MATRICES & DETERMINANTS

Gabriel Cramer

Gabriel Cramer (1704 - 1752) was a Swiss mathematician, born in Geneva. He showed promise in mathematics from an early age. At the age of 18 he received his doctorate. In 1728 he proposed a solution to the St. Petersburg Paradox that came very close to the concept of expected utility theory given ten years later by Daniel Bernoulli. The work by which he is best known for came in his forties. This work is his treatise on algebraic curves "Introduction à l'analyse des lignes courbes algébraique" published in 1750; it contains the earliest demonstration that a curve of the n-th degree is determined by $n(n + 3)/2$ points on it, in general position. He edited the works of the two elder Bernoullis; and wrote on the physical cause of the spheroidal shape of the planets and the motion of their apses (1730), and on Newton's treatment of cubic curves (1746). He was professor at Geneva, and died at Bagnols.

Adapted from A Short Account of the History of Mathematics by W. W. Rouse Ball (4th Edition, 1908).

He was the son of physician Jean Cramer and Anne Mallet Cramer.

IIT-JEE Syllabus
Matrices as a rectangular array of real numbers, equality of matrices, addition, multiplication by a scalar and product of matrices, transpose of a matrix, determinant of a square matrix of order up to three, inverse of a square matrix of order up to three, properties of these matrix operations, diagonal, symmetric and skew-symmetric matrices and their properties, solutions of simultaneous linear equations in two or three variables.

Introduction

Matrix is a rectangular array of real or complex numbers in rows and columns. A matrix is denoted by the capital letters A, B, C etc. If there are m rows and n columns in the matrix, then the matrix is called a $m \times n$ matrix.

Example: $A = \begin{bmatrix} 3 & 2 & 3 \\ 3 & 2 & -4 \end{bmatrix}$ or $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & -4 \end{bmatrix}$

Here A is a 2 × 3 matrix because A has two rows and three columns.

Note: *M*atrix is just an array of numbers and has no numerical value as in case of a determinant.

- **Equality of Two Matrices**

Two $m \times n$ matrices A and B are said to be equal if the corresponding elements of the two matrices are equal

i.e., if $a_{ij} = b_{ij}$ for $i = 1, 2, \ldots\ldots m$; $j = 1, 2, 3, \ldots, n$

i.e., if $a_{11} = b_{11}$, $a_{12} = b_{12}$ etc.

Example: Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & 4 \end{bmatrix}$

Here A and B are equal matrices and we write A = B.

- **Negative of A Matrix**

Let $A = (a_{ij})$ be a $m \times n$ matrix. Then the negative of the matrix A is denoted by – A and is defined as $(-a_{ij})$. Which is also a m × n matrix.

In order to find – A sign of each element of A must be changed.

Example: Let $A = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 2 & -5 \end{bmatrix}$. Then $-A = \begin{bmatrix} -1 & 2 & 0 \\ -3 & -2 & 5 \end{bmatrix}$

Note: If A is a $m \times n$ matrix, then – A is also a $m \times n$ matrix

5.1 Types of Matrices

- **Square Matrix**

A matrix having equal number of rows and columns is called a square matrix. If the matrix A has n rows and n columns is said to be a square matrix of order n.

Example: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}$ is a square matrix of order 3.

- **Horizontal Matrix**

A $m \times n$ matrix is called a horizontal matrix if $m < n$.

Example: $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \end{bmatrix}$. Here A is a horizontal matrix.

- **Vertical Matrix**

A $m \times n$ matrix is called a vertical matrix if $m > n$.

Example : $B = \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}$. Here B is a vertical matrix.

- **Row Matrix**

A matrix having only one row is called a row matrix or a row vector.

Example: A = [1 2 3]. Here A is a row matrix.

- **Column Matrix**

A matrix having only one column is called a column matrix or column vector.

Example: $B = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}$. Here B is a column matrix.

- **Zero or Null Matrix**

A matrix whose each element is zero is called a zero matrix or null matrix.

Example $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Here A and B are zero matrices.

- **Square Matrix**

A matrix containing equal number of rows and columns is called of square matrix.

Example: $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

Here A and B are square matrices of order 2 and 3 respectively

- **Diagonal Matrix**

A square matrix (a_{ij}) is called the diagonal matrix if the diagonal elements are non-zero, whereas all other elements are zero. i.e. $a_{ij} = 0$ for i □ j. and $a_{ij} \neq 0$ for $i = j$.

Example: $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ Here A and B are diagonal matrices.

- **Unit Matrix**

A square matrix is said to be a unit matrix if all the elements along the principal diagonal are unity and all elements not occurring along the principal diagonal are zero. i.e. $a_{ij} = 0$ for $i \neq j$ and $a_{ij} = 1$ for $i = j$.

Example: (i) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (ii) $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A unit matrix of order n is denoted by I_n. Thus in the examples given above $A = I_2$, $B = I_3$

- **Triangular Matrix**

(a) **Lower triangular matrix:** A square matrix (a_{ij}) is called a lower triangular matrix if $a_{ij} = 0$ when $k < j$.

(b) Upper triangular matrix: A square matrix (a_{ij}) is called an upper triangular matrix if a_{ij} = 0 when $i > j$.

Example: *(i)* $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$

Here A and B are lower triangular matrices

(ii) $C = \begin{bmatrix} 1 & -1 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Here C and D are upper triangular matrices.

5.2 Addition of Matrices

Let A and B be two $m \times n$ matrices. The $m \times n$ matrix obtained by adding the corresponding elements of the matrices A and B is called the sum of the matrices A and B and is denoted by $A + B$.

Thus if $A = (a_{ij})$, $B = (b_{ij})$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$

then $A + B = (a_{ij} + b_{ij})$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ is the sum of matrices A and B.

Example: Let $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 4 & 5 \\ 6 & 2 & 9 \end{bmatrix}$

Then $A + B = \begin{bmatrix} 2-3 & 3+4 & 4+5 \\ 0+6 & 2+2 & -1+9 \end{bmatrix} = \begin{bmatrix} -1 & 7 & 9 \\ 6 & 4 & 8 \end{bmatrix}$

- **Properties of Matrix Addition**

Property I

Matrix addition is commutative i.e, if A and B be any two $m \times n$ matrices, then A + B = B + A.

Property II

Matrix addition is associative i.e., if A, B and C be three $m \times n$ matrices, then A + (B + C) = (A + B) + C

Property III

Cancellation laws hold good for addition of matrices i.e., if A, B, C, be any three $m \times n$ matrices, then

(i) A + B = A + C $\Rightarrow$ B = C (left cancellation law)

(ii) B + A = C + A $\Rightarrow$ B = C (right cancellation law)

Property IV

For any matrix A, there exists a null matrix 'O' of the same type such that A + O = A = O + A.

Property V

For any matrix A, there exists a unique matrix B of the same order such that A + B = 0 = B + A.

5.3 Subtraction of Two Matrices

Let A and B be two $m \times n$ matrices. Then the difference of A and B is denoted by A – B and is defined by $A - B = A + (-B)$. Where $A - B$ will also be a $m \times n$ matrix. In order to find $A - B$, the elements of B must be subtracted from the corresponding elements of A.

Examples: Let A = $\begin{bmatrix} 1 & 2 & 3 \\ 3 & -1 & 0 \end{bmatrix}$, B = $\begin{bmatrix} 2 & -1 & 4 \\ 5 & 0 & 6 \end{bmatrix}$ then

$$A - B = \begin{bmatrix} 1-2 & 2+1 & 3-4 \\ 3-5 & -1-0 & 0-6 \end{bmatrix} = \begin{bmatrix} -1 & 3 & -1 \\ -2 & -1 & -6 \end{bmatrix}$$

5.4 Multiplication of A Matrix By A Scalar

Let A be any $m \times n$ matrix and k be any scalar (real or complex number), then the scalar multiple of matrix A by k is denoted by kA or Ak and is defined as the $m \times n$ matrix obtained by multiplying each element of A by k.

Thus if $A = (a_{ij})$; $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$

then $kA = (ka_{ij})$; $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$

Example: Let A = $\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 5 & 4 \end{bmatrix}$, then 3A = $\begin{bmatrix} 3 & 6 \\ 9 & 0 \\ 15 & 12 \end{bmatrix}$

- **Properties of Scalar Multiplication**

(i) If A and B be any two $m \times n$ matrices and k is any scalar, then $k(A + B) = kA + kB$

(ii) If A is any $m \times n$ matrix and a and b are any two scalars, then $(a + b)A = aA + bA$

(iii) If A be any $m \times n$ matrix and k be any scalar, then $(-k)A = -(kA) = k(-A)$

5.5 Multiplication of Two Matrices

Let $A = [a_{ij}]$ be a $m \times n$ matrix and $B = [b_{ij}]$ be a $n \times p$ matrix such that the number of columns in A is equal to the number of rows in B, then the $m \times p$ matrix $C = [c_{ik}]$ such that

$C_{ik} = \sum_{j=1}^{n} a_{ij} b_{ik} = a_{i1} b_{1k} + a_{i2} b_{2k} + a_{i3} b_{3k} + \ldots + a_{in} b_{nk}$ in said to be the product of the matrices A and B in that order and is denoted by AB.

Note :

(i) In the product AB, A is called the pre-factor and B is called the post-factor.

(ii) If the product AB is possible then it is not necessary that the product BA is also possible.

(iii) If A be a $m \times n$ matrix and both AB and BA are defined then B will be a $n \times m$ matrix.

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***Illustration* 1**

(i) A = 3 × 2 matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$, B = 2 × 2 matrix $\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

Then AB = 3 × 2 matrix $\begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \\ a_{31}b_{11}+a_{32}b_{21} & a_{31}b_{12}+a_{32}b_{22} \end{bmatrix}$

(ii) A = 3 × 3 matrix $\begin{bmatrix} 1 & 0 & 5 \\ -1 & 2 & 4 \\ 3 & -2 & 6 \end{bmatrix}$, B = 3 × 2 matrix $\begin{bmatrix} 4 & -1 \\ 2 & -2 \\ 5 & 3 \end{bmatrix}$ then

$$AB = \begin{bmatrix} 1.4+0.2+5.5 & 1(-1)+0(-2)+5.3 \\ (-1)4+2.2+4.5 & (-1)(-1)+2(-2)+4.3 \\ 3.4+(-2)2+6.5 & 3(-1)+(-2)+6.3 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & 14 \\ 20 & 9 \\ 38 & 19 \end{bmatrix}$$

Here BA is not defined because number of columns in B is not equal to the number of rows in A.

❑ **Properties of Matrix Multiplication**

(i) **Matrix multiplication is associative**

i.e., If A, B and C be $m \times n$, $n \times p$ and $p \times q$ matrices, then

A (BC) = (AB) C

(ii) Multiplication of matrices is distributive with respect to addition of matrices i.e., if A is a $m \times n$ matrix and B and C are both $n \times p$ matrices, then A (B + C) = AB + AC

Note: It can be also proved that

(i) If A and B are $m \times n$ matrices and C is a $n \times p$ matrix, then (A + B) C = AC + BC

(ii) If A be a $m \times n$ matrix and I_n be the unit matrix of order n, then $AI_n = I_n A = A$

5.6 Transpose of A Matrix

Let A be any matrix then the matrix obtained by interchanging its rows and columns is called the transpose of A and is denoted by A' or A^T. If A is a $m \times n$ matrix then A' will be a $n \times m$ matrix.

If A = $[a_{ij}]$; $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$

Then A' = $[a_{ji}]$; $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, m$

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 5 \end{bmatrix}$, then $A' = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 3 & 5 \end{bmatrix}$

- **Properties of Transpose of Matrices**

Property I

If A is any matrix, then (A')' = A

Property II

If k is any number real or complex and A be any matrix, then (kA)' = kA'

Property III

(– A)′ = – A′

Property IV

(A + B)' = A' + B'

Property V

If A be a $m \times n$ matrix and B be a $n \times p$ matrix, then (AB)' = B'A'

5.7 Symmetric, Skew Symmetric And Orthogonal Matrices

- **Symmetric Matrix**

A square matrix A = $[a_{ij}]$ is said to be a symmetric matrix, if $a_{ij} = a_{ji}$ for all i and j i.e. A square matrix A is symmetric if and only if A' = A.

Example: $A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 2 & 0 \\ 5 & 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix}$

- **Skew Symmetric Matrix**

A square matrix A = $[a_{ij}]$ is said to be a skew symmetric matrix if $a_{ij} = -a_{ji}$ for all i and j.

Q $a_{ij} = -a_{ji}$ for all i and j

 $a_{ii} = -a_{ii}$ [putting $j = i$]

or $2a_{ii} = 0$ or $a_{ii} = 0$

Thus in a skew symmetric matrix all elements along the principal diagonal are zero. A square matrix A is skew symmetric if and only if A' = – A.

Example: $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$

- **Orthogonal Matrix**

  A matrix A is said to be orthogonal if A'A = I where A' is the transpose of A.

- **Some Results Related to Symmetric And Skew Symmetric Matrices**

(i) If A is any square matrix, then A + A' is a symmetric matrix and A – A' is a skew symmetric matrix.

(ii) Every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew symmetric matrix.

5.8 Determinant of A Square Matrix

Let A be a square matrix of order *n* then the determinant of the matrix A is the value of the determinant whose elements are the corresponding elements of the matrix A and is denoted by | A | or Determinant A. Thus if A = [a_{ij}] be a square matrix of order *n*, then the number

$$\begin{vmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$ is the determinant of the matrix A.

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 3 & 4 & 5 \end{bmatrix}$, then $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 3 & 4 & 5 \end{vmatrix} = 1(-5-8) - 0(10-12) + 3(4+3) = 8$

5.9 Minor And Cofactor of An Element of A Determinant

Let A = [a_{ij}] be a square matrix, then

(i) The minor of the element a_{ij} of | A | is the value of the determinant obtained by deleting its *i*th row and *j*th column and it is denoted by M_{ij}.

(ii) The cofactor of the element a_{ij} of | A | is denoted by the corresponding capital letter C_{ij} and $C_{ij} = (-1)^{i+j} M_{ij}$

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 4 & 5 \\ 0 & -3 & 6 \end{bmatrix}$

In Det. A, minor of 1 = $\begin{vmatrix} 4 & 5 \\ -3 & 6 \end{vmatrix} = 24 + 15 = 39$

Cofactor of 1 = $(-1)^{1+1}$ 39 = 39 [1 lies in the 1st row and 1st column]

Minor of 2 = $\begin{vmatrix} -2 & 5 \\ 0 & 6 \end{vmatrix} = -12 - 0 = -12$

Cofactor of 2 = $(-1)^{1+2}$ (– 12) = 12 [2 occurs in the 1st row and 2nd column]

5.10 Adjoint of A Square Matrix

Let A = [a_{ij}] be a square matrix.

Let B = [A_{ij}] where A_{ij} is the cofactor of the element a_{ij} in the det. A. The transpose B' of the matrix B is called the adjoint of the matrix A and is denoted by adj. A

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}$, then $B = \begin{bmatrix} 15 & -2 & -6 \\ -10 & -1 & 4 \\ -1 & 2 & -1 \end{bmatrix}$

$$\text{adj. } A = \begin{bmatrix} 15 & -10 & -1 \\ -2 & -1 & 2 \\ -6 & 4 & -1 \end{bmatrix}$$

Theorem: If A is any square matrix of order n, then A. (adj A) = (adj A). A = | A | I_n

Proof: Let $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$ Then adj. $A = \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}$

Now A. (adj. A) = $\begin{bmatrix} |A| & 0 & 0 & \dots & 0 \\ 0 & |A| & 0 & \dots & 0 \\ 0 & 0 & |A| & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}$ [a square matrix of order n]

$$|A| \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} = |A| I_n$$

$$\left[\because \sum_{j=1}^{n} a_{ij} C_{ij} = |A| \text{ for } i = 1, 2, \dots, n\right.$$

and $\sum_{j=1}^{n} a_{ij} C_{kj} = 0, i \neq k$

Similarly we can show that (adj A). A = | A | I_n

Thus A (adj. A) = (adj. A) A = | A | I_n

Note:

(i) | A (adj. A) | = | A |^n [| I_n | = 1]

(ii) If | A | □ 0, then | adj. A | = | A |$^{n-1}$

5.11 Non-Singular And Singular Matrices

(i) A square matrix A is said to be a non-singular matrix if | A | □ 0

(ii) A square matrix A is said to be a singular matrix if | A | = 0

5.12 Inverse or Reciprocal of A Square Matrix

Let A be a square matrix of order *n*. Then a matrix B (if such a matrix exists) is called the inverse of A if AB = BA = I_n . Inverse of the square matrix A is denoted by A^{-1}.

- **Existence of The Inverse**

The inverse of a square matrix A exists if and only if A is a non-singular matrix.

If part:

Let A be non-singular square matrix of order *n*. Then | A | □ 0

Let $B = \dfrac{\text{adj. A}}{|A|}$

Then $AB = \dfrac{A(\text{adj. A})}{|A|} = \dfrac{|A|\,I_n}{|A|} = I_n$ [Q A. (adj. A) = | A | I_n] … (i)

Hence B $\dfrac{\text{adj A}}{|A|}$ i.e., B is the inverse of matrix A (by definition of inverse)

Only if part:

Let A be a square matrix of order *n*. Let inverse of A exist. Let B be the inverse of A.

Then by definition of inverse

AB = I_n □ | AB | = | I_n | = 1

or | A | | B | = 1 [Q | AB | = | A | | B |]

□ | A | □ 0, because product | A | | B | is non-zero.

Hence A is non singular

Note:

(i) $A^{-1} = \dfrac{\text{adj A}}{|A|}$

(ii) $AA^{-1} = I_n$ [From (i)]

- **Theorems**

(i) If A and B be any two non-singular matrices, then AB is also a non-singular matrix and

$(AB)^{-1} = B^{-1} A^{-1}$

A, B are non-singular □ | A | □ 0, | B | □ 0

$\therefore$ $|AB| = |A|\,|B| \neq 0$.

Hence AB is non-singular.

Non AB $(B^{-1} A^{-1})$

$= A\{B(B^{-1}A^{-1})\} = A\{(BB^{-1})A^{-1}\}$ [by associative law]

$= A\{I_n A^{-1}\}$ [$BB^{-1} = I_n$]

$= AA^{-1}$ [$I_n A^{-1} = A^{-1}$]

$= I_n$

Hence $B^{-1} A^{-1}$ is the inverse of AB

$\therefore$ $(AB)^{-1} = B^{-1} A^{-1}$

(ii) If A is a non singular matrix, then $(A^{-1})^{-1} = A$

Let A be a square matrix of order *n*,

Then $A^{-1} A = I_n$

$\therefore$ inverse of $A^{-1} = A$

$\therefore$ $(A^{-1})^{-1} = A$

(iii) $I_n^{-1} = I_n$ as $I_n^{-1} I_n = I_n$

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***Illustration* 2**

Find the inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Solution

Given $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

$\therefore$ $|A| = 0(2-3) - 1(1-9) + 2(1-6)$

$= 0 + 8 - 10$

$= -2 \neq 0$

if *C* be the matrix of cofactors of the elements in |**A**|

$\therefore$ $C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

$$C_{11} = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1 \quad ; \quad C_{23} = -\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = 3;$$

$$C_{12} = -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = 8 \quad ; \quad C_{31} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1$$

$$C_{13} = -\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5 \quad ; \quad C_{32} = -\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = 2$$

$$C_{21} = -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1 \quad ; \quad C_{33} = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1$$

$$C_{22} = \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} = -6$$

$$\therefore \quad C = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix}$$

$$\therefore \quad \text{Adj} \quad A = C' = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

Hence, $A^{-1} = \dfrac{Adj\ A}{|A|} = -\dfrac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

5.13 Echelon Form of A Matrix

A matrix A is said to be in echelon form if

(i) every row of A which has all its elements 0, occurs below row which has a non zero element.

(ii) the first non-zero element in each non-zero row is 1

(iii) the number of zeros before the first non-zero element in a row is legs than the number of such zeros in the next row.

Example: *(i)* $\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ *(ii)* $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Note: A row of a matrix is said to be a zero row if all its elements are zero.

5.14 Rank of A Matrix

Let A be a matrix of order $m \times n$. If at least one of its minors of order r is different from zero and all minors of order $(r + 1)$ are zero, then the number r is called the rank of the matrix A and is deonted by □ (A).

Note:

(i) The rank of a zero matrix is zero and the rank of an identity matrix of order n is n.

(ii) The rank of a non-singular matrix of order n is n.

(iii) The rank of a matrix in echelon form is equal to the number of non-zero rows of the matrix.

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***Illustration* 3**

For what values of x the matrix $\begin{vmatrix} 3+x & 5 & 2 \\ 1 & 7+x & 6 \\ 2 & 5 & 3+x \end{vmatrix}$ has the rank 2?

Solution

There is only one 3rd-order minor. In order that the rank may be 2, we must have

$$\begin{vmatrix} 3+x & 5 & 2 \\ 1 & 7+x & 6 \\ 2 & 5 & 3+x \end{vmatrix} = 0 \qquad ..(i)$$

Now, $$\begin{vmatrix} 3+x & 5 & 2 \\ 1 & 7+x & 6 \\ 2 & 5 & 3+x \end{vmatrix} = \begin{vmatrix} 1+x & 0 & -1-x \\ 1 & 7+x & 6 \\ 2 & 5 & 3+x \end{vmatrix} \text{ using } R_1 \to R_1 - R_3$$

$$= \begin{vmatrix} 1+x & 0 & 0 \\ 1 & 7+x & 7 \\ 2 & 5 & 3+x \end{vmatrix}, \text{ using } C_3 \to C_3 + C_1$$

$$= (1 + x) \begin{vmatrix} 7+x & 7 \\ 5 & 5+x \end{vmatrix} = (1 + x) \{(7 + x)(5 + x) - 35\}$$

$$= (1 + x)(x^2 + 12x) = x(1 + x)(x + 12).$$

$\therefore$ (1) holds for $x = 0, -1, -2$

When $x = 0$, the matrix = $\begin{bmatrix} 3 & 5 & 2 \\ 1 & 7 & 6 \\ 2 & 5 & 3 \end{bmatrix}$.

Clearly, a minor $\begin{vmatrix} 3 & 5 \\ 1 & 7 \end{vmatrix} \neq 0$. So, the rank = 2.

When $x = -1$, the matrix = $\begin{bmatrix} 2 & 5 & 2 \\ 1 & 6 & 6 \\ 2 & 5 & 2 \end{bmatrix}$.

Clearly, a minor $\begin{vmatrix} 2 & 5 \\ 1 & 6 \end{vmatrix} \neq 0$. So, the rank = 2.

When $x = -12$, the matrix = $\begin{bmatrix} -9 & 5 & 2 \\ 1 & -5 & 6 \\ 2 & 5 & -9 \end{bmatrix}$.

Clearly, a minor $\begin{vmatrix} -9 & 5 \\ 1 & -5 \end{vmatrix} \neq 0$. So, the rank = 2.

$\therefore$ the matrix has the rank 2 if $x = 0, -1, -12$.

5.15 Elementary Operations or Elementary Transformations of A Matrix

Any of the following operations is called an elementary transformation.

(i) The interchange of any two rows (or columns)

(ii) The multiplication of the elements of any row (or column) by any non-zero number.

(iii) The addition to the elements of any row (or column) the corresponding elements of any other row (or column) multiplied by any number.

Note: An elementary transformation is called a row transformation or column transformation according as it applies to rows or columns.

5.16 Solution of Simultaneous Linear Equations

Let us consider the following system of n linear equations in n unknowns $x_1, x_2, \ldots, x_n$

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad &\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned}\right\} \quad \ldots \text{(i)}$$

If $b_1 = b_2 = \ldots = b_n = 0$, then the system of equations (i) is called a system of homogeneous linear equations and if at least one of $b_1, b_2 \ldots, b_n$ is non zero, then it is called a system of non homogeneous linear equations.

- **Solution of A System of Equations**

  A set of values $\square_1, \square_2, \ldots, \square_n$ of $x_1, x_2, \ldots, x_n$ respectively which satisfy all the equations of the given system of equations is called a solution of the given system of equations.

The system of equations is said to be consistent if its solution exist otherwise it is said to be inconsistent.

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \ B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Then the system of equations (i) can be written as AX = B

Here A is a $n \times n$ matrix and is called the coefficient matrix. Each of X and B is a $n \times 1$ matrix.

I. When system of equations is non-homogeneous:

(i) If | A | $\neq$ 0, the system of equations is consistent and has a unique solution given by $X = A^{-1} B$

(ii) If | A | = 0, the system of equations has no solution or an infinite number of solutions according as (adj A). B is non-zero or zero respectively.

II. When system of equations is homogeneous:

(i) If | A | $\neq$ 0 the system of equations has only trivial solution and number of solutions is one.

(ii) If | A | = 0, the system of equations has non-trival solution and the number of solutions is infinite.

If the system of homogeneous linear equations has number of equations less than the number of unknowns, then it has non-trivial solution.

ILLUSTRATIONS

***Illustration* 4**

Solve the system of equations

$x + 2y + 3z = 1$

$2x + 3y + 2z = 2$

$3x + 3y + 4z = 1$

Solution

We have

$x + 2y + 3z = 1$

$2x + 3y + 2z = 2$

$3x + 3y + 4z = 1$

The given system of equation in the matrix form are written as below:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

AX = B

□ $X = A^{-1}B$

where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$$|A| = 1(12-6) - 2(8-6) + 3(6-9)$$

$$= 6 - 4 - 9$$

$$= 7 \neq 0$$

Let C be the matrix of cofactors of elements in $|A|$

then $C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

Here

$C_{11} \begin{vmatrix} 3 & 2 \\ 3 & 4 \end{vmatrix} = 6$; $C_{12} = -\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} = -2;$ v

$C_{13} = -\begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} = -3$; $C_{21} = -\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = 1$

$C_{22} = -\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} = -5$; $C_{23} = -\begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix} = 3$

$C_{31} = -\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = -5$; $C_{32} = \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} = 4,$

$C_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1$

□ $C = \begin{bmatrix} 6 & -2 & -3 \\ -2 & -5 & 2 \\ -5 & -4 & -1 \end{bmatrix}$

□ Adj $A = \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$

then Adj A) $B = \begin{bmatrix} 2 & -3 & 1 \\ -4 & 6 & -2 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$

Since $X = A^{-1} \cdot B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{7} \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{7} \begin{bmatrix} 3 \\ -8 \\ 2 \end{bmatrix} = \begin{bmatrix} -3/7 \\ 8/7 \\ -2/7 \end{bmatrix}$$

hence $x = -\frac{3}{7},\ y = \frac{8}{7},\ z = -\frac{2}{7}$

PRACTICE EXERCISE

1. Compute the adjoint of the matrix $A = \begin{bmatrix} 1 & 4 & 5 \\ 3 & 2 & 6 \\ 0 & 1 & 0 \end{bmatrix}$ and verify that $A\ (adj\ A) = |A|\ I = (adj\ A)$ A.

2. Using elementary row transformation find the inverse of the matrix $A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix}$.

3. Show that the following system of linear equations is consistent and solve it.

$3x - y - 2z = 2,\ 2x - z = -1,\ 3x - 5y = 3$

Answers

2. $\begin{bmatrix} -5/8 & 5/4 & 1/8 \\ -3/8 & 3/4 & -1/8 \\ -5/4 & 3/2 & 1/4 \end{bmatrix}$

3. $x = -17/8,\ y = -15/8,\ z = -13/4$

5.17 Properties of Determinants

Property I

The value of a determinant remains unaltered, if the rows and columns are interchanged. This is always denoted by $'$ and is also called transpose.

For example : $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Then $D' = D$, D and D' are transposes of each other.

Note: Since the determinant remains unchanged when rows and columns are interchanged, it is obvious that any theorem which is true for 'rows' must also be true for 'columns'.

Property II

If any two rows (or columns) of a determinant be interchanged, the determinant is unaltered in numerical value, but is changed in sign only.

For example : Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_3 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$. Then, $D' = -D$

Property III

If a determinant has two rows (or columns) identical, then its values is zero.

For example : Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$. Then, $D = 0$ (as $R_1 = R_2$)

Property IV

If all the elements of any row (or column) be multiplied by the same number, then the determinant is multiplied by that number.

For example : $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_3 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$. Then $D' = kD$.

Property V

If each elements of any row (or column) can be expressed as a sum of two terms, then the determinant can be expressed as the sum of two determinants.

For example : $\begin{vmatrix} a_1 + x & b_1 + y & c_1 + z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

If each constituent in any row or column consists of r terms then the determinant can be expressed as the sum of r determinants.

Thus, $\begin{vmatrix} a_1 & b_1 & c_1 + \alpha_1 + \beta_1 \\ a_2 & b_2 & c_2 + \alpha_2 + \beta_2 \\ a_3 & b_3 & c_3 + \alpha_3 + \beta_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & \alpha_1 \\ a_2 & b_2 & \alpha_2 \\ a_3 & b_3 & \alpha_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & \beta_1 \\ a_2 & b_2 & \beta_2 \\ a_3 & b_3 & \beta_3 \end{vmatrix}$

Property VI

If a determinant D vanishes for $x = a$, then $(x - a)$ is a factor of D. In other words, if two rows (or two columns) become identical for $x = a$, then $(x - a)$ is factor of D.

In general, if r rows (or r columns) become identical when a is substituted for x, then $(x - a)^{r-1}$ is a factor of D.

For, $r - 1$ of these rows can be replaced by the rows (or columns) in which the elements are the differences of corresponding elements in the original rows (or columns); hence $(x - a)$ is a factor of all the elements in each of these $(r - 1)$ substituted rows (or columns).

5.18 Geometry Involving Determinants

- **Area of Triangle**

Suppose the sides of a triangle lie along the lines whose equations are given by

$a_1x + b_1y + c_1 = 0, a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$

Then the area of the triangle in terms of the coefficients a_1, b_1, c_1.... is given by $\frac{1}{2C_1C_2C_3} \cdot \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$.

Where C_1, C_2, C_3 are cofactors of the elements c_1, c_2, c_3, respectively in the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

- **Condition for General Equation of Second Degree to Represent Pair of Straight Lines**

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

- **Equation of a Line Passing Through Two Given Points**

The equation of a straight line through two points (x_1, y_1) and (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

5.19 Solution of System of Linear Equations

Consider The System of Equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{aligned}\right\} \quad \ldots\text{(i)}$$

A set of values of the variables x, y, z which simultaneously satisfy these three equations is called a solution.

A system of linear equations may have a unique solution, or many solutions, or no solution at all. If it has a solution (whether unique or not) the system is said to be consistent. If it has no solution, it is called an inconsistent system.

If $d_1 = d_2 = d_3 = 0$ in (i), then the system of equations is said to be a homogeneous system. Otherwise it is called a non-homogeneous system of equations.

❑ Solution of System of Linear-Equations in Two Variables By Cramer's Rule

In this section we intend to solve a system of simultaneous linear equations by Cramer's rule named after the Swiss Cramer.

Theorem 1

(Cramer's rule) The solution of the system of simultaneous linear equations

$a_1x + b_1y = c_1$ …(i)

$a_2x + b_2y = c_2$ ...(ii)

is given by $x = \frac{D_1}{D'}$ $y = \frac{D_2}{D'}$ where $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ $D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ and $D_2 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

provided that $D \neq 0$

Proof:

We have $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

So, $xD = x\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1x & b_1 \\ a_2x & b_2 \end{vmatrix}$

$= \begin{vmatrix} a_1x + b_1y & b_1 \\ a_2x + b_2y & b_2 \end{vmatrix}$ [Applying $C_1 \to C_1 + y\, C_2$]

$= \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = D_1$ [Using (i) and (ii)]

Similarly, $yD = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = D_2$

$\therefore$ $x = \frac{D_1}{D}$ and $y = \frac{D_2}{D'}$ provided that $D \neq 0$.

Remark :

Here $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is the determinant of the coefficient matrix $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$.

The determinant D_1 is obtained by replacing first column in D by the column of the right hand side of the given equations.

The determinant D_2 is obtained by replacing the second column in D by the right most column in the given system of equations.

❑ Solution of System of Linear-Equations in Three Variables By Cramer's Rule

Theorem 2

(Cramer's Rule) The solution of the system of linear equations.

$a_1x + b_1y + c_1z = d_1$ …(i)

$a_2x + b_2y + c_2z = d_2$ …(ii)

$a_3x + b_3y + c_3z = d_3$ …(iii)

is given by $x = \frac{D_1}{D'} y = \frac{D_2}{D}$ and $z = \frac{D_3}{D'}$

$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, D_1 \begin{vmatrix} d_1 & b_1 & c1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ and $D = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$ provided that $D \neq 0$.

Proof:

We have

$D = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$ Therefore

$$xD = x\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix} \quad \text{[Applying } C_1 \to C_1 + yC_2 + zC_3\text{]}$$

$$= \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = D_1 \quad \text{[Using } (i)\ (ii) \text{ and} (iii)\text{]}$$

Similarly,

$$yD = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_2 \end{vmatrix} = D_2 \text{ and } zD = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = D_3.$$

$\therefore \quad x = \frac{D_1}{D}, y = \frac{D_2}{D}$ and $z = \frac{D_3}{D'}$ provided that $D \neq 0$.

Remark:

Here D is the determinant of the coefficient matrix. The determinant D_1 is obtained by replacing the elements in first column of D by d_1, d_2, d_3. D_2 is obtained by replacing the elements in the second column of D by d_1, d_2, d_3 and to obtain D_3, replace elements in the third column of D by d_1, d_2, d_3

The above method of solving a system of three linear equations in three unknown can be used exactly the same way to solve a system of n equations in n unknown as stated below.

Theorem 3

(Cramer's Rule) Let there be a system of n simultaneous linear equations n unknown given by

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}\, x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}\, x_n = b_2$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}\, x_n = b_n$$

Let $D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$ and let D_j be the determinant obtained from D after replacing the j^{th} column by $\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$. Then, $x_1 = \frac{D_1}{D},\ x_2 = \frac{D_2}{D},\ \ldots,\ x_n = \frac{D_n}{D},$ provided that $D \neq 0$

- **Condition for Consistency**

Case 1:

For a system of 2 simultaneous linear equations with 2 unknowns

(i) If $D \neq 0$, then the given system of equations is consistent and has a unique solution given by

$$x = \frac{D_1}{D},\ y = \frac{D_2}{D}.$$

(ii) If $D = 0$ and $D_1 = D_2 = 0$, then the system is consistent and has infinitely many solutions.

(iii) If $D = 0$ and one of D_1 and D_2 is non-zero, then tye system is inconsistent.

Case II:

For a system of 3 simultaneous linear equations in three unknowns

(i) If $D \neq 0$, then the given system of equations is consistent and has a unique solution given by

$$x = \frac{D_1}{D'},\ y = \frac{D_2}{D} \text{ and } z = \frac{D_3}{D}.$$

(ii) If $D = 0$ and $D_1 = D_2 = D_3 = 0$, then the given system of equations is consistent with infinitely many solutions.

(iii) If $D = 0$ and at least one of the determinants D_1, D_2, D_3 is non-zero, then the given system of equations is inconsistent.

❑ Algorithm for Solving a System of Simultaneous Linear Equations by Cramer's Rule (Determinant Method)

Step I:

Obtain D, D_1, D_2 and D_3.

Step II:

Find the value of D. If $D \neq 0$, then the system of equations is consistent and has a unique solution. To find the solution, obtain the values of D_1, D_2 an D_3. The solution is given by

$x = \frac{D_1}{D'}$ $y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$.

If $D = 0$, go to step III

Step III:

Find the values of D_1, D_2, D_3. If at least one of these determinants is non-zero, then the system is inconsistent.

If $D_1 = D_2 = D_3 = 0$, then go to step IV.

Step IV:

Take any two equations out of three given equations and shift one of the variables, say z, on the right hand side to obtain two equations in x, y. Solve these two equations by Cramer's rule to obtain x, y in terms of z.

PRACTICE EXERCISE

4. Show that $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$

5. Without expanding evaluate the following determinant. $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$.

6. Solve by Cramer's rule $2x - y = 17$, $3x + 5y = 6$.

Answers

4. 0 **5.** $(a - b)(b - c)(c - a)$ **6.** $x = 7$, $y = -3$

*****

MISCELLANEOUS PROBLEMS

OBJECTIVE TYPE

Example 1

If the value of the determinants $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix}$ is positive, then

(a) $abc > 1$ (b) $abc > -8$ (c) $abc < -8$ (d) $abc > -2$

Solution

We have; $\Delta = \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = abc - (a + b + c(+ 2$

$\therefore \quad \Delta > 0 \Rightarrow abc + 2 > a + b + c$

$\Rightarrow \quad abc + 2 > 3\,(abc)^{1/3} \qquad \left[\because AM > GM \Rightarrow \frac{a+b+c}{3} > (abc)^{1/3}\right]$

$\Rightarrow \quad x^3 + 2 > 3x$, where $x = (abc)^{1/3}$

$\Rightarrow \quad x^3 - 3x + 2 > 0 \Rightarrow (x - 1)^2\,(x + 2) > 0$

$\Rightarrow \quad x + 2 > 0 \Rightarrow x > -2 \Rightarrow (abc)^{1/3} > -2 \Rightarrow abc > -8$

 Ans. (b)

Example 2

The value of the determinant $\begin{vmatrix} \cos\alpha & -\sin\alpha & 1 \\ \sin\alpha & \cos\alpha & 1 \\ \cos(\alpha+\beta) & -\sin(\alpha+\beta) & 1 \end{vmatrix}$ is

(a) independent of α (b) independent of β

(c) independent of α and β (d) none of these

Solution

$$\begin{vmatrix} \cos\alpha & -\sin\alpha & 1 \\ \sin\alpha & \cos\alpha & 1 \\ \cos(\alpha+\beta) & -\sin(\alpha+\beta) & 1 \end{vmatrix} = \begin{vmatrix} \cos\alpha & -\sin\alpha & 1 \\ \sin\alpha & \cos\alpha & 1 \\ 0 & 0 & 1+\sin\beta-\cos\beta \end{vmatrix}$$

Applying $R_3 \rightarrow R_3 - R_1 (\cos\beta) + R_2 (\sin\beta)$

$= (1 + \sin\beta - \cos\beta)(\cos^2\alpha + \sin^2\alpha)$

$= 1 + \sin\beta - \cos\beta$, which is independent of α.

 Ans. (a)

Example 3

If $\begin{vmatrix} a & b & 0 \\ 0 & a & b \\ b & 0 & a \end{vmatrix} = 0$, then

(a) a/b is one of the cube root of unity

(b) a is one of the cube roots of unity

(c) b is one of the cube roots of unity

(d) a/b is one of the cube roots of -1

Solution

We have

$$\begin{vmatrix} a & b & 0 \\ 0 & a & b \\ b & 0 & a \end{vmatrix} = a^3 + b^3$$

$$\therefore \quad \begin{vmatrix} a & b & 0 \\ 0 & a & b \\ b & 0 & a \end{vmatrix} = 0 \Rightarrow a^3 + b^3 = 0 \Rightarrow (a/b)^3 = -1$$

$\Rightarrow$ a/b is one of the cube roots of -1.

 Ans. (d)

Example 4

If ω is an imaginary cube root of unity, then the value of $\begin{vmatrix} 1+\omega & w^2 & -\omega \\ 1+\omega^2 & \omega & -\omega^2 \\ \omega^2+\omega & \omega & -\omega^2 \end{vmatrix}$ is equal to

(a) 0

(b) 2ω

(c) $2\omega^2$

(d) $-3\omega^2$

Solution

Since $1 + \omega + \omega^2 = 0$. Therefore,

$$\begin{vmatrix} 1+\omega & \omega^2 & -\omega \\ 1+\omega^2 & \omega & -\omega^2 \\ \omega^2+\omega & \omega & -\omega^2 \end{vmatrix} = \begin{vmatrix} 1+\omega+\omega^2 & \omega^2 & -\omega \\ 1+\omega^2+\omega & \omega & -\omega^2 \\ \omega^2+2\omega & \omega & -\omega^2 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$

$$= \begin{vmatrix} 0 & \omega^2 & -\omega \\ 0 & \omega & -\omega^2 \\ \omega-1 & \omega & -\omega^2 \end{vmatrix} = (\omega-1)\begin{vmatrix} \omega^2 & -\omega \\ \omega & -\omega^2 \end{vmatrix}$$

$= (\omega - 1)(-\omega^4 + \omega^2) = (\omega - 1)(-\omega + \omega^2)$

$= -\omega^2 + \omega^3 + \omega - \omega^2 = -\omega^2 + (1 + \omega) - \omega^2 = -3\omega^2$

□ **Ans. (d)**

Example 5

If $f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$, then $f(2x) - f(x)$ equals

(a) $a(2a + 3x)$ (b) $ax(2x + 3a)$ (c) $ax(2a + 3x)$ (d) $x(2a + 3x)$

Solution

Applying $R_2 \to R_2 - xR_1$ and $R_3 \to R_3 - xR_2$, we get

$$f(x) = \begin{vmatrix} a & -1 & 0 \\ 0 & a+x & -1 \\ 0 & 0 & a+x \end{vmatrix} = a(a + x)^2$$

$\therefore \quad f(2x) - f(x) = a(a + 2x)^2 - a(a + x)^2 = ax(2a + 3x)$.

□ **Ans. (c)**

Example 6

If α, β, γ are the roots of $x^3 + ax^2 + b = 0$, then the value of $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$ is

(a) $-a^3$ (b) $a^3 - 3b$ (c) a^3 (d) $a^2 - 3b$

Solution

Since α, β, γ are the roots of the given equation, therefore $\alpha + \beta + \gamma = -a$, $\alpha\beta + \beta\gamma + \gamma\alpha = 0$ and $\alpha\beta\gamma = -b$.

Now,

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = -(\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha)$$

$$= -(\alpha + \beta + \gamma)\{(\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \beta\gamma + \gamma\alpha)\}$$

$$= -(-a)\{(a^2 - 0)\} = a^3.$$

 Ans. (c)

Example 7

If α, β and γ are the roots of the equation $x^3 + px + q = 0$ (with $p \neq 0$ and $q \neq 0$), the value of the determinant

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$$

(a) p (b) q (c) $p^2 - 2q$ (d) none of these

Solution

Since α, β, γ are the roots of the given equation, therefore $\alpha + \beta + \gamma = 0$.

Now,

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = \begin{vmatrix} a+\beta+\gamma & \beta & \gamma \\ \alpha+\beta+\gamma & \gamma & \alpha \\ \alpha+\beta+\gamma & \alpha & \beta \end{vmatrix} \quad [\text{Using } C_1 \to C_1 = C_2 + C_3]$$

$$= \begin{vmatrix} 0 & \beta & \gamma \\ 0 & \gamma & \alpha \\ 0 & \alpha & \beta \end{vmatrix} \quad [\because \alpha + \beta + \gamma = 0]$$

$$= 0$$

 Ans. (d)

Example 8

The value of the determinant $\begin{vmatrix} x+2 & x+3 & x+5 \\ x+4 & x+6 & x+9 \\ x+8 & x+11 & x+15 \end{vmatrix}$ is

(a) 2 (b) –2 (c) 3 (d) $x - 1$

Solution

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$, we have Given determinant

$$= \begin{vmatrix} x+2 & x+3 & x+5 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{vmatrix}$$

$$= \begin{vmatrix} x & x & x+1 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{vmatrix}$$ [Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_3 - R_2$]

$$= \begin{vmatrix} x & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$ [Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_2$]

$= -2$ [Expanding along R_3]

$\therefore$ **Ans. (b)**

Example 9

If x, y, z are in A.P., then the value of the determinant $\begin{vmatrix} a+2 & a+3 & a+2x \\ a+3 & a+4 & a+2y \\ a+4 & a+5 & a+2z \end{vmatrix}$ is

(a) 1 (b) 0 (c) $2a$ (d) a

Solution

Since x, y, z are in A.P. Therefore, $x + z - 2y = 0$.

Now,

$$\begin{vmatrix} a+2 & a+3 & a+2x \\ a+3 & a+4 & a+2y \\ a+4 & a+5 & a+2z \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 2(x+z-2y) \\ a+3 & a+4 & a+2y \\ a+4 & a+5 & a+2z \end{vmatrix}$$ Applying $R_1 \to R_1 + R_3 - 2R_2$

$$= \begin{vmatrix} 0 & 0 & 0 \\ a+3 & a+4 & a+2y \\ a+4 & a=5 & a=2z \end{vmatrix}$$ [$\because$ $x + z - 2y = 0$]

$= 0$

$\therefore$ **Ans. (b)**

Example 10

If $\Delta = \begin{vmatrix} \cos(\alpha_1-\beta_1) & \cos(\alpha_1-\beta_2) & \cos(\alpha_1-\beta_3) \\ \cos(\alpha_2-\beta_1) & \cos(\alpha_2-\beta_2) & \cos(\alpha_2-\beta_3) \\ \cos(\alpha_3-\beta_1) & \cos(\alpha_3-\beta_2) & \cos(\alpha_3-\beta_3) \end{vmatrix}$, then Δ equals

(a) $\cos\alpha_1 . \cos\alpha_2 . \cos\alpha_3 . \cos\beta_1 . \cos\beta_2 . \cos\beta_3$

(b)$\cos\alpha_1 + \cos\alpha_1 + \cos\alpha_3 + \cos\beta_1 + \cos\beta_2 + \cos\beta_3$

(c) $\cos(\alpha_1-\beta_1)\cos(\alpha_2-\beta_2)\cos(\alpha_3-\beta_3)$ (d) none of these

Solution

We have

$$\Delta = \begin{vmatrix} \cos\alpha_1 & \sin\alpha_1 & 0 \\ \cos\alpha_2 & \sin\alpha_2 & 0 \\ \cos\alpha_3 & \sin\alpha_3 & 0 \end{vmatrix} \begin{vmatrix} \cos\beta_1 & \sin\beta_1 & 0 \\ \cos\beta_2 & \sin\beta_2 & 0 \\ \cos\beta_3 & \sin\beta_3 & 0 \end{vmatrix}$$

$= 0 \times 0 = 0$

 Ans. (d)

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SUBJECTIVE TYPE

Example 1

If $\begin{bmatrix} x^2-4x & x^2 \\ x^2 & x^3 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -x+2 & 1 \end{bmatrix}$, then $x = \ldots$

Solution

Given, $\begin{bmatrix} x^2-4x & x^2 \\ x^2 & x^3 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -x+2 & 1 \end{bmatrix}$

□ $x^2 - 4x = -3$ □ $x^2 - 4x + 3 = 0$ □ $x = 1, 3$

$x^2 = 1$ □ $x = \pm 1$

$x^2 = -x + 2$ □ $x^2 + x - 2 = 0$ □ $x = -2, 1$

$x^3 = 1$ □ $x = 1$, □, □^{2}

□ common value of x is 1

Example 2

If $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & -3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 & 6 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$, find $4A - 3B$.

Solution

$$4A - 3B = 4\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & -3 & 1 \end{bmatrix} - 3\begin{bmatrix} 4 & 5 & 6 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 8 & 12 \\ -4 & 0 & 8 \\ 4 & -12 & 4 \end{bmatrix} - \begin{bmatrix} 12 & 15 & 18 \\ -3 & 0 & 3 \\ 6 & 3 & 6 \end{bmatrix} = \begin{bmatrix} -8 & -7 & -6 \\ -1 & 0 & 5 \\ -2 & -15 & -2 \end{bmatrix}$$

Example 3

If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$

Find AB and BA and show that AB □ BA.

Solution

A is a 2×3 matrix and B is a 3×2 matrix

□ AB is defined and it will be a 2×2 matrix

Now $\quad AB = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2-8+6 & 3-10+3 \\ -8+8+10 & -12+12+5 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$$

Since B is a 3×2 matrix and A is a 2×3 matrix

$\therefore$ BA is defined and it will be a 3×3 matrix.

Again $\quad BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 2-12 & -4+6 & 6+15 \\ 4-20 & -8+10 & 12+25 \\ 2-4 & -4+2 & 6+5 \end{bmatrix} = \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$$

Clearly $AB \neq BA$

Example 4

Find the transpose and adjoint of the matrix A, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$

Solution

1st part: $A' = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 5 & 4 \\ 3 & 0 & 3 \end{bmatrix}$

2nd part: Let B be the matrix whose elements are cofactors of the corresponding elements of the matrix A. Then

$$B = \begin{bmatrix} 15 & 0 & -10 \\ 6 & -3 & 0 \\ -15 & 0 & 5 \end{bmatrix}$$

$\therefore \quad \text{adj } A = B' = \begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$

Example 5

Compute the inverse of the matrix

$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ and verify that $A^{-1} A = I$.

Solution

Let B be the matrix whose elements are co-factors of the corresponding elements of A, then

$$B = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 6 & -3 \\ -13 & 9 & -1 \end{bmatrix} \quad \square \quad \text{adj } A = B' = \begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 1.2 + 2(-3) + 5.5 = 21$$

$$\square \qquad A^{-1} = \frac{\text{adj A}}{|\text{A}|} = \frac{\text{adj A}}{21} = \begin{bmatrix} 2/21 & 3/21 & -13/21 \\ -3/21 & 6/21 & 9/21 \\ 5/21 & -3/21 & -1/21 \end{bmatrix}$$

Also $$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\square \qquad A^{-1} A = \begin{bmatrix} \frac{2+6+13}{21} & \frac{4+9-13}{21} & \frac{10+3-13}{21} \\ \frac{-3+12-9}{21} & \frac{-6+18+9}{21} & \frac{-15+6+9}{21} \\ \frac{5-6+1}{21} & \frac{10-9-1}{21} & \frac{25-3-1}{21} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{I}$$

Example 6

Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, prove that

$A^2 - 4A - 5I = 0$, hence obtain A^{-1}.

Solution

$$A^2 = A \,.\, A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

Now, $$A^2 - 4A - 5I = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 8-8-0 & 8-8-0 \\ 8-8-0 & 9-4-5 & 8-8-0 \\ 8-8-0 & 8-8-0 & 9-4-5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Thus $A^2 - 4A - 5I = O$ [Here O is the zero matrix]

 $A^{-1}A^2 - 4A^{-1}A - 5A^{-1}I = A^{-1}O = O$

or $(A^{-1}A)A - 4(A^{-1}A) - 5A^{-1}I = O$; or $IA - 4I - 5A^{-1} = O$

$$5A^{-1} = A - 4I = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix} = \begin{bmatrix} -3/5 & 2/5 & 2/5 \\ 2/5 & -3/5 & 2/5 \\ 2/5 & 2/5 & -3/5 \end{bmatrix}$$

Example 7

Solve the following equations by matrix method $5x + 3y + z = 16$, $2x + y + 3z = 19$, $x + 2y + 4z = 25$

Solution

Let $A = \begin{bmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 16 \\ 19 \\ 25 \end{bmatrix}$

Then the matrix equation of the given system of equations become $AX = B$

Now $|A| = \begin{vmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix} = 5(4-6) - 3(8-3) + 1(4-1) = -22 \neq 0$

Hence A is non-singular. Therefore the given system of equations will have the unique solution given by $X = A^{-1}B$

Let C be the matrix whose elements are the cofactors of the corresponding elements of A, then

$$C = \begin{bmatrix} -2 & -5 & 3 \\ -10 & 19 & -7 \\ 8 & -13 & -1 \end{bmatrix}$$

 adj $A = C' = \begin{bmatrix} -2 & -10 & 8 \\ -5 & 19 & -13 \\ 3 & -7 & -1 \end{bmatrix}$

□ □ $A^{-1} = \dfrac{\text{adj A}}{|\text{A}|} = \dfrac{1}{-22}\begin{bmatrix} -2 & -10 & 8 \\ -5 & 19 & -13 \\ 3 & -7 & -1 \end{bmatrix}$

$$\begin{bmatrix} 2/22 & 10/22 & -8/22 \\ 5/22 & -19/22 & 13/22 \\ -3/22 & 7/22 & 1/22 \end{bmatrix}$$

□ $X = A^{-1} B = \begin{bmatrix} 2/22 & 10/22 & -8/22 \\ 5/22 & -19/22 & 13/22 \\ -3/22 & 7/22 & 1/22 \end{bmatrix}\begin{bmatrix} 16 \\ 19 \\ 25 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$

□ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$

□ $x = 1, y = 2, z = 5$

Example 8

Let $A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$, where Show that for $A^n = \begin{bmatrix} a^n & \dfrac{b(a^n - 1)}{(a-1)} \\ 0 & 1 \end{bmatrix}$

Solution

We have to shown by mathematical induction

Step I

For $n = 1$

$$A^1 = \begin{bmatrix} a & b\dfrac{(a-1)}{(a-1)} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

Which is true for $n = 1$

Step II

Assume it is true for $n = k$.

□ $A^k = \begin{bmatrix} a^k & b\dfrac{(a^k - 1)}{(a-1)} \\ 0 & 1 \end{bmatrix}$

Step III

For $n = k + 1$

$$A^{k+1} = A^k \, . \, A$$

$$= \begin{bmatrix} a^k & \frac{b(a^k - 1)}{(a-1)} \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^k.a + 0 & a^k.b + \frac{b(a^k - 1)}{(a-1)}.1 \\ 0 + 0 & 0 + 1.1 \end{bmatrix}$$

$$= \begin{bmatrix} a^{k+1} & b\left\{\frac{a^{k+1} - 1}{(a-1)}\right\} \\ 0 & 1 \end{bmatrix}$$

Hence, it is true for all positive integral values of

Example 9

Determine the values of □□□□□□□ when

$$\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$$ is orthogonal

Solution

Let $A = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$

□ $A' = \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix}$

But given **A** is orthogonal

□ ***AA' = I***

□ $$\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix} \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

□ $$\begin{bmatrix} 4\beta^2 + \gamma^2 & 2\beta^2 - \gamma^2 & 2 - \beta^2 + \gamma^2 \\ 2\beta^2 - \gamma^2 & \alpha^2 + \beta^2 + \gamma^2 & \alpha^2 - \beta^2 - \gamma^2 \\ -2\beta^2\gamma^2 & \alpha^2 - \beta^2 - \gamma^2 & \alpha^2 + \beta^2 + \gamma^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equation the corresponding elements, we have

$4\beta^2 + \gamma^2 = 1$ ... (1)

$2\beta^2 - \gamma^2 = 0$ ... (2)

$\alpha^2 + \beta^2 + \gamma^2 = 1$ ... (3)

From (1) and (2), $6\beta^2 = 1$ □ $\beta^2 = \frac{1}{6}$

and $\gamma^2 = \dfrac{1}{3}$

From (3) $\qquad \alpha^2 = 1 - \beta^2 - \gamma^2$

$$= 1 - \frac{1}{6} - \frac{1}{3}$$

$$= \frac{1}{2}$$

Hence, $\alpha = \pm\dfrac{1}{\sqrt{2}}, \beta = \pm\dfrac{1}{\sqrt{6}}$ and $\gamma = \pm\dfrac{1}{\sqrt{3}}$

Example 9

By the method of matrix inversion, solve the system

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & -1 \end{bmatrix}$$

Solution

We have

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & -1 \end{bmatrix}$$

or $\quad$ ***AX = B*** $\qquad\qquad\qquad$ ... (1)

or $\quad$ $\boldsymbol{X = A^{-1} B}$

Where $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}, X = \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & -1 \end{bmatrix}$

= 1 (– 5 – 7) – 1 (– 2 – 14) + 1 (2 – 10)

= – 12 + 16 – 8

= – 4 □ 0

Let ***C*** be the matrix of cofactors of elements of |***A***|.

□ $\quad C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

$$= \begin{bmatrix} \begin{vmatrix} 5 & 7 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 7 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 2 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 5 & 7 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 16 & -8 \\ 2 & -3 & 1 \\ 2 & -5 & 3 \end{bmatrix}$$

∴ Adj $A = C' = \begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}$

∴ $A^{-1} = -\frac{1}{4}\begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix} = \frac{Adj.(A)}{|A|}$

Now, $A^{-1}B = -\frac{1}{4}\begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix} \times \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & -1 \end{bmatrix}$

$$= -\frac{1}{4}\begin{bmatrix} -4 & 4 \\ -12 & -8 \\ -20 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 5 & 1 \end{bmatrix}$$

From (1)

$\boldsymbol{X = A^{-1}B}$

∴ $\begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 5 & 1 \end{bmatrix}$

On equating the corresponding elements, we have

$x = 1$, $u = -1$

$y = 3$, $v = 2$

$z = 5$, $w = 1$

Example 10

Show that the roots of the equation $\begin{vmatrix} x & m & n & 1 \\ a & x & n & 1 \\ a & b & x & 1 \\ a & b & c & 1 \end{vmatrix} = 0$ are independent of m and n

Solution

$$\Delta = \begin{vmatrix} x-a & m-x & 0 & 0 \\ 0 & x-b & n-x & 0 \\ 0 & 0 & x-c & 0 \\ a & b & c & 1 \end{vmatrix}, \begin{vmatrix} R_1 \to & R_1 - & R_2 \\ R_2 \to & R_2 - & R_3 \\ R_3 \to & R_3 - & R_4 \end{vmatrix}$$

$$= \begin{vmatrix} x-a & m-x & 0 \\ 0 & x-b & n-x \\ 0 & 0 & x-c \end{vmatrix} = (x-a)(x-b)(x-c)$$

$$\therefore \quad \Delta = 0 \Rightarrow (x-a)(x-b)(x-c) = 0$$

whose roots are $x = a, b, c$ and they are independent of m, n

Example 11

Evaluate $\begin{vmatrix} \sqrt{13}+\sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{26} & 5 & \sqrt{10} \\ 3+\sqrt{65} & \sqrt{15} & 5 \end{vmatrix}$

Solution

$$\Delta = \sqrt{5}.\sqrt{5} \begin{vmatrix} \sqrt{13}+\sqrt{3} & 2 & 1 \\ \sqrt{15}+\sqrt{26} & \sqrt{5} & \sqrt{2} \\ 3+\sqrt{65} & \sqrt{3} & \sqrt{5} \end{vmatrix}$$

$$= \begin{vmatrix} \sqrt{13}+\sqrt{3} & 2 & 1 \\ \sqrt{15}+-\sqrt{16} & \sqrt{5}-2\sqrt{2} & 0 \\ 3-\sqrt{15} & \sqrt{3}-\sqrt{5} & 0 \end{vmatrix}, \quad \begin{matrix} R_2 \to R_2 - \sqrt{2}R_1 \\ R_3 \to R_3 - \sqrt{5}R_1 \end{matrix}$$

$$= \begin{vmatrix} \sqrt{15}-\sqrt{16} & \sqrt{5}-2\sqrt{2} \\ 3-\sqrt{15} & \sqrt{3}-2\sqrt{5} \end{vmatrix}$$

$$= 5\sqrt{3} \begin{vmatrix} \sqrt{5}-\sqrt{2} & \sqrt{5}-2\sqrt{2} \\ \sqrt{3}-\sqrt{5} & \sqrt{3}-2\sqrt{5} \end{vmatrix}$$

$$= 5\sqrt{3}\left\{(\sqrt{5}-\sqrt{2})(\sqrt{3}-2\sqrt{5})-(\sqrt{3}-\sqrt{5})(\sqrt{5}-2\sqrt{2})\right\}$$

$$= 5\sqrt{3}\left\{\sqrt{15}-10-\sqrt{6}+2\sqrt{10}-\sqrt{15}+2\sqrt{6}+5-2\sqrt{10}\right\}$$

$$= 5\sqrt{3}(\sqrt{6}-5)$$

Example 12

For a fixed $n \in N$, if

$$D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$$

then $\left\{\dfrac{D}{(n!)^3} - 4\right\}$ is divisible by n

Solution

Here

$$D = n!(n+1)!(n+2)!\begin{vmatrix} 1 & n+1 & (n+2)(n+1) \\ 1 & n+2 & (n+3)(n+2) \\ 1 & n+3 & (n+4)(n+3) \end{vmatrix}$$

$$= n!.(n+1).(n+2)!\begin{vmatrix} 1 & n+1 & (n+2)(n+1) \\ 0 & 1 & (n+2)2 \\ 0 & 1 & (n+3)2 \end{vmatrix}, R_3 \to R_3 - R_2, R_2 \to R_2 - R_1$$

$$= n!(n+1)!(n+2)![(n+3)2-(n+2).2]$$

$$= n!(n+1)!(n+2)!.2$$

$$\therefore \quad \frac{D}{(n!)^3} = 2(n+1)^2(n+2) = 2n^3 + 8n^2 + 10n + 4$$

$\therefore \quad \left\{\frac{D}{(n!)^3} - 4\right\}$ is divisible by n.

Example 13

If $a \neq p, b \neq q, c \neq r$ and $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$ then find the value of $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c}$

Solution

Here, $\Delta = \begin{vmatrix} p & b & c \\ a-p & q-b & 0 \\ 0 & b-q & r-c \end{vmatrix}$, $\begin{matrix} R_3 \to R_3 - R_2 \\ R_2 \to R_2 - R_1 \end{matrix}$

$$= (p-a)(q-b)(r-c)\begin{vmatrix} \frac{p}{p-a} & \frac{b}{q-b} & \frac{c}{r-c} \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix}$$

$$= (p-a)(q-b)(r-c) \times \begin{vmatrix} \frac{p}{p-a} + \frac{b}{q-b} + \frac{c}{r-c} & \frac{b}{q-b} & r - \frac{c}{c} \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} C_1 \to C_1 + C_2 + C_3$$

$$\therefore \quad \Delta = (p-q)(q-b)(r-c)\left(\frac{p}{p-a} + \frac{b}{q-b} + \frac{c}{r-c}\right)$$

= 0, from the question.

But, $a \neq p, b \neq q, c \neq r.$ So $\frac{p}{p-a} + \frac{b}{q-b} + \frac{c}{r-c} = 0$

$$\therefore \quad \frac{p}{p-a}+\left(\frac{b}{q-b}+1\right)+\left(\frac{c}{r-c}+1\right)=1+1$$

or $$\frac{p}{p-a}+\frac{q}{q-b}+\frac{r}{r-c}=2$$

Example 14

Prove that

$$\begin{vmatrix} ax-by-cz & ay+bx & cx+az \\ ay+bx & by-cz-ax & bz+cy \\ cx+az & bz+cy & cz-ax-by \end{vmatrix} = (x^2+y^2+z^2)(a^2+b^2+c^2)(ax+by+cz)$$

Solution

$$\Delta = \frac{1}{a}\begin{vmatrix} a^2x-aby-caz & ay+bx & cx+az \\ a^2y+abx & by-cz-ax & bz+cy \\ acx+a^2z & bz+cy & cz-ax-by \end{vmatrix}$$

$$=\frac{1}{a}\begin{vmatrix} (a^2+b^2+c^2) & ay+bx & cx+az \\ \left(a^2+b^2+c^2\right)y & by-cz-ax & bz+cy \\ \left(a^2+b^2+c^2\right)z & bz+cy & cz-ax-by \end{vmatrix}$$

$$= (a^2+b^2+c^2)\frac{1}{a}\begin{vmatrix} x & ay+bx & cx+az \\ y & by-cz-ax & bz+cy \\ z & bz+cy & cz-ax-by \end{vmatrix}$$

$$= (a^2+b^2+c^2)\frac{1}{ax}\begin{vmatrix} x^2 & axy+bx^2 & cx^2azx \\ y & by-cz-ax & bz+cy \\ z & bz+cy & cz-ax-by \end{vmatrix}$$

$$= (a^2+b^2+c^2)(x^2+y^2+z^2)\frac{1}{ax}\times \begin{vmatrix} x^2+y^2+z^2 & b(x^2+y^2+z^2) & c(x^2+y^2+z^2) \\ y & by-cz-ax & bz+cy \\ z & bz+cy & cz-ax-by \end{vmatrix}$$

$R_1 \to R_1 + yR_2 + zR_3$

$$= (a^2+b^2+c^2)(x^2+y^2=z^2)\frac{1}{ax}\times \begin{vmatrix} 1 & b & c \\ y & by-cz-ax & bz+cy \\ z & bz+cy & cz-ax-by \end{vmatrix}$$

$$=\frac{(a^2+b^2+c^2)(x^2+y^2+z^2)}{ax} \times \begin{vmatrix} 1 & b & c \\ 0 & -cz-ax & bz \\ 0 & cy & -ax-by \end{vmatrix}, \begin{matrix} R_2 \to R_2 - yR_1 \\ R_3 \to R_3 - zR_1 \end{matrix}$$

$$= \frac{(a^2+b^2+c^2)(x^2+y^2+z^2)}{ax} \times \begin{vmatrix} -cz-ax & bz \\ cy & -ax-by \end{vmatrix}$$

$$= \frac{(a^2+b^2+c^2)(x^2+y^2+z^2)}{ax} \times \{acxz+bcyz+a^2x^2+abxy-bcyz\}$$

$$= \frac{(a^2+b^2+c^2)(x^2+y^2+z^2)}{ax} \times \{acxz+bcyz+a^2+x^2+abxy-bcyz\}$$

$$= \frac{(a^2+b^2+c^2)(x^2+y^2+z^2)}{ax} ax(cz+ax+by)$$

$$= (x^2+y^2+z^2)(a^2+b^2+c^2)(ax+by+cz)$$

Example 15

If $x_1 \neq 0; r = 1,2,3$ then prove that

$$\begin{vmatrix} x_1+a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & x_2+a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & x_3+a_3b_3 \end{vmatrix} = x_1x_2x_3\left(1+\frac{a_1b_1}{x_1}+\frac{a_2b_2}{x_2}+\frac{a_3b_3}{x_3}\right)$$

Solution

$$\Delta = \frac{1}{b_1^2}\begin{vmatrix} x_1+a_1b_1 & a_1b_1b_2 & a_1b_1b_3 \\ a_2b_1 & b_1x_2+a_2b_1b_2 & a_2b_1b_3 \\ a_3b_1 & a_3b_1b_2 & b_1x_3+a_3+b_1b_3 \end{vmatrix}$$

$$= \frac{1}{b_1^2}\begin{vmatrix} x_1+a_1b_1 & -b_2x_1 & -b_3x_1 \\ a_2b_1 & b_1x_2 & 0 \\ a_3b_1 & 0 & b_1x_3 \end{vmatrix}, \quad \begin{matrix} C_2 \to C_2 - b_2C_1 \\ C_3 \to C_3 - C_3 - b_3C_1 \end{matrix}$$

Example 16

Prove that $= \begin{vmatrix} \sin\alpha & \sin\beta & \sin\gamma \\ \cos\alpha & \cos\beta & \cos\gamma \\ \sin 2\alpha & \sin 2\beta & \sin 2\gamma \end{vmatrix} = 4\sin\frac{1}{2}(\beta-\gamma).\sin\frac{1}{2}(\gamma-\alpha).\sin\frac{1}{2}(\alpha-\beta)\times$

$[\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)]$

Solution

$$\Delta = \frac{1}{\cos\gamma}\begin{vmatrix} \sin\alpha.\cos\gamma & \sin\beta.\cos\gamma & \sin\gamma.\cos\gamma \\ \cos\alpha & \cos\beta & \cos\gamma \\ \sin 2\alpha & \sin 2\beta & \sin 2\gamma \end{vmatrix} = \frac{1}{\cos\gamma}$$

$$\begin{vmatrix} \sin(\alpha-\gamma) & \sin(\beta-\gamma) & 0 \\ \cos\alpha & \cos\beta & \cos\gamma \\ \sin 2\alpha & \sin 2\beta & \sin 2\gamma \end{vmatrix} = \frac{1}{\cos\gamma} \times \begin{vmatrix} \sin(\alpha-\gamma) & \sin(\beta-\gamma) & 0 \\ \cos\alpha & \cos\beta & \cos\gamma \\ \sin 2\alpha - 2\cos\alpha.\sin\gamma & \sin 2\beta - 2\cos\beta\sin\gamma & 0 \end{vmatrix},$$

$R_1 \to R_1 - \sin\gamma \times R_2, R_3 \to R_3 - 2\sin\gamma \times R_2$

$$\frac{1}{\cos\gamma}(-\cos\gamma)\times\begin{vmatrix}\sin(\alpha-\gamma) & \sin(\beta-\gamma)\\ 2\cos\alpha(\sin\alpha-\sin\gamma) & 2\cos\beta(\sin\beta-\sin\gamma)\end{vmatrix}$$

$$=-2\begin{vmatrix}2\sin\frac{\alpha-\gamma}{2}.\cos\frac{\alpha-\gamma}{2} & 2\sin\frac{\beta-\gamma}{2}.\cos\frac{\beta-\gamma}{2}\\ \cos\alpha.2\cos\frac{\alpha+\gamma}{2}.\sin\frac{\alpha-\gamma}{2} & \cos\beta.2\cos\frac{\beta+\gamma}{2}.\sin\frac{\beta-\gamma}{2}\end{vmatrix}=-4\sin\frac{\alpha-\gamma}{2}\sin\frac{\beta-\gamma}{2}$$

$$\times\begin{vmatrix}\cos\frac{\alpha-\gamma}{2} & \cos\frac{\beta-\gamma}{2}\\ 2\cos\alpha.\cos\frac{\alpha+\gamma}{2} & 2\cos\beta.\cos\frac{\beta+\gamma}{2}\end{vmatrix}==-4\sin\frac{\alpha-\gamma}{2}.\sin\frac{\beta-\gamma}{2}\quad\times\begin{vmatrix}\cos\frac{\alpha-\gamma}{2} & \cos\frac{\beta-\gamma}{2}\\ \cos\frac{\alpha-\gamma}{2}+\cos\frac{3\alpha+\gamma}{2} & \cos\frac{\beta+\gamma}{2}\end{vmatrix}$$

$$R_2\to R_2-R_1\quad=\quad-4\sin\frac{\alpha-\gamma}{2}\sin\frac{\beta-\gamma}{2}\quad\times\times\left\{\cos\frac{\alpha-\gamma}{2}.\cos\frac{3\beta+\gamma}{2}-\cos\frac{\beta-\gamma}{2}.\cos\frac{3\alpha+\gamma}{2}\right\}$$

$$=-2\sin\frac{\alpha-\gamma}{2}\sin\frac{\beta-\gamma}{2}\quad\left\{\left(\cos\frac{\alpha-3\beta-2\gamma}{2}+\cos\frac{a+3\beta}{2}-\cos\frac{3\alpha-\beta+2\gamma}{2}+\cos\frac{3\alpha+\beta}{2}\right)\right\}$$

$$=-2\sin\frac{\alpha-\gamma}{2}\sin\frac{\beta-\gamma}{2}\quad\left\{\left(\cos\frac{\alpha-3\beta-2\gamma}{2}-\cos\frac{3\alpha-\beta+2\gamma}{2}+\cos\frac{\alpha+3\beta}{2}-\cos\frac{3\alpha+\beta}{2}\right)\right\}$$

$$=-2\sin\frac{\alpha-\gamma}{2}.\sin\frac{\beta-\gamma}{2}\times\left\{2\sin(\alpha-\beta).\sin\frac{\alpha+\beta+\gamma}{2}+2\sin\frac{\alpha-\beta}{2}\sin(\alpha+\beta)\right\}$$

$$=-4\sin\frac{\alpha-\gamma}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\alpha-\beta}{2}\times\left\{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha+\beta+2\gamma}{2}+\sin(\alpha+\beta)\right\}$$

$$=-4\sin\frac{\alpha-\gamma}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\alpha-\beta}{2}\times\{\sin(\alpha+\gamma)+\sin(\beta+\gamma)+\sin(\alpha+\beta)\}$$

$$=4\sin\frac{1}{2}(\beta-\gamma).\sin\frac{1}{2}(\gamma-\alpha)\sin\frac{1}{2}(\alpha-\beta)\times\{\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)\}.$$

Example 17

Prove that the following deteminant vanishes if any two of x, y, z are equal or $x + y + z$ = □/2.

$$\begin{vmatrix}\sin x & \sin y & \sin z\\ \cos x & \cos y & \cos z\\ \cos^3 x & \cos^3 y & \cos^3 z\end{vmatrix}$$

Solution

$$\Delta=\cos x\cos y\cos z\begin{vmatrix}\tan x & \tan y & \tan z\\ 1 & 1 & 1\\ \cos^2 x & \cos^2 y & \cos^2 z\end{vmatrix}=\cos x\cos y\cos z$$

$$=-\cos x\cos y\cos z\times\begin{vmatrix}\tan y-\tan x & \tan z-\tan y\\ \cos^2 y-\cos^2 x & \cos^2 z-\cos^2 y\end{vmatrix}$$

$$= -\cos x \cos y \cos z \begin{vmatrix} \dfrac{\sin(y-x)}{\cos x \cos y} & \dfrac{\sin(z-y)}{\cos y.\cos z} \\ \sin^2 x - \sin^2 y & \sin^2 y - \sin^2 z \end{vmatrix}$$

$$= \begin{vmatrix} \cos z.\sin(x-y) & \cos x.\sin(y-z) \\ \sin(x+y).\sin(x-y) & \sin(y+z).\sin(y-z) \end{vmatrix} \qquad \ldots(1)$$

$$= \sin(x-y)\sin(y-z) \begin{vmatrix} \cos z & \cos x \\ \sin(x+y) & \sin(y+z) \end{vmatrix}$$

$$= \sin(x-y)\sin(y-z) \times \{\sin(y+z)\cos z - \sin(x+y)\cos x\}$$

$$= \frac{1}{2}\sin(x-y)\sin(y-z)\,[\{\sin(y+2z)+\sin y\} - \{\sin(y+2x)+\sin y\}]$$

$$= \frac{1}{2}\sin(x-y)\sin(y-z)\,[\sin(y+2z) - \sin(y+2x)]$$

$$= \frac{1}{2}\sin(x-y)\sin(y-z).2\cos(x+y+z).\sin(z-x)$$

$$= \sin(x-y)\sin(y-z)\sin(z-x)\cos(x+y+z).$$

Clearly, □ is zero when any two of *x*, *y*, *z* are equal or $x+y+z=\frac{\pi}{2}$.

Note. The conclusion for the sum can also be made at the stage (1).

Example 18 Prove that in a ΔABC, $\begin{vmatrix} \sin 2A & \sin C & \sin B \\ \sin C & \sin 2B & \sin A \\ \sin B & \sin A & \sin 2C \end{vmatrix} = 0.$

Solution

[We know from properties of triangle in trigonometry that in a □*ABC*,

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ and $\cos A = \frac{b^2+c^2-a^2}{2bc}$, etc.]

$$\Delta = \begin{vmatrix} 2\sin A.\cos A & \sin C & \sin B \\ \sin C & 2\sin B\cos B & \sin A \\ \sin B & \sin A & 2\sin C - \cos C \end{vmatrix}$$

$$= \begin{vmatrix} 2.\dfrac{a}{2R}.\dfrac{b^2+c^2-a^2}{2bc} & \dfrac{c}{2R} & \dfrac{b}{2R} \\ \dfrac{c}{2R} & 2.\dfrac{b}{2R}.\dfrac{c^2+a^2-b^2}{2ca} & \dfrac{a}{2R} \\ \dfrac{b}{2R} & \dfrac{a}{2R} & 2.\dfrac{c}{2R}.\dfrac{a^2+b^2-c^2}{2ab} \end{vmatrix}$$

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Exercise - I

OBJECTIVE TYPE QUESTIONS

Single choice questions

1. If A and B are two matrices such that $AB = B$ and $BA = A$ then $A^2 + B^2$ is equal to

   (a) $2AB$ (b) $2BA$ (c) $A + B$ (d) AB

2. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, the value of A^n is

   (a) $\begin{bmatrix} 3n & -4n \\ n & n \end{bmatrix}$ (b) $\begin{bmatrix} 2+n & 5-n \\ n & -n \end{bmatrix}$ (c) $\begin{bmatrix} 3^n & (-4)^n \\ 1 & (-1)^n \end{bmatrix}$ (d) None of these

3. If $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then A is a

   (a) Diagonal matrix (b) Scalar matrix (c) Nilpotent matrix (d) Idempotent matrix

4. If $A = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix}$, then A^{50} is

   (a) $\begin{bmatrix} 1 & 0 \\ 0 & 50 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 50 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 25 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 25 & 1 \end{bmatrix}$

5. The value of determinant $\begin{vmatrix} \cos\alpha & -\sin\alpha & 1 \\ \sin\alpha & \cos\alpha & 1 \\ \cos(\alpha+\beta) & -\sin(\alpha+\beta) & 1 \end{vmatrix}$ is

   (a) independent of α (b)independent of β

   (c) independent of α and β (d)none of these

6. The values of λ and μ for which the system of equations $x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$ have unique solution are

   (a) $\lambda = 3, \mu \in R$ (b) $\lambda = 3, \mu = 10$ (c) $\lambda \neq 3, \mu = 10$ (d) $\lambda \neq 3, \mu \neq 10$

7. The value of the determinant $\begin{vmatrix} 1 & \omega^6 & \omega^8 \\ \omega^6 & \omega^3 & \omega^7 \\ \omega^8 & \omega^7 & 1 \end{vmatrix}$, where $\omega^3 = 1$, is

   (a) 3 (b) -3 (c) $(1 - \omega)^2$ (d) none of these

8. If $A + B + C = \pi$, then the value of $\begin{vmatrix} \sin(A+B+C) & \sin B & \cos C \\ -\sin B & 0 & \tan A \\ \cos(A+B) & -\tan A & 0 \end{vmatrix}$ is equal to

(a) 0 (b) 1 (c) $2 \sin B \tan A \cos C$ (d) none of these

9. If $f(n) = \begin{vmatrix} n & 1 & 5 \\ n^2 & 2r+1 & 2r+1 \\ n^3 & 3r^2 & 3r+1 \end{vmatrix}$, then $\sum_{r=1}^{r} f(n)$ is

(a) $2\sum_{n=1}^{r} n$ (b) $2\sum_{n=1}^{r} n^2$ (c) $\frac{1}{2}\sum_{n=1}^{r} n^2$ (d) 0

10. The values of λ and μ for which the system of equations $x + y + z = 6$, $x + 2y + 3z = 10$ and $x + 2y + \lambda z = \mu$ have no solution are

(a) $\lambda = 3, \mu = 10$ (b) $\lambda = 3, \mu \neq 10$ (c) $\lambda \neq 3, \mu = 10$ (d) $\lambda \neq 3, \mu \neq 10$

11. If the system of equations $x + ay + az = 0$, $bx + y + bz = 0$ and $cx + cy + z = 0$ where a, b, c are non-zero non unity, has a non-trivial solution, then the value of $\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c}$

(a) –1 (b) 0 (c) 1 (d) $\frac{abc}{a^2+b^2+c^2}$

12. If a, b, c are different, then $\begin{vmatrix} 1 & 1 & 1 \\ (x-a)^2 & (x-b)^2 & (x-c)^2 \\ (x-b)(x-c) & (x-c)(x-a) & (x-a)(x-b) \end{vmatrix}$ vanishes when

(a) $a + b + c = 0$ (b) $x = \frac{1}{3}(a + b + c)$ (c) $x = \frac{1}{2}(a + b + c)$ (d) $x = a + b + c$

13. If a, b, c are sides of a triangle and $\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix} = 0$, then

(a) ΔABC is equilateral (b) ΔABC is right angled isosceles

(c) ΔABC is isosceles (d) none of these

14. For a positive integer n, value of $\Delta = \begin{vmatrix} {}^{n+2}C_2 & {}^{n+3}C_2 & {}^{n+4}C_2 \\ {}^{n+3}C_2 & {}^{n+4}C_2 & {}^{n+5}C_2 \\ {}^{n+4}C_2 & {}^{n+5}C_2 & {}^{n+6}C_2 \end{vmatrix}$ is

(a) 1 (b) –1 (c) $n - 1$ (d) $n^2 + n + 2$

15. If a, b, c are in A.P. with common difference d and $\begin{vmatrix} x+1 & x+a & x+b \\ x+a & x+b & x+c \\ x-b+1 & x-1 & x-a+c \end{vmatrix}$ has absolute value 2, then d is

(a) a (b) –2 (c) ± 1 (d) none of these

More than one correct option

1. If A and B are 3×3 matrices and $|A| \neq 0$, then
   (a) $|AB| = 0 \Rightarrow |B| = 0$ (b) $|AB| \neq 0 \Rightarrow |B| = 0$ (c) $|A^{-1}| = |A|^{-1}$ (d) $|2A| = 2|A|$

2. If A and B are two symmetric matrices of the same order then
   (a) $A + B$ is symmetric (b) $A - B$ is symmetric (c) AB is symmetric (d) BAB is symmetric
3. If A, B and C are three square matrices of the same order, then $AB = AC \Rightarrow B = C$ if
   (a) $|A| \neq 0$ (b) A is invertible (c) A is orthogonal (d) A is symmetric
4. If α, β, γ are three real numbers and $A = \begin{bmatrix} 1 & \cos(\alpha-\beta) & \cos(\alpha-\gamma) \\ \cos(\beta-\alpha) & 1 & \cos(\beta-\gamma) \\ \cos(\gamma-\alpha) & \cos(\gamma-\beta) & 1 \end{bmatrix}$ Then
   (a) A is symmetric (b) A is orthogonal
   (c) A is singular (d) A is not invertible
5. If A is an invertible matrix, then which of the following are true
   (a) $A \neq O$ (b) $Adj.\, A \neq O$ (c) $|A| \neq O$ (d) $A^{-1} = |A|\, Adj.\, A$.
6. If A and B are square matrices of the same order such that $A^2 = A$, $B^2 = B$, $AB = BA = O$, then
   (a) $AB^2 = O$ (b) $(A + B)^2 = A + B$ (c) $(A - B)^2 = A - B$ (d) none of these
7. Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ then
   (a) $A^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} \ \forall\ n \in N$ (b) $\lim_{n\to\infty} \frac{1}{n} A^n = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$
   (c) $\lim_{n\to\infty} \frac{1}{n^2} A^n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (d) none of these
8. If $D_k = \begin{vmatrix} 2^{k-1} & \frac{1}{k(k+1)} & \sin k\theta \\ x & y & z \\ 2^n - 1 & \frac{n}{n+1} & \frac{\sin\left(\frac{n+1}{2}\right)\theta \sin\frac{n}{2}\theta}{\sin\theta/2} \end{vmatrix}$ then $\sum_{k=1}^{n} D_k$ is equal to

(a) 0 (b) independent of n

(c) independent of θ (d) independent of x, y and z

9. The digits A, B C are such that the three digit numbers $A88$, $6B8$, $86C$ are divisible by 72 then the determinant $\begin{vmatrix} A & 6 & 8 \\ 8 & B & 6 \\ 8 & 8 & C \end{vmatrix}$ is divisible by

(a) 72 (b) 144 (c) 288 (d) 216

10. If $a > b > c$ and the system of equations $ax + by + cz = 0$, $bx + cy + az = 0$, $cx + ay + bz = 0$ has a non trivial solution then both the roots of the quadratic equation $at^2 + bt + c = 0$ are

(a) real (b) of opposite sign (c) positive (d) complex

Exercise - II

GENERAL TYPE (ASSERTION & REASON / PASSAGE BASED / MATCHING TYPE QUESTIONS)

Assertion & Reason Type

In the following question, a statement of Assertion (A) is given which is followed by a corresponding statement of reason (R). Mark the correct answer out of the following options/codes.

(a) If both (A) and (R) are true and (R) is the correct explanation of (A).

(b) If both (A) and (R) are true but (R) is not correct explanation of (A).

(c) If (A) is true but (R) is false.

(d) If (A) is false but (R) is true.

1. A: If A and B are two matrices such that $AB = B$, $BA = A$ then $A^2 + B^2 = A + B$.

   R: A and B are idempotent matrices.

2. A: The determinant of a matrix $\begin{bmatrix} 0 & p-q & p-r \\ q-p & 0 & q-r \\ r-p & r-q & 0 \end{bmatrix}$ is zero.

   R: The determinant of a skew symmetric matrix of odd order is zero.

3. A: If A is an $n \times n$ matrix, then det $(mA) = m^n$ det A, where m is any scalar.

   R: If U is a matrix obtained from V by multiplying any row or dolumn by a scalar m, then det $U = m$ det V.

4. A: If A and B are two non singular matrices than $(AB)^{-1} = B^{-1} A^{-1}$.

   R: $Adj\ (AB) = (Adj\ B)\ (Adj\ A)$.

5. A: The inverse of $\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$ does not exist.

   R: The matrix is non-singular.

Passage Based Questions

Passage – I

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and $c_{ij} = (-1)^{i+j}$ (determinant obtained by deleting ith row and jth column),

then $\begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} = \Delta^2$

1. If $\begin{vmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x^2 & 1 & x \end{vmatrix} = 7$ and $\Delta = \begin{vmatrix} x^3-1 & 0 & x-x^4 \\ 0 & x-x^4 & x^3-1 \\ x-x^4 & x^3-1 & 0 \end{vmatrix}$, then

   (a) $\Delta = 7$ (b) $\Delta = 343$ (c) $\Delta = -49$ (d) $\Delta = 49$

2. If $a^3 + b^3 + c^3 - 3\,abc = -5$ then value of $aA + bB + cC$ is

   (a) –5 (b) 5 (c) 25 (d) –25

3. If $3ABC - A^3 - B^3 - C^3 = \pi^2$ and $\Delta = \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix}$. Then $\sin^3(\Delta) + \cos^3(\Delta)$ equals

   (a) 1 (b) –1 (c) 0 (d) none of these

Passage – II

Consider the system of equations $2x + 5y + pz = q$; $x + 2y + 3z = 14$; $x + y + z = 6$. The system is called smart brilliant, good and lazy according as it has solution, unique solution, infinitely many solutions and no solution respectively. Considering the above statement as granted answer the following questions

1. For what values of p and q the system is smart

   (a) p □ 8 or $p = 8$ and $q = 36$ (b) p □□ 8 and $q = 36$

   (c) p □□8 and q □□36 (d) p □□ 8 or $p = 8$ and q □□36

2. For what values of p and q the following system is lazy

   (a) p □ 8 or $p = 8$ and $q = 36$ (b) p □□ 8 and $q = 36$

   (c) p □□8 and q □□36 (d) p □□ 8 or $p = 8$ and q □□36

3. For what values of p and q the above system is brilliant

   (a) p □ 8 or $p = 8$ and $q = 36$ (b) p □□ 8 and $q = 36$

   (c) p = □8 and q □□36 (d) p □□ 8, q is any real number

Matching Type Questions

1. **Column I** **Column II**

	Column I		Column II
(a)	If A = diag (d_1, d_2,d_3,…, d_n) is a diagonal matrix then $\|A\|$ =	(p)	Singular
(b)	If A and B be two non-null square matrices such that AB is a null matrix then A and B both are	(q)	$\|A\|^{n-2} A$
(c)	If A is a non-singular matrix then *adj* (*adj A*) is	(r)	Inconsistent
(d)	Let $AX = B$ be a system of n equations with n-unknown such that $\|A\| = 0$ and (*adj A*)$B \neq 0$ then the system is	(s)	$d_1\, d_2\, d_3 \ldots dn$

(a) (a-p), (b-s), (c-q), (d-r)
(b)(a-q), (b-p), (c-s), (d-r)
(c) (a-s), (b-p), (c-q), (d-r)
(d)(a-r), (b-s), (c-q), (d-r)

2. Match the following for the system of linear equations $\lambda x + y + z = 1$, $x + \lambda y + z = \lambda$, $x + y + \lambda z = \lambda^2$

Column I	Column II
(a) $\lambda = 1$	(p) unique solution
(b) $\lambda \neq 1$	(q) infinite solutions
(c) $\lambda \neq 1$, $\lambda \neq -2$	(r) no solution
(d) $\lambda = -2$	

(a) (a-q), (b-p, r), (c-s), (d-r)
(b) (a-p), (b-s), (c-r), (d-q)
(c) (a-q), (b-p, r), (c-p), (d-r)
(d) (a-r), (b-p), (c-s), (d-q)

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Exercise - III

SUBJECTIVE TYPE

1. Let A = $\begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$ and $(A = I)^{50} - 50A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, find $a + b + c + d$.

2. Find the number of values of λ for which the homogeneous system of equation

   $(a - \lambda)x + by + cz = 0$, $bx + (c - \lambda)y + az = 0$, $cx + ay + (b - \lambda)z = 0$ has a non-trivial solution.

3. If $A = \begin{bmatrix} t & t+1 \\ t-1 & t \end{bmatrix}$ is a matrix such that $AA' = I_2$ then trace of the matrix must be

4. If $2s = a + b + c$ and $\begin{vmatrix} a^2 & (s-a)^2 & (s-a)^2 \\ (s-b)^2 & b^2 & (s-b)^2 \\ (s-c)^2 & (s-c)^2 & c^2 \end{vmatrix} = ks^3 (s - a)(s - b)(s - c)$ then the numerical quantity k should be

5. Let $\Delta(x) = \begin{vmatrix} x-1 & 2x^2-5 & x^3-1 \\ 2x^2+5 & 2x+2 & x^3+3 \\ x^3-1 & x+1 & 3x^2-2 \end{vmatrix}$ and $ax + b$ be the remainder when $\Delta(x)$ is divided by $x^2 - 1$, find $2a + b$.

6. Find the sum of all the values of λ for which the system of equations $\lambda x + y + z = 1$, $x + \lambda y + z = \lambda$, $x + y + \lambda z = \lambda^2$ has infinite number of solutions.

7. If $a \neq p$, $b \neq q$, $c \neq r$ and $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$ then find the value of $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c}$.

8. If x, y, z are cube roots of unity and $D = \begin{vmatrix} x^2+y^2 & z^2 & z^2 \\ x^2 & y^2+z^2 & x^2 \\ y^2 & y^2 & z^2+x^2 \end{vmatrix}$ then real part of D must be.

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Exercise - IV

IIT – JEE PROBLEMS

A. Fill in the blanks

1. Let $p\lambda^4 + q\lambda^3 + r\lambda^2 + s\lambda + t = \begin{vmatrix} \lambda^2+3\lambda & \lambda-1 & \lambda+3 \\ \lambda+1 & -2\lambda & \lambda-4 \\ \lambda-3 & \lambda+4 & 3\lambda \end{vmatrix}$. be an identity in λ, where p, q, r, s and t are constants. Then, the value of t is ______ .

2. The solution set of the equation $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$ is

3. Given that $x = -9$ is a root $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$, the other two roots are ______ and ______ .

4. The value of the determinant $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$ is ______ .

5. For positive number x, y and z, the numerical value of the determinant $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$ is ______

B. True / False

6. The determinant $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ and $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ are not identically equal.

C. Multiple Choice Questions with ONE correct answer

7. Consider the set A of all determinants of order 3 with entries 0 or 1 only. Let B be the subset of A consisting of all determinants with value 1. Let C be the subset of A consisting of all determinants with value –1. Then

(a) C is empty (b) B has as many elements as C

(c) $A = B \cup C$ (d) B has twice as many elements as C

8. $\begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix} = 0$, if

(a) x, y, z are in A.P. (b) x, y, z are in G.P.

(c) x, y, z are in H.P. (d) xy, yz, zx are in A.P.

9. The parameter, on which the value of the determinant

$\begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x & \sin px & \sin(p+d)x \end{vmatrix}$ does not depend upon, is

(a) a (b) p (c) d (d) x

10. If $f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)(x-1) \end{vmatrix}$, then $f(100)$ is equal to

(a) 0 (b) 1 (c) 100 (d) – 100

11. If the system of equations $x - ky - z = 0$, $kx - y - z = 0$, $x + y - z = 0$ has a non-zero solution, then possible values of k are

(a) –1, 2 (b) 1, 2 (c) 0, 1 (d) –1, 1

12. The number of distinct real roots of $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ in the interval $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ is

(a) 0 (b) 2 (c) 1 (d) 3

13. If the system of equations $x + ay = 0$, $az + y = 0$ and $ax + z = 0$ has infinite solutions, then the value of a is

(1) –1 (b) 1 (c) 0 (d) no real values

14. Given $2x - y + 2z = 2$, $x - 2y + z = -4$, $x + y + \lambda z = 4$, then the value of λ such that the given system of equation has no solution, is

(a) 3 (b) 1 (c) 0 (d) –3

15. The determinant $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix}$ is equal to zero, then

(a) a, b, c are in A.P. (b) a, b, c are in G.P.

(c) a, b, c are in H.P. (d) α is a root of the equation $ax^2 + bx + c = 0$

16. If A and B are square matrices of equal degree, then which one is correct among the following

(a) $A + B = B + A$ (b) $A + B = A - B$ (c) $A - B = B - A$ (d) $AB = BA$

17. If $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$, then value of α for which $A^2 = B$, is

(a) 1 (b) –1 (c) 4 (d) no real values

18. $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $|A^3| = 125$, then the value of α is

(a) ± 1 (b) ± 2 (c) ± 3 (d) ± 5

19. If $P = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q = PAP^T$, then $P^T Q^{2005} P$ is

(a) $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2005 \\ 2005 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 2005 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

20. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$, $6A^{-1} = A^2 + cA + dI$, then (c, d) is

(a) (–6, 11) (b) (–11, 6) (c) (11, 6) (d) (6, 11)

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Answers

Exercise - I

Single Choice Questions

1.	(c)	2.	(d)	3.	(a)	4.	(d)	5.	(a)
6.	(a)	7.	(a)	8.	(a)	9.	(b)	10.	(b)
11.	(a)	12.	(b)	13.	(c)	14.	(b)	15.	(c)

More Than One Choice Correct

1.	(a, b, c)	2.	(a, b, d)	3.	(a, b, c)	4.	(a, b, c)	5.	(a, b, c)
6.	(a, b)	7.	(a, b, c)	8.	(a, b, c, d)	9.	(a, b, c)	10	(a, b)

Exercise – II

Assertion and Reason

1.	(a)	2.	(a)	3.	(a)	4.	(a)	5.	(c)

Passage Based Questions

Passage – I

1.	(d)	2.	(a)	3.	(b)

Passage – II

1.	(a)	2.	(a)	3.	(c)

Matching Type Questions

1.	(c)	2.	(c)

Exercise - III

Subjective Type Questions

1.	2	2.	3	3.	0	4.	2	5.	15
6.	1	7.	2	8.	4				

Exercise - IV

IIT-JEE Level Problem

Section - A

1.	0	2.	{–1, 2}	3.	2 and 7	4.	0	5.	0

Section - B

6. False

Section – C

7.	(b)	8.	(b)	9.	(b)	10.	(a)	11.	(d)
12.	(c)	13.	(a)	14.	(b)	15.	(b, e)	16.	(a)
17.	(d)	18.	(c)	19.	(a)	20.	(a)		

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