

LIMITS

2.2.1 LIMIT OF A FUNCTION

Let $y = f(x)$ be a function of x . If at $x = a$, $f(x)$ takes indeterminate form, then we consider the values of the function which are very near to ' a '. If these values tend to a definite unique number as x tends to ' a ', then the unique number so obtained is called the limit of $f(x)$ at $x = a$ and we write it as $\lim_{x \rightarrow a} f(x)$.

(1) **Meaning of ' $x \rightarrow a$ '**: Let x be a variable and a be the constant. If x assumes values nearer and nearer to ' a ' then we say ' x tends to a ' and we write ' $x \rightarrow a$ '. It should be noted that as $x \rightarrow a$, we have $x \neq a$. By ' x tends to a ' we mean that

(i) $x \neq a$ (ii) x assumes values nearer and nearer to ' a ' and

(iii) We are not specifying any manner in which x should approach to ' a '. x may approach to a from left or right as shown in figure.



(2) **Left hand and right hand limit** : Consider the values of the functions at the points which are very near to a on the left of a . If these values tend to a definite unique number as x tends to a , then the unique number so obtained is called left-hand limit of $f(x)$ at $x = a$ and symbolically we write it as $f(a-0) = \lim_{x \rightarrow a^-} f(x) = \lim_{h \rightarrow 0} f(a-h)$

Similarly we can define right-hand limit of $f(x)$ at $x = a$ which is expressed as $f(a+0) = \lim_{x \rightarrow a^+} f(x) = \lim_{h \rightarrow 0} f(a+h)$.

(3) **Method for finding L.H.L. and R.H.L.**

(i) For finding right hand limit (R.H.L.) of the function, we write $x + h$ in place of x , while for left hand limit (L.H.L.) we write $x - h$ in place of x .

(ii) Then we replace x by ' a ' in the function so obtained.

(iii) Lastly we find limit $h \rightarrow 0$.

(4) **Existence of limit** : $\lim_{x \rightarrow a} f(x)$ exists when,

(i) $\lim_{x \rightarrow a^-} f(x)$ and $\lim_{x \rightarrow a^+} f(x)$ exist i.e. L.H.L. and R.H.L. both exists.

(ii) $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$ i.e. L.H.L. = R.H.L.

Note : $\square$ If a function $f(x)$ takes the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at $x = a$, then we say that $f(x)$ is indeterminate or meaningless at $x = a$. Other indeterminate forms are $\infty - \infty, \infty \times \infty, 0 \times \infty, 1^\infty, 0^0, \infty^0$

$\square$ In short, we write L.H.L. for left hand limit and R.H.L. for right hand limit.

□ It is not necessary that if the value of a function at some point exists then its limit at that point must exist.

(5) **Sandwich theorem** : If $f(x)$, $g(x)$ and $h(x)$ are any three functions such that, $f(x) \leq g(x) \leq h(x) \forall x \in$ neighborhood of $x = a$ and $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = l$ (say), then $\lim_{x \rightarrow a} g(x) = l$. This theorem is normally applied when the $\lim_{x \rightarrow a} g(x)$ can't be obtained by using conventional methods as function $f(x)$ and $h(x)$ can be easily found.

Example: 1 If $f(x) = \begin{cases} x, & \text{when } x > 1 \\ x^2, & \text{when } x < 1 \end{cases}$, then $\lim_{x \rightarrow 1} f(x) =$

- (a) x^2 (b) x (c) -1 (d) 1

Solution: (d) To find L.H.L. at $x = 1$ i.e.,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{h \rightarrow 0} f(1-h) = \lim_{h \rightarrow 0} (1-h)^2 = \lim_{h \rightarrow 0} (1+h^2-2h) = 1 \quad \text{i.e.,} \quad \lim_{x \rightarrow 1^-} f(x) = 1 \quad \dots(i)$$

$$\text{Now find R.H.L. at } x = 1 \quad \text{i.e.,} \quad \lim_{x \rightarrow 1^+} f(x) = \lim_{h \rightarrow 0} f(1+h) = 1 \quad \text{i.e.,} \quad \lim_{x \rightarrow 1^+} f(x) = 1 \quad \dots(ii)$$

From (i) and (ii), L.H.L. = R.H.L. $\Rightarrow \lim_{x \rightarrow 1} f(x) = 1$.

Example: 2 $\lim_{x \rightarrow 2} \frac{|x-2|}{x-2} =$

- (a) 1 (b) -1 (c) Does not exist (d) None of these

Solution: (c) L.H.L. = $\lim_{x \rightarrow 2^-} \frac{|x-2|}{x-2} = \lim_{h \rightarrow 0} \frac{|2-h-2|}{2-h-2} = \lim_{h \rightarrow 0} \frac{-h}{-h} = -1 \quad \dots(i)$

$$\text{and, R.H.L.} = \lim_{x \rightarrow 2^+} \frac{|x-2|}{x-2} = \lim_{h \rightarrow 0} \frac{|2+h-2|}{2+h-2} = \lim_{h \rightarrow 0} \frac{h}{h} = 1 \quad \dots(ii)$$

From (i) and (ii) L.H.L. $\neq$ R.H.L. i.e. $\lim_{x \rightarrow 2} \frac{|x-2|}{x-2}$ does not exist.

Example: 3 If $f(x) = \begin{cases} \frac{2}{5-x}, & \text{when } x < 3 \\ 5-x, & \text{when } x > 3 \end{cases}$, then

- (a) $\lim_{x \rightarrow 3^+} f(x) = 0$ (b) $\lim_{x \rightarrow 3^-} f(x) = 0$ (c) $\lim_{x \rightarrow 3^+} f(x) \neq \lim_{x \rightarrow 3^-} f(x)$ (d) None of these

Solution: (c) $\lim_{x \rightarrow 3^+} f(x) = 5-3 = 2$ and $\lim_{x \rightarrow 3^-} f(x) = \frac{2}{5-3} = 1$

Example: 4 Let the function f be defined by the equation $f(x) = \begin{cases} 3x, & \text{if } 0 \leq x \leq 1 \\ 5-3x, & \text{if } 1 < x \leq 2 \end{cases}$, then

- (a) $\lim_{x \rightarrow 1} f(x) = f(1)$ (b) $\lim_{x \rightarrow 1} f(x) = 3$ (c) $\lim_{x \rightarrow 1} f(x) = 2$ (d) $\lim_{x \rightarrow 1} f(x)$ does not exist

Solution: (d) L.H.L. = $\lim_{x \rightarrow 1-0} f(x) = \lim_{h \rightarrow 0} f(1-h) = \lim_{h \rightarrow 0} 3(1-h) = \lim_{h \rightarrow 0} (3-3h) = 3-3.0 = 3$

$$\text{R.H.L.} = \lim_{x \rightarrow 1+0} f(x) = \lim_{h \rightarrow 0} f(1+h) = \lim_{h \rightarrow 0} [5-3(1+h)] = \lim_{h \rightarrow 0} (2-3h) = 2-3.0 = 2$$

Hence $\lim_{x \rightarrow 1} f(x)$ does not exist.

Example: 5 $\lim_{x \rightarrow 0} \frac{|x|}{x} =$

- (a) 1 (b) -1 (c) 0 (d) Does not exist

Solution: (d) $\ominus \lim_{x \rightarrow 0^-} \frac{|x|}{x} = -1$ and $\lim_{x \rightarrow 0^+} \frac{|x|}{x} = 1$, hence limit does not exist.

2.2.2 FUNDAMENTAL THEOREMS ON LIMITS

The following theorems are very useful for evaluation of limits if $\lim_{x \rightarrow 0} f(x) = l$ and $\lim_{x \rightarrow 0} g(x) = m$ (l and m are real numbers) then

- (1) $\lim_{x \rightarrow a} (f(x) + g(x)) = l + m$ (Sum rule)
- (2) $\lim_{x \rightarrow a} (f(x) - g(x)) = l - m$ (Difference rule)
- (3) $\lim_{x \rightarrow a} (f(x) \cdot g(x)) = lm$ (Product rule)
- (4) $\lim_{x \rightarrow a} k f(x) = kl$ (Constant multiple rule)
- (5) $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{l}{m}, m \neq 0$ (Quotient rule)
- (6) If $\lim_{x \rightarrow a} f(x) = +\infty$ or $-\infty$, then $\lim_{x \rightarrow a} \frac{1}{f(x)} = 0$
- (7) $\lim_{x \rightarrow a} \log\{f(x)\} = \log\{\lim_{x \rightarrow a} f(x)\}$
- (8) If $f(x) \leq g(x)$ for all x , then $\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$
- (9) $\lim_{x \rightarrow a} [f(x)]^{g(x)} = \{\lim_{x \rightarrow a} f(x)\}^{\lim_{x \rightarrow a} g(x)}$
- (10) If p and q are integers, then $\lim_{x \rightarrow a} (f(x))^{p/q} = l^{p/q}$, provided $(l)^{p/q}$ is a real number.
- (11) If $\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x)) = f(m)$ provided ' f ' is continuous at $g(x) = m$ e.g. $\lim_{x \rightarrow a} \ln[f(x)] = \ln(l)$, only if $l > 0$.

2.2.3 SOME IMPORTANT EXPANSIONS

In finding limits, use of expansions of following functions are useful :

- (1) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \dots$
- (2) $a^x = 1 + x \log a + \frac{(x \log a)^2}{2!} + \dots$
- (3) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$
- (4) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, |x| < 1$
- (5) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots, \text{ where } |x| < 1$
- (6) $(1+x)^{\frac{1}{x}} = e^{\frac{1}{x} \log(1+x)} = e^{\frac{1}{x} (x - \frac{x^2}{2} + \frac{x^3}{3} - \dots)} = e \left(1 - \frac{x}{2} + \frac{11}{24} x^2 - \dots \right)$
- (7) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$
- (8) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$
- (9) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$
- (10) $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$
- (11) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$
- (12) $\tanh x = x - \frac{x^3}{3} + 2x^5 - \dots$
- (13) $\sin^{-1} x = x + \frac{1}{2} \cdot \frac{x^3}{3!} + \frac{3}{8} \cdot \frac{x^5}{5!} + \dots$
- (14) $\cos^{-1} x = \left(\frac{\pi}{2} \right) - \sin^{-1} x$
- (15) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

2.2.4 METHODS OF EVALUATION OF LIMITS

We shall divide the problems of evaluation of limits in five categories.

(1) **Algebraic limits** : Let $f(x)$ be an algebraic function and 'a' be a real number. Then $\lim_{x \rightarrow a} f(x)$ is known as an algebraic limit.

(i) **Direct substitution method** : If by direct substitution of the point in the given expression we get a finite number, then the number obtained is the limit of the given expression.

(ii) **Factorisation method** : In this method, numerator and denominator are factorised. The common factors are cancelled and the rest outputs the results.

(iii) **Rationalisation method** : Rationalisation is followed when we have fractional powers (like $\frac{1}{2}, \frac{1}{3}$ etc.) on expressions in numerator or denominator or in both. After rationalisation the terms are factorised which on cancellation gives the result.

(iv) **Based on the form when $x \rightarrow \infty$** : In this case expression should be expressed as a function $1/x$ and then after removing indeterminate form, (if it is there) replace $\frac{1}{x}$ by 0.

Step I : Write down the expression in the form of rational function, i.e., $\frac{f(x)}{g(x)}$, if it is not so.

Step II : If k is the highest power of x in numerator and denominator both, then divide each term of numerator and denominator by x^k .

Step III : Use the result $\lim_{x \rightarrow \infty} \frac{1}{x^n} = 0$, where $n > 0$.

Note : $\square$ **An important result** : If m, n are positive integers and $a_0, b_0 \neq 0$ are non-zero real

$$\text{numbers, then } \lim_{x \rightarrow \infty} \frac{a_0 x^m + a_1 x^{m-1} + \dots + a_{m-1} x + a_m}{b_0 x^n + b_1 x^{n-1} + \dots + b_{n-1} x + b_n} = \begin{cases} \frac{a_0}{b_0}, & \text{if } m = n \\ 0, & \text{if } m < n \\ \infty, & \text{if } m > n \end{cases}$$

Example: 6 $\lim_{x \rightarrow 1} (3x^2 + 4x + 5) =$

- (a) 12 (b) -1 (c) Does not exist (d) None of these

Solution: (a) $\lim_{x \rightarrow 1} (3x^2 + 4x + 5) = 3(1)^2 + 4(1) + 5 = 12$.

Example: 7 The value of $\lim_{x \rightarrow 2} \frac{3^{x/2} - 3}{3^x - 9}$ is

- (a) 0 (b) $\frac{1}{3}$ (c) $\frac{1}{6}$ (d) $\ln 3$

Solution: (c) $\lim_{x \rightarrow 2} \frac{3^{x/2} - 3}{(3^{x/2})^2 - (3)^2} = \lim_{x \rightarrow 2} \frac{(3^{x/2} - 3)}{(3^{x/2} - 3)(3^{x/2} + 3)} = \frac{1}{6}$.

Example: 8 The value of $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a}$ is

- (a) 0 (b) na^{n-1} (c) na^n (d) 1

Solution: (b) $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = \lim_{x \rightarrow a} \frac{(x-a)(x^{n-1} + x^{n-2}a + \dots + a^{n-1})}{(x-a)} = \lim_{x \rightarrow a} (x^{n-1} + x^{n-2}a + \dots + a^{n-1}) = n \cdot a^{n-1}.$

Example: 9 $\lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{x+h} - \frac{1}{x} \right]$ equals

- (a) $\frac{1}{2x}$ (b) $-\frac{1}{2x}$ (c) $\frac{1}{x^2}$ (d) $-\frac{1}{x^2}$

Solution: (d) $\lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{x+h} - \frac{1}{x} \right] = \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{x - (x+h)}{(x+h)x} \right] = \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-h}{(x+h)x} \right] = -\frac{1}{x^2}.$

Example: 10 The value of $\lim_{x \rightarrow 0} \frac{\sqrt{1-x^2} - \sqrt{1+x^2}}{x^2}$ is

- (a) 1 (b) -1 (c) -2 (d) 0

Solution: (b) $\lim_{x \rightarrow 0} \frac{(\sqrt{1-x^2} - \sqrt{1+x^2})}{x^2} \cdot \frac{(\sqrt{1-x^2} + \sqrt{1+x^2})}{(\sqrt{1-x^2} + \sqrt{1+x^2})} = \lim_{x \rightarrow 0} \frac{(1-x^2) - (1+x^2)}{x^2(\sqrt{1-x^2} + \sqrt{1+x^2})} = \frac{-2}{2} = -1.$

Example: 11 $\lim_{x \rightarrow 3} \frac{x-3}{\sqrt{x-2} - \sqrt{4-x}}$ equals

- (a) 1 (b) $\frac{3}{2}$ (c) $\frac{1}{4}$ (d) None of these

Solution: (d) $\lim_{x \rightarrow 3} \frac{x-3}{\sqrt{x-2} - \sqrt{4-x}} = \lim_{x \rightarrow 3} \frac{(x-3)(\sqrt{x-2} + \sqrt{4-x})}{(\sqrt{x-2})^2 - (\sqrt{4-x})^2}$
 $= \lim_{x \rightarrow 3} \frac{(x-3)(\sqrt{x-2} + \sqrt{4-x})}{(2x-6)} = \lim_{x \rightarrow 3} \frac{\sqrt{x-2} + \sqrt{4-x}}{2} = \frac{1+1}{2} = 1.$

Example: 12 $\lim_{x \rightarrow \infty} \frac{ax^2 + bx + c}{dx^2 + ex + f} =$

- (a) $\frac{b}{e}$ (b) $\frac{c}{f}$ (c) $\frac{a}{d}$ (d) $\frac{d}{a}$

Solution: (c) Here the expression assumes the form $\frac{\infty}{\infty}$. We note that the highest power of x in both the numerator and denominator is 2. So we divide each terms in both the numerator and denominator by x^2 .

$$\lim_{x \rightarrow \infty} \frac{ax^2 + bx + c}{dx^2 + ex + f} = \lim_{x \rightarrow \infty} \frac{a + \frac{b}{x} + \frac{c}{x^2}}{d + \frac{e}{x} + \frac{f}{x^2}} = \frac{a+0+0}{d+0+0} = \frac{a}{d}.$$

Example: 13 $\lim_{x \rightarrow \infty} \left[\sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x} \right]$ is equal to

- (a) 0 (b) $\frac{1}{2}$ (c) $\log 2$ (d) e^4

Solution: (b) $\lim_{x \rightarrow \infty} \left[\sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x} \right] = \lim_{x \rightarrow \infty} \frac{x + \sqrt{x + \sqrt{x}} - x}{\sqrt{x + \sqrt{x + \sqrt{x}}} + \sqrt{x}} = \lim_{x \rightarrow \infty} \frac{\sqrt{x + \sqrt{x}}}{\sqrt{x + \sqrt{x + \sqrt{x}}} + \sqrt{x}} = \lim_{x \rightarrow \infty} \frac{\sqrt{1 + x^{-1/2}}}{\sqrt{1 + \sqrt{x^{-1} + x^{-3/2}}} + 1} = \frac{1}{2}.$

Example: 14 The values of constants a and b so that $\lim_{x \rightarrow \infty} \left(\frac{x^2+1}{x+1} - ax - b \right) = 0$ is

- (a) $a=0, b=0$ (b) $a=1, b=-1$ (c) $a=-1, b=1$ (d) $a=2, b=-1$

Solution: (b) We have $\lim_{x \rightarrow \infty} \left(\frac{x^2+1}{x+1} - ax - b \right) = 0 \Rightarrow \lim_{x \rightarrow \infty} \frac{x^2(1-a) - x(a+b) + 1-b}{x+1} = 0$

Since the limit of the given expression is zero, therefore degree of the polynomial in numerator must be less than that of denominator. As the denominator is a first degree polynomial. So, numerator must be a constant i.e., a zero degree polynomial. $\therefore 1 - a = 0$ and $a + b = 0 \Rightarrow a = 1$ and $b = -1$. Hence, $a = 1$ and $b = -1$.

Example: 15 $\lim_{x \rightarrow 1} x^x =$

- (a) 1 (b) ∞ (c) Not defined (d) None of these

Solution: (a) $\lim_{x \rightarrow 1} x^x = \left(\lim_{x \rightarrow 1} x \right)^{\lim_{x \rightarrow 1} x} = 1^1 = 1$

Example: 16 $\lim_{x \rightarrow 1} (1+x)^{1/x} =$

- (a) 2 (b) e (c) Not defined (d) None of these

Solution: (a) $\lim_{x \rightarrow 1} (1+x)^{1/x} = \left(\lim_{x \rightarrow 1} (1+x) \right)^{\lim_{x \rightarrow 1} \left(\frac{1}{x} \right)} = 2$

Example: 17 The value of the limit of $\frac{x^3 - x^2 - 18}{x - 3}$ as x tends to 3 is

- (a) 3 (b) 9 (c) 18 (d) 21

Solution: (d) Let $y = \lim_{x \rightarrow 3} \frac{x^3 - x^2 - 18}{x - 3} = \lim_{x \rightarrow 3} (x^2 + 2x + 6) = 9 + 6 + 6 = 21$

Example: 18 The value of the limit of $\frac{x^3 - 8}{(x^2 - 4)}$ as x tends to 2 is

- (a) 3 (b) $\frac{3}{2}$ (c) 1 (d) 0

Solution: (a) $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x^2 - 4} = \lim_{x \rightarrow 2} \frac{(x^2 + 2x + 4)(x - 2)}{(x + 2)(x - 2)} = \lim_{x \rightarrow 2} \frac{x^2 + 2x + 4}{x + 2} = \frac{4 + 4 + 4}{2 + 2} = 3.$

Example: 19 $\lim_{x \rightarrow 0} \frac{x}{\sqrt{1+x} - \sqrt{1-x}}$ is equal to

- (a) $\frac{1}{2}$ (b) 2 (c) 1 (d) 0

Solution: (c) $\lim_{x \rightarrow 0} \left(\frac{x}{\sqrt{1+x} - \sqrt{1-x}} \right) = \lim_{x \rightarrow 0} \left(\frac{x}{\sqrt{1+x} - \sqrt{1-x}} \times \frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right)$
 $= \lim_{x \rightarrow 0} \left(\frac{x(\sqrt{1+x} + \sqrt{1-x})}{1+x-1+x} \right) = \lim_{x \rightarrow 0} \left(\frac{(\sqrt{1+x} + \sqrt{1-x})}{2} \right) = \frac{2}{2} = 1$

Example: 20 $\lim_{x \rightarrow a} \frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}}$ equals

- (a) $\frac{2a}{3\sqrt{3}}$ (b) $\frac{2}{3\sqrt{3}}$ (c) 0 (d) None of these

Solution: (b) $\lim_{x \rightarrow a} \frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} = \lim_{x \rightarrow a} \left(\frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \right) \times \left(\frac{\sqrt{a+2x} + \sqrt{3x}}{\sqrt{a+2x} + \sqrt{3x}} \right) \times \left(\frac{\sqrt{3a+x} + 2\sqrt{x}}{\sqrt{3a+x} + 2\sqrt{x}} \right)$
 $= \lim_{x \rightarrow a} \left\{ \frac{\sqrt{3a+x} + 2\sqrt{x}}{3(\sqrt{a+2x} + \sqrt{3x})} \right\} = \frac{2}{3\sqrt{3}}.$

Example: 21 $\lim_{n \rightarrow \infty} \frac{1^{99} + 2^{99} + 3^{99} + \dots + n^{99}}{n^{100}} =$

- (a) $\frac{99}{100}$ (b) $\frac{1}{100}$ (c) $\frac{1}{99}$ (d) $\frac{1}{101}$

Solution: (b) $\lim_{n \rightarrow \infty} \frac{1^{99} + 2^{99} + 3^{99} + \dots + n^{99}}{n^{100}} = \lim_{n \rightarrow \infty} \sum_{r=1}^n \left(\frac{r^{99}}{n^{100}} \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \left(\frac{r}{n} \right)^{99} = \int_0^1 x^{99} dx = \left[\frac{x^{100}}{100} \right]_0^1 = \frac{1}{100}.$

Example: 22 The values of constants 'a' and 'b' so that $\lim_{x \rightarrow \infty} \left(\frac{x^2-1}{x+1} - ax - b \right) = 2$ is

- (a) $a=0, b=0$ (b) $a=1, b=-1$ (c) $a=1, b=-3$ (d) $a=2, b=-1$

Solution: (c) $\lim_{x \rightarrow \infty} \left(\frac{x^2-1}{x+1} - ax - b \right) = 2 \Rightarrow \lim_{x \rightarrow \infty} x - 1 - ax - b = 2 \Rightarrow \lim_{x \rightarrow \infty} x(1-a) - (1+b) = 2.$

Comparing the coefficient of both sides, $1-a=0$ and $1+b=-2 \Rightarrow a=1, b=-3$

Example: 23 $\lim_{n \rightarrow \infty} \left[\frac{\sum n^2}{n^3} \right] =$

- (a) $-\frac{1}{6}$ (b) $\frac{1}{6}$ (c) $\frac{1}{3}$ (d) $-\frac{1}{3}$

Solution: (c) $\lim_{n \rightarrow \infty} \left[\frac{n(n+1)(2n+1)}{6n^3} \right] = \lim_{n \rightarrow \infty} \frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6} = \frac{1}{3}$

Note : $\square$ Students should remember that,

$$\lim_{n \rightarrow \infty} \frac{\sum n}{n^2} = \frac{1}{2} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\sum n^2}{n^3} = \frac{1}{3}.$$

Example: 24 $\lim_{n \rightarrow \infty} \left[\frac{1}{1-n^2} + \frac{2}{1-n^2} + \dots + \frac{n}{1-n^2} \right]$ is equal to

- (a) 0 (b) $-\frac{1}{2}$ (c) $\frac{1}{2}$ (d) None of these

Solution: (b) $\lim_{n \rightarrow \infty} \left[\frac{1}{1-n^2} + \frac{2}{1-n^2} + \dots + \frac{n}{1-n^2} \right] = \lim_{n \rightarrow \infty} \frac{\sum n}{1-n^2} = \frac{1}{2} \lim_{n \rightarrow \infty} \frac{n^2+n}{1-n^2} = -\frac{1}{2}.$

Example: 25 If $f(x) = \frac{2}{x-3}$, $g(x) = \frac{x-3}{x+4}$ and $h(x) = -\frac{2(2x+1)}{x^2+x-12}$ then $\lim_{x \rightarrow 3} [f(x) + g(x) + h(x)]$ is

- (a) -2 (b) -1 (c) $-\frac{2}{7}$ (d) 0

Solution: (c) We have $f(x) + g(x) + h(x) = \frac{x^2-4x+17-4x-2}{x^2+x-12} = \frac{x^2-8x+15}{x^2+x-12} = \frac{(x-3)(x-5)}{(x-3)(x+4)}$

$$\therefore \lim_{x \rightarrow 3} [f(x) + g(x) + h(x)] = \lim_{x \rightarrow 3} \frac{(x-3)(x-5)}{(x-3)(x+4)} = -\frac{2}{7}.$$

Example: 26 If $\lim_{n \rightarrow \infty} \left[\frac{n!}{n^n} \right]^{1/n}$ equal

- (a) e (b) $\frac{1}{e}$ (c) $\frac{\pi}{4}$ (d) $\frac{4}{\pi}$

Solution: (b) Let $P = \lim_{n \rightarrow \infty} \left(\frac{n!}{n^n} \right)^{1/n} \Rightarrow P = \lim_{n \rightarrow \infty} \left(\frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdot \frac{4}{n} \cdot \dots \cdot \frac{n}{n} \right)^{1/n}$

$$\therefore \log P = \frac{1}{n} \lim_{n \rightarrow \infty} \left(\log \frac{1}{n} + \log \frac{2}{n} + \dots + \log \frac{n}{n} \right) \Rightarrow \log P = \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} \log \frac{r}{n}$$

$$\log P = \int_0^1 \log x \, dx = [x \log x - x]_0^1 = (-1) \Rightarrow P = \frac{1}{e}.$$

Example: 27 If $\lim_{x \rightarrow \infty} \left[\frac{x^3+1}{x^2+1} - (ax+b) \right] = 2$, then

- (a) $a=1$ and $b=1$ (b) $a=1$ and $b=-1$ (c) $a=1$ and $b=-2$ (d) $a=1$ and $b=2$

Solution: (c) $\lim_{x \rightarrow \infty} \left(\frac{x^3+1}{x^2+1} - (ax+b) \right) = 2 \Rightarrow \lim_{x \rightarrow \infty} \left(\frac{x^3(1+\frac{1}{x^3}) - bx^2 - ax + (1-b)}{x^2+1} \right) = 2$

$$\Rightarrow \lim_{x \rightarrow \infty} [x^3(1-a) - bx^2 - ax + (1-b)] = 2(x^2 + 1).$$

Comparing the coefficients of both sides, $1-a=0$ and $-b=2$ or $a=1, b=-2$.

Example: 28 $\lim_{x \rightarrow \infty} \frac{(x+1)^{10} + (x+2)^{10} + \dots + (x+100)^{10}}{x^{10} + 10^{10}}$ is equal to

- (a) 0 (b) 1 (c) 10 (d) 100

Solution: (d) $\lim_{x \rightarrow \infty} \frac{(x+1)^{10} + (x+2)^{10} + \dots + (x+100)^{10}}{x^{10} + 10^{10}} = \lim_{x \rightarrow \infty} \frac{x^{10} \left[\left(1 + \frac{1}{x}\right)^{10} + \left(1 + \frac{2}{x}\right)^{10} + \dots + \left(1 + \frac{100}{x}\right)^{10} \right]}{x^{10} \left[1 + \frac{10^{10}}{x^{10}} \right]} = 100.$

Example: 29 Let $f(x) = 4$ and $f(x) = 4$, then $\lim_{x \rightarrow 2} \frac{xf(2) - 2f(x)}{x-2}$ equals

- (a) 2 (b) -2 (c) -4 (d) 3

Solution: (c) $y = \lim_{x \rightarrow 2} \frac{xf(2) - 2f(x)}{x-2} \Rightarrow y = \lim_{x \rightarrow 2} \frac{xf(2) - 2f(2) + 2f(2) - 2f(x)}{x-2}$
 $\Rightarrow y = \lim_{x \rightarrow 2} \frac{-2f(x) + 2f(2) + xf(2) - 2f(2)}{(x-2)} \Rightarrow y = \lim_{x \rightarrow 2} -2 \frac{[f(x) - f(2)]}{x-2} + \lim_{x \rightarrow 2} \frac{f(2) \cdot (x-2)}{(x-2)}$
 $\Rightarrow y = -2 \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x-2} + f(2) \Rightarrow y = -2 \lim_{x \rightarrow 2} f(x) + f(2) = -8 + 4 = -4.$

(2) **Trigonometric limits** : To evaluate trigonometric limits the following results are very important.

(i) $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 = \lim_{x \rightarrow 0} \frac{x}{\sin x}$

(ii) $\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 = \lim_{x \rightarrow 0} \frac{x}{\tan x}$

(iii) $\lim_{x \rightarrow 0} \frac{\sin^{-1} x}{x} = 1 = \lim_{x \rightarrow 0} \frac{x}{\sin^{-1} x}$

(iv) $\lim_{x \rightarrow 0} \frac{\tan^{-1} x}{x} = 1 = \lim_{x \rightarrow 0} \frac{x}{\tan^{-1} x}$

(v) $\lim_{x \rightarrow 0} \frac{\sin x}{x} = \frac{\pi}{180}$

(vi) $\lim_{x \rightarrow 0} \cos x = 1$

(vii)

$\lim_{x \rightarrow a} \frac{\sin(x-a)}{x-a} = 1$ (viii) $\lim_{x \rightarrow a} \frac{\tan(x-a)}{x-a} = 1$

(ix) $\lim_{x \rightarrow a} \sin^{-1} x = \sin^{-1} a \mid a \mid \leq 1$

(x) $\lim_{x \rightarrow a} \cos^{-1} x = \cos^{-1} a \mid a \mid \leq 1$

(xi) $\lim_{x \rightarrow a} \tan^{-1} x = \tan^{-1} a, -\infty < a < \infty$

(xii) $\lim_{x \rightarrow \infty} \frac{\sin x}{x} = \lim_{x \rightarrow \infty} \frac{\cos x}{x} = 0$

(xiii)

$\lim_{x \rightarrow \infty} \frac{\sin(1/x)}{(1/x)} = 1$

Example: 30 $\lim_{x \rightarrow 1} (1-x) \tan\left(\frac{\pi x}{2}\right) =$

- (a) $\frac{\pi}{2}$ (b) π (c) $\frac{2}{\pi}$ (d) 0

Solution: (c) $\lim_{x \rightarrow 1} (1-x) \tan\left(\frac{\pi x}{2}\right)$, Put $1-x=y \Rightarrow$ as $x \rightarrow 1, y \rightarrow 0$

Thus $\lim_{y \rightarrow 0} y \tan \frac{\pi(1-y)}{2} = \lim_{y \rightarrow 0} \frac{2}{\pi} \cdot \frac{\left(\frac{\pi y}{2}\right)}{\tan\left(\frac{\pi y}{2}\right)} = \frac{2}{\pi} \times 1 = \frac{2}{\pi}.$

Example: 31 $\lim_{x \rightarrow 1} \frac{\sqrt{1-\cos 2(x-1)}}{x-1}$

- (a) Exists and it equal $\sqrt{2}$
 (b) Exists and it equals $-\sqrt{2}$

(c) Does not exist because $x-1 \rightarrow 0$

(d) Does not exist because left hand limit is not equal to right hand limit

Solution: (d) $f(1+) = \lim_{h \rightarrow 0} f(1+h) = \lim_{h \rightarrow 0} \frac{\sqrt{1-\cos 2h}}{h} = \lim_{h \rightarrow 0} \sqrt{2} \frac{\sinh}{h} = \sqrt{2}$

$$f(1-) = \lim_{h \rightarrow 0} f(1-h) = \lim_{h \rightarrow 0} \frac{\sqrt{1-\cos(-2h)}}{-h} = \lim_{h \rightarrow 0} \sqrt{2} \frac{\sinh}{-h} = -\sqrt{2}.$$

$\therefore$ limit does not exist because left hand limit is not equal to right hand limit.

Example: 32 $\lim_{x \rightarrow 0} \frac{(1-\cos 2x) \sin 5x}{x^2 \sin 3x} =$

- (a) $\frac{10}{3}$ (b) $\frac{3}{10}$ (c) $\frac{6}{5}$ (d) $\frac{5}{6}$

Solution: (a) $\lim_{x \rightarrow 0} \frac{2\sin^2 x \sin 5x}{x^2 \sin 3x} \cdot \frac{3x}{3x} \cdot \frac{5x}{5x} \cdot \frac{5x}{3x} = \lim_{x \rightarrow 0} \frac{2\sin^2 x}{x^2} \cdot \frac{3x}{\sin 3x} \cdot \frac{\sin 5x}{5x} \cdot \frac{5x}{3x} = 2 \cdot \frac{5}{3} = \frac{10}{3}.$

Example: 33 $\lim_{x \rightarrow 0} \frac{x^3}{\sin x^2} =$

- (a) 0 (b) $\frac{1}{3}$ (c) 3 (d) $\frac{1}{2}$

Solution: (a) $\lim_{x \rightarrow 0} \frac{x^3}{\sin x^2} = \lim_{x \rightarrow 0} \frac{x^2}{\sin x^2} \cdot x = \left(\lim_{x \rightarrow 0} \frac{x^2}{\sin x^2} \right) \left(\lim_{x \rightarrow 0} x \right) = 1 \cdot 0 = 0.$

Example: 34 $\lim_{x \rightarrow 0} \frac{\sin 3x + \sin x}{x} =$

- (a) $\frac{1}{3}$ (b) 3 (c) 4 (d) $\frac{1}{4}$

Solution: (c) $\lim_{x \rightarrow 0} \frac{\sin 3x + \sin x}{x} = \lim_{x \rightarrow 0} \frac{\sin 3x}{x} + \lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \cdot 3 + \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \cdot 3 + 1 = 4.$

Example: 35 If $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then $\lim_{x \rightarrow 0} f(x) =$

- (a) 1 (b) 0 (c) -1 (d) None of these

Solution: (b) $\lim_{x \rightarrow 0} x \sin \left(\frac{1}{x} \right) = \left(\lim_{x \rightarrow 0} x \right) \left(\lim_{x \rightarrow 0} \sin \frac{1}{x} \right) = 0 \times (\text{A number oscillating between } -1 \text{ and } 1) = 0.$

Example: 36 If $f(x) = \begin{cases} \frac{\sin [x]}{[x]}, & [x] \neq 0 \\ 0, & [x] = 0 \end{cases}$, then $\lim_{x \rightarrow 0} f(x)$ equals

- (a) 1 (b) 0 (c) -1 (d) Does not exist

Solution: (d) In closed interval of $x=0$ at right hand side $[x]=0$ and at left hand side $[x]=-1$. Also $[0]=0$.

Therefore function is defined as $f(x) = \begin{cases} \frac{\sin [x]}{[x]}, & (-1 \leq x < 0) \\ 0, & (0 \leq x < 1) \end{cases}$

$\therefore$ Left hand limit $= \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{\sin [x]}{[x]} = \frac{\sin(-1)}{-1} = \sin 1^c$

Right hand limit $= 0$, Hence, limit doesn't exist.

Example: 37 $\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3} =$

- (a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) $\frac{2}{3}$ (d) None of these

Solution: (a) $\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3} = \lim_{x \rightarrow 0} \frac{\sin x - \sin x \cos x}{x^3 \cos x} = \lim_{x \rightarrow 0} \frac{\sin x \left(2 \sin^2 \frac{x}{2} \right)}{x^3 \cos x} = \lim_{x \rightarrow 0} \left[\frac{\sin x}{x} \cdot \frac{2}{\cos x} \cdot \frac{\sin^2 \frac{x}{2}}{\left(\frac{x}{2} \right)^2} \cdot \frac{1}{4} \right] = \frac{1}{2}$

Example: 38 If $f(x) = \frac{\sin(e^{x-2} - 1)}{\log(x-1)}$, then $\lim_{x \rightarrow 2} f(x)$ is given by

- (a) -2 (b) -1 (c) 0 (d) 1

Solution: (d) $\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} \frac{\sin(e^{x-2} - 1)}{\log(x-1)} = \lim_{t \rightarrow 0} \frac{\sin(e^t - 1)}{\log(t+1)}$. (Putting $x = 2 + t$)

$$= \lim_{t \rightarrow 0} \frac{\sin(e^t - 1)}{e^t - 1} \cdot \frac{e^t - 1}{t} \cdot \frac{t}{\log(t+1)} = \lim_{t \rightarrow 0} \frac{\sin(e^t - 1)}{e^t - 1} \left(\frac{1}{t!} + \frac{t}{2!} + \dots \right) \left[\frac{1}{\left(1 - \frac{1}{2}t + \frac{1}{3}t^2 - \dots \right)} \right]$$

$$= 1.1.1 = 1 \quad [\because \text{As } t \rightarrow 0, e^t - 1 \rightarrow 0, \therefore \frac{\sin(e^t - 1)}{(e^t - 1)} = 1]$$

Example: 39 $\lim_{x \rightarrow \pi/2} \frac{a^{\cot x} - a^{\cos x}}{\cot x - \cos x} =$

- (a) $\log a$ (b) $\log 2$ (c) a (d) $\log x$

Solution: (a) $\lim_{x \rightarrow \pi/2} \left(\frac{a^{\cot x} - a^{\cos x}}{\cot x - \cos x} \right) = \lim_{x \rightarrow \pi/2} a^{\cos x} \left(\frac{a^{\cot x - \cos x} - 1}{\cot x - \cos x} \right)$
 $= a^{\cos(\pi/2)} \lim_{x \rightarrow \pi/2} \left(\frac{a^{\cot x - \cos x} - 1}{\cot x - \cos x} \right) = 1 \log a = \log a.$

Example: 40 If $f(x) = \begin{vmatrix} \sin x & \cos x & \tan x \\ x^3 & x^2 & x \\ 2x & 1 & 1 \end{vmatrix}$, then $\lim_{x \rightarrow 0} \frac{f(x)}{x^2}$ is

- (a) 3 (b) -1 (c) 0 (d) 1

Solution: (d) $f(x) = x(x-1)\sin x - (x^3 - 2x^2)\cos x - x^3 \tan x$

$$= x^2 \sin x - x^3 \cos x - x^3 \tan x + 2x^2 \cos x - x \sin x$$

$$\text{Hence, } \lim_{x \rightarrow 0} \frac{f(x)}{x^2} = \lim_{x \rightarrow 0} \left(\sin x - x \cos x - x \tan x + 2 \cos x - \frac{\sin x}{x} \right) = 0 - 0 - 0 + 2 - 1 = 1.$$

Example: 41 If $f(x) = \cot^{-1} \left(\frac{3x - x^3}{1 - 3x^2} \right)$ and $g(x) = \cos^{-1} \left(\frac{1 - x^2}{1 + x^2} \right)$, then $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{g(x) - g(a)}$, $0 < a < \frac{1}{2}$ is

- (a) $\frac{3}{2(1+a^2)}$ (b) $\frac{3}{2(1+x^2)}$ (c) $\frac{3}{2}$ (d) $-\frac{3}{2}$

Solution: (d) $f(x) = \cot^{-1} \left\{ \frac{3x - x^3}{1 - 3x^2} \right\}$ and $g(x) = \cos^{-1} \left\{ \frac{1 - x^2}{1 + x^2} \right\}$

Put $x = \tan \theta$ in both equation

$$f(\theta) = \cot^{-1} \left\{ \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta} \right\} = \cot^{-1} \{ \tan 3\theta \}$$

$$f(\theta) = \cot^{-1} \cot \left(\frac{\pi}{2} - 3\theta \right) = \frac{\pi}{2} - 3\theta \Rightarrow f(\theta) = -3 \quad \dots\dots(i)$$

$$\text{and } g(\theta) = \cos^{-1} \left\{ \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right\} = \cos^{-1} (\cos 2\theta) = 2\theta \Rightarrow g(\theta) = 2 \quad \dots\dots(ii)$$

$$\text{Now } \lim_{x \rightarrow a} \left(\frac{f(x) - f(a)}{g(x) - g(a)} \right) = \lim_{x \rightarrow a} \left(\frac{f(x) - f(a)}{x - a} \right) \cdot \frac{1}{\lim_{x \rightarrow a} \left(\frac{g(x) - g(a)}{x - a} \right)} = f'(x) \cdot \frac{1}{g'(x)} = -3 \times \frac{1}{2} = -\frac{3}{2}.$$

Example: 42 $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\left[1 - \tan\left(\frac{x}{2}\right)\right][1 - \sin x]}{\left[1 + \tan\left(\frac{x}{2}\right)\right][\pi - 2x]^3}$ is

(a) $\frac{1}{8}$ (b) 0 (c) $\frac{1}{32}$ (d) ∞

Solution: (c) $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)(1 - \sin x)}{(\pi - 2x)^3}$

Let $x = \frac{\pi}{2} + y$, then $y \rightarrow 0 \Rightarrow \lim_{y \rightarrow 0} \frac{\tan\left(\frac{-y}{2}\right)(1 - \cos y)}{(-2y)^3} = \lim_{y \rightarrow 0} \frac{-\tan\frac{y}{2} \cdot 2\sin^2\frac{y}{2}}{(-8)y^3} = \lim_{y \rightarrow 0} \frac{1}{32} \frac{\tan\frac{y}{2}}{\left(\frac{y}{2}\right)} \left[\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right]^2 = \frac{1}{32}.$

Example: 43 If $\lim_{x \rightarrow 0} \frac{[(a-n)x - \tan x] \sin nx}{x^2} = 0$, where n is non-zero real number, then a is equal to

(a) 0 (b) $\frac{n+1}{n}$ (c) n (d) $n + \frac{1}{n}$

Solution: (d) $\lim_{x \rightarrow 0} n \frac{\sin nx}{nx} \cdot \lim_{x \rightarrow 0} \left((a-n)n - \frac{\tan x}{x} \right) = 0 \Rightarrow n[(a-n)n - 1] = 0 \Rightarrow (a-n)n = 1 \Rightarrow a = n + \frac{1}{n}.$

(3) **Logarithmic limits** : To evaluate the logarithmic limits we use following formulae

(i) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \dots \dots$ to ∞ where $-1 < x \leq 1$ and expansion is true only if base is e .

(ii) $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$

(iii)

$\lim_{x \rightarrow e} \log_e x = 1$

(iv)

$\lim_{x \rightarrow 0} \frac{\log(1-x)}{x} = -1$

(v) $\lim_{x \rightarrow 0} \frac{\log_a(1+x)}{x} = \log_a e, a > 0, \neq 1$

Example: 44 $\lim_{h \rightarrow 0} \frac{\log_e(1+2h) - 2\log_e(1+h)}{h^2}$

(a) -1

(b) 1

(c) 2

(d) -2

Solution: (a) $\lim_{h \rightarrow 0} \frac{\log_e(1+2h) - 2\log_e(1+h)}{h^2} = \lim_{x \rightarrow a} \frac{\left(2h - \frac{(2h)^2}{2} + \frac{(2h)^3}{3} - \dots \dots \infty\right) - 2\left(h - \frac{h^2}{2} + \frac{h^3}{3} - \dots \dots\right)}{h^2}$

$= \lim_{h \rightarrow 0} \frac{-h^2 + 2h^3 - \dots}{h^2} = \lim_{h \rightarrow 0} \frac{h^2\{-1 + 2h - \dots\}}{h^2} = \lim_{h \rightarrow 0} \{-1 + 2h + \dots\} = -1.$

Example: 45 $\lim_{x \rightarrow a} \frac{\log\{1+(x-a)\}}{(x-a)} =$

(a) -1

(b) 2

(c) 1

(d) -2

Solution: (c) Let $x - a = y$, when $x \rightarrow a, y \rightarrow 0$,

$\therefore$ The given limit $= \lim_{y \rightarrow 0} \frac{\log(1+y)}{y} = 1.$

Example: 46 $\lim_{h \rightarrow 0} \frac{\log_{10}(1+h)}{h} =$

(a) 1

(b) $\log_{10} e$

(c) $\log_e 10$

(d) None of these

Solution: (b) $\lim_{h \rightarrow 0} \frac{\log_e(1+h)}{h} \cdot \frac{1}{\log_e 10} = \log_{10} e.$

Example: 47 If $\lim_{x \rightarrow 0} \frac{\log 8 + x - \log 8 - x}{x} = k$ then the value of k is

(a) 0

(b) $-\frac{1}{3}$

(c) $\frac{2}{3}$

(d) $-\frac{2}{3}$

Solution: (c) $\lim_{x \rightarrow 0} \frac{\log(3+x) - \log(3-x)}{x} = \lim_{x \rightarrow 0} \frac{\log\left(\frac{3+x}{3-x}\right)}{x} = \lim_{x \rightarrow 0} \frac{\log\left(\frac{1+(x/3)}{1-(x/3)}\right)}{x}$
 $= \lim_{x \rightarrow 0} \frac{\log(1+(x/3))}{x} - \lim_{x \rightarrow 0} \frac{\log(1-(x/3))}{x} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}.$

(4) Exponential limits :

(i) **Based on series expansion :** We use $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \dots \dots \infty$

To evaluate the exponential limits we use the following results –

(a) $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$

(b) $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log_e a$

(c) $\lim_{x \rightarrow 0} \frac{e^{\lambda x} - 1}{x} = \lambda \quad (\lambda \neq 0)$

(ii) **Based on the form 1^∞ :** To evaluate the exponential form 1^∞ we use the following results.

(a) If $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$, then $\lim_{x \rightarrow a} \{1 + f(x)\}^{1/g(x)} = e^{\lim_{x \rightarrow a} \frac{f(x)}{g(x)}}$, or

when $\lim_{x \rightarrow a} f(x) = 1$ and $\lim_{x \rightarrow a} g(x) = \infty$. Then $\lim_{x \rightarrow a} \{f(x)\}^{g(x)} = \lim_{x \rightarrow a} [1 + f(x) - 1]^{g(x)} = e^{\lim_{x \rightarrow a} (f(x)-1)g(x)}$

(b) $\lim_{x \rightarrow 0} (1+x)^{1/x} = e$ (c) $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$ (d) $\lim_{x \rightarrow 0} (1+\lambda x)^{1/x} = e^\lambda$ (e) $\lim_{x \rightarrow \infty} \left(1 + \frac{\lambda}{x}\right)^x = e^\lambda$

Note : $\square \lim_{x \rightarrow \infty} a^x = \begin{cases} \infty, & \text{if } a > 1 \\ 0, & \text{if } a < 1 \end{cases}$ i.e., $a^\infty = \infty$, if $a > 1$ and $a^\infty = 0$ if $a < 1$.

Example: 48 $\lim_{x \rightarrow 0} \frac{e^{\alpha x} - e^{\beta x}}{x} =$

(a) $\alpha + \beta$

(b) $\frac{1}{\alpha} + \beta$

(c) $\alpha^2 - \beta^2$

(d) $\alpha - \beta$

Solution: (d) $\lim_{x \rightarrow 0} \frac{e^{\alpha x} - e^{\beta x}}{x} = \lim_{x \rightarrow 0} \frac{(e^{\alpha x} - 1) - (e^{\beta x} - 1)}{x} = \lim_{x \rightarrow 0} \frac{e^{\alpha x} - 1}{x} - \lim_{x \rightarrow 0} \frac{e^{\beta x} - 1}{x} = \alpha - \beta.$

Example: 49 The value of $\lim_{x \rightarrow 0} \frac{e^x - (1+x)}{x^2}$ is

(a) 0

(b) $\frac{1}{2}$

(c) 1

(d) $\frac{1}{4}$

Solution: (b) $\lim_{x \rightarrow 0} \frac{e^x - (1+x)}{x^2} = \lim_{x \rightarrow 0} \frac{(1+x+\frac{x^2}{2!}+\dots) - (1+x)}{x^2} = \lim_{x \rightarrow 0} \frac{x^2\left(\frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \dots\right)}{x^2} = \frac{1}{2!} = \frac{1}{2}.$

Example: 50 $\lim_{x \rightarrow 0} \frac{a^x - 1}{\sqrt{1+x} - 1}$ is equal to

(a) $2\log_e a$

(b) $\frac{1}{2}\log_e a$

(c) $a\log_e 2$

(d) None of these

Solution: (a) $\lim_{x \rightarrow 0} \frac{a^x - 1}{\sqrt{1+x} - 1} = \lim_{x \rightarrow 0} \frac{a^x - 1}{\sqrt{1+x} - 1} \cdot \frac{\sqrt{1+x} + 1}{\sqrt{1+x} + 1} = \lim_{x \rightarrow 0} \frac{(a^x - 1)(\sqrt{1+x} + 1)}{1+x-1} = \lim_{x \rightarrow 0} \left(\frac{a^x - 1}{x}\right) \cdot (\sqrt{1+x} + 1)$
 $= \left(\lim_{x \rightarrow 0} \frac{a^x - 1}{x}\right) \cdot \left(\lim_{x \rightarrow 0} (\sqrt{1+x} + 1)\right) = (\log_e a) \cdot (2) = 2\log_e a.$

Example: 51 The value of $\lim_{x \rightarrow \infty} \left(\frac{x+3}{x+1}\right)^{x+2}$ is

(a) e^4

(b) 0

(c) 1

(d) e^2

Solution: (d) $\lim_{x \rightarrow \infty} \left(\frac{x+3}{x+1} \right)^{x+2} = \lim_{x \rightarrow \infty} \left(1 + \frac{2}{x+1} \right)^{\frac{x+1}{2} \cdot (x+2) \cdot \frac{2}{(x+1)}} = \lim_{x \rightarrow \infty} \left(1 + \frac{2}{x+1} \right)^{\frac{x+1}{2} \cdot 2 \cdot \left(\frac{1+\frac{2}{x+1}}{1+\frac{1}{x+1}} \right)} = e^{2 \lim_{x \rightarrow \infty} \left[\left(1 + \frac{2}{x} \right) / \left(1 + \frac{1}{x} \right) \right]} = e^2.$

Alternative method : $\lim_{x \rightarrow \infty} \left(\frac{x+3}{x+1} \right)^{x+2} = \lim_{x \rightarrow \infty} \left(1 + \frac{2}{x+1} \right)^{x+2} = e^{\lim_{x \rightarrow \infty} \frac{2}{x+1} (x+2)} = e^{\lim_{x \rightarrow \infty} 2 \cdot \frac{1+\frac{2}{x+1}}{1+\frac{1}{x+1}}} = e^2$

Example: 52 If a, b, c, d are positive, then $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{a+bx} \right)^{c+dx}$

(a) $e^{d/b}$

(b) $e^{c/a}$

(c) $e^{(c+d)/(a+b)}$

(d) e

Solution: (a) $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{a+bx} \right)^{c+dx} = \lim_{x \rightarrow \infty} \left\{ \left(1 + \frac{1}{a+bx} \right)^{a+bx} \right\}^{\frac{c+dx}{a+bx}} = e^{d/b} \left\{ \lim_{x \rightarrow \infty} \left(1 + \frac{1}{a+bx} \right)^{a+bx} = e \text{ and } \lim_{x \rightarrow \infty} \frac{c+dx}{a+bx} = \frac{d}{b} \right\}$

Alternative method : $e^{\lim_{x \rightarrow \infty} \left(\frac{1}{a+bx} \right) \left(\frac{c+dx}{1} \right)} = e^{d/b}.$

Example: 53 $\lim_{x \rightarrow 0} x^x =$

(a) 0

(b) 1

(c) e

(d) None of these

Solution: (b) Let $y = x^x \Rightarrow \log y = x \log x$; $\therefore \lim_{y \rightarrow 0} \log y = \lim_{x \rightarrow 0} x \log x = 0 = \log 1 \Rightarrow \lim_{x \rightarrow 0} x^x = 1$

Example: 54 The value of $\lim_{x \rightarrow 0} \frac{(1+x)^{1/x} - e + \frac{1}{2}ex}{x^2}$ is

(a) $\frac{11e}{24}$

(b) $\frac{-11e}{24}$

(c) $\frac{e}{24}$

(d) None of these

Solution: (a) $(1+x)^{1/x} = e^{\frac{1}{x} \log(1+x)} = e^{\frac{1}{x} \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \right)} = e^{1 - \frac{x}{2} + \frac{x^2}{3} - \dots} = e e^{-\frac{x}{2} + \frac{x^2}{3} - \dots}$

$$= e \left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \dots \right) + \frac{1}{2!} \left(-\frac{x}{2} + \frac{x^2}{3} - \dots \right)^2 + \dots \right] = e \left[1 - \frac{x}{2} + \frac{11}{24}x^2 - \dots \right]$$

$$\therefore \lim_{x \rightarrow 0} \frac{(1+x)^{1/x} - e + \frac{ex}{2}}{x^2} = \frac{11e}{24}$$

Example: 55 $\lim_{x \rightarrow 0} \frac{(1+x)^{1/x} - e}{x}$ equals

(a) $\pi/2$

(b) 0

(c) $2/e$

(d) $-e/2$

Solution: (d) $(1+x)^{1/x} = e^{\frac{1}{x} [\log(1+x)]} = e^{\frac{1}{x} \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \right)} = e^{\left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \dots \right)} = e e^{\left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \dots \right)}$

$$= e \left[1 + \frac{\left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \dots \right)}{1!} + \frac{\left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \dots \right)^2}{2!} + \dots \right] = \left[e - \frac{ex}{2} + \frac{11e}{24}x^2 - \dots \right]$$

$$\therefore \lim_{x \rightarrow 0} \frac{(1+x)^{1/x} - e}{x} = \lim_{x \rightarrow 0} \frac{e - \frac{ex}{2} + \frac{11e}{24}x^2 - \dots - e}{x} \Rightarrow \lim_{x \rightarrow 0} \left(-\frac{e}{2} - \frac{11e}{24}x + \dots \right) = -\frac{e}{2}.$$

Example: 56 $\lim_{m \rightarrow \infty} \left(\cos \frac{x}{m} \right)^m =$

(a) 0

(b) e

(c) $1/e$

(d) 1

Solution: (d) $\lim_{m \rightarrow \infty} \left(\cos \frac{x}{m} \right)^m = \lim_{m \rightarrow \infty} \left[1 + \left(\cos \frac{x}{m} - 1 \right) \right]^m = \lim_{m \rightarrow \infty} \left[1 - \left(-\cos \frac{x}{m} + 1 \right) \right]^m$

$$= \lim_{m \rightarrow \infty} \left[1 - 2 \sin^2 \frac{x}{2m} \right]^m = e^{\lim_{m \rightarrow \infty} \left(-2 \sin^2 \frac{x}{2m} \right) m} = e^{\lim_{m \rightarrow \infty} -2 \left(\frac{\sin \frac{x}{2m}}{\frac{x}{2m}} \right)^2 \left(\frac{x^2}{4m^2} \right) m} = e^{-2 \lim_{m \rightarrow \infty} \frac{x^2}{4m}} = e^0 = 1.$$

Example: 57 $\lim_{n \rightarrow \infty} \left(\frac{n^2 - n + 1}{n^2 - n - 1} \right)^{n(n-1)} =$

(a) e

(b) e^2

(c) e^{-1}

(d) 1

Solution: (b) $\lim_{n \rightarrow \infty} \left(\frac{n^2 - n + 1}{n^2 - n - 1} \right)^{n(n-1)} = \lim_{n \rightarrow \infty} \left(\frac{n(n-1) + 1}{n(n-1) - 1} \right)^{n(n-1)} = \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{1}{n(n-1)} \right)^{n(n-1)}}{\left(1 - \frac{1}{n(n-1)} \right)^{n(n-1)}} = \frac{e}{e^{-1}} = e^2.$

Alternative Method: $\lim_{n \rightarrow \infty} \left(1 + \frac{2}{n^2 - n - 1} \right)^{n(n-1)} = e^{\lim_{n \rightarrow \infty} \frac{2n(n-1)}{n^2 - n - 1}} = e^2.$

(5) L' Hospital's rule : If $f(x)$ and $g(x)$ be two functions of x such that

(i) $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$

(ii) Both are continuous at $x = a$

(iii) Both are differentiable at $x = a$.

(iv) $f(x)$ and $g'(x)$ are continuous at the point $x = a$, then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ provided that $g'(a) \neq 0$

Note : $\square$ The above rule is also applicable if $\lim_{x \rightarrow a} f(x) = \infty$ and $\lim_{x \rightarrow a} g(x) = \infty$.

$\square$ If $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ assumes the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and $f(x), g(x)$ satisfy all the condition

embodied in L' Hospital rule, we can repeat the application of this rule on $\frac{f(x)}{g'(x)}$ to get,

$\lim_{x \rightarrow a} \frac{f(x)}{g'(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g''(x)}$. Sometimes it may be necessary to repeat this process a number of times till our goal of evaluating limit is achieved.

Example: 58 $\lim_{x \rightarrow 0} \frac{1 - \cos mx}{1 - \cos nx} =$

(a) m/n

(b) n/m

(c) $\frac{m^2}{n^2}$

(d) $\frac{n^2}{m^2}$

Solution: (c) $\lim_{x \rightarrow 0} \frac{1 - \cos mx}{1 - \cos nx} = \lim_{x \rightarrow 0} \left\{ \frac{2 \sin^2 \frac{mx}{2}}{2 \sin^2 \frac{nx}{2}} \right\} = \lim_{x \rightarrow 0} \left[\left\{ \frac{\sin \frac{mx}{2}}{\frac{mx}{2}} \right\}^2 \cdot \frac{m^2 x^2}{4} \cdot \frac{1}{\left\{ \frac{\sin \frac{nx}{2}}{\frac{nx}{2}} \right\}^2} \cdot \frac{4}{n^2 x^2} \right] = \frac{m^2}{n^2} \times 1 = \frac{m^2}{n^2}$

Trick : Apply L-Hospital rule ,

$$\lim_{x \rightarrow 0} \frac{1 - \cos nx}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{n \sin nx}{n \sin x} = \lim_{x \rightarrow 0} \frac{n^2 \cos nx}{n^2 \cos x} = \frac{n^2}{n^2}.$$

Example: 59 The integer n for which $\lim_{x \rightarrow 0} \frac{(\cos x - 1)(\cos x - e^x)}{x^n}$ is a finite non-zero number is

- (a) 1 (b) 2 (c) 3 (d) 4

Solution: (c) n cannot be negative integer for then the limit = 0

$$\begin{aligned} \text{Limit} &= \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2} \cdot \frac{e^x - \cos x}{x^{n-2}}}{2^2 \left(\frac{x}{2}\right)^2} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{e^x - \cos x}{x^{n-2}} \quad (n \neq 1 \text{ for then the limit} = 0) \\ &= \frac{1}{2} \lim_{x \rightarrow 0} \frac{e^x + \sin x}{(n-2)x^{n-3}}. \text{ So, if } n=3, \text{ the limit is } \frac{1}{2(n-2)} \text{ which is finite. If } n=4, \text{ the limit is} \\ &\text{infinite.} \end{aligned}$$

Example: 60 Let $f: R \rightarrow R$ be such that $f(1) = 3$ and $f'(1) = 6$. Then $\lim_{x \rightarrow 0} \left\{ \frac{f(1+x)}{f(1)} \right\}^{\frac{1}{x}}$ equals

- (a) 1 (b) $e^{1/2}$ (c) e^2 (d) e^3

Solution: (c) $\lim_{x \rightarrow 0} \left\{ \frac{f(1+x)}{f(1)} \right\}^{\frac{1}{x}} = e^{\lim_{x \rightarrow 0} \frac{1}{x} [\log f(1+x) - \log f(1)]} = e^{\lim_{x \rightarrow 0} \frac{f'(1+x) / f(1+x)}{1}} = e^{\frac{f'(1)}{f(1)}} = e^{6/3} = e^2.$

Example: 61 $\lim_{\alpha \rightarrow \pi/4} \frac{\sin \alpha - \cos \alpha}{\alpha - \pi/4} =$

- (a) $\sqrt{2}$ (b) $1/\sqrt{2}$ (c) 1 (d) None of these

Solution: (a) $\lim_{\alpha \rightarrow \pi/4} \frac{\sin \alpha - \cos \alpha}{\alpha - \pi/4} \left(\frac{0}{0} \text{ form} \right) = \lim_{\alpha \rightarrow \pi/4} \frac{\cos \alpha + \sin \alpha}{1} \quad (\text{By 'L' Hospital rule})$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

Example: 62 $\lim_{x \rightarrow a} \frac{x^3 - a^3}{x^2 - a^2} =$

- (a) 0 (b) Not defined (c) $2a$ (d) $\frac{3a}{2}$

Solution: (d) $\lim_{x \rightarrow a} \frac{x^3 - a^3}{x^2 - a^2} \left(\frac{0}{0} \text{ form} \right) = \lim_{x \rightarrow a} \frac{3x^2}{2x} \quad (\text{By 'L' Hospital rule}) = \frac{3a^2}{2a} = \frac{3a}{2}.$

Example: 63 $\lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} =$

- (a) $1/2\sqrt{x}$ (b) $1/2\sqrt{h}$ (c) Zero (d) None of these

Solution: (a) $\lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \lim_{h \rightarrow 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} =$

$$\frac{1}{2\sqrt{x}}$$

Trick : Applying 'L' Hospital's rule, [Differentiating N^r and D^r with respect to h]

We get, $\lim_{h \rightarrow 0} \frac{\frac{1}{2\sqrt{x+h}} - 0}{1} = \frac{1}{2\sqrt{x}}.$

Example: 64 $\lim_{\alpha \rightarrow \beta} \frac{\sin^2 \alpha - \sin^2 \beta}{\alpha^2 - \beta^2}$

- (a) 0 (b) 1 (c) $\frac{\sin \beta}{\beta}$ (d) $\frac{\sin 2\beta}{2\beta}$

Solution: (d) $\lim_{\alpha \rightarrow \beta} \frac{\sin^2 \alpha - \sin^2 \beta}{\alpha^2 - \beta^2} = \lim_{\alpha - \beta \rightarrow 0} \frac{\sin(\alpha + \beta) \sin(\alpha - \beta)}{(\alpha + \beta)(\alpha - \beta)} = \lim_{\alpha - \beta \rightarrow 0} \frac{\sin(\alpha - \beta)}{(\alpha - \beta)} \cdot \lim_{\alpha - \beta \rightarrow 0} \frac{\sin(\alpha + \beta)}{(\alpha + \beta)} = \lim_{\alpha \rightarrow \beta} \frac{\sin(\alpha + \beta)}{(\alpha + \beta)} = \frac{\sin 2\beta}{2\beta}.$

Trick : By L' Hospital's rule, $\lim_{\alpha \rightarrow \beta} \frac{2\sin \alpha \cos \alpha}{2\alpha} = \frac{\sin 2\beta}{2\beta}.$

Example: 65 $\lim_{x \rightarrow 0} \frac{\tan 2x - x}{3x - \sin x}$ equals

(a) 2/3

(b) 1/3

(c) 1/2

(d) 0

Solution: (c) $\lim_{x \rightarrow 0} \frac{\tan 2x - x}{3x - \sin x} = \lim_{x \rightarrow 0} \left[\frac{2 \tan 2x}{3 - \frac{\sin x}{x}} - 1 \right] = \frac{1}{2}.$

Example: 66 If $G(x) = -\sqrt{25 - x^2}$, then $\lim_{x \rightarrow 1} \frac{G(x) - G(1)}{x - 1}$ equals

(a) 1/24

(b) 1/5

(c) $-\sqrt{24}$

(d) None of these

Solution: (d) $\lim_{x \rightarrow 1} \frac{G(x) - G(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{-\sqrt{25 - x^2} + \sqrt{24}}{x - 1}$ [Multiply both numerator and denominator by $(\sqrt{24} + \sqrt{25 - x^2})$]

$$= \lim_{x \rightarrow 1} \frac{x + 1}{\sqrt{24} + \sqrt{25 - x^2}} = \frac{1}{\sqrt{24}}$$

Alternative method: By L'-Hospital rule, $\lim_{x \rightarrow 1} \frac{G'(x)}{1} = \lim_{x \rightarrow 1} \frac{-1(-2x)}{2\sqrt{25 - x^2}} = \frac{1}{\sqrt{24}}$

Example: 67 If $f(a) = 2, f'(a) = 1, g(a) = 1, g'(a) = 2$, then $\lim_{x \rightarrow a} \frac{g(x)f(a) - g(a)f(x)}{x - a}$ equals

(a) -3

(b) $\frac{1}{3}$

(c) 3

(d) $-\frac{1}{3}$

Solution: (c) Applying L - Hospital's rule, we get, $\lim_{x \rightarrow a} \frac{g(x)f(a) - g(a)f(x)}{x - a} = \lim_{x \rightarrow a} \frac{g'(x)f(a) - g(a)f'(x)}{1} = g'(a)f(a) - g(a)f'(a) = 2 \times 2 - 1 \times (1) = 3.$

Example: 68 $\lim_{x \rightarrow 0} \frac{(1 + x)^n - 1}{x} =$

(a) n

(b) 1

(c) -1

(d) None of these

Solution: (a) $\lim_{x \rightarrow 0} \frac{(1 + nx + {}^nC_2 x^2 + \dots \text{higher powers of } x \text{ to } x^n) - 1}{x} = n$

Trick : Apply L- Hospital rule.

Example: 69 $\lim_{x \rightarrow 0} \frac{\sin x + \log(1 - x)}{x^2}$ is equal to

(a) 0

(b) $\frac{1}{2}$

(c) $-\frac{1}{2}$

(d) None of these

Solution: (c) Apply L- Hospital rule, we get, $\lim_{x \rightarrow 0} \frac{\cos x - \frac{1}{1-x}}{2x} = \lim_{x \rightarrow 0} \frac{-\sin x - \frac{1}{(1-x)^2}}{2} = -\frac{1}{2}$

Alternative method : $\lim_{x \rightarrow 0} \frac{\sin x + \log(1 - x)}{x^2} = \lim_{x \rightarrow 0} \left(\frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots}{x^2} \right) + \lim_{x \rightarrow 0} \left(\frac{-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots}{x^2} \right)$
 $\left(\ominus \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \text{ and } \log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots \right)$

Hence, $\lim_{x \rightarrow 0} \frac{-\frac{x^2}{2} - x^3 \left(\frac{1}{3!} + \frac{1}{3} \right) - \frac{x^4}{4} \dots}{x^2} = -\frac{1}{2}$.

Example: 70 $\lim_{x \rightarrow 0} \frac{xe^x - \log(1+x)}{x^2}$ equals

(a) $\frac{2}{3}$

(b) $\frac{1}{3}$

(c) $\frac{1}{2}$

(d) $\frac{3}{2}$

Solution: (d) Let $y = \lim_{x \rightarrow 0} \frac{xe^x - \log(1+x)}{x^2}$ $\left(\frac{0}{0} \text{ form} \right)$

Applying L-Hospital's rule, $y = \lim_{x \rightarrow 0} \frac{e^x + xe^x - \frac{1}{1+x}}{2x}$ $\left(\frac{0}{0} \text{ form} \right)$

$$y = \lim_{x \rightarrow 0} \frac{1}{2} \left[e^x + e^x + xe^x + \frac{1}{(1+x)^2} \right] = \lim_{x \rightarrow 0} \frac{1}{2} [1+1+0+1] = \frac{3}{2}$$

Example: 71 $\lim_{x \rightarrow 0} \frac{\sin^{-1} x - \tan^{-1} x}{x^3}$ is equal to

(a) 0

(b) 1

(c) -1

(d) $\frac{1}{2}$

Solution: (d) $\lim_{x \rightarrow 0} \frac{\sin^{-1} x - \tan^{-1} x}{x^3}$ $\left(\frac{0}{0} \text{ form} \right)$

Applying L-Hospital's rule,

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{\sqrt{1-x^2}} - \frac{1}{1+x^2}}{3x^2} \quad \left(\frac{0}{0} \text{ form} \right)$$

$$= \lim_{x \rightarrow 0} \frac{\frac{-1}{2} \times \frac{-2x}{(1-x^2)^{3/2}} + \frac{2x}{(1+x^2)^2}}{6x} = \lim_{x \rightarrow 0} \frac{1}{6} \left[\frac{1}{(1-x^2)^{3/2}} + \frac{2}{(1+x^2)^2} \right] = \frac{1}{2}$$

Example: 72 $\lim_{x \rightarrow 1} \frac{1 + \log x - x}{1 - 2x + x^2} =$

(a) 1

(b) -1

(c) 0

(d) $-\frac{1}{2}$

Solution: (d) Applying L-Hospital's rule, $\lim_{x \rightarrow 1} \frac{1 + \log x - x}{1 - 2x + x^2} = \lim_{x \rightarrow 1} \frac{\frac{1}{x} - 1}{-2 + 2x} = \lim_{x \rightarrow 1} \frac{1-x}{2x(x-1)}$

Again applying L-Hospital's rule, we get $\lim_{x \rightarrow 1} \frac{-1}{4x-2} = -\frac{1}{2}$

Example: 73 $\lim_{x \rightarrow 0} \frac{4^x - 9^x}{x(4^x + 9^x)} =$

(a) $\log\left(\frac{2}{3}\right)$

(b) $\frac{1}{2} \log\left(\frac{3}{2}\right)$

(c) $\frac{1}{2} \log\left(\frac{3}{2}\right)$

(d) $\log\left(\frac{3}{2}\right)$

Solution: (a) $y = \lim_{x \rightarrow 0} \frac{4^x - 9^x}{x(4^x + 9^x)}$ $\left(\frac{0}{0} \text{ form} \right)$

Using L-Hospital's rule, $y = \lim_{x \rightarrow 0} \frac{4^x \log 4 - 9^x \log 9}{(4^x + 9^x) + x(4^x \log 4 + 9^x \log 9)} \Rightarrow y = \frac{\log 4 - \log 9}{2} \Rightarrow$

$$y = \frac{\log\left(\frac{2}{3}\right)^2}{2} = \log\frac{2}{3}$$

Example: 74 If $f(a) = 2$, $f'(a) = 1$, $g(a) = -3$, $g'(a) = -1$, then $\lim_{x \rightarrow a} \frac{f(x)g(x) - f(a)g(a)}{x-a} =$

(a) 1

(b) 6

(c) -5

(d) -1

Solution: (a) $\lim_{x \rightarrow a} \frac{f(a)g(x) - f(x)g(a)}{x - a} \left(\frac{0}{0} \text{ form} \right)$

Using L-Hospital's rule, $\lim_{x \rightarrow a} \frac{f(a)g'(x) - f'(x)g(a)}{1 - 0} = f(a) \times g'(a) - f'(a) \times g(a) = 2 \times (-1) - 1 \times (-3) = 1.$

Example: 75 The value of $\lim_{x \rightarrow 7} \frac{2 - \sqrt{x-3}}{x^2 - 49}$ is

- (a) $\frac{2}{9}$ (b) $-\frac{2}{49}$ (c) $\frac{1}{56}$ (d) $-\frac{1}{56}$

Solution: (d) Applying L-Hospital's rule, $\lim_{x \rightarrow 7} \frac{0 - \frac{1}{2\sqrt{x-3}}}{2x} = \lim_{x \rightarrow 7} \frac{-1}{4x\sqrt{x-3}} = \frac{-1}{4.7\sqrt{7-3}} = \frac{-1}{56}.$

Example: 76 Let $f(a) = g(a) = k$ and their n^{th} derivatives $f^n(a), g^n(a)$ exist and are not equal for some n . If

$\lim_{x \rightarrow a} \frac{f(a)g(x) - f(x)g(a)}{g(x) - f(x)} = 4$, then the value of k is

- (a) 4 (b) 2 (c) 1 (d) 0

Solution: (a) $\lim_{x \rightarrow a} \frac{k g(x) - k f(x)}{g(x) - f(x)} = 4$

By L-Hospital's rule, $\lim_{x \rightarrow a} k \left[\frac{g'(x) - f'(x)}{g'(x) - f'(x)} \right] = 4, \therefore k = 4.$

Example: 77 The value of $\lim_{x \rightarrow 0} \left(\frac{\int_0^{x^2} \sec^2 t dt}{x \sin x} \right)$ is

- (a) 3 (b) 2 (c) 1 (d) 0

Solution: (c) $\lim_{x \rightarrow 0} \frac{\frac{d}{dx} \int_0^{x^2} \sec^2 t dt}{\frac{d}{dx} (x \sin x)} = \lim_{x \rightarrow 0} \frac{\sec^2 x^2 \cdot 2x}{\sin x + x \cos x}$ (By L' - Hospital's rule)

$$= \lim_{x \rightarrow 0} \frac{2 \sec^2 x^2}{\left(\frac{\sin x}{x} + \cos x \right)} = \frac{2 \times 1}{1 + 1} = 1.$$

Example: 78 $\lim_{x \rightarrow \pi/6} \left[\frac{3 \sin x - \sqrt{3} \cos x}{6x - \pi} \right]$

- (a) $\sqrt{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) $-\sqrt{3}$ (d) $-\frac{1}{\sqrt{3}}$

Solution: (b) Using L-Hospital's rule, $\lim_{x \rightarrow \pi/6} \frac{3 \cos x + \sqrt{3} \sin x}{6} = \frac{3 \cdot \frac{\sqrt{3}}{2} + \sqrt{3} \cdot \frac{1}{2}}{6} = \frac{1}{\sqrt{3}}.$

Example: 79 Given that $f(2) = 6$ and $f(1) = 4$, then $\lim_{h \rightarrow 0} \frac{f(2h+2+h^2) - f(2)}{f(h-h^2+1) - f(1)} =$

- (a) Does not exist (b) $-\frac{3}{2}$ (c) $\frac{3}{2}$ (d) 3

Solution: (d) $\lim_{h \rightarrow 0} \frac{f(2h+2+h^2) - f(2)}{f(h-h^2+1) - f(1)} = \lim_{h \rightarrow 0} \frac{f(2h+2+h^2)(2+2h)}{f(h-h^2+1)(1-2h)} = \frac{6 \times 2}{4 \times 1} = 3.$

ASSIGNMENT

BASIC LIMITS

Basic Level

1. $\lim_{x \rightarrow \infty} \frac{(3x-1)(2x+5)}{(x-3)(3x+7)}$ is equal to
 (a) 3 (b) 2 (c) -2 (d) 1
2. $\lim_{x \rightarrow \infty} \frac{2x^2+3x+4}{3x^2+3x+4}$ is equal to
 (a) $\frac{2}{3}$ (b) 1 (c) 0 (d) ∞
3. $\lim_{x \rightarrow a} f(x) \cdot g(x)$ exists if
 (a) $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exists (b) $\lim_{x \rightarrow a} f(x)g(x)$ exists
 (c) $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ exists (d) $\lim_{x \rightarrow a} f(x)g\left(\frac{1}{x}\right)$ exists
4. $\lim_{x \rightarrow \infty} \left[x - \sqrt{x^2 + x} \right] =$
 (a) $\frac{1}{2}$ (b) 1 (c) $-\frac{1}{2}$ (d) 0
5. If $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ exists then
 (a) Both $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ must exist (b) $\lim_{x \rightarrow a} f(x)$ not exist but $\lim_{x \rightarrow a} g(x)$ exists
 (c) Neither $\lim_{x \rightarrow a} f(x)$ nor $\lim_{x \rightarrow a} g(x)$ exists (d) $\lim_{x \rightarrow a} f(x)$ exist but $\lim_{x \rightarrow a} g(x)$ does not exist
6. Which of the following statement is not correct
 (a) $\lim_{x \rightarrow c} [f(x) + g(x)] = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x)$ (b) $\lim_{x \rightarrow c} [f(x) - g(x)] = \lim_{x \rightarrow c} f(x) - \lim_{x \rightarrow c} g(x)$
 (c) $\lim_{x \rightarrow c} [f(x) \cdot g(x)] = \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} g(x)$ (d) $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)}$
7. $\lim_{x \rightarrow 3^-} \frac{|x-3|}{x-3} =$
 (a) 1 (b) -1 (c) 0 (d) Does not exist
8. $\lim_{x \rightarrow 3^+} \frac{|x-3|}{x-3} =$
 (a) 1 (b) -1 (c) 0 (d) Does not exist
9. If $\lim_{x \rightarrow a} \phi(x) = a^3, a \neq 0$ then $\lim_{x \rightarrow a} \phi\left(\frac{x}{a}\right)$ is equal to
 (a) a^2 (b) $\frac{1}{a^2}$ (c) $\frac{1}{a^3}$ (d) a^3
10. If $\lim_{x \rightarrow a} \frac{a^x - x^a}{x^x - a^a} = -1$, then
 (a) $a = 1$ (b) $a = 0$ (c) $a = e$ (d) None of these
11. $\lim_{x \rightarrow 4} \left[\frac{x^{3/2} - 8}{x - 4} \right] =$
 (a) $\frac{3}{2}$ (b) 3 (c) $\frac{2}{3}$ (d) $\frac{1}{3}$

12. $\lim_{x \rightarrow 0} \left[\frac{\sqrt{a+x} - \sqrt{a-x}}{x} \right] =$
 (a) 1 (b) 0 (c) $\sqrt{a}$ (d) $\frac{1}{\sqrt{a}}$
13. $\lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 + 5x + 8}}{4x + 5}$ is equal to
 (a) $-1/2$ (b) 0 (c) $1/2$ (d) 1
14. $\lim_{x \rightarrow 4} \frac{3 - \sqrt{5+x}}{x-4}$ is equal to
 (a) $1/6$ (b) $-1/6$ (c) 0 (d) 1
15. $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 - 1}}{2x + 1}$ is equal to
 (a) 1 (b) 0 (c) -1 (d) $\frac{1}{2}$
16. $\lim_{x \rightarrow 2} \frac{x^2 - 3x + 2}{x^2 + x - 6}$ equals
 (a) $1/5$ (b) $2/5$ (c) 1 (d) 5
17. The value of $\lim_{\theta \rightarrow 0} \left(\frac{\sin \theta / 4}{\theta} \right)$ is
 (a) 0 (b) $\frac{1}{4}$ (c) 1 (d) Note in existence
18. $\lim_{x \rightarrow 0} \frac{x^2 - 2x}{2 \sin x}$ equals
 (a) 1 (b) -1 (c) 0 (d) None of these
19. $\lim_{x \rightarrow 0} \frac{\sin(2+x) - \sin(2-x)}{x} =$
 (a) $\sin 2$ (b) $2 \sin 2$ (c) $2 \cos 2$ (d) 2
20. $\lim_{n \rightarrow \infty} (3^n + 4^n)^{\frac{1}{n}} =$
 (a) 3 (b) 4 (c) ∞ (d) e
21. True statement for $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{2+3x} - \sqrt{2-3x}}$ is
 (a) Does not exist (b) Lies between 0 and $\frac{1}{2}$
 (c) Lies between $\frac{1}{2}$ and 1 (d) Greater than 1
22. $\lim_{x \rightarrow 1} \frac{(2x-3)(\sqrt{x}-1)}{2x^2 + x - 3} =$
 (a) $-\frac{1}{10}$ (b) $\frac{1}{10}$ (c) $-\frac{1}{8}$ (d) None of these
23. $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 8x + 3} - \sqrt{x^2 + 4x + 3}) =$
 (a) 0 (b) ∞ (c) 2 (d) $\frac{1}{2}$
24. $\lim_{x \rightarrow 0} \left(\frac{x^o}{\sin x^o} \right)$ equals
 (a) 1 (b) $\frac{\pi}{180}$ (c) $\frac{180}{\pi}$ (d) None of these

Advance Level

25. $\lim_{n \rightarrow \infty} \left[\frac{1^3 + 2^3 + 3^3 + \dots + n^3}{n^4} \right] =$
 (a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) None of these
26. $\lim_{n \rightarrow \infty} \left[\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{3n} \right] =$
 (a) 0 (b) $\log_e 4$ (c) $\log_e 3$ (d) $\log_e 2$
27. $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k}{n^2 + k^2} =$
 (a) $\left(\frac{1}{2}\right) \log 2$ (b) $\log 2$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$
28. The $\lim_{x \rightarrow 0} (\cos x)^{\cot x}$ is
 (a) 0 (b) 1 (c) $\frac{1}{3}$ (d) $\frac{2}{3}$
29. $\lim_{x \rightarrow 0} \frac{x \tan 2x - 2x \tan x}{(1 - \cos 2x)^2} =$
 (a) 2 (b) -2 (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$
30. If $f(x) = \sqrt{\frac{x - \sin x}{x + \cos^2 x}}$, then $\lim_{x \rightarrow \infty} f(x)$ is
 (a) 0 (b) ∞ (c) 1 (d) Not exist
31. If $f(x) = \begin{cases} x & ; x < 0 \\ 1 & ; x = 0 \\ x^2 & ; x > 0 \end{cases}$, then, $\lim_{x \rightarrow 0} f(x) =$
 (a) 0 (b) 1 (c) 2 (d) Does not exist
32. If $f(x) = \begin{cases} \sin x & , x \neq n\pi \\ 0 & , \text{otherwise} \end{cases}, n \in \mathbb{Z}$ $g(x) = \begin{cases} x^2 + 1 & , x \neq 0, 2 \\ 4 & , x = 0 \\ 5 & , x = 2 \end{cases}$, then $\lim_{x \rightarrow 0} g\{f(x)\} =$
 (a) 1 (b) 0 (c) $\frac{1}{2}$ (d) $\frac{1}{4}$
33. $\lim_{n \rightarrow \infty} \frac{1^p + 2^p + 3^p + \dots + n^p}{n^{p+1}} =$
 (a) $\frac{1}{p+1}$ (b) $\frac{1}{1-p}$ (c) $\frac{1}{p} - \frac{1}{(p-1)}$ (d) $\frac{1}{p+2}$
34. $\lim_{n \rightarrow \infty} \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n}$ equals
 (a) 2 (b) -1 (c) 1 (d) 3
35. $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{\sin^{-1} x} =$
 (a) 2 (b) 1 (c) -1 (d) None of these
36. $\lim_{x \rightarrow 0} \cos \frac{1}{x}$
 (a) Is continuous at $x=0$ (b) Differentiable at $x=0$ (c) Does not exist (d) None of these
37. If $x_n = \frac{1-2+3-4+5-6+\dots-2n}{\sqrt{n^2+1} + \sqrt{4n^2-1}}$, then $\lim_{n \rightarrow \infty} x_n$ is equal to
 (a) $\frac{1}{3}$ (b) $-\frac{1}{3}$ (c) $\frac{2}{3}$ (d) 1

38. $\lim_{n \rightarrow \infty} \left[\frac{1}{n} + \frac{1}{\sqrt{n^2 + n}} + \frac{1}{\sqrt{n^2 + 2n}} + \dots + \frac{1}{\sqrt{n^2 + (n-1)n}} \right]$ is equal to
 (a) $2 + 2\sqrt{2}$ (b) $2\sqrt{2} - 2$ (c) $2\sqrt{2}$ (d) 2
39. $\lim_{n \rightarrow \infty} \frac{1}{1^3 + n^3} + \frac{4}{2^3 + n^3} + \dots + \frac{1}{2n}$ is equal to
 (a) $\frac{1}{3} \log_e 3$ (b) $\frac{1}{3} \log_e 2$ (c) $\frac{1}{3} \log_e \frac{1}{3}$ (d) None of these
40. The value of $\lim_{n \rightarrow \infty} \left[\frac{n}{1+n^2} + \frac{n}{4+n^2} + \frac{n}{9+n^2} + \dots + \frac{1}{2n} \right]$ is equal to
 (a) e (b) $\frac{1}{e}$ (c) $\frac{\pi}{4}$ (d) $\frac{4}{\pi}$
41. $\lim_{x \rightarrow \infty} \frac{x^n}{e^x} = 0$ for
 (a) No value of n (b) n is any whole number (c) $n = 0$ only (d) $n = 2$ only

EXPONENTIAL AND LOGARITHMIC LIMITS

Basic Level

42. $\lim_{x \rightarrow 0} \left[\tan\left(\frac{\pi}{4} + x\right) \right]^{1/x}$ is equal to
 (a) e^{-1} (b) e (c) e^2 (d) $\sqrt{e}$
43. $\lim_{x \rightarrow 0} \left(\frac{3^x - 1}{x} \right)$ equals
 (a) $\log 3$ (b) $3 \log 3$ (c) $2 \log 3$ (d) None of these
44. $\lim_{x \rightarrow 0} \frac{\cos(\sin x) - 1}{x^2} =$
 (a) 1 (b) -1 (c) $1/2$ (d) $-1/2$
45. $\lim_{x \rightarrow 0} \frac{e^{1/x}}{\frac{1}{1+e^x}} =$
 (a) 0 (b) 1 (c) Does not exist (d) None of these
46. $\lim_{x \rightarrow 0} \frac{a^{\sin x} - 1}{b^{\sin x} - 1} =$
 (a) $\frac{a}{b}$ (b) $\frac{b}{a}$ (c) $\frac{\log a}{\log b}$ (d) $\frac{\log b}{\log a}$
47. $\lim_{x \rightarrow 0} \left(\frac{e^x - 1}{x} \right) =$
 (a) $\frac{1}{2}$ (b) ∞ (c) 1 (d) 0
48. $\lim_{x \rightarrow \infty} \left(\frac{x+3}{x+1} \right)^{x+1} =$
 (a) e^2 (b) e^3 (c) e (d) e^{-1}
49. $\lim_{x \rightarrow 0} (1 - ax)^{\frac{1}{x}} =$
 (a) e (b) e^{-a} (c) 1 (d) e^a

Advance Level

50. $\lim_{x \rightarrow \infty} \left(\frac{x-3}{x+2} \right)^x$ is equal to
 (a) e (b) e^{-1} (c) e^{-5} (d) e^5
51. $\lim_{x \rightarrow 0} \left(\frac{1+5x^2}{1+3x^2} \right)^{1/x^2}$ is
 (a) e^2 (b) e (c) e^{-1} (d) None of these
52. The value of $\lim_{x \rightarrow \infty} \left(\frac{x^2-2x+1}{x^2-4x+2} \right)^x$ is equal to
 (a) e^2 (b) e^{-2} (c) e^6 (d) None of these
53. $\lim_{x \rightarrow \infty} \left[1 + \frac{1}{mx} \right]^x$ equal to
 (a) $e^{1/m}$ (b) $e^{-1/m}$ (c) e^m (d) m^e
54. $\lim_{x \rightarrow \infty} \left(\frac{x^2+5x+3}{x^2+x+3} \right)^x =$
 (a) e^4 (b) e^2 (c) e^3 (d) e
55. $\lim_{x \rightarrow 0} \frac{x2^x - x}{1 - \cos x}$ is equal to
 (a) $\log 2$ (b) $\log 4$ (c) 0 (d) None of these
56. $\lim_{x \rightarrow 0} \left(\frac{a^x - b^x}{x} \right) =$
 (a) $\log \left(\frac{b}{a} \right)$ (b) $\log \left(\frac{a}{b} \right)$ (c) $\frac{a}{b}$ (d) $\log a^b$
57. $\lim_{x \rightarrow \infty} \frac{\log x^n - [x]}{[x]}, n \in \mathbb{N}$ ($[x]$ denotes greatest integer less than or equal to x)
 (a) Has value -1 (b) Has value 0 (c) Has value 1 (d) Does not exist
58. $\lim_{x \rightarrow 0} \frac{\sin(\cos^2 x)}{x^2} =$
 (a) $-\pi$ (b) π (c) $\frac{\pi}{2}$ (d) 1
59. $\lim_{x \rightarrow 0} \left\{ \frac{\sin x - x + \frac{x^3}{6}}{x^5} \right\} =$
 (a) $\frac{1}{120}$ (b) $-\frac{1}{120}$ (c) $\frac{1}{20}$ (d) None of these
60. $\lim_{x \rightarrow 0} \frac{\sqrt{\frac{1}{2}(1 - \cos 2x)}}{x} =$
 (a) 1 (b) -1 (c) 0 (d) Does not exist
61. $\lim_{x \rightarrow \pi/4} \frac{\sqrt{2} \cos x - 1}{\cot x - 1} =$

- (a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{1}{2\sqrt{2}}$ (d) 1

62. $\lim_{x \rightarrow a} \frac{\cos x - \cos a}{\cot x - \cot a} =$

- (a) $\frac{1}{2} \sin^3 a$ (b) $\frac{1}{2} \operatorname{cosec}^2 a$ (c) $\sin^3 a$ (d) $\operatorname{cosec}^3 a$

Advance Level

63. $\lim_{x \rightarrow \infty} \sqrt{\frac{x + \sin x}{x - \cos x}} =$

- (a) 0 (b) 1 (c) -1 (d) None of these

64. The value of $\lim_{n \rightarrow \infty} \cos\left(\frac{x}{2}\right) \cos\left(\frac{x}{4}\right) \cos\left(\frac{x}{8}\right) \dots \cos\left(\frac{x}{2^n}\right)$ is

- (a) 1 (b) $\frac{\sin x}{x}$ (c) $\frac{x}{\sin x}$ (d) None of these

65. If x is a real number in $[0, 1]$, then the value of $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} [1 + \cos^{2m}(n! \pi x)]$ is given by

- (a) 2 or 1 according as x is rational or irrational
(b) 1 or 2 according as x is rational or irrational
(c) 1 for all x (d) 2 or 1 for all x

66. $\lim_{x \rightarrow 0} \frac{1 - \cos 5x}{2^x - 3^x}$ is equal to

- (a) 0 (b) 1 (c) 5 (d) None of these

67. If $\lim_{x \rightarrow 0} \frac{x^n - \sin x^n}{x - \sin^n x}$ is non zero definite, then n must be

- (a) 1 (b) 2 (c) 3 (d) None of these

68. The values of a and b such that $\lim_{x \rightarrow 0} \frac{x(1 + a \cos x) - b \sin x}{x^3} = 1$, are

- (a) $\frac{5}{2}, \frac{3}{2}$ (b) $\frac{5}{2}, -\frac{3}{2}$ (c) $-\frac{5}{2}, -\frac{3}{2}$ (d) None of these

69. $\lim_{h \rightarrow 0} \frac{2 \left[\sqrt{3} \sin\left(\frac{\pi}{6} + h\right) - \cos\left(\frac{\pi}{6} + h\right) \right]}{\sqrt{3}h(\sqrt{3} \cosh - \sinh)} =$

- (a) $-\frac{2}{3}$ (b) $-\frac{3}{4}$ (c) $-2\sqrt{3}$ (d) $\frac{4}{3}$

70. If $f(x) = \int \frac{2 \sin x - \sin 2x}{x^3} dx$ then $\lim_{x \rightarrow 0} f(x)$ equals

- (a) 0 (b) 1 (c) -1 (d) 1/2

71. $\lim_{x \rightarrow 0} \frac{x}{\tan^{-1}(2x)}$ is equal to

- (a) 0 (b) 1 (c) $\frac{1}{2}$ (d) None of these

72. $\lim_{x \rightarrow 0} \frac{e^{\tan x} - e^x}{\tan x - x}$ is equal to

- (a) 0 (b) 1 (c) e (d) $\frac{1}{e}$

73. $\lim_{x \rightarrow \pi/2} (\sec x - \tan x)$ equals

- (a) 0 (b) 1 (c) -1 (d) None of these

74. $\lim_{x \rightarrow 0} \frac{2\sin^2 3x}{x^2} =$
 (a) 0 (b) 1 (c) 18 (d) 36
75. $\lim_{x \rightarrow 0} \frac{\sin 2x}{x} =$
 (a) 0 (b) 1 (c) 1/2 (d) 2
76. If $f(1) = 1$ and $f'(1) = 4$, then the value of $\lim_{x \rightarrow 1} \frac{\sqrt{f(x)} - 1}{\sqrt{x} - 1}$ is
 (a) 9 (b) 4 (c) 12 (d) 1
77. $\lim_{x \rightarrow 0} \frac{\log_e(1+x)}{3^x - 1} =$
 (a) $\log_e 3$ (b) 0 (c) 1 (d) $\log_3 e$
78. $\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{\sin x} =$
 (a) 0 (b) 1 (c) 2 (d) Non existent
79. $\lim_{x \rightarrow -2} \frac{\sin^{-1}(x+2)}{x^2 + 2x} =$
 (a) 0 (b) ∞ (c) $-\frac{1}{2}$ (d) None of these
80. $\lim_{x \rightarrow a} \frac{(x^{-1} - a^{-1})}{x - a} =$
 (a) $\frac{1}{a}$ (b) $-\frac{1}{a}$ (c) $\frac{1}{a^2}$ (d) $-\frac{1}{a^2}$
81. $\lim_{x \rightarrow 1} \frac{\log x}{x-1}$ is equal to
 (a) 1 (b) 0 (c) -1 (d) 1/2
82. $\lim_{x \rightarrow 1} \frac{1 - x^{-1/3}}{1 - x^{-2/3}} =$
 (a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) $-\frac{2}{3}$

Advance Level

83. The value of $\lim_{x \rightarrow 0} \frac{x \cos x - \log(1+x)}{x^2}$ is
 (a) $\frac{1}{2}$ (b) 0 (c) 1 (d) None of these
84. $\lim_{x \rightarrow \frac{\pi}{4}} \left(\frac{1 - \tan x}{1 - \sqrt{2} \sin x} \right)$ equals
 (a) 0 (b) 1 (c) -2 (d) 2
85. $\lim_{x \rightarrow 0} \frac{2^x - 1}{\sqrt{1+x} - 1}$ equals
 (a) $\log 2$ (b) $\log 4$ (c) $\log 3$ (d) None of these
86. $\lim_{x \rightarrow 0} \left[\frac{\sin(x+a) + \sin(a-x) - 2\sin a}{x \sin x} \right]$ is equal to
 (a) $\sin a$ (b) $-\sin a$ (c) 1 (d) 0
87. $\lim_{x \rightarrow \pi/2} \frac{\int_{\pi/2}^x t dt}{\sin(2x - \pi)}$ is equal to
 (a) ∞ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{8}$
88. $\lim_{y \rightarrow 0} \frac{(x+y)\sec(x+y) - x\sec x}{y} =$
 (a) $\sec x (x \tan x + 1)$ (b) $x \tan x + \sec x$ (c) $x \sec x + \tan x$ (d) None of these

89. $\lim_{x \rightarrow \frac{\pi}{3}} \frac{\sin\left(\frac{\pi}{3} - x\right)}{2\cos x - 1}$ is equal to
 (a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\sqrt{3}$ (d) $\frac{2}{\sqrt{3}}$
90. $\lim_{x \rightarrow 0} \frac{e^{x^2} - \cos x}{x^2}$ is equal to
 (a) $\frac{3}{2}$ (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) None of these
91. $\lim_{x \rightarrow 0} \frac{x \cos x - \sin x}{x^2 \sin x}$ equals
 (a) $\frac{1}{3}$ (b) $-\frac{1}{3}$ (c) 3 (d) -3

ANSWER

ASSIGNMENT (BASIC & ADVANCE LEVEL)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
b	a	a	c	a	d	b	a	d	a	b	d	c	b	d	a	b	b	c	b
21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
b	a	c	a	c	c	a	b	c	c	d	a	a	c	b	c	b	b	b	c
41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
b	c	a	d	d	c	c	a	b	c	a	a	a	a	b	b	a	b	a	d
61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
b	c	b	b	a	a	a	c	d	b	c	b	a	c	d	b	d	c	c	d
81	82	83	84	85	86	87	88	89	90	91									
a	b	a	d	b	b	c	a	b	a	b									