$

$\Rightarrow (a^3 + 8a - 1)^2 - 8(a^2 - 4a) \geq 0$ and $\frac{a^2 - 4a}{2} < 0$

$\Rightarrow a^2 - 4a < 0$ [$\because$ $a^2 - 4a < 0 \Rightarrow (a^3 + 8a - 1)^2 - 8(a^2 - 4a) \geq 0$]

$\Rightarrow 0 < a < 4$.

 **Ans. (b)**

**Example 3**

Let $\alpha, \beta$ be the roots of $ax^2 + bx + c = 0$; $\gamma, \delta$ be the roots of $px^2 + qx + r = 0$; and $D_1, D_2$ the respective discriminants of the equations. If $\alpha, \beta, \gamma$ and $\delta$ are in A.P., then $D_1 : D_2 =$

(a) $\frac{a^2}{b^2}$ (b) $\frac{a^2}{p^2}$ (c) $\frac{b^2}{q^2}$ (d) $\frac{c^2}{r^2}$

**Solution**

We have

$$\alpha+\beta=\frac{-b}{a}, \alpha\ \beta=\frac{c}{a'}$$

$$\gamma+\delta=-\frac{q}{p} \text{ and } \gamma\delta=\frac{r}{p}.$$

Now,

$\alpha, \beta, \gamma, \delta$ are in AP $\Rightarrow$ $\beta-\alpha=\delta-\gamma$

$\Rightarrow$ $(\beta-\alpha)^2=(\delta-\gamma)^2 \Rightarrow (\beta+\alpha)^2-4\alpha\beta=(\gamma+\delta)^2-4\gamma\delta$

$\Rightarrow$ $\dfrac{b^2}{a^2}-\dfrac{4c}{a}=\dfrac{q^2}{p^2}-\dfrac{4r}{p} \Rightarrow \dfrac{b^2-4ac}{a^2}=\dfrac{q^2-4rp}{p^2}$

$\Rightarrow$ $\dfrac{D_1}{a^2}=\dfrac{D_2}{p^2} \Rightarrow \dfrac{D_1}{D_2}=\dfrac{a^2}{p^2}$.

 **Ans. (b)**

**Example 4**

If $\alpha$, $\beta$ are the roots of $a\,x^2+bx+c=0$ and $\alpha+h$, $\beta+h$ are the roots of $px^2+qx+r=0$, then $h=$

(a) $\left(\dfrac{b}{a}-\dfrac{q}{p}\right)$ (b) $\dfrac{1}{2}\left(\dfrac{b}{a}-\dfrac{q}{p}\right)$ (c) $-\dfrac{1}{2}\left(\dfrac{a}{b}-\dfrac{p}{q}\right)$ (d) none of these

**Solution**

We have

$\alpha+\beta=-\dfrac{b}{a}, \alpha\beta=\dfrac{c}{a},$ and

$\alpha+h+\beta+h=-\dfrac{q}{p}$ and $(\alpha+h)(\beta+h)=\dfrac{r}{p}$.

$\Rightarrow$ $\alpha+\beta+2h=-\dfrac{q}{p} \Rightarrow -\dfrac{b}{a}+2h=\dfrac{-q}{p}$ [$\because$ $\alpha+\beta=-\dfrac{b}{a}$]

$\Rightarrow$ $h=\dfrac{1}{2}\left(\dfrac{b}{a}-\dfrac{q}{p}\right)$

 **Ans. (b)**

**Example 5**

If both the roots of the equation $x^2-6ax+2-2a+9a^2=0$ exceed 3, then

(a) $a>\dfrac{9}{11}$ (b) $a\geq\dfrac{11}{9}$ (c) $a>\dfrac{11}{9}$ (d) $a<\dfrac{11}{9}$

**Solution**

Let $f(x)=x^2-6ax+2-2a+9a^2$, and let $\alpha$ be the roots of $f(x)=0$. Then equation $f(x)=0$ will have its both roots greater then 3, if

(i) Disc $\geq 0$ (ii) $\alpha + \beta > 6$ (iii) $f(3) > 0$

Now, Disc $\geq 0 \Rightarrow 36a^2 - 8 + 8a - 36a^2 \geq 0$

$\Rightarrow -8 + 8a \geq 0 \Rightarrow a \geq 1$ …(i)

$\alpha + \beta > 6 \Rightarrow 6a > 6 \Rightarrow a > 1$ …(ii)

and, $f(3) > 0 \Rightarrow 9 - 18a + 2 - 2a + 9a^2 > 0$

$\Rightarrow 9a^2 - 20a + 11 > 0 \Rightarrow 9a^2 - 11a - 9a + 11 > 0$

$\Rightarrow (9a - 11)(a - 1) > 0 \Rightarrow a < 1$ or $a > \frac{11}{9}$ …(iii)

From (i), (ii) and (iii), we get $a > 11/9$.

 **Ans. (c)**

**Example 6**

If every pair from among the equations $x^2 + px + qr = 0$, $x^2 + qx + rp = 0$ and $x^2 + rx + pq = 0$ has a common root, then the product of three common roots is

(a) $pqr$ (b) $2pqr$ (c) $p^2 q^2 r^2$ (d) None of these

**Solution**

we have

$\alpha\beta = qr$, $\beta\gamma = rp$ and $\gamma\alpha = pq$

$\Rightarrow (\alpha\beta)(\beta\gamma)(\gamma\alpha) = (qr)(rp)pq)$

$\Rightarrow (\alpha\beta\gamma)^2 = (pqr)^2 \Rightarrow \alpha\beta\gamma = pqr$

$\therefore$ **Ans. (a)**

**Example 7**

If $a \in Z$ and the equation $(x - a)(x - 10) + 1 = 0$ has integral roots, then the values of $'a'$ are

(a) 10,8 (b)12, 10 (c) 12, 8 (d) None of these

**Solution**

Since $a$ and $x$ are integers. Therefore,

$(x - a)(x - 10) + 1 = 0$

$\Rightarrow (x - a)(x - 10) = -1$

$\Rightarrow (x - a = 1$ and $x - 10 = -1)$ or $(x - a = -1$ and $x - 10 = 1)$

$\Rightarrow (x = 9$ and $a = 8)$ or $(x = 11$ and $a = 12)$

$\Rightarrow a = 8$ or $a = 12$.

$\therefore$ **Ans. (c)**

**Example 8**

If $m \in Z$ and the equation $mx^2 + (2m - 1)x + (m - 2) = 0$ has rational roots, then $m$ is of the form

(a) $n(n + 2), n \in Z$ (b) $n(n + 1), n \in Z$ (c) $n(n - 2), n \in Z$ (d) None of these

**Solution**

The discriminant $D$ of the given equation is given by $D = (2m - 1)^2 - 4m\,(m - 2) = 4m + 1$

If the given equation has rational roots, then the discriminant should be a perfect square of a rational number, say $a$. i.e., $4m + 1 = a^2 \Rightarrow a^2$ is an integer [$\because$ $4m + 1$ is an integer]

$\Rightarrow$ $a$ is an integer

Now, $4m + 1 = a^2$

$\Rightarrow$ $4m = (a^2 - 1) \Rightarrow 4m = (a - 1)(a + 1)$

$\Rightarrow$ $(a - 1)(a + 1)$ is an even integer of the form $4m$

$\Rightarrow$ $a - 1$ and $a + 1$ are even integers [$\because$ $4m$ is an even integer]

$\Rightarrow$ $a$ is an odd integer

Let $a = 2n + 1$, where $n \in Z$ then,

$a^2 = 4m + 1 \Rightarrow (2n + 1)^2 = 4m + 1$

$\Rightarrow$ $m = n\,(n + 1)$, where $n \in Z$.

$\therefore$ **Ans. (b)**

**Example 9**

The integer $'k'$ for which the inequality $x^2 - 2\,(4k - 1)\,x + 15k^2 - 2k - 7 > 0$ is valid for any $x$, is

(a) 2 (b) 3 (c) 4 (d) none of these

**Solution**

Let $f(x) = x^2 - 2\,(4k - 1)x + 15k^2 - 2k - 7$. Then,

$f(x) > 0 \Rightarrow$ Disc $< 0$ [$\because$ coeff. of $x^2 > 0$]

$\Rightarrow$ $4\,(4k - 1)^2 - 4\,(15k^2 - 2k - 7) < 0$

$\Rightarrow$ $k^2 - 6k + 8 < 0 \Rightarrow 2 < k < 4$.

$\therefore$ **Ans. (b)**

**Example 10**

The values of $'a'$ for which $2x^2 - 2\,(2a + 1)\,x + a\,(a + 1) = 0$ may have one root less than $a$ and other root greater than $'a'$ are given by

(a) $1 > a > 0$ (b) $-1 < a < 0$ (c) $a \geq 0$ (d) $a > 0$ or $a < -1$

**Solution**

The given condition suggests that $a$ lies between the roots. Let $f(x) = 2x^2 - 2\,(2a + 1)\,x + a\,(a + 1)$. For $a$ to lie between the roots, we must have

Disc $\geq 0$ and $f(a) < 0$

Now, Disc $\geq 0 \Rightarrow 4\,(2a + 1)^2 - 8a\,(a + 1) \geq 0$

$\Rightarrow$ $8\left[a^2 + a + \frac{1}{2}\right] \geq 0$, which is always true

$\therefore$ $f(a) < 0 \Rightarrow 2a^2 - 2a\,(2a + 1) + a\,(a + 1) < 0$

$\Rightarrow \quad -a^2 - a < 0 \Rightarrow a^2 + a > 0$

$\Rightarrow \quad a > 0$ or $a < -1$.

$\therefore$ **Ans. (d)**

**Example 11**

If $x^3 + 3x^2 - 9x + c$ is of the form $(x - \alpha)^2 (x - \beta)$ then $c =$

(a) $-5$ (b) 27 (c) $-27$ (d) 0

**Solution**

If $f(x) = x^3 + 3x^2 - 9x + c = (x - \alpha)^2 (x - \beta)$, then $(x - \alpha)$ is a factor of order 2. So, $x - \alpha$ is a factor of order one of $f'(x)$ i.e. $3x^2 + 6x - 9$.

Now, $f'(x) = 3x^2 + 6x - 9 = 3(x^2 + 2x - 3) = 3(x + 3)(x - 1)$

$\therefore \quad f'(x) = 0 \Rightarrow x = 1$ or $x = -3$.

This shows that either $\alpha = 1$ or $\alpha = -3$

If $\alpha = 1$, then as $\alpha$ is a root of $x^3 + 3x^2 - 9x + c = 0$. Therefore,

$$1 + 3 - 9 + c = 0 \Rightarrow c = 5$$

If $\alpha = -3$, then as $\alpha$ is a root of $x^3 + 3x^2 - 9x + c = 0$.

Therefore, $-27 + 27 + 27 + c = 0 \Rightarrow c = -27$.

$\therefore$ **Ans. (c)**

**Example 12**

The number of real roots of $(6 - x)^4 + (8 - x)^4 = 16$ is

(a) 0 (b) 2 (c) 4 (d) None of these

**Solution**

Let $y = 7 - x$. Then the given equation become.

$$(y + 1)^4 + (y - 1)^4 = 16 \Rightarrow y^4 + 6y - 7 = 0$$

$\Rightarrow (y^2 - 1)(y^2 + 7) = 0 \Rightarrow y^2 - 1 = 0$ $\quad [\because y^2 + 7 \neq 0]$

$\Rightarrow y = \pm 1$

$\Rightarrow 7 - x = \pm 1 \Rightarrow x = 6, 8$.

$\therefore$ **Ans. (b)**

## SUBJECTIVE TYPE

**Example 1**

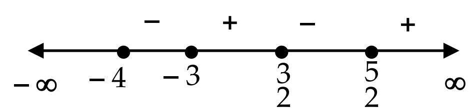
Solve: $\dfrac{8x^2+16x-51}{(2x-3)\,(x+4)} > 3$

**Solution**

$$\frac{8x^2+16x-51}{(2x-3)\,(x+4)} > 3 \;\Rightarrow\; \frac{8x^2+16x-51}{(2x-3)\,(x+4)} - 3 > 0$$

$$\Rightarrow \quad \frac{8x^2+16x-51-3(2x-3)\,(x+4)}{(2x-3)(x+4)} > 0$$

$$\Rightarrow \quad \frac{2x^2+x-15}{(2x-3)\,(x+4)} > 0 \;\Rightarrow\; \frac{(2x-5)\,(x+3)}{(2x-3)\,(x+4)} > 0$$




Critical points are: $x = -4, -3, 3/2, 5/2$

$$\therefore \quad \frac{(2x-5)(x+3)}{(2x-3)\,(x+4)} > 0$$

$$\Rightarrow \quad x \in (-\infty, -4) \cup (-3, 3/2) \cup (5/2, \infty)$$

**Example 2**

Find all the values of $a$, so that 6 lies between the roots of the equation $x^2 + 2(a-3)\,x + 9 = 0$.

**Solution**

Let $f(x) = x^2 + 2\,(a-3)\,x + 9$. If 6 lies between the roots of $f(x) = 0$, then we must have the following.

*(i)* Disc > 0, and (ii) $f(6) < 0$ ($\because$ coeff. of $x^2$ is positive).

Now,

Disc > 0 $\Rightarrow$ $4\,(a-3)^2 - 36 > 0 \;\Rightarrow\; (a-3)^2 - 9 > 0$

$\Rightarrow \quad a^2 - 6a > 0$

$\Rightarrow \quad a\,(a-6) > 0 \;\Rightarrow a < 0$ or $a > 6$

and, $f(6) < 0 \;\Rightarrow\; 36 + 12\,(a-3) + 9 < 0$

$\Rightarrow \quad 12a + 9 < 0 \;\Rightarrow\; a < -\dfrac{3}{4}$

From (i) and (ii), we get : $a < -3/4$ i.e. $a \in (-\infty, -3/4)$.

**Example 3**

Find the value of $k$, so that the equations $2x^2 + kx - 5 = 0$ and $x^2 - 3x - 4 = 0$ have one root in common.

**Solution**

Let $\alpha$ be the common root of the two equations

Then,

$2\alpha^2 + k\alpha - 5 = 0$

$\alpha^2 - 3\alpha - 4 = 0$

Solving these two equations, we get

$$\frac{\alpha^2}{-4k-15} = \frac{\alpha}{-5+8} = \frac{1}{-6-k} \quad \Rightarrow \alpha^2 = \frac{4k+15}{k+6} \text{ and } \alpha = \frac{-3}{k+6}$$

$$\Rightarrow \left(\frac{-3}{k+6}\right) = \frac{4k+15}{k+6} \qquad [\because \alpha^2 = (\alpha)^2]$$

$\Rightarrow$ $(4k + 15)(k + 6) = 9 \ \ 4k^2 + 39k + 81 = 0$

$\Rightarrow$ $k = -3$ or $k = -\frac{27}{4}$.

**Example 4**

If $\frac{a_1}{a_2}, \frac{b_1}{b_2}, \frac{c_1}{c_2}$ are in AP and the equations $a_1x^2 + 2b_1x + c_1 = 0$ and $a_2x^2 + 2b_2x + c_2 = 0$ have a common root, prove that $a_2, b_2, c_2$ are in G.P.

**Solution**

It is given that the equation $a_1x^2 + 2b_1x + c_1 = 0$ and $a_2x^2 + 2b_2x + c_2 = 0$ have a common root. So, let $\alpha$ be the common root.

$\therefore$ $a_1\alpha^2 + 2b_1\alpha + c_1 = 0$

and, $a_2\alpha^2 + 2b_2\alpha + c_2 = 0$

$$\Rightarrow \frac{\alpha^2}{2(b_1c_2 - b_2c_1)} = \frac{\alpha}{c_1a_2 - c_2a_1} = \frac{1}{2(a_1b_2 - a_2b_1)}$$

$$\Rightarrow \alpha^2 = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \text{ and } \alpha = \frac{c_1a_2 - c_2a_1}{2(a_1b_2 - a_2b_1)}$$

Now, $\alpha^2 = (\alpha)^2$

$$\Rightarrow \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} = \left\{\frac{c_1a_2 - c_2a_1}{2(a_1b_2 - a_2b_1)}\right\}^2$$

$\Rightarrow$ $4(b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1) = (c_1a_2 - c_2a_1)^2$

$$\Rightarrow 4b_2c_2\left(\frac{b_1}{b_2} \cdot \frac{c_1}{c_2}\right) . a_2b_2\left(\frac{a_1}{a_2} - \frac{b_1}{b_2}\right) = (c_2a_2)^2\left(\frac{c_1}{c_2} - \frac{a_1}{a_2}\right)^2$$

$$\Rightarrow 4a_2b_2^2c_2\left(\frac{b_1}{b_2} - \frac{c_1}{c_2}\right)\left(\frac{a_1}{a_2} - \frac{b_1}{b_2}\right) = a_2^2c_2^2\left(\frac{c_1}{c_2} - \frac{a_1}{a_2}\right)^2$$

It is given that $\frac{a_1}{a_2}, \frac{b_1}{b_2}, \frac{c_1}{c_2}$ are in A.P. Let $D$ be the common difference of this A.P. Then,

$$\therefore \frac{b_1}{b_2} - \frac{a_1}{a_2} = \frac{c_1}{c_2} - \frac{b_1}{b_2} = D \text{ and } \frac{c_1}{c_2} - \frac{a_1}{a_2} = 2D.$$

Putting these values in (i), we get

$4a_2b_2^2c_2 \times (-D) \times (-D) = a_2^2c_2^2(2D)^2$

$\Rightarrow \quad b_2^2 = a_2c_2 \;\Rightarrow\; a_2, b_2, c_2$ are in GP.

**Example 5**

If $x$ is real, prove that $\frac{x^2+34x-71}{x^2+2x-7}$ can not have value between 5 and 9.

**Solution**

Let $y=\frac{x^2-34x-71}{x^2+2x-7}$ . Then,

$x^2(y-1)+2x(y-17)-(7y-71)=0$

Since $x$ is real. Therefore, the above equation has real roots.

So,

Disc $\geq 0 \;\Rightarrow\; 4(y-17)^2+4(y-1)(7y-71)\geq 0$

$\Rightarrow \quad y^2-14y+45\geq 0 \;\Rightarrow\; (y-5)(y-9)\geq 0$

$\Rightarrow \quad y\leq 5$ or $y\geq 9$

Hence, the values of the given expression cannot lies between 5 and 9.

**Example 6**

Find the values of $a$ for which the expression $\frac{ax^2+3x-4}{3x-4x^2+a}$ assumes all real values for real values of $x$.

**Solution**

Let $y=\frac{ax^2+3x-4}{3x-4x^2+a}$. Then,

$x^2(a+4y)+3(1-y)x-(4+ay)=0$

Since $x$ is real. So, roots of the above equation are real.

$\therefore$ Disc $\geq 0$ for all $y\in R$

$\Rightarrow \; 9(1-y)^2+4(a+4y)(4+ay)\geq 0$ for all $y\in R$

$\Rightarrow \; (9+16a)y^2+(4a^2+46)y+(9+16a)\geq 0$ for $y\in R$

$\Rightarrow \; 9+16a>0$ and Disc $\leq 0$

Now, Disc $\leq 0$

$\Rightarrow \; (4a^2+46)^2-4(9+16a)(9+16a)\leq 0$

$\Rightarrow \; 4(2a^2+23)^2-4(9+16a)^2\leq 0 \;\Rightarrow\; (2a^2+23)^2-(9+16a)^2\leq 0$

$\Rightarrow \; (2a^2+16a+32)(2a^2-16a+14)\leq 0$

$\Rightarrow \; 4(a+4)^2(a^2-8a+7)\leq 0 \;\Rightarrow\; a^2-8a+7\leq 0$

$\Rightarrow \; (a-1)(a-7)\leq 0 \;\Rightarrow\; 1\leq a\leq 7.$

Thus, $9+16a>0$ and $1\leq a\leq 7$. $\Rightarrow\; 1\leq a\leq 7$.

Thus, the expression will assume all real values for all real values of $x$, if $a\in[1, 7]$.

**Example 7**

For what values of $m$ can the expression $2x^2 + mxy + 3y^2 - 5y - 2$ be expressed as the product of two linear factors?

**Solution**

Comparing the given expression with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$, we have

$a = 2$, $h = m/2$, $b = 3$, $c = -2$, $f = -5/2$, $g = 0$

The given expression is resolvable into linear factors, if

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \Rightarrow -12 - \frac{25}{2} + 2\left(\frac{m^2}{4}\right) = 0$$

$\Rightarrow \quad m^2 = 49 \Rightarrow \quad m = \pm 7.$

**Example 8**

If the expression $ax^2 + by^2 + cz^2 + 2ayz + 2bzx + 2cxy$ can be resolved into rational factors, prove that $a^3 + b^3 + c^3 = 3abc$.

**Solution**

$ax^2 + by^2 + cz^2 + 2ayz + 2bzx + 2cxy$

$$= z^2\left[a\left(\frac{x}{z}\right)^2 + b\left(\frac{y}{z}\right)^2 + c + 2a\left(\frac{y}{z}\right) + 2b\left(\frac{x}{z}\right) + 2c\left(\frac{x}{z}\right)\left(\frac{y}{z}\right)\right]$$

$= z^2\,[aX^2 + bY^2 + 2cXY + 2bX + 2aY + c] \qquad \ldots(i)$

where $X = \frac{x}{z}$ and $Y = \frac{y}{z}$.

The given expression can be resolved into rational factors if the expression within brackets in (i) is expressible into rational factors, the condition for which is

$abc + 2\,abc - a\,.\,a^2 - b.\,b^2 - c\,.\,c^2 = 0 \Rightarrow a^3 + b^3 + c^3 = 3\,abc.$

**Example 9**

Let $a$, $b$, $c$ be real numbers and $a$ □ 0. If □ is a root of $a^2x^2 + bx + c = 0$, □ is a root of $a^2x^2 - bx - c = 0$ and 0 < □ < □, then show that the equation $a^2x^2 + 2bx + 2c = 0$ has a root □ that always satisfies □ < □ < □.

***Solution***

$a^2$□$^2$ + $b$□ + $c = 0$ $\qquad \ldots (i)$

$a^2$□$^2$ – $b$□ – $c = 0$ $\qquad \ldots (ii)$

Let $f(x) = a^2x^2 + 2bx + 2c$

$f$(□) = $a^2$□$^2$ + 2 ($b$□ + $c$) = $a^2$□$^2$ – 2$a^2$□$^2$ = – $a^2$□$^2$.

= – $a^2$□$^2$ = – ive by ($i$)

$f$ (□) = $a^2$□$^2$ + 2($b$□ + $c$) = $a^2$□$^2$ + 2$a^2$□$^2$

$= 3a^2 □^2$ = + ive by (*ii*)

Since $f$ (□) and $f$ (□) are of opposite signs we know from theory of equations that a root □ of the equation $f(x) = 0$ lies between □ and □

□ □□□□□□□□□□

**Example 10**

A polynomial of degree $n > 3$ leaves remainders 2, 1, – 1 when divided by $(x - 1)$, $(x + 2)$ and $(x + 1)$ respectively. What will be the remainder if the polynomial is divided by $(x^2 - 1)(x + 2)$?

**Solution**

Let the polynomial be $f(x)$, which when divided by ($x$ – □), leaves remainder $q$ (say),

$f(x)$ □ $p$ ($x$ – □) + $q$ … (*i*)

For □ = 1, $q = 2$. Thus,

$f(x) = p(x - 1) + 2$ … (*ii*)

$x = 1$ □ $f(1) = 2$

Similarly with $(x + 2)$ and $(x + 1)$

$f(-2) = 1, f(-1) = -1$ … (*iii*)

Since $(x^2 - 1)(x + 2)$ is a cubic, the remainder when $f(x)$ is divided by $(x^2 - 1)(x + 2)$ should be of the form $ax^2 + bx + c$

□ $f(x)$ □ $(x^2 - 1)(x + 2) . l + (ax^2 + bx + c)$

Putting $x = 1, -2, -1$ respectively, we get

$f(1) = a + b + c = 2$ From (*ii*)

$f(-2) = 4a - 2b + c = 1$ From (*iii*)

$f(-1) = a - b + c = -1$ From (*iii*)

Solving these, $a = \frac{7}{6},\ b = \frac{3}{2},\ c = -\frac{2}{3}$

□ the remainder is = $\frac{7}{6}x^2 + \frac{3}{2}x - \frac{2}{3}$

**Example 11**

If $a < b < c < d$ then show that $(x - a)(x - c) +$ □$(x - b)(x - d) = 0$ has real roots.

***Solution***

Let $f(x) = (x - a)(x - c) +$ □ $(x - b)(x - d)$

Now $f(b) = (b - a)(b - c) < 0$

$f(d) = (d - a)(d - c) > 0$

□ at least one real root in $(b, d)$

Since one root is real □ both roots are real as coefficient are real.

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**Exercise - I**

## OBJECTIVE TYPE QUESTIONS

**Single choice questions**

1. If $\alpha,\beta$ are roots of the equation $ax^2 + 3x + 2 = 0$ ($a < 0$), then $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$ is greater than

   (a) 0 (b) 1 (c) 2 (d) None of these

2. If the product of the roots of the equation $x^2 - 2\sqrt{2}\ kx + 2e^{2\log k} - 1 = 0$ is 31, then the roots of the equation are real for $k$ equal to

   (a) –1 (b) 2 (c) 3 (d) 4

3. If one root of the equation $ax^2 + bx + c = 0$ be the square of the other, then

   (a) $a^3 + bc(b + c) = 3abc$ (b) $b^3 + ac(a + c) = 3abc$ (c) $c^3 + ab(a + b) = 3abc$ (d) None of these

4. The equation $ax^2 + bx + a = 0$ ($a, b \in R$) and $x^3 - 2x^2 + 2x - 1 = 0$ have 2 roots common. Then $a + b$ must be equal to

   (1) 1 (b) –1 (c) 0 (d) None of these

5. The value of $k$ for which both the roots of the equation $4x^2 - 20kx + (25k^2 + 15k - 66) = 0$ are les than 2, lies in

   (a) (4/5, 2) (b) (0, 2) (c) (–1, 4/5) (d) $(-\infty, -1)$

6. If $x + 1$ is a factor of $x^4 + (p - 3)x^3 - (3p - 5)x^2 + (2p - 9)x + 6$, then the value of $p$ is

   (a) – 4 (b) 0 (c) 4 (d) 2

7. The equation $|x - x^2 - 1| = |2x - 3 - x^2|$ has

   (a) infinitely many solutions (b) one solution
   (c) two solutions (d) no solution

8. If $\alpha$, $\beta$ are the roots of the equation $ax^2 + bx + c = 0$, then the roots of the equation $ax^2 - bx(x - 1) + c(x - 1)^2 = 0$ are

   (a) $\alpha - 1, \beta - 1$ (b) $\frac{\alpha}{1+\alpha}, \frac{\beta}{1+\beta}$ (c) $\frac{\alpha+1}{\alpha}, \frac{\beta+1}{\beta}$ (d) None of these

9. The product of real roots of the equation $|2x + 3|^2 - 3|2x + 3| + 2 = 0$ is

   (a) 5/4 (b) 5/2 (c) 5 (d) 2

10. If the roots of the equation $ax^2 + bx + c = 0$ are of the form $\frac{k+1}{k}$ and $\frac{k+2}{k+1}$, then $(a + b + c)^2$ is equal to

   (a) $b^2 - 4ac$ (b) $b^2 - 2ac$ (c) $2b^2 - ac$ (d) $\sum a^2$

11. If sin θ and cos θ are the roots of the equation $lx^2 + mx + n = 0$, then

(a) $l^2 - m^2 + 2ln = 0$ (b) $l^2 + m^2 + 2ln = 0$

(c) $l^2 - m^2 - 2ln = 0$ (d) $l^2 + m^2 - 2ln = 0$

12. The equation $x^{\frac{3}{4}(\log_2 x)^2 + \log_2 x - \frac{5}{4}} = \sqrt{2}$ has

(a) only one real root (b) exactly three real roots

(c) exactly one rational root (d) non-real roots

13. The set of real roots of the equation $\log_{(5x+4)}(2x+3)^3 - \log_{(2x+3)}(10x^2 + 23x + 12) = 1$ is

(a) {–1} (b) {–3/5} (c) empty set (d) {– 1/3}

14. The values of '*a*' for which $(a^2 - 1)x^2 + 2(a - 1)x + 2$ is positive for any *x* are

(a) $a > 1$ (b) $a < 1$ (c) $a > -3$ (d) $a < -3$ or $a > 1$

15. The number of roots of the equation $x^2 + x + 3 + 2 \sin x = 0$, $x \in [-\pi, \pi]$ is

(a) 2 (b) 3 (c) 4 (d) None of these

**More than one correct option**

1. If the equation $x^2 + bx + c = 0$ and $bx^2 + cx + 1 = 0$ have a common root then

(a) $b + c + 1 = 0$ (b) $b^2 + c^2 + 1 = bc$

(c) $(b - c)^2 + (b - 1)^2 + (c - 1)^2 = 0$ (d) $b + c + 1 = bc$

2. The quadratic equation $x^2 - (m - 3)x + m = 0$ has

(a) real distinct roots if and only if $m \in (-\infty, 1) \cup (9, \infty)$

(b) both positive roots if and only if $m \in (9, \infty)$

(c) both negative roots if and only if $m \in (0, 1)$

(d) at least one positive root if and only if $m \in (9, \infty)$s

3. Let *a*, *b*, *c* be three distinct real numbers such that each of the expression $ax^2 + bx + c$, $bx^2 + cx + a$ and $cx^2 + ax + b$ is positive for each $x \in R$ and let $\alpha = \frac{bc + ca + ab}{a^2 + b^2 + c^2}$, then

(a) $\alpha < 4$ (b) $\alpha < 1$ (c) $\alpha > 1/4$ (d) $\alpha > 1$

4. If *a*, *b*, *c* are in G.P., then the equation $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root then

(a) $\frac{a}{d}, \frac{b}{e}, \frac{c}{f}$ are in H.P. (b) $\frac{a}{d}, \frac{b}{e}, \frac{c}{f}$ are in G.P.

(c) $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in A.P. (d) $\frac{a}{b}, \frac{b}{e}, \frac{c}{f}$ are in A.P.

5. For real *x*, the expression $\frac{(x-a)(x-b)}{x-c}$ will assume all real values provided

(a) $a > b > c$ (b) $a < b < c$ (c) $a > c > b$ (d) $a < c < b$

6. If $x^2 + mx + 1 = 0$ and $(b - c)x^2 + (c - a)x + (a - b) = 0$ have both the roots common, then

(a) $m = -2$ (b) $m = -1$

(c) $a, b, c$ are in A.P. (d) $a, b, c$ are in H.P.

7. If $\alpha$, $\beta$ are the roots of the equation $ax^2 + 2bx + c = 0$ and $\alpha + h$, $\beta + h$ are the roots of the equation $Ax^2 + 2Bx + C = 0$, then

(a) $h = \frac{b}{a} - \frac{B}{A}$ (b) $\frac{b^2 - ac}{B^2 - AC} = \frac{a^2}{A^2}$ (c) $h = \frac{Ac + aC}{A + a}$ (d) $\frac{b^2 - ac}{B^2 - AC} = \frac{a}{A}$

8. Let $\alpha$, $\beta$ be the roots of $x^2 - 4x + A = 0$ and $\gamma$, $\delta$ be the roots of $x^2 - 36x + B = 0$. If $\alpha, \beta, \gamma, \delta$ forms an increasing G.P., then

(a) $B = 81$ A (b) $A = 3$ (c) $B = 243$ (d) $A + B = 251$

9. If $abc < 0$, then the equation $ax^2 + 2(b + c - a)x + bc = 0$

(a) real roots (b) one positive and one negative root

(c) both positive roots (d) both negative roots

10. If equation $ax^2 + bx + 5 = 0$ does not have real and distinct roots then (where $a, b, \in R$)

(a) $5a + b \geq -1$ (b) $a > 0$ (c) a < 0 (d) $a + b + 5 \geq 0$

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**Exercise - III**

## ASSERTION & REASON , COMPREHENSION & MATCHING TYPE

**Assertion and Reason Based Questions**

In the following question, a statement of Assertion (A) is given which is followed by a corresponding statement of reason (R). Mark the correct answer out of the following options/codes.

(a) If both (A) and (R) are true and (R) is the correct explanation of (A).

(b) If both (A) and (R) are true but (R) is not correct explanation of (A).

(c) If (A) is true but (R) is false.

(d) If (A) is false but (R) is true.

1. A: Roots of $x^2 - 2\sqrt{2}x - 34 = 0$ are rational.

   R: Discriminant of $x^2 + 2\sqrt{2}x - 34 = 0$ is a perfect square.

2. A: If $\alpha$, $\beta$ are roots of equation $x^2 - 3x + a = 0$, $a \in R$ and $\alpha < 1 < \beta$, then $a < 2$.

   R: If $f(x) = x^2 - 3x + a$, then $f(1) < 0 \Rightarrow a < 2$.

3. A: Both roots of $2x^2 + \lambda^2 x + 5 = 0$ cannot be positive, if real.

   R: Both roots of $ax^2 + bx + c = 0$ may be positive if $a$ and $c$ have same sign and $D > 0$.

4. A: If $a$ and $b$ integers and roots of $x^2 + ax + b = 0$ are rational, then they must be integers.

   R: If the co-efficient of $x^2$ in a quadratic equation is unity, then its roots must be integers.

5. A: The quadratic equation $(a-b)x^2 + (b-c)x + (c-a) = 0$ have one root $x = 1$

   R. If sum of coefficients in a quadratic equation vanishes then its one root is $x = 1$.

***Passage Based Questions***

**Passage – I**

Consider the inequality, $9^x - a \cdot 3^x - a + 3 \le 0$, where '$a$' is a real parameter. Then the values of '$a$' for which the given inequality has,

1. At least one negative solution, if

   (a) $a \in (2, 3)$ (b) $a \in (2, \infty)$ (c) $a \in (-\infty, 2)$ (d) $a \in (-\infty, 3)$

2. At least one positive solution if

   (a) $a \in (3, \infty)$ (b) $a \in (2, 3)$ (c) $a \in (3, 9)$ (d) $a \in (2, \infty)$

3. At least one real solution, if

(a) $(2, \infty)$ (b) $(0, \infty)$ (c) $(-\infty, \infty)$ (d) $(3, \infty)$

**Passage – II**

Let $f(x) = ax^2 + bx + c$ be a quadratic expression and $y = f(x)$ has graph as shown in figure.

1. Which of the following is false

(a) $ab > 0$ (b) $abc < 0$ (c) $a\ c < 0$ (d) $bc < 0$

2. Which of the following is true

(a) $a + b + c > 0$ (b) $a - b + c > 0$ (c) $a + 3b + 9c < 0$ (d) $a - 3b + 9c > 0$

3. If $f(x)$ is an integer whenever $x$ is integer, then which of the following is always correct?

(a) $a$ is an integer (b) $2a$ is an integer (c) $b$ is an integer (d) $\frac{c}{2}$ is an integer

**Matching Type Questions**

1. The value of $k$ for which the equation $kx^3 - 3x^2 + 1 = 0$ has

| Column I | Column II |
|---|---|
| (a) all three real roots | (p) $\lvert k\rvert > 2$ |
| (b) two equal roots | (q) $k = -2, 2$ |
| (c) exactly one real roots | (r) $\lvert k\rvert < 2$ |

(a) (a-r,q), (b-q), (c-p) (b)(a-q), (b-r), (c-p)

(c) (a-r), (b-q), (c-p) (d)(a-p), (b-q), (c-r)

2 The set of value(s) of $k \in R$ for which

| Column I | Column II |
|---|---|
| (a) $kx^2 - (k + 1)x + 2k - 1 = 0$ has no real roots | (p) $\{1, -2\}$ |
| (b) $x^2 - 2(4k - 1)x + 15k^2 - 2k - 7 > 0$ for each $x$ | (q) $(-\infty, -1/7) \cup (1, \infty)$ |
| (c) Sum of the roots of $x^2 + (2 - k - k^2)x - k^2 = 0$ is zero | (r) $\{-4\}$ |
| (d) The roots of $x^2 + (2k - 1)x + k^2 + 2 = 0$ are in the ratio 1 : 2 | (s) $(2, 4)$ |

(a) (a-s), (b-q ), (c-p), (d-r) (b)(a-q), (b-s ), (c-p), (d-r)

(c) (a-p), (b-r), (c-s), (d-q) (d)(a-r), (b-q), (c-p), (d-s)

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**Exercise - III**

## SUBJECTIVE TYPE

1. Prove that the roots of the equation (x – a) (x – b) + (x – b) (x – c) + (x – c) (x – a) = 0 are equal if and only if a = b = c.

2. Let $x^2 - (m - 3)x + m = 0 (m \in R)$ be a quadratic equation. Find the value of m for which the roots are
   - (a) Real and distinct.
   - (b) Equal.
   - (c) Not real.
   - (d) Opposite in sign.
   - (e) Equal in magnitude but opposite in sign.
   - (f) Positive
   - (g) Negative
   - (h) Such that at least one is positive.
   - (i) One root is smaller than 2 and the other root is greater than 2.
   - (j) Both the roots are greater than 2.
   - (k) Both the roots are smaller than 2.
   - (l) Exactly one root lies in the interval (1, 2).
   - (m) Both the roots lies in the interval (1, 2).
   - (n) At least one root lies in the interval (1, 2).
   - (o) One root is greater than 2 and the other root is smaller than 1.

3. If the roots of the quadratic equation $p(q-r)x^2 + q(r-p)x + r(p-q) = 0$ be equal, then prove that $\frac{1}{r} + \frac{1}{p} = \frac{2}{q}$.

4. If f(x) is a quadratic expression such that $f(x) > 0 \ \forall \ x \in R$, and if $g(x) = f(x) + f'(x) + f''(x)$, then prove that $g(x) > 0 \ \forall \ x \in R$..

5. Solve $4^x + 6^x = 9^x$.

6. Find the values of 'a' for which $4^t - (a-4)2^t + \frac{9}{4}a < 0 \quad \forall \quad t \in (1, 2)$

7. Find the values of 'a' for which the equation $\left(x^2+x+2\right)^2 - (a-3)\left(x^2+x+2\right)\left(x^2+x+1\right) + (a-4)\left(x^2+x+1\right)^2 = 0$ has at least one real root.

8. Find a for which $x^2 - x + a - 3 < 0$ for at least one negative x.

9. If the quadratic equation $ax^2 + bx + c = 0$ has real roots of opposite sign in the interval (–2, 2), then prove that $1 + \frac{c}{4a} - \left|\frac{b}{2a}\right| > 0$.

10. For what integral values of 'a' the equation $x^2 - x(1-a) - (a+2) = 0$ has integral roots.

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**Exercise - IV**

## IIT – JEE PROBLEMS

**A. Fill in the blanks**

1. If $2+i\sqrt{3}$ is a root of the equation $x^2 + px + q = 0$ where $p$ and $q$ are real, then $(p, q)$ =(____ , ____).
2. If the products of the roots of the equation $x^2 - 2kx + 2e^{2 \ln k} - 1 = 0$ is 7, then the roots are real for $k$ = _____.
3. The system of equations $\lambda x + y + z = 0$, $-x + \lambda y + z = 0$, $-x - y + \lambda z = 0$ will have a non-zero solution if real values of $\lambda$ are given by ______ .
4. If the equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ $(a \neq b)$ have a common root, then the numerical value of $a + b$ is ______ .
5. The solution of the equation $\log_7 \log_5 \left(\sqrt{x+5} + \sqrt{x}\right) = 0$ is ______ .
6. If $\alpha$, $\beta$, $\gamma$ are the cube roots of $p$, $p < 0$, then for any $x$, $y$ and $z$, $\dfrac{x\alpha + y\beta + z\gamma}{x\beta + y\gamma + z\alpha} =$ ______ .
7. The sum of all the real roots of the equations $|x - 2|^2 + |x - 2| - 2 = 0$ is ______ .

**B. True / False**

8. If $x - r$ is a factor of the polynomial $f(x) = a_n x^n + \ldots + a_0$, repeated $m$ times $(1 < m \leq n)$, then $r$ is a root of $f'(x) = 0$ repeated $m$ times.
9. The equation $2x^2 + 3x + 1 = 0$ has an irrational root.
10. If $a < b < c < d$, then the roots of the equation $(x - a)(x - c) + 2(x - b)(x - d) = 0$ are real and distinct.
11. If $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + dx + c$, where $ac \neq 0$, then $P(x)\,Q(x)$ has at least two real roots.

**C. Multiple Choice Questions with ONE correct answer**

12. Let$a > 0$, $b > 0$ and $c > 0$. Then both the roots of the equation $ax^2 + bx + c = 0$
    (a) are real and negative (b)have negative real parts
    (c) have positive real parts (d)none of these
13. Both the roots of the equation
    $(x - b)(x - c) + (x - a)(x - c) + (x - a)(x - b) = 0$ are always
    (a) positive (b) negative (c) real (d) none of these
14. The number of real solutions of the equation $|x|^2 - 3|x| + 2 = 0$
    (a) 4 (b) 1 (c) 3 (d) 4

15. If $x_1, x_2, \ldots, x_n$ are any real numbers and $n$ is any positive integer, then

(a) $n\sum_{i=1}^{n} x_i^2 < \left(\sum_{i=1}^{n} x_i\right)^2$ (b) $\sum_{i=1}^{n} x_i^2 \geq \left(\sum_{i=1}^{n} x_i\right)$ (c) $\sum_{i=1}^{n} x_i^2 \geq n\left(\sum_{i=1}^{n} x_i\right)^2$ (d) none of these

16. The largest interval for which $x^{12} - x^9 + x^4 - x + 1 > 0$ is

(a) $-4 < x \leq 0$ (b) $0 < x < 1$ (c) $-100 < x < 100$ (d) $-\infty < x < \infty$

17. If $a + b + c = 0$, then the quadratic equation $3ax^2 + 2bx + c = 0$ has

(a) at least one root in (0, 1) (b)one root in (2, 3) and the other in (–2, –1)

(c) imaginary roots (d) none of these

18. The equation $x - \frac{2}{x-1} = 1 - \frac{2}{x-1}$ has

(a)no root (b)one root (c)two equal root (d)infinitely many roots

19. If $a$, $b$ and $c$ are distinct positive numbers, then the expression $(b + c - a)(c + a - b)(a + b - c) - abc$ is

(a) positive (b) negative (c) non-positive (d) non-negative

20. If $\alpha$ and $\beta$ are the roots of $x^2 + px + q = 0$ and $\alpha^4, \beta^4$ are the roots of $x^2 - rx + s =$ , then the equation $x^2 - 4\,qx + 2q^2 - r = 0$ has always

(a) two real roots (b) two positive roots

(c) two negative roots (d) one positive and one negative root

21. Let $a, b, c$ be real numbers, $a \neq 0$. If a root of $a^2x^2 + bx + c = 0$, $\beta$ is the root of $a^2x^2 - bx - c = 0$ and $0 < \alpha < \beta$, then the equation $a^2x^2 + 2bx + 2c = 0$ has a root $\gamma$ that always satisfies.

(a) $\gamma = \frac{\alpha+\beta}{2}$ (b) $\gamma = a + \frac{\beta}{2}$ (c) $\gamma = \alpha$ (d) $\alpha < \gamma < \beta$

22. Let $\alpha, \beta$ be the roots of the equation $(x - a)(x - b) = c$, $c \neq 0$ Then the roots of the equation $(x - \alpha)(x - \beta) + c = 0$ are

(a) $a, c$ (b) $b, c$ (c) $a, b$ (d) $a + c, b + c$

23. The equation $\sqrt{x+1} - \sqrt{x-1} = \sqrt{4x-1}$ has

(a) no solution (b) one solution

(c) two solution (d) more than two solutions

24. In a triangle $PQR$, $\angle R = \frac{\pi}{2}$, if $\tan\left(\frac{P}{2}\right)$ and $\tan\left(\frac{Q}{2}\right)$ are the roots of the equation $ax^2 + bx + c = 0 (a \neq 0)$, then

(a) $a + b = c$ (b) $b + c = a$ (c) $a + c = b$ (d) $b = c$

25. If the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are real and less than 3, then

(a) $a < 2$ (b) $2 \leq a \leq 3$ (c) $3 < a \leq 4$ (d) $a > 4$

26. If $\alpha$ and $\beta$ $(\alpha < \beta)$ are the roots of the equation $x^2 + bx + c = 0$, where $c < 0 < b$, then

(a) $0 < \alpha < \beta$ (b) $\alpha < 0 < \beta < |\alpha|$ (c) $\alpha < \beta < 0$ (d) $\alpha < 0 < |\alpha| < \beta$

27. If $b > a$, then the equation $(x - a)(x - b) - 1 = 0$ has

(a) both roots in $(a, b)$ (b) both roots in $(-\infty, a)$

(c) both roots in $(b, +\infty)$ (d) one root in $(-\infty, a)$ and the other in $(b, +\infty)$

28. For the equation $3x^2 + px + 3 = 0$, $p > 0$, if one of the root is square of the other, then $p$ is equal to

(a) 1/3 (b) 1 (c) 3 (d) 2/3

29. The number of values of $k$ for which the system of equations $(k + 1)x + 8y = 4k$, $kx + (k + 3)y = 3k - 1$ has infinitely many solution, is

(a) 0 (b) 1 (c) 2 (d) infinite

30. The set of all real numbers $x$ for which $x^2 - |x + 2| + x > 0$ is

(a) $(-\infty, -2) \cup (2, \infty)$ (b) $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

(c) $(-\infty, -1) \cup (1, \infty)$ (d) $(\sqrt{2}, \infty)$

31. For all '$x$' $x^2 + 2ax + (10 - 3a) > 0$, then the interval in which '$a$' lies is

(a) $a < -5$ (b) $-5 < a < 2$ (c) $a > 5$ (d) $2 < a < 5$

32. If one root is square of the other root of the equation $x^2 + px + q = 0$, then the relation between $p$ and $q$ is

(a) $p^3 - (3p - 1)q + p^2 = 0$ (b) $p^3 - q(3p + 1) + q^2 = 0$

(c) $p^3 + q(3p - 1) + q^2 = 0$ (d) $p^3 + q(3p + 1) + q^2 = 0$

33. If $a$, $b$, $c$ are the sides of a triangle $ABC$ such that $x^2 - 2(a + b + c)x + 3\lambda(ab + bc + ca) = 0$ has real roots, then

(a) $\lambda < \frac{4}{3}$ (b) $\lambda > \frac{5}{3}$ (c) $\lambda \in \left(\frac{4}{3}, \frac{5}{3}\right)$ (d) $\lambda \in \left(\frac{1}{3}, \frac{5}{3}\right)$

**D. Subjective Problems**

1. If the quadratic equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$, $(a \neq b)$ have a common root, then the numerical value of $a + b$ is.

2. If $ax^2 - bx + 5 = 0$ does not have 2 distinct real roots, then find the minimum value of $5a + b$.

3. If $x^2 - 10ax - 11b = 0$ have roots $c$ and and $d$. $x^2 - 10cx - 11d = 0$ have roots $a$ and $b$. Find $a + b + c + d$

4. $a$, $b$, $c$ are positive integers. The equation $ax^2 + bx + c = 0$ has one root in $(-1, 0)$ and other root in $(0, 1)$. If the least value of $a + c$ is $b + k$, then $k$ must be equal to.

5. For what values of $m$, does the system of equations $3x + my = m$, $2x - 5y = 20$ has solution satisfying the conditions $x > 0$, $y > 0$ ?

6. If one root of the quadratic equation $ax^2 + bx + c = 0$ is equal to the nth power of the other, then show that $\left(ac^n\right)^{\frac{1}{n+1}} + \left(a^n c\right)^{\frac{1}{n+1}} + b = 0$.

7. Find all real values of which satisfy $x^2 - 3x + 2 > 0$ and $x^2 - 2x - 4 \leq 0$.

8. If $a > 0$, $b > 0$ and $c > 0$ prove that $(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9$.

9. Solve for $x$: $\left(5+2\sqrt{6}\right)^{x^2-3}+\left(5-2\sqrt{6}\right)^{x^2-3}=10$

10. For $a \le 0$, determine all real roots of the equation $x^2 - 2a|x - a) - 3a^2 = 0$.

11. Solve: $|x^2 + 4x + 3| + 2x + 5 = 0$

12. Find the set of all $x$ for which $\dfrac{2x}{2x^2+5x+2} > \dfrac{1}{x+1}$.

13. Let $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$ be the roots of $ax^2 + bx + c = 0$ and $px^2 + qx + r = 0$ respectively. If the system of equations $\alpha_1 y + \alpha_2 z = 0$ and $\beta_1 y + \beta_2 z = 0$ has a non trivial solution, then prove that $\dfrac{b^2}{q^2} = \dfrac{ac}{pr}$

14. Let $a$, $b$, $c$ be real. If $ax^2 + bx + c = 0$ has two real roots $\alpha$ and $\beta$, where $\alpha < -1$ and $\beta > 1$, then show that $1+\dfrac{c}{a}+\left|\dfrac{b}{a}\right|<0$.

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# ANSWERS

## Exercise - I

**Only One Option is correct**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. | (d) | 2. | (d) | 3. | (b) | 4. | (c) | 5. | (d) |
| 6. | (c) | 7. | (b) | 8. | (b) | 9. | (b) | 10. | (a) |
| 11. | (a) | 12. | (b) | 13. | (d) | 14. | (d) | 15. | (d) |

**More Than One Choice Correct**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. | (a,c ) | 2. | (a, b, c, d) | 3. | (b, c) | 4. | (a, c) | 5. | (c, d) |
| 6. | (a, c) | 7. | (a, b) | 8. | (a, b, c) | 9. | (a, b) | 10. | (a, b, d) |

## Exercise - II

**Assertion and Reason**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. | (d) | 2. | (b) | 3. | (b) | 4. | (c) | 5. | (a) |

**Passage Based Questions**

**Passage – I**

| | | | | | |
|---|---|---|---|---|---|
| 1. | (a) | 2. | (d) | 3. | (d) |

**Passage – II**

| | | | | | |
|---|---|---|---|---|---|
| 1. | (b) | 2. | (d) | 3. | (b) |

**Matching Type Questions**

| | | | |
|---|---|---|---|
| 1. | (a) | 2. | (b) |

## Exercise - III

**Subjective Type Questions**

2. (a) $m \in (-\infty, 1) \cup (9, \infty)$ (b) $m \in \{1.9\}$ (c) $m \in (1, 9)$

(d) $m < 0$ (e) $m \in \phi$ (f) $m \in [9, \infty)$

(g) $m \in (0, 1]$ (h) $m \in (-\infty, 0) \cup [9, \infty)$ (i) m 10

(j) $m \in [9, 10)$ (k) $m \in (-\infty, 1]$ (l) $m > 10$

(m) $m \in \phi$ (n) $m \in (10, \infty)$ (o) $m \in \phi$

5. $x = \dfrac{In(\sqrt{5}-1) - In(2)}{In2 - In3}$

6. No such a exist

7. $5 < a \le \dfrac{19}{3}$

8. $-\infty < a < 3$

10. $-2, 0$

## Exercise - IV

### IIT-JEE Level Problem

### Section – A

1. (–4, 7) 2. $k = 2$ 3. 0 4. –1 5. 4
6. $\omega^2$ 7. 4

### Section – B

8. False 9. False 10. False 11. False

### Section – C

| | | | | |
|---|---|---|---|---|
| 12. (b) | 13. (c) | 14. (a) | 15. (d) | 16. (d) |
| 17. (a) | 18. (a) | 19. (b) | 20. (d) | 21. (d) |
| 22. (c) | 23. (a) | 24. (a) | 25. (a) | 26. (b) |
| 27. (d) | 28. (c) | 29. (b) | 30. (b) | 31. (b) |
| 32. (a) | 33. (a) | | | |

### Section – D

1. –1 2. –1 3. 1210 4. 1

5. $m \in \left(-\infty, \frac{-15}{2}\right) \cup (30, \infty)$ 7. $x\ ke \left[1-\sqrt{5}, 1\right] \cup \left(1 | \sqrt{5}, 2\right)$

9. $x \in (\pm 2, \pm \sqrt{2})$ 10. $x = \{a(1 - \sqrt{2}), a(\sqrt{6} - 1)\}$ 11. $x = -4, (-1 - \sqrt{3})$

12. $x \in (-2, -1) \cup \left(-\frac{2}{3}, -\frac{1}{2}\right)$

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