LOGARITHMS

1.1.1 Definition

"The Logarithm of a given number to a given base is the index of the power to which the base must be raised in order to equal the given number."

If $a > 0$ and $\neq 1$, then logarithm of a positive number N is defined as the index x of that power of 'a' which equals N *i.e.*, $\log_a N = x$ iff $a^x = N \Rightarrow a^{\log_a N} = N, a > 0, a \neq 1$ and $N > 0$

It is also known as fundamental logarithmic identity.

The function defined by $f(x) = \log_a x,\ a > 0, a \neq 1$ is called logarithmic function.

Its domain is $(0, \infty)$ and range is R. a is called the base of the logarithmic function.

When base is 'e' then the logarithmic function is called natural or Napierian logarithmic function and when base is 10, then it is called common logarithmic function.

Note : ❑ The logarithm of a number is unique *i.e.* No number can have two different log to a given base.

❑ $\log_e a = \log_e 10 . \log_{10} a$ or $\log_{10} a = \dfrac{\log_e a}{\log_e 10} = 0.434 \log_e a$

1.1.2 Characteristic and Mantissa

(1) The integral part of a logarithm is called the characteristic and the fractional part is called mantissa.

$$\log_{10} N = \underset{\text{Characteristics}}{\underset{\downarrow}{\text{integer}}} + \underset{\text{Mantissa}}{\underset{\downarrow}{\text{fraction}(+ve)}}$$

(2) The mantissa part of log of a number is always kept positive.

(3) If the characteristics of $\log_{10} N$ be n, then the number of digits in N is $(n+1)$

(4) If the characteristics of $\log_{10} N$ be $(-n)$ then there exists $(n-1)$ number of zeros after decimal part of N.

Example: 1 For $y = \log_a x$ to be defined 'a' must be

(a) Any positive real number (b) Any number

(c) $\geq e$ (d) Any positive real number $\neq 1$

Solution: (d) It is obvious (Definition).

Example: 2 Logarithm of $32\sqrt[5]{4}$ to the base $2\sqrt{2}$ is

(a) 3.6 (b) 5 (c) 5.6 (d) None of these

Solution: (a) Let x be the required logarithm , then by definition $(2\sqrt{2})^x = 32\sqrt[5]{4}$

$(2.2^{1/2})^x = 2^5.2^{2/5}$; $\therefore\ 2^{\frac{3x}{2}} = 2^{5+\frac{2}{5}}$

Here, by equating the indices, $\dfrac{3}{2}x = \dfrac{27}{5}$, $\therefore x = \dfrac{18}{5} = 3.6$

1.1.3 Properties of Logarithms

Let m and n be arbitrary positive numbers such that $a > 0,\ a \neq 1,\ b > 0,\ b \neq 1$ then

(1) $\log_a a = 1,\ \log_a 1 = 0$

(2) $\log_a b . \log_b a = 1 = \log_a a = \log_b b \Rightarrow \log_a b = \dfrac{1}{\log_b a}$

(3) $\log_c a = \log_b a . \log_c b$ or $\log_c a = \dfrac{\log_b a}{\log_b c}$

(4) $\log_a (mn) = \log_a m + \log_a n$

(5) $\log_a \left(\dfrac{m}{n}\right) = \log_a m - \log_a n$

(6) $\log_a m^n = n \log_a m$

(7) $a^{\log_a m} = m$

(8) $\log_a \left(\dfrac{1}{n}\right) = -\log_a n$

(9) $\log_{a^\beta} n = \dfrac{1}{\beta} \log_a n$

(10) $\log_{a^\beta} n^\alpha = \dfrac{\alpha}{\beta} \log_a n$, $(\beta \neq 0)$

(11) $a^{\log_c b} = b^{\log_c a}$, $(a, b, c > 0$ and $c \neq 1)$

Example: 3 The number $\log_2 7$ is

(a) An integer (b) A rational number
(c) An irrational number (d) A prime number

Solution: (c) Suppose, if possible, $\log_2 7$ is rational, say p/q where p and q are integers, prime to each other.

Then, $\dfrac{p}{q} = \log_2 7 \Rightarrow 7 = 2^{p/q} \Rightarrow 2^p = 7^q$,

Which is false since L.H.S is even and R.H.S is odd. Obviously $\log_2 7$ is not an integer and hence not a prime number

Example: 4 If $\log_7 2 = m$, then $\log_{49} 28$ is equal to

(a) $2(1+2m)$ (b) $\dfrac{1+2m}{2}$ (c) $\dfrac{2}{1+2m}$ (d) $1+m$

Solution: (b) $\log_{49} 28 = \dfrac{\log 28}{\log 49} = \dfrac{\log 7 + \log 4}{2\log 7} = \dfrac{\log 7}{2\log 7} + \dfrac{\log 4}{2\log 7} = \dfrac{1}{2} + \dfrac{1}{2}\log_7 4 = \dfrac{1}{2} + \dfrac{1}{2}.2\log_7 2 = \dfrac{1}{2} + \log_7 2 = \dfrac{1}{2} + m = \dfrac{1+2m}{2}$

Example: 5 If $\log_e \left(\dfrac{a+b}{2}\right) = \dfrac{1}{2}(\log_e a + \log_e b)$, then relation between a and b will be

(a) $a = b$ (b) $a = \dfrac{b}{2}$ (c) $2a = b$ (d) $a = \dfrac{b}{3}$

Solution: (a) $\log_e \left(\dfrac{a+b}{2}\right) = \dfrac{1}{2}(\log_e a + \log_e b) = \dfrac{1}{2}\log_e (ab) = \log_e \sqrt{ab}$

$\Rightarrow \dfrac{a+b}{2} = \sqrt{ab} \Rightarrow a + b = 2\sqrt{ab} \Rightarrow \left(\sqrt{a} - \sqrt{b}\right)^2 = 0 \Rightarrow \sqrt{a} - \sqrt{b} = 0 \Rightarrow a = b$

Example: 6 If $\log_{10} 3 = 0.477$, the number of digits in 3^{40} is

(a) 18 (b) 19 (c) 20 (d) 21

Solution: (c) Let $y = 3^{40}$ is

Taking log both the sides, $\log y = \log 3^{40} \Rightarrow \log y = 40 \log 3 \Rightarrow \log y = 19.08$

$\therefore$ Number of digits in $y = 19 + 1 = 20$

Example: 7 Which is the correct order for a given number α in increasing order

(a) $\log_2 \alpha, \log_3 \alpha, \log_e \alpha, \log_{10} \alpha$ (b) $\log_{10} \alpha, \log_3 \alpha, \log_e \alpha, \log_2 \alpha$

(c) $\log_{10} \alpha, \log_e \alpha, \log_2 \alpha, \log_3 \alpha$ (d) $\log_3 \alpha, \log_e \alpha, \log_2 \alpha, \log_{10} \alpha$

Solution: (b) Since 10, 3, *e*, 2 are in decreasing order

Obviously, $\log_{10} \alpha, \log_3 \alpha, \log_e \alpha, \log_2 \alpha$ are in increasing order.

1.1.4 Logarithmic Inequalities

(1) If $a > 1, p > 1 \Rightarrow \log_a p > 0$

(2) If $0 < a < 1, p > 1 \Rightarrow \log_a p < 0$

(3) If $a > 1, 0 < p < 1 \Rightarrow \log_a p < 0$

(4) If $p > a > 1 \Rightarrow \log_a p > 1$

(5) If $a > p > 1 \Rightarrow 0 < \log_a p < 1$

(6) If $0 < a < p < 1 \Rightarrow 0 < \log_a p < 1$

(7) If $0 < p < a < 1 \Rightarrow \log_a p > 1$

(8) If $\log_m a > b \Rightarrow \begin{cases} a > m^b, \text{ if } m > 1 \\ a < m^b, \text{ if } 0 < m < 1 \end{cases}$

(9) $\log_m a < b \Rightarrow \begin{cases} a < m^b, \text{ if } m > 1 \\ a > m^b, \text{ if } 0 < m < 1 \end{cases}$

(10) $\log_p a > \log_p b \Rightarrow a \geq b$ if base *p* is positive and >1 or $a \leq b$ if base *p* is positive and < 1 *i.e.*, $0 < p < 1$

In other words, if base is greater than 1 then inequality remains same and if base is positive but less than 1 then the sign of inequality is reversed.

Example: 8 If $x = \log_3 5,\ y = \log_{17} 25$, which one of the following is correct

(a) $x < y$ (b) $x = y$ (c) $x > y$ (d) None of these

Solution: (c) $y = \log_{17} 25 = 2 \log_{17} 5$

$\therefore \frac{1}{y} = \frac{1}{2} \log_5 17$

$\frac{1}{x} = \log_5 3 = \frac{1}{2} \log_5 9$

Clearly $\frac{1}{y} > \frac{1}{x}$, $\therefore x > y$

Example: 9 If $\log_{0.3}(x-1) < \log_{0.09}(x-1)$, then *x* lies in the interval

(a) $(2, \infty)$ (b) (– 2, –1) (c) (1, 2) (d) None of these

Solution: (a) $\log_{0.3}(x-1) < \log_{(0.3)^2}(x-1) = \frac{1}{2}\log_{0.3}(x-1)$

$\therefore \quad \frac{1}{2}\log_{0.3}(x-1) < 0$

or $\log_{0.3}(x-1) < 0 = \log 1$ or $(x-1) > 1$ or $x > 2$

As base is less than 1, therefore the inequality is reversed, now $x>2 \Rightarrow x$ lies in $(2, \infty)$.

ASSIGNMENT

Properties Of Logarithm

Basic Level

1. $\log ab - \log |b| =$

(a) $\log a$ (b) $\log |a|$ (c) $-\log a$ (d) None of these

2. The value of $\sqrt{(\log_{0.5}^2 4)}$ is

(a) –2 (b) $\sqrt{(-4)}$ (c) 2 (d) None of these

3. The value of $\log_3 4 \log_4 5 \log_5 6 \log_6 7 \log_7 8 \log_8 9$ is

(a) 1 (b) 2 (c) 3 (d) 4

4. $\log_7 \log_7 \sqrt{7(\sqrt{7\sqrt{7}})} =$

(a) $3\log_2 7$ (b) $1 - 3\log_3 7$ (c) $1 - 3\log_7 2$ (d) None of these

5. The value of $81^{(1/\log_5 3)} + 27^{\log_9 36} + 3^{4/\log_7 9}$ is equal to

(a) 49 (b) 625 (c) 216 (d) 890

6. $7\log\left(\frac{16}{15}\right) + 5\log\left(\frac{25}{24}\right) + 3\log\left(\frac{81}{80}\right)$ is equal to

(a) 0 (b) 1 (c) $\log 2$ (d) $\log 3$

7. If $\log_4 5 = a$ and $\log_5 6 = b$, then $\log_3 2$ is equal to

(a) $\frac{1}{2a+1}$ (b) $\frac{1}{2b+1}$ (c) $2ab+1$ (d) $\frac{1}{2ab-1}$

8. If $\log_k x \cdot \log_5 k = \log_x 5, k \neq 1, k > 0$, then x is equal to

(a) k (b) $\frac{1}{5}$ (c) 5 (d) None of these

9. If $\log_5 a \cdot \log_a x = 2$, then x is equal to

(a) 125 (b) a^2 (c) 25 (d) None of these

10. If $a^2 + 4b^2 = 12ab$, then $\log(a+2b)$ is

(a) $\frac{1}{2}[\log a + \log b - \log 2]$ (b) $\log\frac{a}{2} + \log\frac{b}{2} + \log 2$ (c) $\frac{1}{2}[\log a + \log b + 4\log 2]$ (d) $\frac{1}{2}[\log a - \log b + 4\log 2]$

11. If $A = \log_2 \log_2 \log_4 256 + 2\log_{\sqrt{2}} 2$, then A *is* equal to

(a) 2 (b) 3 (c) 5 (d) 7

12. If $\log_{10} x = y$, then $\log_{1000} x^2$ is equal to

(a) y^2 (b) $2y$ (c) $\frac{3y}{2}$ (d) $\frac{2y}{3}$

13. If $x=\log_a(bc), y=\log_b(ca), z=\log_c(ab)$, then which of the following is equal to 1

(a) $x+y+z$ (b) $(1+x)^{-1}+(1+y)^{-1}+(1+z)^{-1}$ (c) xyz (d) None of these

14. If $a=\log_{24}12, b=\log_{36}24$ and $c=\log_{48}36$, then 1+*abc* is equal to

(a) $2ab$ (b) $2ac$ (c) $2bc$ (d) 0

15. If $a^x=b, b^y=c, c^z=a$, then value of *xyz* is

(a) 0 (b) 1 (c) 2 (d) 3

16. If $\log x:\log y:\log z=(y-z):(z-x):(x-y)$ then

(a) $x^y.y^z.z^x=1$ (b) $x^xy^yz^z=1$ (c) $\sqrt[x]{x}\sqrt[y]{y}\sqrt[z]{z}=1$ (d) None of these

17. $\log_4 2-\log_8 2+\log_{16}2-....\text{to }\infty$ is

(a) e^2 (b) ln $2+1$ (c) ln $2-1$ (d) $1-\ln 2$

18. If $\log_{10}2=0.30103, \log_{10}3=0.47712$, the number of digits in $3^{12}\times 2^8$ is

(a) 7 (b) 8 (c) 9 (d) 10

19. $\sum_{r=1}^{89}\log_3(\tan r^o)$

(a) 3 (b) 1 (c) 2 (d) 0

20. $\sum_{n=1}^{n}\frac{1}{\log_{2^n}(a)}=$

(a) $\frac{n(n+1)}{2}\log_a 2$ (b) $\frac{n(n+1)}{2}\log_2 a$ (c) $\frac{(n+1)^2n^2}{4}\log_2 a$ (d) None of these

21. Which of the following is not true

(a) $\log(1+x)<x$ for $x>0$ (b) $\frac{x}{1+x}<\log(1+x)$ for $x>0$ (c) $e^x>1+x$ for $x>0$ (d) $e^x<1-x$ for $x>0$

22. The solution of the equation $\log_7\log_5(\sqrt{x^2+5+x})=0$

(a) $x=2$ (b) $x=3$ (c) $x=4$ (d) $x=-2$

Advance Level

23. $\log_4 18$ is

(a) A rational number (b) An irrational number (c) A prime number (d) None of these

24. The value of $(0.05)^{\log_{\sqrt{20}}(0.1+0.01+0.001+......)}$ is

(a) 81 (b) $\frac{1}{81}$ (c) 20 (d) $\frac{1}{20}$

25. If *a*, *b*, *c* are distinct positive numbers, each different from 1, such that

$[\log_b a \log_c a - \log_a a] + [\log_a b \log_c b - \log_b b] + [\log_a c \log_b c - \log_c c] = 0$, then abc =

(a) 1 (b) 2 (c) 3 (d) None of these

26. If $\log_{12} 27 = a$, then $\log_6 16 =$

(a) $2.\dfrac{3-a}{3+a}$ (b) $3.\dfrac{3-a}{3+a}$ (c) $4.\dfrac{3-a}{3+a}$ (d) None of these

27. If $n = 1983!$, then the value of expression $\dfrac{1}{\log_2 n} + \dfrac{1}{\log_3 n} + \dfrac{1}{\log_4 n} + ... + \dfrac{1}{\log_{1983} n}$ is equal to

(a) –1 (b) 0 (c) 1 (d) 2

28. If $\dfrac{\log x}{b-c} = \dfrac{\log y}{c-a} = \dfrac{\log z}{a-b}$, then which of the following is true

(a) $xyz = 1$ (b) $x^a y^b z^c = 1$ (c) $x^{b+c} y^{c+a} z^{a+b} = 1$ (d) $xyz = x^a y^b z^c$

29. If $x_n > x_{n-1} > ... > x_2 > x_1 > 1$ then the value of $\log_{x_1} \log_{x_2} \log_{x_3} .....\log_{x_n} x_n^{x_{n-1}^{\cdot^{\cdot^{x_1}}}}$ is equal to

(a) 0 (b) 1 (c) 2 (d) None of these

30. The number of solution of $\log_2(x+5) = 6 - x$ is

(a) 2 (b) 0 (c) 3 (d) None of these

31. The number of real values of the parameter *k* for which $(\log_{16} x)^2 - \log_{16} x + \log_{16} k = 0$ with real coefficients will have exactly one solution is

(a) 2 (b) 1 (c) 4 (d) None of these

32. If $x^{\frac{3}{4}(\log_3 x)^2 + \log_3 x - \frac{5}{4}} = \sqrt{3}$ then *x* has

(a) One positive integral value (b) One irrational value

(c) Two positive rational values (d) None of these

LOGARITHMIC INEQUALITIES

Basic Level

33. If $x = \log_5(1000)$ and $y = \log_7(2058)$ then

(a) $x > y$ (b) $x < y$ (c) $x = y$ (d) None of these

34. The number $\log_{20} 3$ lies in

(a) $\left(\dfrac{1}{4}, \dfrac{1}{3}\right)$ (b) $\left(\dfrac{1}{3}, \dfrac{1}{2}\right)$ (c) $\left(\dfrac{1}{2}, \dfrac{3}{4}\right)$ (d) $\left(\dfrac{3}{4}, \dfrac{4}{5}\right)$

35. If $\dfrac{1}{\log_3 \pi} + \dfrac{1}{\log_4 \pi} > x$, then *x* be

(a) 2 (b) 3 (c) 3.5 (d) π

36. If $\log_{1/\sqrt{2}} \sin x > 0, x \in [0, 4\pi]$, then the number of values of x which are integral multiples of $\frac{\pi}{4}$, is

(a) 4 (b) 12 (c) 3 (d) None of these

37. The set of real values of x satisfying $\log_{1/2}(x^2 - 6x + 12) \geq -2$ is

(a) $(-\infty, 2]$ (b) $[2, 4]$ (c) $[4, +\infty)$ (d) None of these

38. The set of real values of x for which $2^{\log_{\sqrt{2}}(x-1)} > x + 5$ is

(a) $(-\infty, -1) \cup (4, +\infty)$ (b) $(4, +\infty)$ (c) $(-1, 4)$ (d) None of these

Advance Level

39. Solution set of inequality $\log_{10}(x^2 - 2x - 2) \leq 0$ is

(a) $[-1, 1-\sqrt{3}]$ (b) $[1+\sqrt{3}, 3]$ (c) $[-1, 1-\sqrt{3}) \cup (1+\sqrt{3}, 3]$ (d) None of these

40. If $\frac{1}{2} \leq \log_{0.1} x \leq 2$ then........

(a) The maximum value of x is $\frac{1}{\sqrt{10}}$ (b) x lies between $\frac{1}{100}$ and $\frac{1}{\sqrt{10}}$

(c) x does not lie between $\frac{1}{100}$ and $\frac{1}{\sqrt{10}}$ (d) The minimum value of x is $\frac{1}{100}$

41. If $\log_{0.04}(x-1) \geq \log_{0.2}(x-1)$ then x belongs to the interval

(a) $(1, 2]$ (b) $(-\infty, 2]$ (c) $[2, +\infty)$ (d) None of these

42. The set of real values of x for which $\log_{0.2} \frac{x+2}{x} \leq 1$ is

(a) $\left(-\infty, -\frac{5}{2}\right] \cup (0, +\infty)$ (b) $\left[\frac{5}{2}, +\infty\right)$ (c) $(-\infty, -2) \cup (0, +\infty)$ (d) None of these

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ANSWER

LOGARITHMS

ASSIGNMENT (BASIC & ADVANCE LEVEL)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
b	c	b	c	d	c	d	b, c	c	c	c	d	b	c	b	b	d	c	d	a
21	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
d	c	b	a	a	c	c	a,b,c,d	b	d	b	a,b,c	a	b	a	a	b	b	c	a,b,d

41	42
c	a