POLYNOMIALS

2
CHAPTER

CONTENTS

➢ INTRODUCTION

Algebra is that branch of mathematics which treats the relation of numbers.

➢ CONSTANTS AND VARIABLES

In algebra, two types of symbols are used: constants and variable (literals).

◈ Constant :

It is a symbol whose value always remains the same, whatever the situation be.

For example: 5, –9, $\frac{3}{8}$, π, $\frac{7}{15}$, etc.

◈ Variable :

It is a symbol whose value changes according to the situation.

For example : x, y, z, ax, a + x, 5y, – 7x, etc.

➢ ALGEBRAIC EXPRESSION

(a) An algebraic expression is a collection of terms separated by plus (+) or minus (–) sign. For example : 3x + 5y, 7y – 2x, 2x – ay + az, etc.

(b) The various parts of an algebraic expression that are separated by '+' or '–' sign are called terms.

For example :

	Algebraic expression	**No. of terms**	**Terms**
(i)	–32x	1	–32x
(ii)	2x + 3y	2	2x and 3y
(iii)	ax – 5y + cz	3	ax, –5y and cz
(iv)	$\frac{3}{x} + \frac{y}{7} - \frac{xy}{8} + 9$	4	$\frac{3}{x}, \frac{y}{7}, -\frac{xy}{8}$ and 9 & so on.

Types of Algebraic Expressions :

(i) Monomial : An algebraic expression having only one term is called a monomial. For ex. 8y, –7xy, $4x^2$, abx, etc. 'mono' means 'one'.

(ii) Binomial : An algebraic expression having two terms is called a binomial. For ex. 8x + 3y, 8x + 3, 8 + 3y, a + bz, 9 – 4y, $2x^2 - 4z$, $6y^2 - 5y$, etc. 'bi' means 'two'.

(iii) Trinomial : An algebraic expression having three terms is called a trinomial. For ex. ax – 5y + 8z, $3x^2 + 4x + 7$, $9y^2 - 3y + 2x$, etc. 'tri means 'three'.

(iv) Multinomial : An algebraic expression having two or more terms is called a multinomial.

➢ FACTORS AND COEFFICIENTS

◈ Factor :

Each combination of the constants and variables, which form a term, is called a factor.

For examples :

(i) 7, x and 7x are factors of 7x, in which 7 is constant (numerical) factor and x is variable (literal) factor.

(ii) In $\mathbf{-5x^2y}$, the numerical factor is –5 and literal factors are : x, y, xy, x^2 and x^2y.

◈ Coefficient :

Any factor of a term is called the coefficient of the remaining term.

For example :

(i) In **7x** ; 7 is coefficient of x

(ii) In $\mathbf{-5x^2y}$**;** 5 is coefficient of $-x^2y$; –5 is coefficient of x^2y.

Ex. 1 Write the coefficient of :

(i) x^2 in $3x^3 - 5x^2 + 7$

(ii) xy in 8xyz

(iii) –y in $2y^2 - 6y + 2$

(iv) x^0 in $3x + 7$

Sol. (i) –5

(ii) 8z

(iii) 6

(iv) Since $x^0 = 1$, Therefore

$3x + 7 = 3x + 7x^0$

coefficient of x^0 is 7.

➢ DEGREE OF A POLYNOMIAL

The greatest power (exponent) of the terms of a polynomial is called degree of the polynomial.

For example :

(a) In polynomial $5x^2 - 8x^7 + 3x$:

(i) The power of term $5x^2 = 2$

(ii) The power of term $-8x^7 = 7$

(iii) The power of 3x = 1

Since, the greatest power is 7, therefore degree of the polynomial $5x^2 - 8x^7 + 3x$ is 7

(b) The degree of polynomial :

(i) $4y^3 - 3y + 8$ is 3

(ii) 7p + 2 is 1($p = p^1$)

(iii) $2m - 7m^8 + m^{13}$ is 13 and so on.

❖ EXAMPLES ❖

Ex.2 Find which of the following algebraic expression is a polynomial.

(i) $3x^2 - 5x$ (ii) $x + \frac{1}{x}$

(iii) $\sqrt{y} - 8$ (iv) $z^5 - \sqrt[3]{z} + 8$

Sol. (i) $3x^2 - 5x = 3x^2 - 5x^1$

It is a polynomial.

(ii) $x + \frac{1}{x} = x^1 + x^{-1}$

It is not a polynomial.

(iii) $\sqrt{y} - 8 = y^{1/2} - 8$

Since, the power of the first term ($\sqrt{y}$) is $\frac{1}{2}$, which is not a whole number.

(iv) $z^5 - \sqrt[3]{z} + 8 = z^5 - z^{1/3} + 8$

Since, the exponent of the second term is 1/3, which in not a whole number. Therefore, the given expression is not a polynomial.

Ex.3 Find the degree of the polynomial :

(i) $5x - 6x^3 + 8x^7 + 6x^2$

(ii) $2y^{12} + 3y^{10} - y^{15} + y + 3$

(iii) x

(iv) 8

Sol. (i) Since the term with highest exponent (power) is $8x^7$ and its power is 7.

$\therefore$ The degree of given polynomial is 7.

(ii) The highest power of the variable is 15

$\Rightarrow$ degree = 15.

(iii) $x = x^1 \Rightarrow$ degree is 1.

(iv) $8 = 8x^0 \Rightarrow$ degree = 0

➢ TYPES OF POLYNOMIALS

(A) Based on degree :

If degree of polynomial is

			Examples
1.	One	Linear	$x + 3,\ y - x + 2,\ \sqrt{3}\ x - 3$
2.	Two	Quadratic	$2x^2 - 7, \frac{1}{3}x^2 + y^2 - 2xy, x^2 + 1 + 3y$
3.	Three	Cubic	$x^3 + 3x^2 - 7x + 8,\ 2x^2 + 5x^3 + 7,$
4.	Four	bi-quadratic	$x^4 + y^4 + 2x^2y^2,\ x^4 + 3, \ldots$

(B) Based on Terms :

If number of terms in polynomial is

			Examples
1.	One	Monomial	$7x,\ 5x^9,\ \frac{7}{3}x^{16},\ xy, \ldots\ldots$
2.	Two	Binomial	$2 + 7y^6,\ y^3 + x^{14},\ 7 + 5x^9, \ldots$
3.	Three	Trinomial	$x^3 - 2x + y,\ x^{31} + y^{32} + z^{33}, \ldots..$

Note : (1) Degree of constant polynomials

(**Ex.**5, 7, –3, 8/5, …) is zero.

(2) Degree of zero polynomial (zero = 0 = zero polynomial) is not defined.

➢ POLYNOMIAL IN ONE VARIABLE

If a polynomial has only one variable then it is called polynomial in one variable.

Ex. $P(x) = 2x^3 + 5x - 3$	Cubic trinomial
$Q(x) = 7x^7 - 5x^5 - 3x^3 + x + 3$	polynomial of degree 7
$R(y) = y$	Linear, monomial
$S(t) = t^2 + 3$	Quadratic Binomial

Note : General form of a polynomial in one variable x of degree 'n' is $a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0,\ a_n \neq 0$, where $a_n, a_{n-1}, \ldots a_2, a_1, a_0$ all are constants.

$\therefore$ for linear $ax + b, \quad a \neq 0$

for quadratic $ax^2 + bx + c, \quad a \neq 0$

for cubic $ax^3 + bx^2 + cx + d, \ a \neq 0$

➢ REMAINDER THEOREM

(i) Remainder obtained on dividing polynomial p(x) by x – a is equal to p(a) .

(ii) If a polynomial p(x) is divided by (x + a) the remainder is the value of p(x) at x = –a.

(iii) (x – a) is a factor of polynomial p(x) if p(a) = 0

(iv) (x + a) is a factor of polynomial p(x) if p(–a) = 0

(v) (x – a)(x – b) is a factor of polynomial p(x),

if p(a) = 0 and p(b) = 0.

❖ EXAMPLES ❖

Ex.4 Find the remainder when $4x^3 - 3x^2 + 2x - 4$ is divided by

(a) x – 1 (b) x + 2 (c) $x + \frac{1}{2}$

Sol. Let $p(x) = 4x^3 - 3x^2 + 2x - 4$

(a) When p(x) is divided by (x – 1), then by remainder theorem, the required remainder will be p(1)

$p(1) = 4(1)^3 - 3(1)^2 + 2(1) - 4$

$= 4 \times 1 - 3 \times 1 + 2 \times 1 - 4$

$= 4 - 3 + 2 - 4 = -1$

(b) When p(x) is divided by (x + 2), then by remainder theorem, the required remainder will be p (–2).

$p(-2) = 4(-2)^3 - 3(-2)^2 + 2(-2) - 4$

$= 4 \times (-8) - 3 \times 4 - 4 - 4$

$= -32 - 12 - 8 = -52$

(c) When p(x) is divided by, $\left(x + \frac{1}{2}\right)$ then by remainder theorem, the required remainder will be

$$p\left(-\frac{1}{2}\right) = 4\left(-\frac{1}{2}\right)^3 - 3\left(-\frac{1}{2}\right)^2 + 2\left(-\frac{1}{2}\right) - 4$$

$$= 4 \times \left(-\frac{1}{8}\right) - 3 \times \frac{1}{4} - 2 \times \frac{1}{2} - 4$$

$$= -\frac{1}{2} - \frac{3}{4} - 1 - 4 = \frac{1}{2} - \frac{3}{4} - 5$$

$$= \frac{-2 - 3 - 20}{4} = \frac{-25}{4}$$

➢ VALUES OF A POLYNOMIAL

For a polynomial $f(x) = 3x^2 - 4x + 2$.

To find its value at $x = 3$;

replace x by 3 everywhere.

So, the value of $f(x) = 3x^2 - 4x + 2$ at $x = 3$ is

$f(3) = 3 \times 3^2 - 4 \times 3 + 2$

$= 27 - 12 + 2 = 17.$

Similarly, the value of polynomial

$f(x) = 3x^2 - 4x + 2$,

(i) at $x = -2$ is $f(-2) = 3(-2)^2 - 4(-2) + 2$

$= 12 + 8 + 2 = 22$

(ii) at $x = 0$ is $f(0) = 3(0)^2 - 4(0) + 2$

$= 0 - 0 + 2 = 2$

(iii) at $x = \frac{1}{2}$ is $f\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right) + 2$

$= \frac{3}{4} - 2 + 2 = \frac{3}{4}$

Ex.5 Find the value of the polynomial $5x - 4x^2 + 3$ at:

(i) $x = 0$ (ii) $x = -1$

Sol. Let $p(x) = 5x - 4x^2 + 3$.

(i) At $x = 0$, $p(0) = 5 \times 0 - 4 \times (0)^2 + 3$

$= 0 - 0 + 3 = 3$

(ii) At $x = -1$, $p(-1) = 5(-1) - 4(-1)^2 + 3$

$= -5 - 4 + 3 = -6$

➢ ZEROES OF A POLYNOMIAL

If for $x = a$, the value of the polynomial $p(x)$ is 0 i.e., $p(a) = 0$; then $x = a$ is a zero of the polynomial $p(x)$.

For example :

(i) For polynomial $p(x) = x - 2$; $p(2) = 2 - 2 = 0$

$\therefore$ $x = 2$ or simply 2 is a zero of the polynomial

$p(x) = x - 2$.

(ii) For the polynomial $g(u) = u^2 - 5u + 6$;

$g(3) = (3)^2 - 5 \times 3 + 6 = 9 - 15 + 6 = 0$

$\therefore$ 3 is a zero of the polynomial $g(u)$

$= u^2 - 5u + 6$.

Also, $g(2) = (2)^2 - 5 \times 2 + 6 = 4 - 10 + 6 = 0$

$\therefore$ 2 is also a zero of the polynomial

$g(u) = u^2 - 5u + 6$

(a) Every linear polynomial has one and only one zero.

(b) A given polynomial may have more than one zeroes.

(c) If the degree of a polynomial is n; the largest number of zeroes it can have is also n.

For example :

If the degree of a polynomial is 5, the polynomial can have at the most 5 zeroes; if the degree of a polynomial is 8; largest number of zeroes it can have is 8.

(d) A zero of a polynomial need not be 0.

For example : If $f(x) = x^2 - 4$,

then $f(2) = (2)^2 - 4 = 4 - 4 = 0$

Here, zero of the polynomial $f(x) = x^2 - 4$ is 2 which itself is not 0.

(e) 0 may be a zero of a polynomial.

For example : If $f(x) = x^2 - x$,

then $f(0) = 0^2 - 0 = 0$

Here 0 is the zero of polynomial

$f(x) = x^2 - x$.

❖ EXAMPLES ❖

Ex.6 Verify whether the indicated numbers are zeroes of the polynomial corresponding to them in the following cases :

(i) $p(x) = 3x + 1$, $x = -\frac{1}{3}$

(ii) $p(x) = (x + 1)(x - 2)$, $x = -1, 2$

(iii) $p(x) = x^2$, $x = 0$

(iv) $p(x) = \lambda x + m$, $x = -\frac{m}{\lambda}$

(v) $p(x) = 2x + 1$, $x = \frac{1}{2}$

Sol. (i) $p(x) = 3x + 1$

$\Rightarrow p\left(-\frac{1}{3}\right) = 3 \times -\frac{1}{3} + 1 = -1 + 1 = 0$

$\therefore$ $x = -\frac{1}{3}$ is a zero of $p(x) = 3x + 1$.

(ii) $p(x) = (x + 1)(x - 2)$

$\Rightarrow p(-1) = (-1 + 1)(-1 - 2) = 0 \times -3 = 0$

and, $p(2) = (2 + 1)(2 - 2) = 3 \times 0 = 0$

$\therefore$ $x = -1$ and $x = 2$ are zeroes of the given polynomial.

(iii) $p(x) = x^2 \Rightarrow p(0) = 0^2 = 0$

$\therefore$ $x = 0$ is a zero of the given polynomial

(iv) $p(x) = \lambda x + m \Rightarrow p\left(-\frac{m}{\lambda}\right) = \lambda\left(-\frac{m}{\lambda}\right) + m$

$= -m + m = 0$

$\therefore$ $x = -\frac{m}{\lambda}$ is a zero of the given polynomial.

(v) $p(x) = 2x + 1 \Rightarrow p\left(\frac{1}{2}\right) = 2 \times \frac{1}{2} + 1$

$= 1 + 1 = 2 \neq 0$

$\therefore$ $x = \frac{1}{2}$ is not a zero of the given polynomial.

Ex.7 Find the zero of the polynomial in each of the following cases :

(i) $p(x) = x + 5$ (ii) $p(x) = 2x + 5$

(iii) $p(x) = 3x - 2$

Sol. To find the zero of a polynomial p(x) means to solve the polynomial equation $p(x) = 0$.

(i) For the zero of polynomial $p(x) = x + 5$

$p(x) = 0 \Rightarrow x + 5 = 0 \Rightarrow x = -5$

$\therefore$ $x = -5$ is a zero of the polynomial $p(x) = x + 5$.

(ii) $p(x) = 0 \Rightarrow 2x + 5 = 0$

$\Rightarrow 2x = -5$ and $x = \frac{-5}{2}$

$\therefore$ $x = \frac{-5}{2}$ is a zero of $p(x) = 2x + 5$.

(iii) $p(x) = 0 \Rightarrow 3x - 2 = 0$

$\Rightarrow 3x = 2$ and $x = \frac{2}{3}$.

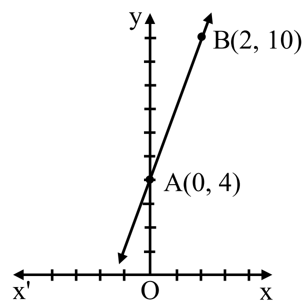
$\therefore$ $x = \frac{2}{3}$ is zero of $p(x) = 3x - 2$

GEOMETRIC MEANING OF THE ZEROES OF A POLYNOMIAL

Let us consider linear polynomial ax + b. The graph of y = ax + b is a straight line.

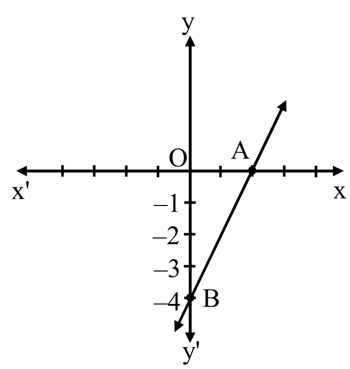
For example : The graph of $y = 3x + 4$ is a straight line passing through (0, 4) and (2, 10).

x	0	2
y = 3x + 4	4	10
Points	A	B



(i) Let us consider the graph of $y = 2x - 4$ intersects the x-axis at x = 2. The zero $2x - 4$ is 2. Thus, the zero of the polynomial $2x - 4$ is the x-coordinate of the point where the graph $y = 2x - 4$ intersects the x-axis.

x	2	0
y = 2x – 4	0	– 4
Points	A	B



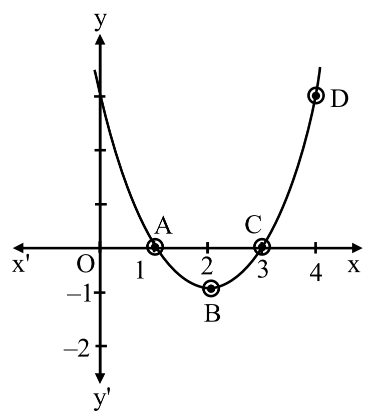
(ii) A general equation of a linear polynomial is ax + b. The graph of y = ax + b is a straight line which intersects the x-axis at $\left(\frac{-b}{a}, 0\right)$.

Zero of the polynomial ax + b is the x-coordinate of the point of intersection of the graph with x-axis.

(iii) Let us consider the quadratic polynomial $x^2 - 4x + 3$. The graph of $x^2 - 4x + 3$ intersects the x-axis at the point (1, 0) and (3, 0). Zeroes of the polynomial $x^2 - 4x + 3$ are the x-coordinates of the points of intersection of the graph with x-axis.

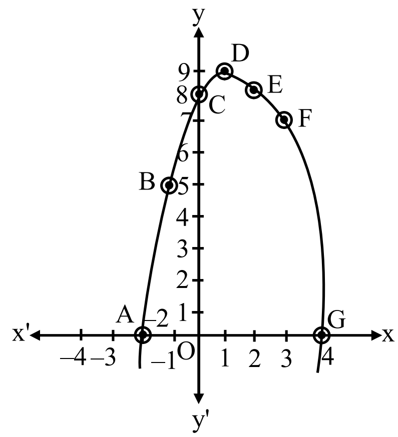
x	1	2	3	4	5
$y = x^2 - 4x + 3$	0	−1	0	3	8
Points	A	B	C	D	E

The shape of the graph of the quadratic polynomials is ∪ and the curve is known as parabola.



(iv) Now let us consider one more polynomial $-x^2 + 2x + 8$. Graph of this polynomial intersects the x-axis at the points (4, 0), (–2, 0). Zeroes of the polynomial $-x^2 + 2x + 8$ are the x-coordinates of the points at which the graph intersects the x-axis. The shape of the graph of the given quadratic polynomial is ∩ and the curve is known as parabola.

x	−2	−1	0	1	2	3	4
y	0	5	8	9	8	7	0
Points	A	B	C	D	E	F	G



The zeroes of a quadratic polynomial $ax^2 + bx + c$ he x-coordinates of the points where the graph of $y = ax^2 + bx + c$ intersects the x-axis.

Cubic polynomial : Let us find out geometrically how many zeroes a cubic has.

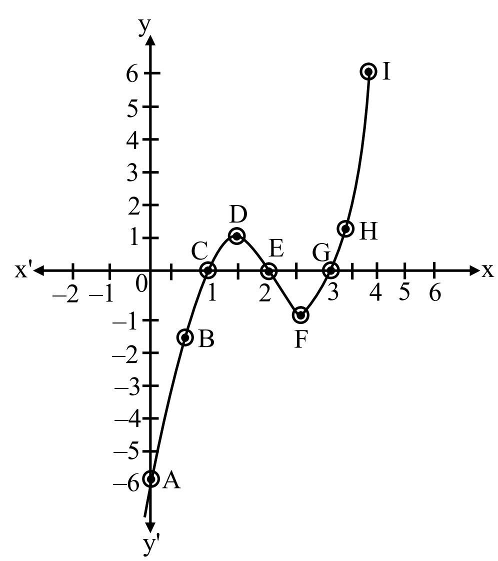
Let consider cubic polynomial

$x^3 - 6x^2 + 11x - 6$.

x	0	0.5	1	1.5	2	2.5	3	3.5	4
$y = x^3 - 6x^2 + 11x - 6$	−6	−1.875	0	0.375	0	−0.375	0	1.875	6
Points	A	B	C	D	E	F	G	H	I

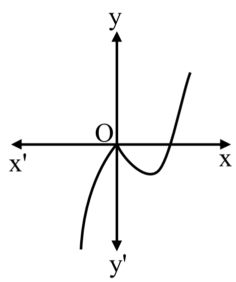
Case 1 :

The graph of the cubic equation intersects the x-axis at three points (1, 0), (2, 0) and (3, 0). Zeroes of the given polynomial are the x-coordinates of the points of intersection with the x-axis.



Case 2 :

The cubic equation $x^3 - x^2$ intersects the x-axis at the point (0, 0) and (1, 0). Zero of a polynomial $x^3 - x^2$ are the x-coordinates of the point where the graph cuts the x-axis.

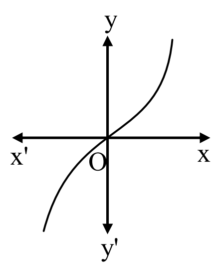


Zeroes of the cubic polynomial are 0 and 1.

Case 3 :

$y = x^3$

Cubic polynomial has only one zero.

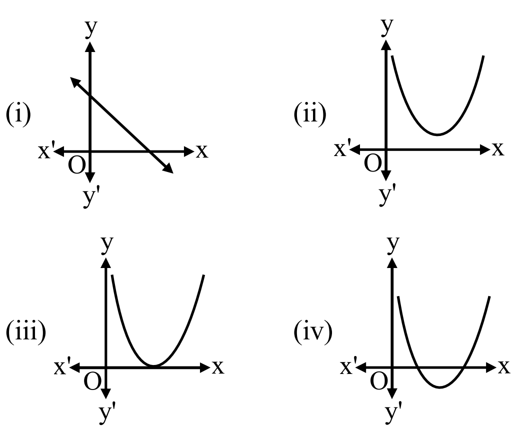


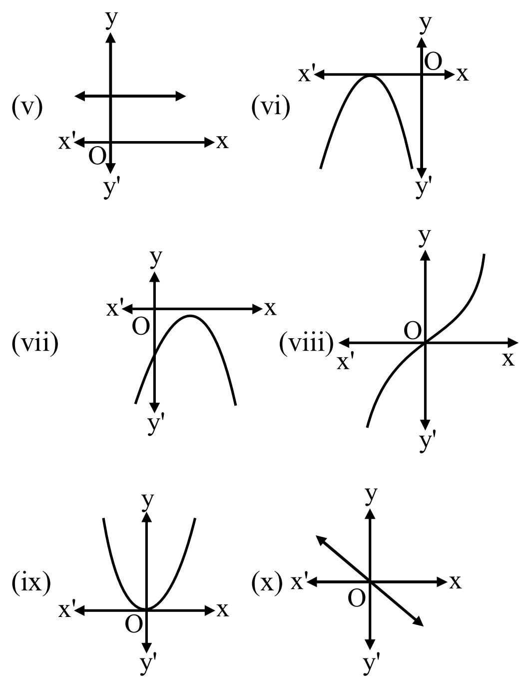
In brief : A cubic equation can have 1 or 2 or 3 zeroes or any polynomial of degree three can have at most three zeroes.

Remarks : In general, polynomial of degree n, the graph of $y = p(x)$ passes x-axis at most at n points. Therefore, a polynomial p(x) of degree n has at most n zeroes.

❖ EXAMPLES ❖

Ex.8 Which of the following correspond to the graph to a linear or a quadratic polynomial and find the number of zeroes of polynomial.





Sol. (i) The graph is a straight line so the graph is of a linear polynomial. The number of zeroes is one as the graph intersects the x-axis at one point only.

(ii) The graph is a parabola. So, this is the graph of quadratic polynomial. The number of zeroes is zero as the graph does not intersect the x-axis.

(iii) Here the polynomial is quadratic as the graph is a parabola. The number of zeroes is one as the graph intersects the x-axis at one point only (two coincident points).

(iv) Here, the polynomial is quadratic as the graph is a parabola. The number of zeroes is two as the graph intersects the x-axis at two points.

(v) The polynomial is linear as the graph is straight line. The number of zeroes is zero as the graph does not intersect the x-axis.

(vi) The polynomial is quadratic as the graph is a parabola. The number of zeroes is 1 as the graph intersects the x-axis at one point (two coincident points) only.

(vii) The polynomial is quadratic as the graph is a parabola. The number of zeroes is zero, as the graph does not intersect the x-axis.

(viii) Polynomial is neither linear nor quadratic as the graph is neither a straight line nor a parabola is one as the graph intersects the x-axis at one point only.

(ix) Here, the polynomial is quadratic as the graph is a parabola. The number of zeroes is one as the graph intersects the x-axis at one point only (two coincident points).

(x) The polynomial is linear as the graph is a straight line. The number of zeroes is one as the graph intersects the x-axis at only one point.

➢ RELATIONSHIP BETWEEN THE ZEROES AND THE COEFFICIENTS OF A POLYNOMIAL.

Consider quadratic polynomial

$P(x)=2x^2-16x+30.$

Now, $2x^2 - 16x + 30 = (2x - 6)(x - 3)$

$= 2(x - 3)(x - 5)$

The zeroes of P(x) are 3 and 5.

Sum of the zeroes

$$= 3 + 5 = 8 = \frac{-(-16)}{2} = -\left[\frac{\text{coefficient of } x}{\text{coefficient of } x^2}\right]$$

Product of the zeroes

$$= 3 \times 5 = 15 = \frac{30}{2} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

So if $ax^2 + bx + c$, $a \neq 0$ is a quadratic polynomial and α, β are two zeroes of polynomial then $\boxed{\alpha + \beta = -\frac{b}{a}}$, $\boxed{\alpha\beta = \frac{c}{a}}$

❖ EXAMPLES ❖

Ex.9 Find the zeroes of the quadratic polynomial $6x^2 - 13x + 6$ and verify the relation between the zeroes and its coefficients.

Sol. We have, $6x^2 - 13x + 6 = 6x^2 - 4x - 9x + 6$

$= 2x(3x - 2) - 3(3x - 2)$

$= (3x - 2)(2x - 3)$

So, the value of $6x^2 - 13x + 6$ is 0, when $(3x - 2) = 0$ or $(2x - 3) = 0$ i.e.,

When $x = \frac{2}{3}$ or $\frac{3}{2}$

Therefore, the zeroes of $6x^2 - 13x + 6$ are

$\frac{2}{3}$ and $\frac{3}{2}$.

Sum of the zeroes

$$= \frac{2}{3} + \frac{3}{2} = \frac{13}{6} = \frac{-(-13)}{6} = \frac{-\text{coefficient of } x}{\text{coefficient of } x^2}$$

Product of the zeroes

$$= \frac{2}{3} \times \frac{3}{2} = \frac{6}{6} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

Ex.10 Find the zeroes of the quadratic polynomial $4x^2 - 9$ and verify the relation between the zeroes and its coefficients.

Sol. We have,

$4x^2 - 9 = (2x)^2 - 3^2 = (2x - 3)(2x + 3)$

So, the value of $4x^2 - 9$ is 0, when

$2x - 3 = 0$ or $2x + 3 = 0$

i.e., when $x = \frac{3}{2}$ or $x = -\frac{3}{2}$.

Therefore, the zeroes of $4x^2 - 9$ are $\frac{3}{2}$ & $-\frac{3}{2}$.

Sum of the zeroes

$$= \frac{3}{2} - \frac{3}{2} = 0 = \frac{-(0)}{4} = \frac{-\text{coefficient of } x}{\text{coefficient of } x^2}$$

Product of the zeroes

$$= \left(\frac{3}{2}\right)\left(-\frac{3}{2}\right) = \frac{-9}{4} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

Ex.11 Find the zeroes of the quadratic polynomial $9x^2 - 5$ and verify the relation between the zeroes and its coefficients.

Sol. We have,

$9x^2 - 5 = (3x)^2 - (\sqrt{5})^2 = (3x - \sqrt{5})(3x + \sqrt{5})$

So, the value of $9x^2 - 5$ is 0,

when $3x - \sqrt{5} = 0$ or $3x + \sqrt{5} = 0$

i.e., when $x = \frac{\sqrt{5}}{3}$ or $x = \frac{-\sqrt{5}}{3}$.

Sum of the zeroes

$$= \frac{\sqrt{5}}{3} - \frac{\sqrt{5}}{3} = 0 = \frac{-(0)}{9} = \frac{-\text{coefficient of } x}{\text{coefficient of } x^2}$$

Product of the zeroes

$$= \left(\frac{\sqrt{5}}{3}\right)\left(\frac{-\sqrt{5}}{3}\right) = \frac{-5}{9} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

Ex.12 If α and β are the zeroes of $ax^2 + bx + c$, $a \neq 0$ then verify the relation between the zeroes and its coefficients.

Sol. Since α and β are the zeroes of polynomial $ax^2 + bx + c$.

Therefore, $(x - \alpha)$, $(x - \beta)$ are the factors of the polynomial $ax^2 + bx + c$.

$\Rightarrow$ $ax^2 + bx + c = k(x - \alpha)(x - \beta)$

$\Rightarrow$ $ax^2 + bx + c = k\{x^2 - (\alpha + \beta)x + \alpha\beta\}$

$\Rightarrow ax^2 + bx + c = kx^2 - k(\alpha + \beta)x + k\alpha\beta$ ...(1)

Comparing the coefficients of x^2, x and constant terms of (1) on both sides, we get

$a = k$, $b = -k(\alpha + \beta)$ and $c = k\alpha\beta$

$\Rightarrow$ $\alpha + \beta = -\frac{b}{k}$ and $\alpha\beta = \frac{c}{k}$

$\alpha + \beta = \frac{-b}{a}$ and $\alpha\beta = \frac{c}{a}$ $[\because k = a]$

Sum of the zeroes $= \frac{-b}{a} = \frac{-\text{coefficient of } x}{\text{coefficient of } x^2}$

Product of the zeroes $= \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$

Ex. 13 Prove relation between the zeroes and the coefficient of the quadratic polynomial $ax^2 + bx + c$.

Sol. Let α and β be the zeroes of the polynomial $ax^2 + bx + c$

$$\therefore \quad \alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad ....(1)$$

$$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad ....(2)$$

By adding (1) and (2), we get

$$\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2b}{2a} = -\frac{b}{a} = \frac{\text{coefficient of } x}{\text{coefficient of } x^2}$$

Hence, sum of the zeroes of the polynomial $ax^2 + bx + c$ is $-\frac{b}{a}$

By multiplying (1) and (2), we get

$$\alpha\beta = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \frac{(-b)^2 - \sqrt{(b^2 - 4ac)^2}}{4a^2} = \frac{b^2 - b^2 + 4ac}{4a^2}$$

$$= \frac{4ac}{4a^2} = \frac{c}{a}$$

$$= \frac{\text{constant term}}{\text{coefficient of } x^2}$$

Hence, product of zeroes $= \frac{c}{a}$

> In general, it can be proved that if α, β, γ are the zeroes of a cubic polynomial $ax^3 + bx^2 + cx + d$, then
>
> $$\alpha + \beta + \gamma = \frac{-b}{a}$$
>
> $$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$
>
> $$\alpha\beta\gamma = \frac{-d}{a}$$
>
> Note, $\frac{b}{a}$, $\frac{c}{a}$ and $\frac{d}{a}$ are meaningful because $a \neq 0$.

Ex.14 find the zeroes of the quadratic polynomial $x^2 - 2x - 8$ and verify a relationship between zeroes and its coefficients.

Sol. $x^2 - 2x - 8 = x^2 - 4x + 2x - 8$

$= x(x - 4) + 2(x - 4) = (x - 4)(x + 2)$

So, the value of $x^2 - 2x - 8$ is zero when $x - 4 = 0$ or $x + 2 = 0$ i.e., when $x = 4$ or $x = -2$.

So, the zeroes of $x^2 - 2x - 8$ are 4, – 2.

Sum of the zeroes

$= 4 - 2 = 2 = \frac{-(-2)}{1} = \frac{-\text{coefficient of } x}{\text{coefficient of } x^2}$

Product of the zeroes

$= 4(-2) = -8 = \frac{-8}{1} = \frac{\text{constant term}}{\text{coefficient of } x^2}$

Ex.15 Verify that the numbers given along side of the cubic polynomials are their zeroes. Also verify the relationship between the zeroes and the coefficients. $2x^3 + x^2 - 5x + 2$; $\frac{1}{2}$, 1, – 2

Sol. Here, the polynomial p(x) is

$2x^3 + x^2 - 5x + 2$

Value of the polynomial $2x^3 + x^2 - 5x + 2$

when x = 1/2

$$= 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 - 5\left(\frac{1}{2}\right) + 2 = \frac{1}{4} + \frac{1}{4} - \frac{5}{2} + 2 = 0$$

So, 1/2 is a zero of p(x).

On putting x = 1 in the cubic polynomial

$2x^3 + x^2 - 5x + 2$

$= 2(1)^3 + (1)^2 - 5(1) + 2 = 2 + 1 - 5 + 2 = 0$

On putting x = – 2 in the cubic polynomial

$2x^3 + x^2 - 5x + 2$

$= 2(-2)^3 + (-2)^2 - 5(-2) + 2$

$= -16 + 4 + 10 + 2 = 0$

Hence, $\frac{1}{2}$, 1, – 2 are the zeroes of the given polynomial.

Sum of the zeroes of p(x)

$$= \frac{1}{2} + 1 - 2 = -\frac{1}{2} = \frac{-\text{coefficient of } x^2}{\text{coefficient of } x^3}$$

Sum of the products of two zeroes taken at a time

$$= \frac{1}{2} \times 1 + \frac{1}{2} \times (-2) + 1 \times (-2)$$

$$= \frac{1}{2} - 1 - 2 = -\frac{5}{2} = \frac{\text{coefficient of } x}{\text{coefficient of } x^3}$$

Product of all the three zeroes

$$= \left(\frac{1}{2}\right) \times (1) \times (-2) = -1$$

$$= \frac{-(2)}{2} = \frac{-\text{constant term}}{\text{coefficient of } x^3}$$

➤ SYMMETRIC FUNCTIONS OF ZEROS OF A QUADRATIC POLYNOMIAL.

◈ Symmetric function :

An algebraic expression in α and β, which remains unchanged, when α and β are interchanged is known as symmetric function in α and β.

For example, $\alpha^2 + \beta^2$ and $\alpha^3 + \beta^3$ etc. are symmetric functions. Symmetric function is to be expressed in terms of (α + β) and αβ. So, this can be evaluated for a given quadratic equation.

◈ Some useful relations involving α and β :

1. $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
2. $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$
3. $\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta) = (\alpha + \beta)\sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$
4. $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$
5. $\alpha^3 - \beta^3 = (\alpha - \beta)^3 + 3\alpha\beta(\alpha - \beta)$
6. $\alpha^4 + \beta^4 = [(\alpha + \beta)^2 - 2\alpha\beta]^2 - 2(\alpha\beta)^2$
7. $\alpha^4 - \beta^4 = (\alpha^2 + \beta^2)(\alpha^2 - \beta^2)$ then use (1) and (3)

❖ EXAMPLES ❖

Ex.16 If α and β are the zeroes of the polynomial $ax^2 + bx + c$. Find the value of

(i) $\alpha - \beta$ (ii) $\alpha^2 + \beta^2$.

Sol. Since α and β are the zeroes of the polynomial $ax^2 + bx + c$.

$$\therefore \quad \alpha + \beta = -\frac{b}{a}; \quad \alpha\beta = \frac{c}{a}$$

(i) $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$$= \left(-\frac{b}{a}\right)^2 - \frac{4c}{a} = \frac{b^2}{a^2} - \frac{4c}{a} = \frac{b^2 - 4ac}{a^2}$$

$$\alpha - \beta = \frac{\sqrt{b^2 - 4ac}}{a}$$

(ii) $\alpha^2 + \beta^2 = \alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta$

$= (\alpha + \beta)^2 - 2\alpha\beta$

$$= \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right) = \frac{b^2 - 2ac}{a^2}$$

Ex.17 If α and β are the zeroes of the quadratic polynomial $ax^2 + bx + c$. Find the value of

(i) $\alpha^2 - \beta^2$ (ii) $\alpha^3 + \beta^3$.

Sol. Since α and β are the zeroes of $ax^2 + bx + c$

$\therefore \quad \alpha + \beta = \frac{-b}{a}, \qquad \alpha\beta = \frac{c}{a}$

(i) $\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta)$

$= -\frac{b}{a}\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

$= -\frac{b}{a}\sqrt{\left(\frac{-b}{a}\right)^2 - 4\frac{c}{a}} = -\frac{b}{a}\sqrt{\frac{b^2-4ac}{a^2}}$

$= -\frac{b\sqrt{b^2-4ac}}{a^2}$

(ii) $\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 + \beta^2 - \alpha\beta)$

$= (\alpha + \beta)[(\alpha^2 + \beta^2 + 2\alpha\beta) - 3\alpha\beta]$

$= (\alpha + \beta)[(\alpha + \beta)^2 - 3\alpha\beta]$

$= \frac{-b}{a}\left[\left(\frac{-b}{a}\right)^2 - \frac{3c}{a}\right]$

$= \frac{-b}{a}\left[\frac{b^2}{a^2} - \frac{3c}{a}\right] = \frac{-b}{a}\left(\frac{b^2-3ac}{a^2}\right)$

$= \frac{-b^3 + 3abc}{a^3}$

➤ TO FORM A QUADRATIC POLYNOMIAL WITH THE GIVEN ZEROES

Let zeroes of a quadratic polynomial be α and β.

$\therefore \quad x = \alpha, \qquad x = \beta$

$x - \alpha = 0, \quad x - \beta = 0$

The obviously the quadratic polynomial is

$(x - \alpha)(x - \beta)$

i.e., $\quad x^2 - (\alpha + \beta)x + \alpha\beta$

$x^2 - \text{(Sum of the zeroes) } x + \text{Product of the zeroes}$

❖ EXAMPLES ❖

Ex.18 Form the quadratic polynomial whose zeroes are 4 and 6.

Sol. Sum of the zeroes = 4 + 6 = 10

Product of the zeroes = 4 × 6 = 24

Hence the polynomial formed

$= x^2$ – (sum of zeroes) x + Product of zeroes

$= x^2 - 10x + 24$

Ex.19 Form the quadratic polynomial whose zeroes are –3, 5.

Sol. Here, zeroes are – 3 and 5.

Sum of the zeroes = – 3 + 5 = 2

Product of the zeroes = (–3) × 5 = – 15

Hence the polynomial formed

$= x^2$ – (sum of zeroes) x + Product of zeroes

$= x^2 - 2x - 15$

Ex.20 Find a quadratic polynomial whose sum of zeroes and product of zeroes are respectively-

(i) $\frac{1}{4}, -1$ (ii) $\sqrt{2}, \frac{1}{3}$ (iii) $0, \sqrt{5}$

Sol. Let the polynomial be $ax^2 + bx + c$ and its zeroes be α and β.

(i) Here, $\alpha + \beta = \frac{1}{4}$ and $\alpha \cdot \beta = -1$

Thus the polynomial formed

$= x^2$ – (Sum of zeroes) x + Product of zeroes

$= x^2 - \left(\frac{1}{4}\right)x - 1 = x^2 - \frac{x}{4} - 1$

The other polynomial are $k\left(x^2 - \frac{x}{4} - 1\right)$

If k = 4, then the polynomial is $4x^2 - x - 4$.

(ii) Here, $\alpha + \beta = \sqrt{2}$, $\alpha\beta = \frac{1}{3}$

Thus the polynomial formed

$= x^2$ – (Sum of zeroes) x + Product of zeroes

$= x^2 - (\sqrt{2})x + \frac{1}{3}$ or $x^2 - \sqrt{2}x + \frac{1}{3}$

Other polynomial are $k\left(x^2 - \sqrt{2}x + \frac{1}{3}\right)$

If k = 3, then the polynomial is

$3x^2 - 3\sqrt{2}x + 1$

(iii) Here, $\alpha + \beta = 0$ and $\alpha.\beta = \sqrt{5}$

Thus the polynomial formed

$= x^2$ – (Sum of zeroes) x + Product of zeroes

$= x^2 - (0)x + \sqrt{5} = x^2 + \sqrt{5}$

Ex.21 Find a cubic polynomial with the sum of its zeroes, sum of the products of its zeroes taken two at a time, and product of its zeroes as 2, – 7 and –14, respectively.

Sol. Let the cubic polynomial be

$ax^3 + bx^2 + cx + d$

$\Rightarrow \; x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a}$ ....(1)

and its zeroes are α, β and γ, then

$\alpha + \beta + \gamma = 2 = -\frac{b}{a}$

$\alpha\beta + \beta\gamma + \alpha\gamma = -7 = \frac{c}{a}$

$\alpha\beta\gamma = -14 = -\frac{d}{a}$

Putting the values of $\frac{b}{a}$, $\frac{c}{a}$ and $\frac{d}{a}$ in (1),

we get

$x^3 + (-2)x^2 + (-7)x + 14$

$\Rightarrow \; x^3 - 2x^2 - 7x + 14$

Ex.22 Find the cubic polynomial with the sum, sum of the product of its zeroes taken two at a time and product of its zeroes as 0, –7 and –6 respectively.

Sol. Let the cubic polynomial be

$ax^3 + bx^2 + cx + d$

$\Rightarrow \; x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a}$ ....(1)

and its zeroes are α, β, γ. Then

$\alpha + \beta + \gamma = 0 = -\frac{b}{a}$

$\alpha\beta + \beta\gamma + \alpha\gamma = -7 = \frac{c}{a}$

$\alpha\beta\gamma = -6 = \frac{-d}{a}$

Putting the values of $\frac{b}{a}$, $\frac{c}{a}$ and $\frac{d}{a}$ in (1),

we get

$x^3 - (0)x^2 + (-7)x + (-6)$

or $x^3 - 7x + 6$

Ex.23 If α and β are the zeroes of the polynomials $ax^2 + bx + c$ then form the polynomial whose zeroes are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

Sol. Since α and β are the zeroes of $ax^2 + bx + c$

So $\alpha + \beta = \frac{-b}{a}$, $\alpha\beta = \frac{c}{a}$

Sum of the zeroes $= \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta}$

$= \frac{-\frac{b}{a}}{\frac{c}{a}} = -\frac{b}{c}$

Product of the zeroes

$= \frac{1}{\alpha} \cdot \frac{1}{\beta} = \frac{1}{\frac{c}{a}} = \frac{a}{c}$

But required polynomial is

x^2 – (sum of zeroes) x + Product of zeroes

$\Rightarrow \; x^2 - \left(\frac{-b}{c}\right)x + \left(\frac{a}{c}\right)$

or $x^2 + \frac{b}{c}x + \frac{a}{c}$

or $c\left(x^2 + \frac{b}{c}x + \frac{a}{c}\right)$

$\Rightarrow \; cx^2 + bx + a$

Ex.24 If α and β are the zeroes of the polynomial $x^2 + 4x + 3$, form the polynomial whose zeroes are $1 + \frac{\beta}{\alpha}$ and $1 + \frac{\alpha}{\beta}$.

Sol. Since α and β are the zeroes of the polynomial $x^2 + 4x + 3$.

Then, $\alpha + \beta = -4$, $\alpha\beta = 3$

Sum of the zeroes

$= 1 + \frac{\beta}{\alpha} + 1 + \frac{\alpha}{\beta} = \frac{\alpha\beta + \beta^2 + \alpha\beta + \alpha^2}{\alpha\beta}$

$= \frac{\alpha^2 + \beta^2 + 2\alpha\beta}{\alpha\beta} = \frac{(\alpha + \beta)^2}{\alpha\beta} = \frac{(-4)^2}{3} = \frac{16}{3}$

Product of the zeroes

$= \left(1+\frac{\beta}{\alpha}\right)\left(1+\frac{\alpha}{\beta}\right) = 1 + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{\alpha\beta}{\alpha\beta}$

$= 2 + \frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{2\alpha\beta+\alpha^2+\beta^2}{\alpha\beta}$

$= \frac{(\alpha+\beta)^2}{\alpha\beta} = \frac{(-4)^2}{3} = \frac{16}{3}$

But required polynomial is

x^2 – (sum of zeroes) x + product of zeroes

or $x^2 - \frac{16}{3}x + \frac{16}{3}$ or $k\left(x^2 - \frac{16}{3}x + \frac{16}{3}\right)$

or $3\left(x^2 - \frac{16}{3}x + \frac{16}{3}\right)$ (if k = 3)

$\Rightarrow 3x^2 - 16x + 16$

➤ WORKING RULE TO DIVIDE A POLYNOMIAL BY ANOTHER POLYNOMIAL

Step 1:

First arrange the term of dividend and the divisor in the decreasing order of their degrees.

Step 2 :

To obtain the first term of quotient divide the highest degree term of the dividend by the highest degree term of the divisor.

Step 3 :

To obtain the second term of the quotient, divide the highest degree term of the new dividend obtained as remainder by the highest degree term of the divisor.

Step 4 :

Continue this process till the degree of remainder is less than the degree of divisor.

❖ Division Algorithm for Polynomial

If p(x) and g(x) are any two polynomials with

$g(x) \neq 0$, then we can find polynomials q(x) and r(x) such that

$p(x) = q(x) \times g(x) + r(x)$

where r(x) = 0 or degree of r(x) < degree of g(x).

The result is called Division Algorithm for polynomials.

Dividend = Quotient × Divisor + Remainder

❖ EXAMPLES ❖

Ex.25 Divide $3x^3 + 16x^2 + 21x + 20$ by $x + 4$.

Sol.

$$\begin{array}{r|l}
 & 3x^2 + 4x + 5 \\
\hline
x+4 & 3x^3 + 16x^2 + 21x + 20 \\
 & 3x^3 + 12x^2 \\
 & - \quad - \\
\hline
 & 4x^2 + 21x + 20 \\
 & 4x^2 + 16x \\
 & - \quad - \\
\hline
 & 5x + 20 \\
 & 5x + 20 \\
 & - \quad - \\
\hline
 & 0
\end{array}$$

First term of q(x) $= \frac{3x^3}{x} = 3x^2$

Second term of q(x) $= \frac{4x^2}{x} = 4x$

Third term of q(x) $= \frac{5x}{x} = 5$

Quotient $= 3x^2 + 4x + 5$

Remainder = 0

Ex.26 Apply the division algorithm to find the quotient and remainder on dividing p(x) by g(x) as given below :

$p(x) = x^3 - 3x^2 + 5x - 3,\ g(x) = x^2 - 2$

Sol. We have,

$p(x) = x^3 - 3x^2 + 5x - 3$ and $g(x) = x^2 - 2$

$$\begin{array}{r|l}
 & x - 3 \\
\hline
x^2-2 & x^3 - 3x^2 + 5x - 3 \\
 & x^3 \qquad - 2x \\
 & - \qquad + \\
\hline
 & -3x^2 + 7x - 3 \\
 & -3x^2 \qquad + 6 \\
 & + \qquad - \\
\hline
 & 7x - 9
\end{array}$$

First term of quotient is $\frac{x^3}{x^2} = x$

Second term of quotient is $\frac{-3x^2}{x^2} = -3$

We stop here since

degree of $(7x - 9)$ < degree of $(x^2 - 2)$

So, quotient = x – 3, remainder = 7x – 9

Therefore,

Quotient × Divisor + Remainder

$= (x - 3)(x^2 - 2) + 7x - 9$

$= x^3 - 2x - 3x^2 + 6 + 7x - 9$

$= x^3 - 3x^2 + 5x - 3$ = Dividend

Therefore, the division algorithm is verified.

Ex.27 Apply the division algorithm to find the quotient and remainder on dividing p(x) by g(x) as given below

$p(x) = x^4 - 3x^2 + 4x + 5,\ g(x) = x^2 + 1 - x$

Sol. We have,

$p(x) = x^4 - 3x^2 + 4x + 5,\ g(x) = x^2 + 1 - x$

$$
\begin{array}{r|l}
 & x^2 + x - 3 \\
x^2 - x + 1 & x^4 - 3x^2 + 4x + 5 \\
 & x^4 - x^3 + x^2 \\
 & - \ \ + \ \ - \\
\hline
 & x^3 - 4x^2 + 4x + 5 \\
 & x^3 - x^2 + x \\
 & - \ \ + \ \ - \\
\hline
 & -3x^2 + 3x + 5 \\
 & -3x^2 + 3x - 3 \\
 & + \ \ - \ \ + \\
\hline
 & 8
\end{array}
$$

We stop here since

degree of (8) < degree of $(x^2 - x + 1)$.

So, quotient $= x^2 + x - 3$, remainder $= 8$

Therefore,

Quotient × Divisor + Remainder

$= (x^2 + x - 3)(x^2 - x + 1) + 8$

$= x^4 - x^3 + x^2 + x^3 - x^2 + x - 3x^2 + 3x - 3 + 8$

$= x^4 - 3x^2 + 4x + 5$ = Dividend

Therefore the Division Algorithm is verified.

Ex.28 Check whether the first polynomial is a factor of the second polynomial by applying the division algorithm. $t^2 - 3;\ 2t^4 + 3t^3 - 2t^2 - 9t - 12$.

Sol. We divide $2t^4 + 3t^3 - 2t^2 - 9t - 12$ by $t^2 - 3$

$$
\begin{array}{r|l}
 & 2t^2 + 3t + 4 \\
t^2 - 3 & 2t^4 + 3t^3 - 2t^2 - 9t - 12 \\
 & 2t^4 \qquad - 6t^2 \\
 & - \qquad\quad + \\
\hline
 & 3t^3 + 4t^2 + 9t - 12 \\
 & 3t^3 \qquad - 9t \\
 & - \qquad\quad + \\
\hline
 & 4t^2 \qquad - 12 \\
 & 4t^2 \qquad - 12 \\
 & - \qquad\quad + \\
\hline
 & 0
\end{array}
$$

Here, remainder is 0, so $t^2 - 3$ is a factor of $2t^4 + 3t^3 - 2t^2 - 9t - 12$.

$2t^4 + 3t^3 - 2t^2 - 9t - 12 = (2t^2 + 3t + 4)(t^2 - 3)$

Ex.29 Obtain all the zeroes of $3x^4 + 6x^3 - 2x^2 - 10x - 5$, if two of its zeroes are $\sqrt{\frac{5}{3}}$ and $-\sqrt{\frac{5}{3}}$.

Sol. Since two zeroes are $\sqrt{\frac{5}{3}}$ and $-\sqrt{\frac{5}{3}}$,

$x = \sqrt{\frac{5}{3}},\ x = -\sqrt{\frac{5}{3}}$

$\Rightarrow \left(x - \sqrt{\frac{5}{3}}\right)\left(x + \sqrt{\frac{5}{3}}\right) = x^2 - \frac{5}{3}$ or $3x^2 - 5$

is a factor of the given polynomial.

Now, we apply the division algorithm to the given polynomial and $3x^2 - 5$.

$$
\begin{array}{r|l}
 & x^2 + 2x + 1 \\
3x^2 - 5 & 3x^4 + 6x^3 - 2x^2 - 10x - 5 \\
 & 3x^4 \qquad - 5x^2 \\
 & - \qquad\quad + \\
\hline
 & 6x^3 + 3x^2 - 10x - 5 \\
 & 6x^3 \qquad - 10x \\
 & - \qquad\quad + \\
\hline
 & 3x^2 \qquad - 5 \\
 & 3x^2 \qquad - 5 \\
 & - \qquad\quad + \\
\hline
 & 0
\end{array}
$$

So, $3x^4 + 6x^3 - 2x^2 - 10x - 5$

$= (3x^2 - 5)(x^2 + 2x + 1) + 0$

Quotient $= x^2 + 2x + 1 = (x + 1)^2$

Zeroes of $(x + 1)^2$ are $-1, -1$.

Hence, all its zeroes are $\sqrt{\frac{5}{3}}, -\sqrt{\frac{5}{3}}, -1, -1$.

Ex.30 On dividing $x^3 - 3x^2 + x + 2$ by a polynomial g(x), the quotient and remainder were $x - 2$ and $-2x + 4$, respectively. Find g(x).

Sol. $p(x) = x^3 - 3x^2 + x + 2$

$q(x) = x - 2$ and $r(x) = -2x + 4$

By Division Algorithm, we know that

$p(x) = q(x) \times g(x) + r(x)$

Therefore,

$x^3 - 3x^2 + x + 2 = (x - 2) \times g(x) + (-2x + 4)$

$\Rightarrow x^3 - 3x^2 + x + 2 + 2x - 4 = (x - 2) \times g(x)$

$\Rightarrow \quad g(x) = \dfrac{x^3 - 3x^2 + 3x - 2}{x - 2}$

On dividing $x^3 - 3x^2 + 3x - 2$ by $x - 2$, we get g(x)

$$\begin{array}{r|l}
 & x^2 - x + 1 \\
\hline
x-2 & x^3 - 3x^2 + 3x - 2 \quad \text{First term of quotient is } \frac{x^3}{x} = x \\
 & x^3 - 2x^2 \\
 & - \quad + \\
\hline
 & -x^2 + 3x - 2 \quad \text{Second term of quotient is } \frac{-x^2}{x} = -x \\
 & -x^2 + 2x \\
 & + \quad - \\
\hline
 & x - 2 \quad \text{Third term of quotient is } \frac{x}{x} = 1 \\
 & x - 2 \\
 & - \quad + \\
\hline
 & 0
\end{array}$$

Hence, $g(x) = x^2 - x + 1$.

Ex.31 Give examples of polynomials p(x), q(x) and r(x), which satisfy the division algorithm and

(i) deg p(x) = deg q(x)

(ii) deg q(x) = deg r(x)

(iii) deg q(x) = 0

Sol. (i) Let $q(x) = 3x^2 + 2x + 6$, degree of q(x) = 2

$p(x) = 12x^2 + 8x + 24$, degree of p(x) = 2

Here, deg p(x) = deg q(x)

(ii) $p(x) = x^5 + 2x^4 + 3x^3 + 5x^2 + 2$

$q(x) = x^2 + x + 1$, degree of q(x) = 2

$g(x) = x^3 + x^2 + x + 1$

$r(x) = 2x^2 - 2x + 1$, degree of r(x) = 2

Here, deg q(x) = deg r(x)

(iii) Let $p(x) = 2x^4 + 8x^3 + 6x^2 + 4x + 12$

q(x) = 2, degree of q(x) = 0

$g(x) = x^4 + 4x^3 + 3x^2 + 2x + 6$

r(x) = 0

Here, deg q(x) = 0

Ex.32 If the zeroes of polynomial $x^3 - 3x^2 + x + 1$ are a – b, a , a + b. Find a and b.

Sol. $\because$ a – b, a, a + b are zeros

$\therefore$ product $(a - b)\,a(a + b) = -1$

$\Rightarrow (a^2 - b^2)\,a = -1$ …(1)

and sum of zeroes is $(a - b) + a + (a + b) = 3$

$\Rightarrow 3a = 3 \Rightarrow a = 1$ …(2)

by (1) and (2)

$(1 - b^2)1 = -1$

$\Rightarrow 2 = b^2 \Rightarrow b = \pm\sqrt{2}$

$\therefore$ a = –1 & $b = \pm\sqrt{2}$ **Ans.**

Ex.33 If two zeroes of the polynomial $x^4 - 6x^3 - 26x^2 + 138x - 35$ are $2 \pm \sqrt{3}$, find other zeroes.

Sol. $\because$ $2 \pm \sqrt{3}$ are zeroes.

$\therefore$ $x = 2 \pm \sqrt{3}$

$\Rightarrow$ $x - 2 = \pm\sqrt{3}$ (squaring both sides)

$\Rightarrow$ $(x - 2)^2 = 3 \Rightarrow x^2 + 4 - 4x - 3 = 0$

$\Rightarrow$ $x^2 - 4x + 1 = 0$, is a factor of given polynomial

$\therefore$ other factors

$= \dfrac{x^4 - 6x^3 - 26x^2 + 138x - 35}{x^2 - 4x + 1}$

$$\begin{array}{r|l}
 & x^2 - 2x - 35 \\
\hline
x^2 - 4x + 1 & x^4 - 6x^3 - 26x^2 + 138x - 35 \\
 & x^4 - 4x^3 + x^2 \\
 & - \quad + \quad - \\
\hline
 & -2x^3 - 27x^2 + 138x - 35 \\
 & -2x^3 + 8x^2 - 2x \\
 & + \quad - \quad + \\
\hline
 & -35x^2 + 140x - 35 \\
 & -35x^2 + 140x - 35 \\
 & + \quad - \quad + \\
\hline
 & \times
\end{array}$$

$\therefore$ other factors $= x^2 - 2x - 35$

$= x^2 - 7x + 5x - 35 = x(x - 7) + 5(x - 7)$

$= (x - 7)(x + 5)$

$\therefore$ other zeroes are $(x - 7) = 0 \Rightarrow x = 7$

$x + 5 = 0 \Rightarrow x = -5$ **Ans.**

Ex.34 If the polynomial $x^4 - 6x^3 + 16x^2 - 25x + 10$ is divided by another polynomial $x^2 - 2x + k$, the remainder comes out to be x + a, find k & a.

Sol.

$$
\begin{array}{r|l}
x^2 - 2x + k & x^4 - 6x^3 + 16x^2 - 25x + 10 \quad (x^2 - 4x + (8 - k)) \\
 & x^4 - 2x^3 + x^2k \\
 & - \quad + \quad - \\ \hline
 & -4x^3 + x^2(16 - k) - 25x + 10 \\
 & -4x^3 + x^2(8) - 4xk \\
 & + \quad - \quad + \\ \hline
 & x^2[8 - k] + x[4k - 25] + 10 \\
 & x^2[8 - k] - 2x[8 - k] + k(8 - k) \\
 & - \quad + \quad - \\ \hline
 & x[4k - 25 + 16 - 2k] + 10 - 8k + k^2
\end{array}
$$

According to questions, remainder is $x + a$

$\therefore$ coefficient of $x = 1$

$\Rightarrow 2k - 9 = 1$

$\Rightarrow k = (10/2) = 5$

Also constant term = a

$\Rightarrow k^2 - 8k + 10 = a \Rightarrow (5)^2 - 8(5) + 10 = a$

$\Rightarrow a = 25 - 40 + 10$

$\Rightarrow a = -5$

$\therefore k = 5, a = -5$ **Ans.**

EXERCISE # 1

A. Very Short Answer Type Questions

Factorize each of the following expression

Q.1 $x^2 - x - 42$

Q.2 $6 - 5y - y^2$

Q.3 $a^2 + 46a + 205$

Q.4 $ab + ac - b^2 - bc$

Q.5 $p^4 - 81q^4$

Use remainder theorem to find remainder, when p(x) is divided by q(x) in following questions.

Q.6 $p(x) = 2x^2 - 5x + 7, q(x) = x - 1$

Q.7 $p(x) = x^9 - 5x^4 + 1$, $q(x) = x + 1$

Q.8 $p(x) = 2x^3 - 3x^2 + 4x - 1$, $q(x) = x + 2$

B. Short Answer Type Questions

Q.9 Find positive square root of $36x^2 + 60x + 25$

Q.10 Simplify : $\sqrt{2a^2 + 2\sqrt{6}ab + 3b^2}$

Q.11 $(x^2 + 4y)^2 + 21(x^2 + 4y) + 98$

Q.12 Find the value of k if $(x - 2)$ is a factor of $2x^3 - 6x^2 + 5x + k$.

Q.13 Find the value of k if $(x + 3)$ is a factor of $3x^2 + kx + 6$.

Q.14 $p(x) = 3x^6 - 7x^5 + 7x^4 - 3x^3 + 2x^2 - 2, q(x) = x - 1$

Q.15 For what value of k is $y^3 + ky + 2k - 2$ exactly divisible by $(y + 1)$?

C. Long Answer Type Questions

Q.16 If $x + 1$ and $x - 1$ are factors of $mx^3 + x^2 - 2x + n$, find the value of m and n.

Q.17 Find the zeros of the polynomial $f(x) = 2x^2 + 5x - 12$ and verify the relation between its zeroes and coefficients.

Q.18 Find the zeroes of the polynomial $f(x) = x^2 - 2$ and verify the relation between its zeroes and coefficients.

Q.19 Obtain the zeroes of the quadratic polynomial $\sqrt{3}x^2 - 8x + 4\sqrt{3}$ and verify the relation between its zeroes and coefficients.

Q.20 Find a cubic polynomial with the sum of its zeroes, sum of the products of its zeroes taken two at a time and the product of its zeroes as 2, –7 and –14 respectively.

Q.21 Find a cubic polynomial whose zeroes are 3, 5 and – 2.

Q.22 Divide $5x^3 - 13x^2 + 21x - 14$ by $(3 - 2x + x^2)$ and verify the division algorithm.

Q.23 What real number should be subtracted from the polynomial $(3x^3 + 10x^2 - 14x + 9)$ so that $(3x - 2)$ divides it exactly?

Q.24 Find all the zeroes of $(2x^4 - 3x^3 - 5x^2 + 9x - 3)$, it being given that two of its zeroes are $\sqrt{3}$ and $-\sqrt{3}$.

ANSWER KEY

A. VERTY SHORT ANSWER TYPE :

1. $(x + 6)(x - 7)$ **2.** $(6 + y)(1 - y)$ **3.** $(a + 41)(a + 5)$ **4.** $(a - b)(b + c)$

5. $(p + 3q)(p - 3q)(p^2 + 9q^2)$ **6.** 4 **7.** -5 **8.** -37

B. SHORT ANSWER TYPE :

9. $6x + 5$ **10.** $(\sqrt{2}\,a + \sqrt{3}\,b)$ **11.** $(x^2 + 4y + 7)(x^2 + 4y + 14)$ **12.** -2

13. 11 **15**. 3

C. LONG ANSWER TYPE :

16. $m = 2,\ n = -1$ **17.** $-4, \frac{3}{2}$ **18.** $-\sqrt{2}, \sqrt{2}$ **19.** $2\sqrt{3}, \frac{2}{\sqrt{3}}$

20. $x^3 - 2x^2 - 7x + 14$ **21.** $x^3 - 6x^2 - x + 30$ **22.** quotient = $5x - 3$, Remainder = -5

23. 5 **24.** $\sqrt{3}, -\sqrt{3}, 1, \frac{1}{2}$

EXERCISE # 2

Q.1 If $\left(x+\frac{1}{x}\right) = 3$, then find value of $\left(x^2+\frac{1}{x^2}\right)$.

Q.2 If $\left(x-\frac{1}{x}\right) = \frac{1}{2}$, then find $\left(4x^2+\frac{4}{x^2}\right)$.

Q.3 If $\left(x+\frac{1}{x}\right) = 4$, then find $\left(x^4+\frac{1}{x^4}\right)$.

Q.4 If (x – 2) is a factor of (x^2 + 3qx – 2q), then find the value of q.

Q.5 If $x^3 + 6x^2 + 4x + k$ is exactly divisible by (x + 2), then find the value of k.

Q.6 Let f(x) = $x^3 - 6x^2 + 11x - 6$. Then, which one of the following is not factor of f(x) ?

(A) x – 1 (B) x – 2

(C) x + 3 (D) x – 3

Q.7 If $x^{100} + 2x^{99} + k$ is divisible by (x + 1), then find the value of k.

Q.8 On dividing (x^3 – 6x + 7) by (x + 1), find the remainder.

Q.9 Find the value of expression ($16x^2$ + 24x + 9) for $x = -\frac{3}{4}$.

Q.10 If $2x^3 + 5x^2 - 4x - 6$ is divided by 2x + 1, then find remainder.

Q.11 If p(x) = x^2 – 2x – 3, then find
(i) p(3); (ii) p(–1)

Q.12 Find the zeros of the quadratic polynomial ($6x^2$ – 7x – 3) and verify the relation between its zeros and coefficients.

Q.13 Find the zeros of the quadratic polynomial ($5u^2$ + 10u) and verify the relation between the zeros and the coefficients.

Q.14 Find the quadratic polynomial whose zeros are $\frac{2}{3}$ and $\frac{-1}{4}$. Verify the relation between the coefficients and the zeros of the polynomial.

Q.15 Find the quadratic polynomial, sum of whose zeros is 8 and their product is 12. Hence, find the zeros of the polynomial.

Q.16 Find the quadratic polynomial , the sum of whose zeros is –5 and their product is 6. Hence, find the zeros of the polynomial.

Q.17 Find the quadratic polynomial, the sum of whose zeros is 0 and their product is –1. Hence, find the zeros of the polynomial.

Q.18 Find a quadratic polynomial whose one zero is $5 + \sqrt{7}$.

Q.19 On dividing ($x^3 - 3x^2$ + x + 2) by a polynomial g(x), the quotient and remainder are (x – 2) and (–2x + 4) respectively. Find g(x).

Q.20 If the polynomial ($x^4 + 2x^3 + 8x^2$ + 12x + 18) is divided by another polynomial (x^2 + 5), the remainder comes out to be (px + q). Find the value of p and q.

Q.21 Obtain all zeros of the polynomial ($2x^3$ – 4x – x^2 + 2), if two of its zeros are $\sqrt{2}$ and $-\sqrt{2}$.

Q.22 If 1 and –2 are two zeros of the polynomial ($x^3 - 4x^2$ – 7x + 10), find its third zero.

Q.23 Find all the zeros of the polynomial ($2x^4 - 11x^3 + 7x^2$ + 13x – 7), it being given that two if its zeros are ($3 + \sqrt{2}$) and ($3 - \sqrt{2}$).

Q.24 If α, β are the zeros of the polynomial f(x) = x^2 – 5x + k such that α – β = 1, find the value of k.

Q.25 Show that the polynomial f(x) = $x^4 + 4x^2 + 6$ has no zero.

Q.26 Use remainder theorem to find the value of k, it being given that when $x^3 + 2x^2$ + kx + 3 is divided by (x – 3), then the remainder is 21.

ANSWER KEY

1. 7	**2.** 9	**4. – 1** **5.** – 8	**6.** (C)
7. 1	**8.** 12	**9.** 0	**10.** – 3
11. (i) 0 , (ii) 0	**12.** $\frac{3}{2}, -\frac{1}{3}$	**13.** – 2, 0	**14**. $12x^2 - 5x - 2$
15. $(x^2 - 8x + 12)$, {6, 2}		**16.** $(x^2 + 5x + 6)$, {– 3, – 2}	
17. $(x^2 - 1)$, {1, –1}	**18.** $x^2 - 10x + 18$	**19.** $x^2 - x + 1$	**20.** p = 2, q = 3
21. $\sqrt{2}, -\sqrt{2}, \frac{1}{2}$	**22.** 5	**23.** $(3+\sqrt{2}),(3-\sqrt{2})$, $\frac{1}{2}$, – 1	
24. k = 6	**26.** k = – 9		