PERMUTATION & COMBINATION

Frank Plumpton Ramsey

Although Ramsey was a lecturer in mathematics, he produced work in a remarkable range of topics over a short period. His second simplification is to suggest simplifying Russell's theory of types by regarding certain semantic paradoxes as linguistic. He accepted Russell's solution to remove the logical paradoxes of set theory arising from, for example, "the set of all sets which are not members of themselves". However, the semantic paradoxes such as "this is a lie" are, Ramsey claims, quite different and depend on the meaning of the word "lie". These he removed with his reinterpretation that removed the axiom of reducibility.

His second paper on mathematics *On a problem of formal logic* was read to the London Mathematical Society on 13 December 1928 and published in the *Proceedings of the London Mathematical Society* in 1930. This examines methods for determining the consistency of a logical formula and it includes some theorems on combinatorics which have led to the study of a whole new area of mathematics called Ramsey theory. The combinatorics was introduced by Ramsey to solve a special case of the decision problem for the first-order predicate calculus.

Ramsey made a systematic attempt to base the mathematical theory of probability on the notion of partial belief. This work on probability, and also important work on economics, came about mainly because Ramsey was a close friend of Keynes. Being a friend of Keynes certainly did not stop Ramsey attacking Keynes' work, however, and in *Truth and probability* , which Ramsey published in 1926, he argues against Keynes' ideas of an a priori inductive logic. Ramsey's arguments convinced Keynes who then abandoned his own ideas.

Ramsey suffered an attack of jaundice and was taken to Guy's Hospital in London for an operation. He died following the operation.

Introduction

By the term permutation we mean arrangement of distinct or non-distinct things whereas by combination we mean selection of some or all things from distinct or non-distinct things. The term and notation of factorial will often be used in this chapter.

- **Factorial**

  The continued product of first n natural numbers is called the factorial of n and is denoted by $n!$. Thus $n! = 1.2.3.4\ldots..n$

  The definition of factorial to zero or negative integer are as follows

  $n! = n(n-1)!$

$(n-1)! = \frac{n!}{n}$, put $n = 1$, $0! = \frac{1!}{1} = 1$

$\frac{1}{(n-1)!} = \frac{n}{n!}$, put $n = 0$, $\frac{1}{(-1)!} = \frac{0}{0!} = \frac{0}{1} = 0$

put $n = -1$ $\frac{1}{(-2)!} = \frac{-1}{(-1)!} = -1 \times 0 = 0$

$\therefore \quad \frac{1}{(\text{ve integer})!} = 0$.

2.1 Exponent of prime *p* in *n*!

Let p be a given prime and n any positive integer, then the maximum power of a prime 'p' in $n!$ is

$\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] .... + \left[\frac{n}{p^s}\right]$, where [.] denotes the greatest integer function and S is such that $p^s \le n \le p^{s+1}$.

The above formula does not work for composite numbers. For example if we have to find the maximum power of 6 present in *32!*, then the answer is not $\left[\frac{32}{6}\right] + \left[\frac{32}{6^2}\right] + ... = 5$, as 5 is the number of integral multiples of 6 in *1, 2, ..., 32;* and 6 can be obtained on multiplying 2 by 3 also. Hence for the required number, we find the maximum powers of 2 and 3 (say r and s) present in 32!. Using the above formula $r = 31$ and $s = 14$. Hence 2 and 3 will be combined (to form 6) 14 times. Thus maximum power of 6 present in 32! is 14.

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***Illustration* 1**

Find the exponent of 3 in 100!.

Solution

Let $E_p(n)$ denote the exponent of p in n.

Then, $E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + ... + \left[\frac{n}{p^s}\right]$,

where s is the largest positive integer such that $p^s \le n < p^{s+1}$

Here, $n = 100$, $p = 3$ $3^4 < 100 < 3^5$ □ $s = 4$.

So, $E_3(100!) = \left[\frac{100}{3}\right] + \left[\frac{100}{3^2}\right] + \left[\frac{100}{3^3}\right] + \left[\frac{100}{3^4}\right]$

$= 33 + 11 + 3 + 1 = 48$

Hence, the exponent of 3 in 100! is 48

***Illustration* 2**

Find the number of zeros at the end of 100!.

Solution

In terms of prime factors 100! Can be written as $2^a 3^b 5^c$, $7^d \ldots$

Now, $$E_2(100!) = \left[\frac{100}{2}\right] + \left[\frac{100}{2^2}\right] + \left[\frac{100}{2^3}\right] + \left[\frac{100}{2^4}\right] + \left[\frac{100}{2^5}\right] + \left[\frac{100}{2^6}\right]$$

$$= 50 + 25 + 12 + 6 + 3 + 1 = 97$$

and, $E_5(100!) = \left[\frac{100}{5}\right] + \left[\frac{100}{5^2}\right] = 20 + 4 = 24$

Therefore, $100! = 2^{97} \times 3^b \times 5^{24} \times 7^d \times \ldots = 2^{73} \times (2 \times 5)^{24} \times 7^d \times \ldots$

$$= 10^{24} \times 2^{73} \times 3^b \times 7^d \times \ldots$$

Thus, the number of zeros at the end of 100! is 24.

2.2 Fundamental Principle of Counting

- **Fundamental Principle of Addition**

  Suppose a work *A* can occur in *m* ways and *B* can occur in *n* ways and both cannot occur simultaneously. Then *A* or *B* can occur in (*m* + *n*) ways. This rule is also applicable for two or more excusive works.

  ***For example*:** If we have to find the number of two digit numbers which either ends in 7 or are even numbers. The number of two digit numbers ending in 7 is 9 and number of two digit even numbers is 45. Hence required number is 9 + 45 = 54

- **Fundamental Principle of Multiplication**

  Suppose there are two works *A* and *B*. Let *A* can occur in *m* ways and for each way of occurrence of *A*, *B* can occur in *n* ways, then both *A* and *B* can occur in *mn* ways.

  ***For example*:** Let there be two questions *A* and *B* which can be solved by 2 and 3 methods respectively. Then both *A* and *B* can be solved in $2 \times 3 = 6$ ways.

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***Illustration* 3**

There are 4 students for Physics, 6 students for Chemistry and 7 students for Mathematics gold-medals. In how many way one of these gold-medals be awarded.

Solution

There are Physics, Chemistry and Mathematics gold medal can be awarded is 4,6, and 7 ways respectively. Hence by addition rule number of ways of awarding one of those gold-medals = 4+6+7 = 17 ways.

***Illustration* 4**

Find the number of flinching arrangements of six subjects in 2 periods

Solution

There are six ways to filling the first period. For the second period we are left with remaining subjects and hence there are 5 ways of filling the second period. Hence by multiplication rule total number of ways = $6 \times 5 = 30$

2.3 Permutations

Each of the arrangements in a definite order which can be made by taking some or all of the things is called permutation.

(i) Let r and n be positive integers such that $1 \leq r \leq n$ then, the number of all permutations of n distinct objects taken r at a time is given by

$$n(n-1)(n-2).....(n-(r-1) = \frac{n!}{(n-r)!} = {}^nP_r$$

(ii) The number of permutations of n distinct things taken all at a time is given by ${}^nP_n = n!$

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***Illustration* 5**

In how many ways three different rings can be worn in four fingers with at most one in each finger?

Solution

The total number ways is same as the number of arrangements of 4 fingers, taken 3 at a time. So, required number of ways

$$= {}^4P_3 = \frac{4!}{(4-3)!} = \frac{4!}{1!} = 4! = 24.$$

***Illustration* 6**

How many different signals can be made by 5 flags from 8 flags of different colours?

Solution

The total number of signals is the number of arrangements of 8 flags by taking 5 flags at a time.

Hence, required number of signals

$$= {}^8P_5 = \frac{8!}{(8-5)!} = \frac{8!}{3!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3!}{3!} = 6720$$

***Illustration* 7**

In an examination hall there are four rows of chairs. Each row has 8 chairs one behind the other. There are two classes sitting for the examination with 16 students in each class. It is desired that in each row, all students belong to the same class and that no two adjacent rows are allotted to the same class. In how many ways can these 32 students be seated?

Solution

Let the two classes be C_1 and C_2 and the four rows be R_1, R_2, R_3, R_4. There are 16 students in each class. So, there are 32 students. According to the given conditions there are two different ways in which 32 students can be seated :

	R_1	R_2	R_3	R_4
I	C_1	C_2	C_1	C_2
II	C_2	C_1	C_2	C_1

Since the seating arrangement can be completed by using any one of these two ways. So, by the fundamental principle of addition, Total no. of seating arrangements

= No. of arrangements in I case + No. of arrangements in II case.

Now, 16 students of class C_1 can be seated in 16 chairs in ${}^{16}P_{16} = 16!$ ways.

And, 16 students of class C_2 can be seated in 16 chairs in ${}^{16}P_{16} = 16!$ Ways.

Hence, the total number of seating arrangements

$$= (16! \times 16!) + (16! \times 16!) = 2\,(16! \times 16!)$$

2.4 PERMUTATION UNDER GIVEN CONDITIONS

Permutation of One or All of the Items

When either repetitions of item are allowed or a particular item is ignored in each arrangement.

(i) The number of all permutations of n different objects taken 'r' at a time, when a particular object is to be always included in each arrangement is $r\ {}^{n-1}P_{r-1}$.

(ii) The number of permutation of n different things taken 'r' at a time, when a particular object is never taken in each arrangement is ${}^{n-1}P_r$

(iii) The number of permutations of n different things taken 'r' at a time when two specified objects always occur together is $2\,!\,(r-1)\ {}^{n-2}P_{r-2}$.

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***Illustration* 8**

How many even numbers are there with three digits such that if 5 is one of the digits, then 7 is the next digit?

Solution

We have to determine the total number of even numbers formed by using the given condition. So, at units place we can use one of the digits 0, 2, 4, 6, 8. If 5 is at ten's place then, as per the given condition, 7 should be at unit's place. In such a case the number will not be an even number. So, 5 cannot be at ten's and one's places. Hence, 5 can be only at hundred's place. Now two cases arise.

Case I: When 5 is at hundred's place. If 5 is a hundred's place, then 7 will be at ten's place. So, unit's place can be filled in 5 ways by using the digits 0, 2, 4, 6, 8. So, total number of even numbers = 1 × 1 × 5 = 5.

Case II. When 5 in not at hundred's place. Now, hundred's place can be filled in 8 ways (0 and 5 cannot be used at hundred's place). In ten's place we can use any one of the en digits except 5. So, ten's place can be filled in 9 ways. At unit's place we have to use one of the even digits 0, 2, 4, 6, 8. So, units place can be filled in 5 ways. So, total number of even numbers = 8 × 9 × 5 = 360.

Hence, the total number of required even numbers = 360 + 5 = 365.

***Illustration* 9**

How many numbers between 400 and 1000 can be formed with the digits 0, 2, 3, 4, 5, 6, if no digit is repeated in the same number?

Solution

Number between 400 and 1000 consist of three digits with digit at hundred's place greater than or equal to 4. Hundred's place can be filled, by using the digits 4, 5, 6 in 3 ways. Now, ten's and unit's places can be filled by the remaining 5 digits in $^5P_2$ ways.

Hence, the required number of numbers $= 3 \times {}^5P_2 = 3 \times \frac{5!}{3!} = 3 \times 20 = 60$

***Illustration* 10**

Find the number of ways in which 5 boys and 5 girls be seated in a row so that

(*i*) No two girls may sit together.

(*ii*) All the girls sit together and all the boys sit together

(*iii*) All the girls are never together.

Solution

(i) 5 boys can be seated in a row in $^5P_5 = 5!$ ways. Now, in the 6 gaps 5 girls can be arranged in $^6P_5$ ways. Hence, the number of ways in which no two girls sit together $= 5! \times {}^6P_5 = 5! \times 6!$

(ii) The two groups of girls and boys can be arranged in 2! ways 5 girls can be arranged among themselves in 5! Ways. Similarly, 5 boys can be arranged among themselves in 5! ways. Hence, by the fundamental principle of counting, the total number of requisite seating arrangements = 2! (5! × 5!) = $2(5!)^2$.

(iii) The total number of ways in which all the girls are never together.

= Total number of arrangements – Total number of arrangements in which all the girls are always together = 10! – 5! × 6!

***Illustration* 11**

In a class of 10 students there are 3 girls A, B, C. In how many different ways can they be arranged in a row such that no two of the three girls are consecutive.

Solution

There are 7 boys and 3 girls. Seven boys can be arranged in a row in ${}^7P_7 = 7!$ ways. Now, we have 8 places in which we can arrange 3 girls in ${}^8P_3$ ways.

Hence, by fundamental principle of counting, the number of arrangements $= 7! \times {}^8P_3 = 7! \times 336$.

2.5 Permutations of Object not all Distinct

So far we have discussed about those objects that were distinct, now we will take objects that are not all distinct.

(i) The number of mutually distinguishable permutations of n things, taken all at a time of which 'p_1' are alike of first kind and 'p_2' are alike of second kind, such that $p_1 + p_2 = n$ is $\frac{n!}{p_1!p_2!}$

(ii) The number of permutations of 'n' distinct objects of which p_1 are alike of first kind, p_2 are alike of second kind and all other distinct, such that $p_1 + p_2 < n$, is $\frac{n!}{p_1!p_2!}$

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***Illustration* 12**

How many different words can be formed by using all the letters of the word 'ALLAHABAD'?

(i) In how many of them vowels occupy the even positions?

(ii) In how many of them both L do not come together?

Solution

There are 9 letters in the word 'ALLAHABAD' out of which 4 are A's, 2 are L's and the rest are all distinct.

So, the requisite number of words $= \frac{9!}{4!2!} = 7560$

(i) There are 4 vowels and all are alike i.e. 4 A's. Also, there are 4 even places viz 2^{nd}, 4^{th}, 6^{th} and 8^{th}. So, these 4 even places can be occupied by 4 vowels in $\frac{4!}{4!} = 1$ way. Now, we are left with 5 places in which 5 letters, of which two are alike (2 L's) and other distinct, can be arranged in $\frac{5!}{2!}$ ways.

(ii) Considering both *L* together and treating them as one letter we have 8 letters out of which *A* repeats 4 times and others are distinct. These 8 letters can be arranged in $\frac{8!}{4!}$ ways.

So, the number of words in which both L come together $=\frac{8!}{4!}=1680.$

Hence, the number of words in which both *L* do not come together

= Total no. of words – No. of words in which both *L* come together = 7560 – 1680 = 5880.

***Illustration* 13**

If all the letters of the word 'AGAIN' be arranged as in a dictionary, what is the fiftieth word?

Solution

In dictionary the words at each stage are arranged in alphabetical order. Starting with the letter *A*, an arranging the other four letters GAIN, we obtain 4! = 24 words.

Thus, there are 24 words which start with A. These are the first 24 words.

Then, starting with *G*, and arranging the other four letters *A*, *A*, *I*, *N* in different ways, we obtain $\frac{4!}{2!}=\frac{24}{2}=12$ words.

Thus, there are 12 words, which start with G.

Now, we start with *I*. The remaining 4 letters *A*, *G*, *A*, *N* can be arranged in $\frac{4!}{2!}=12$ ways.

So, there are 12 words, which start with I.

Thus, we have so far constructed 48 words.

The 49^{th} word is NAAGI and hence the 50^{th} word is NAAIG.

***Illustration* 14**

How many numbers greater than a million can be formed with the digits 2, 3, 0, 3, 4, 2, 3?

Solution

Any number of greater than a million will contain all the seven digits.

Now, we have to arrange these seven digits, out of which 2 occurs twice, 3 occurs twice and the rest are distinct.

The number of such arrangements $=\frac{7!}{2!\times 3!}=420$

These arrangements also include those numbers which contain 0 at the million's place.

Keeping 0 fixed at the millionth place, we have 6 digits out of which 2 occurs twice, 3 occurs thrice and the rest are distinct.

These arrangements also include those numbers which contain 0 at the million's place.

Keeping 0 fixed at the millionth place, we have 6 digits out of which 2 occurs twice, 3 occurs thrice and the rest are distinct.

These 6 digits can be arranged in $\frac{6!}{2! \times 3!} = 60$ ways

Hence, the number of required numbers = 420 – 60 = 360.

2.6 Permutations When Object can Repeat

Number of permutations of *n* different objects taken *r* at a time when thing could be repeated any number of times in n^r.

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***Illustration* 15**

How many numbers of 3 digits can be formed with the digits 1, 2, 3, 4, 5 when digits may be repeated?

Solution

The unit's place can be filled in 5 ways. Since, the repetition of digits is allowed, therefore ten's place can be filled in 5 ways and hundred's place can also be filled in 5 ways. Therefore, by the fundamental principle of counting, the required number of three digit numbers = 5 × 5 × 5 = 125.

***Illustration* 16**

Find the number of numbers of 5 digits that can be formed with the digits 0, 1, 2, 3, 4 if the digits can be repeated in the same number.

Solution

In a five digit number 0 cannot be put in ten thousand's place. So, the number of ways of filling up the ten thousand's place = 4.

Since the repetition of digits is allowed, therefore each of the other places can be filled in 5 ways.

So, the required number of numbers = 4 × 5 × 5 × 5 × 5 = 2500.

***Illustration* 17**

In how many ways can 5 letters be posted in 4 letter boxes?

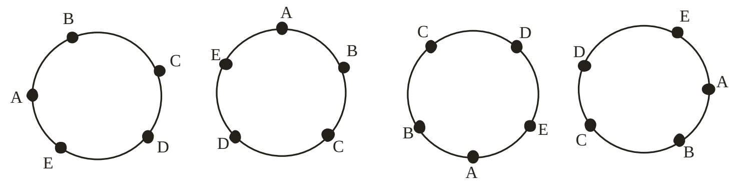
Solution

Since each letter can be posted in any one of the four letter boxes. So, a letter can be posted in 4 ways. Since there are 5 letters and each letter can be posted in 4 ways. So, total number of ways in which all the five letters can be posted = 4 × 4 × 4 × 4 × 4 = 4^5.

2.7 Circular Permutations

- **Arrangement Round a Circular Table**

A circular table has no head and arrangements like these in the figure given below are considered identical.



If n persons are arranged in a straight line, there are $n!$ different ways in which this can be done. Whereas n persons sit a round a circular table, each circular arrangement will be equivalent to n arrangements in a line, so there are $(n-1)!$ different arrangements of n persons around a circle.

In this case the number of circular permutation m clockwise and anticlockwise are distinct.

$\therefore$ the number of circular permutations of n different persons taken all at a time is $(n-1)!$

- **Arrangement of Beads (All Distinct) Around a Circular Wire**

A circular wire differ from a circular table because when we turn it over we see that the other side presents an arrangement of different beads different from that on the first side.

If the wire on the left is turned over we obtain the arrangements on the right. Thus two different arrangements for beads around a circular wire are not distinct.

Thus number of ways in this case $\frac{1}{2}(n-1)!$ In this case the number of circular permutation in clockwise and anticlockwise are not distinct.

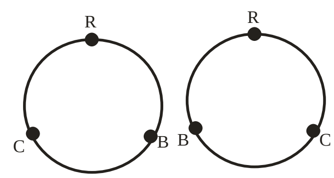


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***Illustration* 18**

Find the number of ways of arranging four boys and four girls in a circle, so that boys and girls are alternate.

Solution

Four boys, say $b_{1,}b_{2,}b_3$ and b_4 can be arranged in a circle in (4-1)! = 6 ways. Four each choice of these ways, there are 4! = 24 choices for girls to arrange them in the circle in the required way (here the number of ways of arranging the girls will not be (4-1)!, as positions are named, as $b_{1,}$ $b_{2,}$ between $b_{2,}$ and $b_{3,}$ etc.). Hence required number of ways is $6\times24=144$

***Illustration* 19**

Consider 21 different pearls on a necklace. How many ways can the pearls in on this necklace such that 3 specific pearls always remain together ?

Solution

After fixing the places of three pearls. Treating 3 specific pearls = 1 units, so we have now 18 pearls + 1 unit = 19 and the number of arrangement will be (19-1)! = 18! Also, the number of ways of 3 pearls can be arranged between themselves is 3! = 6. Since there is no distinction between the clockwise and anticlockwise arrangements. So the required number of arrangements = $\frac{1}{2}18!.6 = 3(18!)$

2.8 Dearrangement

If *n* things are arranged in a row, the number of ways in which these can be deranged so that no one of them occupies it *s* original place is

$$n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+......+(-1)^n\frac{1}{n!}\right)$$

If *r* things goes to wrong place out of *n* things then *(n – r)* things goes to original place *(r < n)*

If D_n = No. of ways. If all *n* things goes to wrong place

If D_n = No. of ways. If all *r* things goes to wrong place

Then, $D_n = {}^nC_{n-r}D_r$

$$D_n = r!\left(1-\frac{1}{1!}+\frac{1}{2!}+......+(-1)^n\frac{1}{n!}\right)$$

PRACTICE EXERCISE

1. If three dice are thrown find the number of ways of getting

   (a) Same number on all of them

   (b) Different number on all of them

2. How many 3 digit numbers can be made b y using the digits 1, 2, 3, 7, 9 when

   (i) repetition of a digit is allowed

   (ii) repetition of a digit is not allowed

3. There are 8 flags of different colour available at a mast. How many different signals can be given by flying flags in a line if

   (i) exactly 3 flags are to be used

   (ii) any number of flags are to be used

4. It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible?

5. Determine the number of natural numbers smaller than 10^4, in the decimal notation of which all the digits are distinct.
6. In how many ways can we arrange the letters of word PROPORTION taken all at a time
7. In how many ways can 20 person sit around a circle such that there is exactly one person between *A* and *B*.
8. How many arrangements can be made with the letters of the word 'MATHEMATICS'? In how many of them vowels are together?
9. In how many can the letters of the word "INTERMEDIATE" be arranged so that

   (i) the vowels always occupy even places?

   (ii) the relative order of vowels and consonants do not alter

   (iii) the order of vowels do not change?
10. There are 5 gentlemen and 4 ladies to dine at a round table. In how many ways can they seat themselves that no two ladies are together?

Answers

1. (i) 6 (ii) 120 **2.** (i) 125 (ii) 60 **3.** **4.** 2880 **5.** 5274

6. $\frac{10!}{3!2!\,2!}$ **7.** 1800 **8.** 4989600, 120960

9. (i) 21600 (ii) 21600 (iii) 332640 **10.** 2880

2.9 Combinations

Different selections made by taking some or all of objects irrespective of their arrangements.

Different selections made by taking some or all of objects irrespective of their arrangements.

2.10 Combination Selection of some items from distinct items

- **Selection from different *n* items taken *r* at a time**

  The number of all combinations of n distinct things taken '*r*' at a time is given by

$$^{n}C_{r}=\frac{n!}{(n-r)!r!}$$

- **Properties of $^{n}C_{r}$**

  (i) $^{n}C_{r}={}^{n}C_{n-r}$

  (ii) $^{n}C_{x}={}^{n}C_{y}\Rightarrow x=y$ or $x+y=n$

  (iii) Let *n* and *r* be non-negative integers such that $r\le n$. Then,

  $^{n}C_{r}+{}^{n}C_{r-1}={}^{n+1}C_{r}$

  (*iv*) Let *n* and *r* be non-negative integers such that $1\le r\le n$

Then $^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1}$

(v) If $1 \le r \le n$, then $n.\,{}^{n-1}C_{r-1} = (n-r+1)\,{}^nC_{r-1}$

(vi) If n is even, then greatest value of $^nC_r\ (0 \le r \le n)$ is $^nC_{n/2}$

(vii) if n is odd, then greatest value of $^nC_r (0 \le r \le n)$ is $\frac{^nC_{n+1}}{2}$ or $\frac{^nC_{n-1}}{2}$

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***Illustration* 20**

A box contains 5 different red and 6 different white balls. In how many ways can 6 balls be selected so that there are at least two balls of each colour?

Solution

The selection of 6 balls, consisting of at least two balls of each colour from 5 red and 6 white balls, can be made in the following ways :

(i) By selecting 2 red balls out of 5 and 4 white balls out of 6. This can be done in $^5C_2 \times {}^6C_4$ ways.

(ii) By selecting 3 red balls out of 5 and 3 white balls out of 6. This can be done in $^5C_3 \times {}^6C_3$ ways.

(iii) By selecting 4 red balls out of 5 and 2 white balls out of 6. This can be done in $^5C_4 \times {}^6C_2$ ways.

Since the selection of 6 balls can be completed in any one of the above ways. Hence, by the fundamental principle of addition, the total number of ways to select the balls

$= {}^5C_2 \times {}^6C_4 + {}^5C_3 \times {}^6C_3 + {}^5C_4 \times {}^6C_2$

$= 10 \times 15 + 10 \times 20 + 5 \times 15 = 425$

***Illustration* 21**

A committee of 5 is to be formed out of 6 gents and 4 ladies. In how many ways this can be done, when

(i) at least two ladies are included *(ii)* at most two ladies are included?

Solution

(i) A committee of 5 persons, consisting of at least two ladies, can be formed in the following ways :

(i) Selecting 2 ladies out of 4 and 3 gents out of 6. This can be done in $^4C_2 \times {}^6C_3$ ways.

(ii) Selecting 3 ladies out of 4 and 2 gents out of 6. This can be done in $^4C_3 \times {}^6C_2$ ways.

(iii) Selecting 4 ladies out of 4 and 1 gent out of 6. This can be done in $^4C_4 \times {}^6C_1$ ways.

Since the committee is formed in each case, therefore, by the fundamental principle of addition, the total number of ways of forming the committee

$= {}^4C_2 \times {}^6C_3 + {}^4C_3 \times {}^6C_2 + {}^4C_4 \times {}^6C_1$

$= 120 + 60 + 6 = 186$

(i) A committee of 5 persons, consisting of at most two ladies, can be constituted in the following ways :

(ii) selecting 5 gents only out of 6. This can be done in ${}^6C_5$ ways.

(iii) Selecting 3 gents only out of 6 and two ladies out of 4. This an be done is ${}^6C_3 \times {}^4C_2$ ways.

Since the committee is formed in each case, so, the total number of ways of forming the committee

$= {}^6C_5 + {}^6C_4 \times {}^4C_1 + {}^6C_3 \times {}^4C_2 = 6 + 60 + 120 = 186$

***Illustration* 22**

How many diagonals are there in a polygon with *n* sides?

Solution

A polygon of *n* sides has *n* vertices. By joining any two vertices of a polygon, we obtain either a side or a diagonal of the polygon. Number of line segments obtained by joining the vertices of a *n*

***Illustration* 23**

There are 10 points in a plane, no three of which are in the same straight line, excepting 4 points, which are collinear. Find the (*i*) number of straight lines obtained from the pairs of these points; (*ii*) number of triangles that can be formed with the vertices as these points.

Solution

(i) Number of straight lines formed joining the 10 points, taking 2 at a time $= {}^{10}C_2 = \frac{10!}{2!8!} = 45$

Number of straight lines formed by joining the four points, taking 2 at a time $= {}^4C_2 = \frac{4!}{2!2!} = 6$

But, 4 collinear points, when joined pair wise give only one line.

$\therefore$ Required number of straight lines = 45 – 6 + 1 = 40.

(ii) Number of triangles formed by joining the points, taking 3 at a time $= {}^{10}C_3 = \frac{10!}{3!7!} = 120$

Number of triangles formed by joining the 4 points, taken 3 at a time = ${}^4C_3 = {}^4C_1 = 4$.

But, 4 collinear points cannot form a triangle when taken 3 at a time. So,

Required number of triangles = 120 – 4 = 116.

***Illustration* 24**

From a class of 25 students, 10 are to be chosen for an excursion party. There are students who decide that either all of them will join or none of them will join. In how many ways can they be chosen?

Solution

We have the following possibilities:

(i) Three particular students join the excursion party.

In this case, we have to choose 7 students from the remaining 22 students. This can be done in ${}^{22}C_7$ ways.

(ii) Three particular students do not join the excursion party.

In this case, we have to choose 10 students from the remaining 22 students. This can be done in ${}^{22}C_{10}$ ways.

Hence, the required number of ways = ${}^{22}C_7 + {}^{22}C_{10} = 817190$.

***Illustration* 25**

In how many ways can 21 identical books on English and 19 identical books on Hindi be placed in a row on a shelf so that two books on Hindi may not be together?

Solution

In order that no two books on Hindi are together, we must first arrange all books in English in a row. Since all English books are identical, so they can be arranged in a row in only one way as shown below :

$$\times E \times E \times E \times E \times \ldots \times E \times E$$

Here E denotes the position of an English book and × that of a Hindi book.

Since there are 21 books on English, the number places mark × are therefore 22.Now, 19 books on Hindi are to be arranged in these 22 places so that no two of them are together. Out of 22 places 19 places for Hindi books can be chosen in ${}^{22}C_{19}$ ways. Since all books on Hindi are identical, so 19 books on Hindi can be arranged in 19 chosen places in only one way. Hence, the required number of ways = $1 \times {}^{22}C_{19} \times 1 = 1540$.

2.11 Selection from distinct items

The number of ways of selecting one or more items from a group of n distinct items is $2^n - 1$.

Out of n items one item can be selected in ${}^nC_1$ ways two item can be selected in ${}^nC_2$ ways and so on.

Thus, the required number of ways is

$${}^nC_1 + {}^nC_2 + {}^nC_3 + ... + {}^nC_n$$

$$= ({}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n) - {}^nC_0 = 2^n - 1$$

2.12 Selection from identical items

The number of ways of selecting one or more items from a group of n distinct items is $2^n - 1$.

(*i*) The number of ways of selecting 'r' items out of 'n' identical item is 1.

(*ii*) The total number of ways of selecting at least one item from n identical items is $(n + 1)$.

(*iii*) Total number of ways of selecting some or all of $p_1 + p_2 + p_3$ items, where p_1 are alike of first kind, p_2 are alike of second kind and p_3 are alike of third kind is $[(p_1 + 1)(p_2 + 1)(p_3 + 1)]-1$.

2.13 Selection of items from group of identical and non-identical items

The total number of ways of selecting one or more items from p_1, identical items of first kind, p_2 identical items of second kind, p_3 identical items of third kind and n non-identical items is

$$(p_1 + 1)(p_2 + 1)(p_3 + 1) . 2^n - 1$$

ILLUSTRATIONS

***Illustration* 26**

Find the total number of proper factors of 7875.

Solution

We have : $7875 = 3^2 \times 5^3 \times 7^1$

The total number of ways of selecting some or all out of two 3's, three 5's and one 7's is

$$(2 + 1)(3 + 1)(1 + 1) - 1 = 23$$

But this includes the given number itself. Therefore, the required number of proper factors is 22.

***Illustration* 27**

Find the number of factors (excluding 1 and the expression itself) of the product of $a^7b^4c^3$ def where a, b, c, d, e, f are all prime numbers.

Solution

The total number of factors of the product $a^7b^4c^3$ def is equal to the number of ways of selecting at least one from seven a's, four b's, three c's, one d's, one e's and one f's. The number of such ways is

$$(7 + 1)(4 + 1)(3 + 1)(2)(2)(2) - 1 = 1279$$

But this includes the given product. Hence, the required number of factor is 1279 – 1 = 1278.

2.14 DIVISION OF ITEMS INTO GROUPS

- **Division of Items into Groups of Unequal Size**

(i) The number of ays in which ($m + n + p$) items can be divided into unequal groups containing m, n, p items is

$$^{m+n+p}C_m . ^{n+p}C_n = \frac{(m+n+p)!}{m!n!p!}$$

(ii) The number of ways to distribute ($m + n + p$) items amongst three persons in the groups containing m, n and p items is

= (no. of ways to divide) × (no. of groups)

$$= \frac{(m+n+p)!}{m!n!p!} + 3!$$

- **Division of Groups of Equal Size**

(i) The number of ways in which pq different items can be divided equally into p groups, each containing q items and the order of the group is not important, is

$$\left(\frac{(pq)!}{(q!)^p}\right).\frac{1}{p!}$$

(ii) The number of ways in which pq different items can be divided equally into p groups, each containing q items and the order of the group is important, is

$$\left(\frac{(pq)!}{(q!)^p} \times \frac{1}{p!}\right).p! = \frac{(pq)!}{(q!)^p}.$$

ILLUSTRATIONS

***Illustration* 28**

In how many ways can a pack of 52 cards be divided equally among four players in order?

Solution

Here 52 cards are to be divided into four equal groups and the order of the groups is important. So, required number of ways $\left(\frac{52!}{(13!)^4 4!}\right) 4! = \frac{52!}{(13!)^4}$

ALITER For the first player we have $^{52}C_{13}$ choices, for the second player $^{39}C_{13}$ choices, for the third player $^{26}C_{13}$ choices and for the last player we have $^{13}C_{13}$ choices. Hence, the total number of ways

$$= {}^{52}C_{13} \times {}^{39}C_{13} \times {}^{26}C_{13} \times {}^{13}C_{13} \times = \frac{52!}{(13!)^4}$$

***Illustration* 29**

In how many ways can a pack of 52 cards be divided equally into four groups?

Solution

The total number of ways is $\frac{52!}{(13!)^4 4!}$

2.15 DIVISION OF IDENTICAL OBJECTS INTO GROUPS

(i) The total number of ways of dividing n identical objects into r groups, if blank groups are allowed is ${}^{n+r-1}C_{r-1}$

Since the total number of items is n. So the required number of ways is the co-efficient of x^n in the product.

Coefficient of x^n in $(x^0 + x + x^2 + x^3 + ... + x^n)^r$

= Co-efficient of x^n in $\left(\frac{1-x^{n+1}}{1-x}\right)^r$

= Co-efficient of x^n in $(1-x^{n+1})^r(1-x)^{-r}$

= Co-efficient of x^n in $(1 - x)^{-r}$

= $\frac{(r+1)(r+2)...(r+n-1)}{1.2.3......r} = {}^{n+r-1}C_{r-1}$

(ii) The number of ways in which n identical items can be divided into r groups such that blank groups are not allowed is ${}^{n-1}C_{r-1}$

The required number of ways is the co-efficient of x^n in $(x + x^2 + ….. x^n)^r$

= co-efficient of x^{n-r} in $(1 + x + x^2 + ….. + x^{n-1})^r$

= co-efficient of x^{n-r} in $(1 - x)^{-r}$

$= {}^{n-1}C_{r-1}$

(*iii*) The number of ways in which n identical items can be divided into r groups, so that no groups contains less than m items and more than k ($m < k$) is

Co-efficient of x^n in the expansion $(x^m + x^{m+1} + ... + x^k)^r$

ILLUSTRATIONS

***Illustration* 30**

Find the number of ways of distributing 5 identical balls into three boxes so that no box is empty and each box being large enough to accommodate all balls.

Solution

The required number of ways is the number of ways of distributing 5 items among 3 persons so that a person receives at least one item = ${}^{5-1}C_{3-1} = {}^4C_2 = 6$.

***Illustration* 31**

Find the number of non-negative integral solutions of the equation $x_1 + x_2 + x_3 + 4x_4 = 20$.

Solution

The total number of non-negative integral solutions of the given equation

= Coefficient of x^{20} in $(x^0 + x^1 + x^2 + …)^3 (x^0 + x^4 + x^8 + x^{16} + …)$

$= \quad$ Coefficient of x^{20} in $\left(\frac{1}{1-x}\right)^3 \left(\frac{1}{1-x^4}\right)$

$= \quad$ Coefficient of x^{20} in $[(1-x)^{-3}(1-x^4)^{-1}]$

$= \quad$ Coefficient of x^{20} in $[(1-x)^{-3}(1+x^4+x^8+x^{12}+x^{16}+x^{20}+\ldots)]$

$= \quad$ Coefficient of x^{20} in $(1-x)^{-3}$ + Coeff. X^{16} in $(1-x)^{-3}$ + Coeff. of x^{12} in $(1-x)^{-3}$ +…+ Coeff. of x^0

in $(1-x)^{-3}$

$= \quad {}^{20+3-1}C_{3-1} + {}^{16+3-1}C_{3-1} + {}^{12+3-1}C_{3-1} + {}^{8+3-1}C_{3-1} + {}^{4+3-1}C_{3-1} + 1$

$= \quad {}^{22}C_2 + {}^{18}C_2 + {}^{14}C_2 + {}^{10}C_2 + {}^{6}C_2 + 1$

$= \quad 231 + 153 + 91 + 45 + 15 + 1$

$= \quad 536.$

***Illustration* 32**

How many integral solutions are there to the system of equations $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ and $x_1 + x_2 + x_3 = 5$ when $x_k \geq 0$?

Solution

We have : $x_1 + x_2 + x_3 + x_4 + x_5 = 20$

and, $x_1 + x_2 + x_3 = 5$

These two equations reduce to

$x_4 + x_5 = 15$ …(i)

and, $x_1 + x_2 + x_3 = 5$ …(ii)

Since corresponding to each solution of (i) there are solutions of equation (ii). So, total number of solutions of the given system of equations

= No. of solutions of (i) × No. of solutions of (ii)

$= {}^{15+2-1}C_{2-1} \times {}^{5+3-1}C_{3-1} = {}^{16}C_1 \times {}^{7}C_2 = 336.$

***Illustration* 33**

How many integers between 1 and 10^6 have the sum of their digits equal to 18?

Solution

Let $n = x_6\ x_5\ x_4\ x_2\ x_1$ be a number between 1 and 10^6, where $x_1, x_2, x_3, \ldots x_6$ are digits at one's, tens, hundreds…places. Clearly $0 \leq x_i \leq 9$, $i = 1, 2, \ldots, 6$.

The total number of values of n is the number of solutions of the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 18$, where $0 \leq x_i \leq 9$; $i = 1, 2, \ldots, 6$

The number of solutions of the above equation is

Coefficient of x^{18} in $(x^0 + x^1 + x^2 + \ldots + x^9)^6$

= Coefficient of x^{18} in $\left(\frac{1-x^{10}}{1-x}\right)^6$

= Coefficient of x^{18} in $(1 – x^{10})^6$ $(1 – x)^{–6}$

= Coefficient of x^{18} in $(1 – x)^{–6}$

= ${}^6C_1$ × Coefficient of x^8 ion $(1 – x)^{–6}$

= ${}^{18+6-1}C_{6-1} - {}^6C_1 \times {}^{8+6-1}C_{6-1}$

= ${}^{23}C_5 - 6 \times {}^{13}C_5$

= 33649 – 7722

= 25927.

***Illustration* 34**

Three six faced die are thrown together. Find the total number of getting k as the sum if

(i) $3 \le k \le 8$ (ii) $9 \le k \le 14$

Solution

Let x_1, x_2 and x_3 be the numbers on the upper faces of three dice. Then, the required number of ways is equal to the number of solutions of the equation

$x_1 + x_2 + x_3 = k$, where $1 \le x_1, x_2, x_3 \le 6$

The number of solutions of this equation is equal to

Coefficient of x^k in $(x + x^2 +\ldots+ x^6)^3$

= Coefficient of $x^{k–3}$ in $(1 + x +\ldots+ x^5)^3$

= Coefficient of $x^{k–3}$ in $(1 + x +\ldots+ x^5)^3$

= Coefficient of $x^{k–3}$ in $\left(\frac{1-x^6}{3}\right)^3$

= Coefficient of $x^{k–3}$ in $(1 – x^6)^3$ $(1 – x)^{–3}$

(i) If $3 \le k \le 8$, then $0 \le k – 3 \le 5$

∴ Coefficient of $x^{k–3}$ in $(1 – x^6)^3$ $(1 – x)^{–3}$

= Coefficient of $x^{k–3}$ in $(1 – x)^{–3}$

= ${}^{k–3+3–1}C_{3–1}$

= ${}^{k–1}C_2$.

Hence, required number of ways = ${}^{k–1}C_2$.

(ii) If $9 \le k \le 14$, then $6 \le k – 3 \le 11$

∴ Coefficient of $x^{k–3}$ in $(1 – x^6)^3$ $(1 – x)^{–3}$

= Coefficient of $x^{k–3}$ in $({}^3C_0 - {}^3C_1\,x^6 + {}^3C_2\,x^{12} - {}^3C_2\,x^{18})(1-x)^{-3}$

= ${}^3C_0$ Coefficient of $x^{k–3}$ in $(1 – x)^{–3}$

– ${}^3C_1$ Coefficient of $x^{k–9}$ in $(1 – x)^{–3}$

$= \quad {}^{3}C_0 \times {}^{k-3+3-1}C_{3-1} - {}^{3}C_1 \times {}^{k-9+3-1}C_{3-1}$

$= \quad {}^{k-1}C_2 - 3 \times {}^{k-7}C_2$

$= \quad 21k - k^2 - 83.$

PRACTICE EXERCISE

11. From 6 boys and 5 girls we have to make a committee of 4 students consisting at least one girl. How many ways can we do this?
12. A committee of 12 is to be formed from 9 women and 8 men. In how many ways this can be done if at least five women have to be include in a committee? In how many of these committees (i) the women are in majority (ii) the men are in majority?
13. A box contains 5 different red and 6 different white balls. In how many ways can 6 balls be selected so that there are at least two balls of each colour?
14. There are 10 points in plane, no three of which are in the same straight line, excepting 4 points, which are collinear. Find the

    (i) number of straight lines obtained from the pairs of these points

    (ii) number of triangles that can be formed with the vertices as these points.
15. In how many ways can a student choose 5 courses out of 9 courses if 2 courses are compulsory for every student?
16. From a class of 12 boys and 10 girls, 10 students are to be chosen for a competition, at least including 4 boys and 4 girls. The 2 girls who won the prizes last year should be included. In how many ways can the selection be made?
17. A student has to answer 10 questions, choosing at least 4 from each of part *A* and part *B*. If three are 6 questions in part *A* and 7 in part *B*, in how many ways can the student choose 10 questions?
18. A candidate is required to answer 7 questions out of 12 questions which are divided into two groups, each containing 6 questions. He is not permitted to attempt more than 5 questions from either group. In how many ways can be choose the 7 questions?
19. In how many ways can a football team of 11 players be selected from 16 players? How many of these will (i) include 2 particular players? (ii) exclude 2 particular players?

Answers

11. 315 **12.** 1008 **13.** 425 **14.** 116 **15.** 35 **16.** 104874 **17.** 266

18. 780

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MISCELLANEOUS PROBLEMS

OBJECTIVE TYPE

Example 1

In a plane there are 37 straight lines, of which 13 pass through the point *A* and 11 pass through the point *B*. Besides, no three lines pass through one point, no line passes through both *A* and *B*, and no two are parallel. Then the number of intersection points the lines have is equal to

(a) 700 (b) 605 (c) 535 (d) none of these

Solution

In the general position, 37 straight lines have ${}^{37}C_2$ points of intersection. But 13 straight lines passing through the point *A* yield one intersection point instead of ${}^{13}C_2$ and 11 straight lines passing through the point *B* yield one intersection point instead of ${}^{11}C_2$.

$\therefore$ The lines ${}^{37}C_2 - {}^{13}C_2 - {}^{11}C_2 + 2$ points of intersection.

i.e $666 - 78 - 55 + 2, i.e. 535$

$\therefore$ **Ans. (c)**

Example 2

Ten persons are arranged in a row. The number of ways of selecting four persons so that no two persons sitting next to each other are selected is

(a) 34 (b) 36 (c) 35 (d) none of these

Solution

To each selection of 4 persons we associated binary sequence of the form 1001001010 where 1 (0) at i^{th} place means the i^{th} persons is selected (not selected)

There exists one-to-one correspondence between the set of selections of 4 persons and set of binary sequence containing 6 zeros and 4 ones.

We are interested in the binary sequences in which no 2 ones are consecutive. We first arrange 6 zeros.

000000

This can be done in just one way. Now, 4 ones can be arranged at any of the 4 places marked with a cross in the following arrangement.

$\times 0 \times 0 \times 0 \times 0 \times 0 \times 0$

We can arrange 4 1's at 7 places in ${}^{7}C_4 = 35$ ways.

$\therefore$ **Ans. (c)**

Example 3

The number of ways in which 20 different things can be divided into three sets of 7,7 and 6 things.

(a) $\frac{20!}{7!7!6!2!}$ (b) $\frac{20!}{7!7!6!}$ (c) 1 (d) $\frac{20!}{7!7!6!3!}$

Solution

First select '6' thing out of $'20' = {}^{20}C_6$

After that '14' things are to be divided into two groups of '7' each. This can be done is $\frac{{}^{14}C_7}{2!}$

Hence the total number of ways = $\frac{{}^{20}C_6 \times {}^{14}C_7}{2!} = \frac{20!}{6!7!7!2!}$

$\therefore$ **Ans. (a)**

Example 4

The number of ways in which 3 noys and 4 girls be seated around a circular table if no two boys sit together is

(a) 36 (b) 3!4! (c) $7! - 3! \times 4!$ (d) 18

Solution

First make the 4 girls sit around a circular table. This can be done in (4-1)! = 3! =6

After that '3' boys can occupy any three crosses. This can be done in ${}^4P_3 = 4!$ways.

$\therefore$ **Ans. (b)**

Example 5

The number of ways of selecting at least one fruit out of 4 Apples, 3 Bananas and 1 each of Mango, Pineapple and Guava are

(a) 159 (b) 12 (c) $4! \times 3! - 1$ (d) none of these

Solution

Among '4' apples, we can select either 0,1,2,3, or all the four apples i.e. we have total '5' choices. Similarly for '3' bananas, we have four choices and for 3 other fruits, we have '2' choices each. Hence total ways = $5 \times 4 \times 2 \times 2 \times 2 = 160$. But this includes a case of selecting '0' apples, '0' of every other fruit Hence total number of ways = 160-1 =159 ways

$\therefore$ **Ans. (a)**

Example 6

The number of arrangements of the letters of the word BANANA in which the two N's do not appear adjacently is

(a) 40 (b) 60 (c) 80 (d)100

Solution

Required number = $\frac{6!}{213!}-\frac{5!}{3!}=40$

$\therefore$ **Ans. (a)**

Example 7

The number of ways in which 7 different books can be given to 5 students if each can receive none, one or more things are

(a) 5^7 (b) 7^5 (c) ${}^{11}C_5$ (d) 12!

Solution

First book can be given to any of the five students, similarly other six books also have 5 choices. Hence the total number of ways = 5^3. **Ans. (a)**

Example 8

The number of positive integral solution of the equation xyz = 3000 are

(a) 200 (b) 300 (c) 3000 (d)50

Solution

$xyz = 3\times 2^3 \times 5^3$

Now the power of '3' is 1, then 3 can be distributed among three variables in '3' ways. Similarly the power of '2' is 2, hence 2 can be distribute among *x*,*y* and *z* in '10' ways and same is the case of 5^3.

Hence total ways = $3\times 10\times 10 = 300$

$\therefore$ **Ans. (b)**

Example 9

The sum of the divisors of $2^5.3^4.5^2$ is:

(a) $3^3.7^1.11^2$ (b) $3^2.7^1.11^2.31$ (c) $3.7.11.31$ (d) none of these

Solution

Any divisor of $2^5.3^4.5^2$ is of the form $2^a.3^b.5^c$ where $0\le a\le 5, 0\le b\le 4$ and $0\le c\le 2$.

Hence, the sum of the divisors

$$\sum_{a=0}^{5}\sum_{b=0}^{4}\sum_{c=0}^{2} 2^a 3^b 5^c = (1+2+....+2^5)(1+3+....+3^4)(1+5+5^2)$$

$$= \left(\frac{2^6-1}{2-1}\right)\left(\frac{3^5-1}{3-1}\right)\left(\frac{5^3-1}{5-1}\right)$$

$$= 63\times 121\times 31 = 3^2.7^1.11^2.31$$

$\therefore$ **Ans. (b)**

SUBJECTIVE TYPE

Example 1

A student is allowed to select at most *n* books from a collection of *(2n+1)* books. If the total number of ways in which he can select a books is 63. Find the value of *n*.

Solution

Given students select at most *n* books from a collection *(2n+1)* books. It means that he select one

book or two book or three book or… or *n* books.

$\Rightarrow \quad {}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + ...{}^{2n+1}C_n = 63$

But we know that,

$\Rightarrow \quad {}^{2n+1}C_6 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 ...{}^{2n+1}C_{n+1} = 2^{2n+1}$

Now, ${}^{2n+1}C_0 + {}^{2n+1}C_{2n+1} = 1$

$$2 + ({}^{2n+1}C_1 + {}^{2n+1}C_2 + ...{}^{2n+1}C_n) + \ ({}^{2n+1}C_{n+1} + {}^{2n+1}C_{n+2} + ...{}^{2n+1}C_{2n}) = 2^{2n+1}$$

$\Rightarrow \quad 2 + ({}^{2n+1}C_1 + {}^{2n+1}C_2 + ...{}^{2n+1}C_n) = 2^{2n+1}$

$$2 + 2.63 = ^{2n+1}$$

$$n = 3$$

Example 2

Given five different green dyes, four different blue dyes and three different red dyes, dow many combination of dyes can be chosen taking at least one green, one blue dye ?

Solution

Any one dye of a particular colour can be either chosen or not; and, thus there are 2 waysin

which each one may be dealt with.

Number of ways of selection so that at least one green dye is included = 2^5-1=31

(l is subtracted to correspond to the case when none of the green dyes is chosen)

A similar argument may be advanced in respect of other two colours also.

Number of combinations = $(2^5-1)\ (2^4-1)\ (2^3)$. = $31 \times 15 \times 8 = 3720$

Example 3

Find the number of different words ending and beginning with a consonant which can be made out of letters of the word EQUATION.

Solution

Total – 8 letters

2– consonants

5- Vowels

The consonants are to occupy 1st and last place and it can be done is $^3P_2$ ways we will now be left with 5 vowels and 1 consonant i.e 6 letters which can be arranged in 6! Ways. Hence the number of words under given conditions is

$$^3P_2 \times 6! = 4320$$

Example 4

The number of times of the digits 3 will be written when listing the integers from 1 to 1000 is

(a) 269 (b) 300 (c) 271 (d) 302

Solution

Since 3 does not occur in 1000, we have to count the number of times 3 occurs when we list the integers from 1 to 999. Any number between 1 and 999 is of the form *xyz* where $0 \le x, y, z, \le 9$. Let us first count the numbers in which 3 occurs exactly once. Since 3 can occurs exactly at two places in $(^3C_2)(9) = 3 \times 9$ such number.

Lastly, 3 can occur in all three digits in one number only. Hence the number of times 3 occurs is $1 \times (3 \times 9^2) + 2 \times (3 \times 9) + 3 \times 1 = 300$

Hence (b) is the correct answer.

Example 5

A man has 7 relatives – 4 ladies and 3 gentlemen. His wife has also 7 relatives-3 ladies and 4 gentlemen. In how many ways can the couple invite for a dinner party, 3 ladies and 3 gentlemen so that are 3 of the man's relatives and 3 of the wife's relative?

Solution

The possible number of ways can be listed as follows.

Man		**Wife**		**Number of ways**
Ladies (4)	**Gentlemen (3)**	**Ladies (3)**	**Gentlemen(4)**	
3	0	0	3	$^4C_3 . ^4C_3 = 16$
2	1	1	2	$(^4C_3 . ^4C_1)^2 = 324$
1	2	2	1	$(^4C_1 . ^3C_1) = 144$
0	3	3	0	$(^3C_3 . ^3C_3) = 1$
			Total	**= 485**

Example 6

15 persons, amongst whom are *A*, *B* and *C* are to speak at a function. Find in how many ways can the speech be done if *A* wants to speak before *B* and *B* is to speak before *C*?

Solution

Leaving out *A*,*B* and *C* there are 12 persons and these 12 may be made to speak in 12! ways. Corresponding to each one of these ways to speaking by 12 persons – there are 13 places between any two of these 12 - *A*,*B* and *C* may speak

(i) in one of the 13 places in the order *A*,*B*,*C* one after the other but together.

(ii) in two of the 13 places in the order *AB* and *C* or *A* and *BC*.

(iii) in three of the 13 places in the order *A*,*B*,*C* but two of them always separated and hence the number of ways.

$$= 12!\{{}^{13}C_1 + 2.{}^{13}C_2 + {}^{13}C_3\}$$

Example 7

All possible two-factor products are formed from the numbers 1,2,3 ….100. How many multiples out of the total obtained are multiples of 3?

Solution

The total number of two-factor products = ${}^{100}C_2$

Out of the numbers *1,2,3,….100;* the multiples of 3 are *3,6,9,…99;* i.e. there are 33 multiples of 3, and therefore there are 67 non- multiples of 3.

$\therefore$ The number of two-factor products which are not multiples of 3 = ${}^{67}C_2$

$\therefore$ The required number = ${}^{100}C_2$ - ${}^{67}C_2$

$$= 4950 – 2211 = 2739$$

Alternatively, the number of two-factor products formed when both factors are multiples of $3 = {}^{33}C_2$ and the number of two-factors products formed when one is a multiple of 3 and the other a non-multiple of $3 = {}^{33}C_1 \times {}^{67}C_1$ one is a

In either case the product is a multiple of 3.

$\therefore$ The required number $= {}^{33}C_2 + 33 \times 67$
$= 528 + 2211 = 2739$

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Exercise – I

OBJECTIVE TYPE QUESTIONS

Multiple Choice questions with ONE option correct

1. The number of ways a mixed double game can be arranged from amongst 9 couples if no husband and wife play in the same game is
   (a) 756 (b)1512 (c)3024 (d)None of these.
2. If the letters of the world *MOTHER* are arranged in all possible orders and these words are written as in a dictionary, then the rank of the word *MOTHER* will be
   (a) 240 (b) 261 (c) 308 (d) 309
3. A closet has 5 pairs of shoes. The number of ways in which 4 shoes can be drawn from it such that there will be no complete pair is
   (a) 80 (b) 160
   (c) 200 (d) None of the foregoing numbers
4. The number of six digit numbers in which digits are in ascending order must be
   (a) 48 (b) 84 (c) 120 (d) 126
5. Let p, q $\in\{1,2,3,4\}$. The number of equations of the form $px^2+qx+1=0$ having real roots must be
   (a) 15 (b) 9 (c) 7 (d) 8
6. The total number of permutations of 4 letters that can be made out of the letters of the word EXAMINATION is
   (a) 2454 (b) 2436 (c) 2545 (d)none of these
7. Ten different letters of an alphabet are give, words with five letters are formed from these given letters. Then the number of words which have at least one letter repeated is
   (a) 69760 (b) 30240 (c) 99784 (d)none of these
8. In an examination there are three multiple choice questions and each questioin has 4 choices. Number of ways in which a student can fail to get all answers correct is
   (a) 11 (b) 12 (c) 27 (d) 63
9. If the letters of the word LATE be permuted and the words so formed be arranged as in a dictionary. Then the rank of LATE is
   (a) 12 (b) 13 (c) 14 (d) 15
10. Number of ways in which Rs. 18 can be distributed amongst four persons such that no body receives less than Rs. 4 is
    (a) 4^2 (b) 2^4 (c) 4! (d)none of these
11. There were two women participating in a chess tournament. Every participant played two games with the other participants. The number of games that the men played between themselves proved to exceed by 66 the number of games that the men played with the women.

The number of participants is

(a) 6 (b) 11 (c) 13 (d)none of these

12. At an election there are five candidates and three members to be elected, and an elector may vote for any number of candidates not greater than the number to be elected. Then the number of ways in which an elector may vote is

(a) 25 (b) 30 (c) 32 (d)none of these

13. The number of all four digit numbers which are divisible by 4 that can be formed from the digits 1,2,3,4 and 5 is

(a) 125 (b) 30 (c) 95 (d)none of these

14. The number of ways in which 5 persons P, Q, R, S and T can be seated in a ring so that P sits between Q and R is

(a) 120 (b) 4 (c) 24 (d) 9

15. The number of integers which lie between 1 and 10^6 and which have sum of the digits as 12 must be

(a) 8550 (b) 5382 (c) 6062 (d) 8055

Multiple Choice questions with ONE or MORE THAN ONE option correct

1. Sanjay has 10 friends among whom two are married to each other, he wishes to invite 5 of them for a party. If the married couple refuse to attend separately then the number of different ways in which he can invite five friends is

(a) ${}^8C_5$ (b) $2\times{}^8C_3$ (c) ${}^{10}C_5-2\times{}^8C_3$ (d)none of these

2. In a chess tournament when the participants were to play one game with another, two chess players fell ill, having played 3 games each. If the total number of games played is 84, the number of participants at the beginning was

(a) 15 (b) ${}^6C_2$ (c) 30 (d) 48

3. There are 10 points in a plane, no three are collinear, except 4 which are collinear. All points are jointed. Let L be the number of different straight lines and T be the number of different triangles, then

(a) T = 120 (b) L = 40 (c) T = 3L - 4 (d)none of these

4. Eight chairs are numbered from 1 to 8. Two women and three men wish to occupy one chair each. First the women choose the chairs from amongst the chairs 2 to 4 and then the men select from the remaining chairs. The numbered of possible arrangements are

(a) ${}^4C_2{}^6C_3\,2!3!$ (b) ${}^4P_2{}^4P_3$ (c) ${}^4P_2{}^6P_3$ (d) ${}^4C_2{}^6C_3$

5. The number of ways of selecting 3 pairs from 8 distinct objects

(a) 420 (b) 105

(c) 21 (d) $({}^8C_2,{}^6C_2,{}^4C_2)/\ 3!.$

6. The number of ways in which we can make a garland with five flowers of one kind and three flowers of another kind must be

(a) 5

(b) 7

(c) $\frac{11}{5!7!}$

(d) Number of integer solutions of the equation $x+y+z=5$ where $x \le y \le z$

7. The number of six digits numbers which have digits in descending order must be

(a) 1 (b) 6! – 5! (c) $^{10}C_6$ (d) 210

8. The number of permutations of 1,2,3,…8 taken all at a time such that product of any two consecutive digits is even, must be

(a) $^5C_4.(4!)^2$ (b) $(4!)^2$ (c) $^5P_4.4!$ (d) 576

9. The number of six digit number in which exactly 2 digits are even must be

(a) 14×5^6 (b) 17×5^6

(c) 21×5^6 (d) must be less than 250000.

10. The number of ways of selecting 3 pairs from 8 distinct objects

(a) 420 (b) 105

(c) 21 (d) $(^8C_2, ^6C_2, ^4C_2)/\ 3!.$

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Exercise - II

ASSERTION & REASON , COMPREHENSION & MATCHING TYPE

Assertion & Reason Type

(A) Both *A* and *R* are true and *R* is correct explanation of *A*.

(B) Both *A* and *R* are true but *R* is not correct explanation of *A*.

(C) *A* is true, *R* is false.

(D) *A* is false, *R* is true.

1. A : The number of ways of distributing *n* identical objects in *r* distinct boxes is ${}^{n+r-1}C_{r-1}$

   R : The number of arrangement of *n* objects of one kind and *r-1* objects of another kind in a line must be $\frac{(n+r-1)!}{n!(r-1)!}$

2. A : The number of selections of objects from *n* types of objects where each may be taken as often as we please must be ${}^{n+r-1}C_r$

   R : The number of non-negative integer solutions of $x_1+x_2+...+x_n=r$ is ${}^{n+r-1}C_{n-1}$

3. A : The number of selections of four letters taken from the word PARALLEL must be 15

   R : Coefficient of x^4 in the expansion of $(1-x)^{-3}$ is 15

Passage Based Questions

Passage – I

It can be observed that largest power of prime *p* contained in *n!* must be equal to (where *[x]* denotes greatest integer $\le x$). The result intuitively follows since in the product $1\times2\times3\times4\times...\times n$ there are $\left[\frac{n}{p}\right]$ integers divisible by p. Among these $\left[\frac{n}{p}\right]$ integers there are $\left[\frac{n}{p^2}\right]$ integers which are divisible by p^2 and so on. If we isolate the power of each prime contained in any number *N* then *N* can be written as $N=2^{\alpha_1}3^{\alpha_2}5^{\alpha_3}7^{\alpha_4}....$ where a_i are non-negative integers.

1. The power of 7 contained in ${}^{1000}C_{500}$ must be equal to

   (a) 3 (b) 2 (c) 1 (d) 0

2. If 50! Is computed, the number of zero at the end must be equal to

   (a) 11 (b) 12 (c) 13 (d) 15

3. 20! Will end in several zeros. The last non-zero digit in 20! Must be equal to

   (a) 2 (b) 4 (c) 6 (d) 8

Passage – II

When m equally spaced parallel lines are interested by n equally spaced parallel lines a network of rectangles or squares is formed. The number of rectangles must be ${}^mC_2 . {}^nC_2$. The squares have several dimensions. The smallest square is of dimension, the number of such squares are $(m-1)\times(n-1)$

1. If $m<n$ the dimension of a largest possible square must be

   (a) m (b) n (c) $m-1$ (d) $n-1$

2. The number of squares on a chess board must be

   (a) *64* (b) *204* (c) *285* (d)*none of these*

3. Total number of squares, when m such lines (as described in comprehension) are interested by n such lines must be $(m<n)$

   (a) mn (b) $\frac{m(m-1)n(n-1)}{4}$

   (c) $\frac{1}{6}n(n-1)(3m-n-1)$ (d) $\frac{1}{6}m(n-1)(3m-n-1)$

Passage – III

Suppose a lot contains n_1 objects of one kind, n_2 objects of another kind and so on such that n_1 $n_1+n_2+....+n_k=n$ then the number of permutation of r objects from this lot must be equal to $r!$ times

Coeff. of x^r in $\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+.....+\frac{x^{n_1}}{n_2!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+.....+\frac{x^{n2}}{n_2!}\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+.....+\frac{x^{nk}}{n_k!}\right)$

1. If $n_1 = n_2 = ... = n_k = 1$, then number of permutations of r objects must be

   (a) ${}^np_r$ (b) $r!^n p_r$ (c) ${}^np_r$ (d)none of these

2. The number of permutations of letters of the word PARALLEL taken four at a time must be

   (a) 216 (b) 244 (d) 286 (d)1680

3. If $n_1+n_2+....+n_k=r$, then number of permutations must be

   (a) $r!$ (b) $(k+r)!$ (c) $\frac{r!}{n_1!n_2!...n_k!}$ (d)none of these

Matching Type Questions

1. **Column I** | **Column II**

	Column I	Column II
(A)	Number of five digit numbers whose digits are in descending order	(P) 216
(B)	Number of five digit numbers having only two different digits 1 and 2	(Q) 252
(C)	Number of five digit numbers divisible by 3 and formed by numbers 0,1,2,3,4,5 without repetitions and Y is ahead of Z.	(R) 30

(a) A-R, B-P, C-Q (b) A-Q, B-R, C- P

(c) A-P, B-R, C-Q (d) None of these

2. On a train route, there are 10 stations :

	Column I	Column II
(A)	Number of ways of selecting 3 stoppages	(P) 56
(B)	Number of ways of choosing 3 stoppages such that no 2 of them are adjacent	(Q) 144
(C)	Number of ways of selecting stoppages such that no 2 are consecutive	(R) 120

(a) A-P, B-R, C- Q (b) A-Q, B-R, C- P

(c) A-Q, B-P, C-Q (d) A-R, B-P, C-Q

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Exercise - III

SUBJECTIVE TYPE

1. How many even numbers of four digits can be formed with the digits 0,1,2,3,4,5 and 6; no digit being used more than once ?
2. A gentleman has 6 friends to invite. In how many ways can he send invitation cards to them if he has three servants to carry the cards.
3. In how many ways can 10 examination papers be arranged so that the best and the worst papers never come together.
4. How many words can be formed with the letters of the word PATALIPUTRA without changing the relative order of the vowels and consonants ?
5. In how many ways 5 Indians and 4 Englishmen can be seated at a round table if
   (i) there is no restriction?
   (ii) All the four Englishmen sit together?
   (iii) All the four Englishmen do not sit together?
   (iv) No two Englishmen sit together?
6. How many integral solutions are possible for the equation $n \times y = 100$ if n and y are natural numbers.
7. Find the number of ways in which n different beads can be arranged to form a necklace.
8. (a) Find the number of words of 4 letters that can be made with the letters of the word COMBINATION.
   (b) Find the number of words of 5 letters that can be made with the letters of the word PROPOSITION.
9. There are 5 mangoes and 4 apples. In how many different ways can a selection of fruits be made if
   (a) fruits of the same kind are different
   (b) fruits of the same kind are identical?
10. In an examination a candidate has to pass in each of the papers. If the total number of different ways in which the candidate can fail is 63, then find the number of papers in the examination.
11. A dictionary is made of the words that can be made by arranging the letters of the word PARKAR. What is the position of the word “PARKAR” in that dictionary if words are printed in the same order as that of an ordinary dictionary?
12. Straight lines are drawn by joining m points on a straight line to n points on another line. Then excluding the given points, prove that the lines drawn will intersect at

$\frac{1}{2}mn(m-1)(n-1)$ points if no two lines drawn are parallel and no three lines are concurrent.

13. Find the sum of all natural numbers n such that $1000 \le n \le 4000$ and that can be made with the digits 0,1,2,3,4 if repetition of digits in the same number is allowed.
14. If $X_k \ge 0; k = 1,2,3,4,5$ and $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ and $x_1 + x_2 + x_3 = 5$ then find the number of integral solutions of $(x_1 + x_2 + x_3 + x_4 + x_5)$.
15. Find the number of numbers of six digits that can be made with the digits 1,2,3,4 if all the digits are to appear in the same number at least once.
16. Five balls are to be placed in three boxes. Each can hold all the five balls. In how many different ways can we place the balls so that no box remains empty, if
    (i) balls and boxes are all different
    (ii) balls are identical but boxes are different
    (iii) balls are different but boxes are identical
    (iv) balls as well as boxes are identical
    (v) balls as well as boxes are identical but boxes are kept in a row?
17. Find the total number of selections of 5 letters from five As, four Bs, three Cs and two Ds. Also find the number of 5-letter words that can be made.
18. How many words can be made with letters of the word INTERMEDIATE if
    (i) the word neither begin with I nor end with E
    (ii) the vowels and consonants alternate in the words
    (iii) the vowels are always consecutive
    (iv) no vowel is between two consonants
    (v) the relative order of vowels and consonants does not change
    (vi) the order of vowels does not change?
19. In a plane there are two sets of parallel lines, one of m lines and the other of n lines. If the lines of one set cut the lines of the other set, how many different parallelograms will be formed?
20. There are n persons sitting around a round table. Prove that the number of different ways in which 3 persons can be selected so that no two of them are neighbours, is $\frac{1}{6}n(n-4)(n-5)$
21. There are 100 white and 100 black balls marked 1,2,3,…100. In how many ways can all the 200 balls be arranged so that the neighbouring balls are of different colours?

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Exercise - IV

IIT – JEE PROBLEMS

A. Fill in the blanks

1. In a certain test, a_1 students gave wrong answers to at least I question where $i = 1$, ______,k. No student gave more than k wrong answers. The total number of wrong answer given is ______ .
2. Let a be a set of n distinct elements. Then the total number of distinct functions from A to A is… and out of these ______ are onto functions.
3. Total number of ways in which six'+' and four '-' signs can be arranged in a line such that no two '-' signs occur together Is ______ .
4. There are four balls of different colours and four boxes of colour, same as those of the balls. The number of ways in which the balls, one each in a box, could be placed such that a ball does not go to a box of its own colour is______ .

B. True/False

5. The product of any r consecutive natural numbers is always divisible by $r!$.

C. Multiple Choice Questions with ONE correct answer

6. Ten different letters of an alphabet are given. Words with five letters are formed from three given letters. Then the number of words which have at least one letter repeated as:

   (a) 69760 (b) 30240 (c) 99748 (d)none of these

7. The value of the expression ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$ is equal to:

   (a) ${}^{47}C_5$ (b) ${}^{52}C_5$ (c) ${}^{52}C_4$ (d)none of these

8. An n-digit number is a positive number with exactly n digits. Nine hundred distinct n-digit numbers are to be formed using only the three digits 2, 5 and 7. The smallest value of n for which this is possible, is:

   (a) 6 (b) 7 (c) 8 (d) 9

9. Number of divisors of the form $(4n+2), n \ge 0$ of the integer 240 is:

   (a) 4 (b) 8 (c) 10 (d) 3

10. How many different nine digit numbers can be formed from the number 22 33 55 888 be rearranging its digits so that the odd digits occupy even positions:

    (a) 16 (b) 36 (c) 60 (d) 180

11. The number of arrangements of the letters of the word from *BANANA* in which the two *N'* s do not appear adjacently is:

(a) 40 (b) 60 (c) 80 (d) 100

12. If r,s,t are prime numbers and *p,q* are the positive integers such that LCM of *p,q is* $r^2 s^4 t^2$, then the number of ordered pairs *(p,q)* is:

(a) 252 (b) 254 (c) 225 (d) 224

D. Subjective Questions

13. Five balls of different colours are to be placed in three boxes of different sizes. Each box can hold all five. In how many different ways can we place the balls so that no box remains empty ?

14. *mn* squares of equal size are arranged to form a rectangle of dimension *m* by *n* where *m* and *n* are natural numbers. Two squares will be called 'neighbours' if they have exactly one common side. A natural number is written in each square such that the number in written any square is the arithmetic mean of the numbers written in its neighbouring squares. Slow that this is possible only if all the numbers used are equal.

15 *m* men and *n* women are to be seated in a row so that no two women sit together. If *m* > *n* then show that the number of ways in which they can be seated is

$$\frac{m!(m+1)1}{(m-n+1)!}$$

16. A b ox contains two white balls, three black balls and four red balls. In how many ways can three balls be drawn from the box if at least one black ball is to be included in the draw ?

17. A committee of 12 is to be formed from 9 women and 8 men. In how many ways this can be done if at least five women have to be included in a committee ? In how many of these committees

(i) The women are in majority ?

(ii) The men are in majority ?

18. Using Permutation or otherwise, prove that $\frac{n^2!}{(n!)^n}$ is an integer, where *n* is a positive integer.

19. Two planes P_1 and P_2 pass through origin. Two lines L_1 and L_2 also passing through origin are such that L_1 lies on P_1 but not on P_2, L_2 lies on P_2 but not on P_1, *A,B,C* are three points other than origin, then prove that the permutation [*A'B'C'*] of [*ABC*] exists. Such that :

(i) *A* lies on L_1, *B* lies on P_1 not on L_1, *C* does not lie on P_1

(ii) *A'* lies on L_2, *B* lies on P_2 not on L_2, *C*' does not lies on P_2

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Answers

Exercise - I

Only One Option is correct

1. (b)	2. (d)	3. (a)	4. (b)	5. (c)
6. (a)	7. (a)	8. (d)	9. (c)	10. (d)
11. (c)	12. (a)	13. (a)	14. (b)	15. (c)

More Than One Choice Correct

1. (b, c)	2. (a, c)	3. (a, c)	4. (a, c)	5. (a, d
6. (a, d)	7. (c, d)	8. (a, c)	9. (a, d)	10. (a, d)

Exercise - II

Assertion and Reason

1. 1 2. 1 3. 4 4. 2

Passage – I

1. (d) 2. (b) 3. (a)

Passage – II

1. (c) 2. (b) 3. (d)

Passage – III

1. (a) 2. (c) 3. (c)

Matching Type Answers

1. (b) 2. (d)

Exercise - III

Subjective Type

1. 420 2. 729 3. $10! - 9! \times 2!$ 4. 3600

5. (i) 8!, (ii) 5! 4!, (iii) 8!, –5!, 4!, (iv) 4! 5! 6. 9 7. $\frac{(n-1)!}{2}$

8. (a) 2454, (b) 6890 9. (a) $2^9 - 1$ (b) 29 10. 6

11. 99th 13. 837250 14. 336 15. 1560

16. (i) 150, (ii) 6, (iii) 50, (iv) 2, (vi) 6 17. 41,901

18. (i) $\frac{83}{24} \times 10!$, (ii) 43200 (iii) 151200, (iv) 151200, (v) 21600, (vi) $\frac{20!}{6!\,2!}$

19. $\frac{mn\,(m-1)\,(n-1)}{4}$ 21. $2\,(100!)^2$

Exercise - IV

IIT-JEE Level Problem

Section - A

1. 2^{n-1} 2. $n^n, \sum_{r=1}^{n}(-1)^{n-r\,n}C_r(r)^n$ 3. 35 4. 9

Section - B

5. True

Section - C

6. (a) 7. (c) 8. (b) 9. (a) 10. (c)
11. (a) 12. (c)

Section - D

13. (300) 15. (64) 16. (6062, 2702, 1008)

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