# INVERSE TRIGNOMETRICAL FUNCTIONS

The inverse of a function $f: A \to B$ exists if $f$ is one-one onto *i.e.*, a bijection and is given by $f(x) = y \Rightarrow f^{-1}(y) = x$.

Consider the sine function with domain $R$ and range [–1, 1]. Clearly this function is not a bijection and so it is not invertible. If we restrict the domain of it in such a way that it becomes one–one, then it would become invertible. If we consider sine as a function with domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and co-domain [–1, 1], then it is a bijection and therefore, invertible. The inverse of sine function is defined as $\sin^{-1} x = \theta \Leftrightarrow \sin\theta = x$, where $\theta \leftarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $x \leftarrow [-1, 1]$.

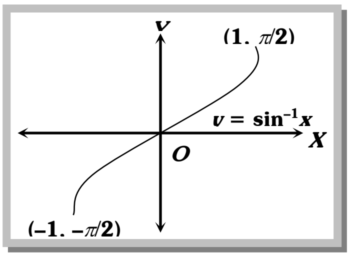
## 5.1 Properties of Inverse Trigonometric Functions

(1) **Meaning of inverse function**

(i) $\sin\theta = x \Rightarrow \sin^{-1} x = \theta$ (ii) $\cos\theta = x \Rightarrow \cos^{-1} x = \theta$ (iii) $\tan\theta = x \Rightarrow \tan^{-1} x = \theta$

(iv) $\cot\theta = x \Rightarrow \cot^{-1} x = \theta$ (v) $\sec\theta = x \Rightarrow \sec^{-1} x = \theta$ (vi) $\text{cosec}\,\theta = x \Rightarrow \text{cosec}^{-1} x = \theta$

(2) **Domain and range of inverse functions**



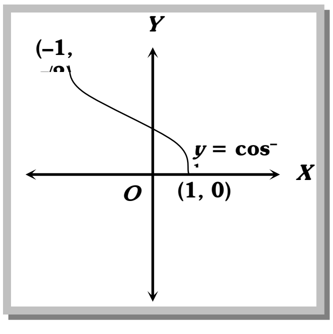

(i) If $\sin y = x$, then $y = \sin^{-1} x$, under certain condition.

$-1 \le \sin y \le 1$; but $\sin y = x$. $\therefore -1 \le x \le 1$

Again, $\sin y = -1 \Rightarrow y = -\frac{\pi}{2}$ and $\sin y = 1 \Rightarrow y = \frac{\pi}{2}$.

Keeping in mind numerically smallest angles or real numbers. $\therefore -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

These restrictions on the values of $x$ and $y$ provide us with the domain and range for the function $y = \sin^{-1} x$.




*i.e.*, Domain : $x \in [-1, 1]$

Range: $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) Let $\cos y = x$, then $y = \cos^{-1} x$, under certain conditions $-1 \le \cos y \le 1$

$\Rightarrow -1 \le x \le 1$

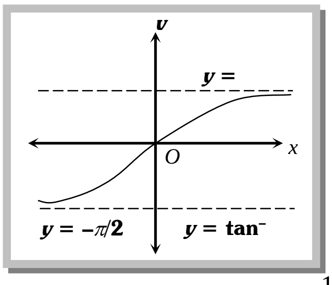
$\cos y = -1 \Rightarrow y = \pi$

$\cos y = 1 \Rightarrow y = 0$

$\therefore$ $0 \le y \le \pi$ {as cos $x$ is a decreasing function in $[0, \pi]$;

hence $\cos\pi \le \cos y \le \cos 0$

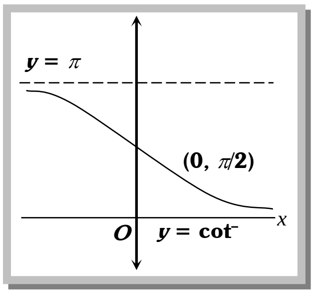
These restrictions on the values of $x$ and $y$ provide us the domain and range for the function $y = \cos^{-1} x$.




*i.e.* Domain: $x \in [-1, 1]$

Range : $y \in [0, \pi]$

(iii)If $\tan y = x$, then $y = \tan^{-1} x$, under certain conditions.




Here, $\tan y \in R \Rightarrow x \in R,\ -\infty < \tan y < \infty \Rightarrow -\frac{\pi}{2} < y < \frac{\pi}{2}$

Thus, Domain $x \in R$;

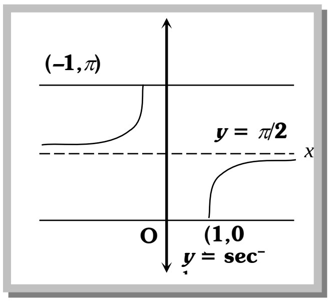
Range $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(iv) If $\cot y = x$, then $y = \cot^{-1} x$

under certain conditions, $\cot y \in R \Rightarrow x \in R$;

$-\infty < \cot y < \infty \Rightarrow 0 < y < \pi$

These conditions on $x$ and $y$ make the function, $\cot y = x$ one-one and onto so that the inverse function exists. *i.e.*, $y = \cot^{-1} x$ is meaningful.



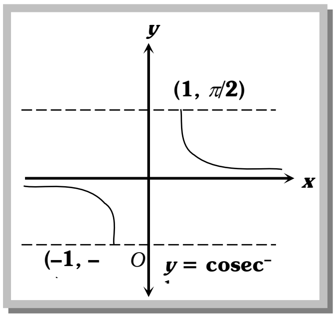

$\Rightarrow$ Domain : $x \in R$

Range : $y \in (0, \pi)$

(v) If $\sec y = x$, then $y = \sec^{-1} x$, where $|x| \geq 1$ and $0 \leq y \leq \pi, y \neq \frac{\pi}{2}$

Here, Domain: $x \in R - (-1, 1)$

Range: $y \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$




(vi) If $\operatorname{cosec} y = x$, then $y = \operatorname{cosec}^{-1} x$

Where $|x| \geq 1$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$

Here, Domain $\in R - (-1, 1)$

Range $\in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

| Function | Domain ($D$) | Range ($R$) |
|---|---|---|
| $\sin^{-1} x$ | $-1 \leq x \leq 1$ or $[-1, 1]$ | $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ or $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ |
| $\cos^{-1} x$ | $-1 \leq x \leq 1$ or $[-1, 1]$ | $0 \leq \theta \leq \pi$ or $[0, \pi]$ |
| $\tan^{-1} x$ | $-\infty < x < \infty$ *i.e.*, $x \in R$ or $(-\infty, \infty)$ | $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ or $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ |
| $\cot^{-1} x$ | $-\infty < x < \infty$ *i.e.*, $x \in R$ or $(-\infty, \infty)$ | $0 < \theta < \pi$ or $(0, \pi)$ |
| $\sec^{-1} x$ | $x \leq -1, x \geq 1$ or $(-\infty, -1] \cup [1, \infty)$ | $\theta \neq \frac{\pi}{2}, 0 \leq \theta \leq \pi$ or $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ |
| $\operatorname{cosec}^{-1} x$ | $x \leq -1, x \geq 1$ or $(-\infty, -1] \cup [1, \infty)$ | $\theta \neq 0, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ or $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ |

(3) $\sin^{-1}(\sin\theta) = \theta$, Provided that $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $\cos^{-1}(\cos\theta) = \theta$, Provided that $0 \leq \theta \leq \pi$

$\tan^{-1}(\tan\theta) = \theta$, Provided that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\cot^{-1}(\cot\theta) = \theta$, Provided that $0 < \theta < \pi$

$\sec^{-1}(\sec\theta) = \theta$, Provided that $0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$

$\text{cosec}^{-1}(\text{cosec}\,\theta) = \theta$, Provided that $-\frac{\pi}{2} \le \theta < 0$ or $0 < \theta \le \frac{\pi}{2}$

(4) $\sin(\sin^{-1} x) = x$, Provided that $-1 \le x \le 1$, $\cos(\cos^{-1} x) = x$, Provided that $-1 \le x \le 1$

tan $(\tan^{-1} x) = x$, Provided that $-\infty < x < \infty$ $\cot(\cot^{-1} x) = x$, Provided that $-\infty < x < \infty$

$\sec(\sec^{-1} x) = x$, Provided that $-\infty < x \le -1$ or $1 \le x < \infty$

$\text{cosec}(\text{cosec}^{-1} x) = x$, Provided that $-\infty < x \le -1$ or $1 \le x < \infty$

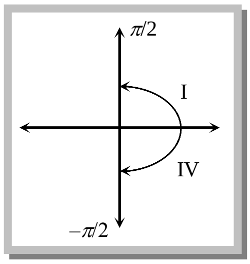
(5) $\sin^{-1}(-x) = -\sin^{-1} x$ $\cos^{-1}(-x) = \pi - \cos^{-1} x$, $\tan^{-1}(-x) = -\tan^{-1} x$

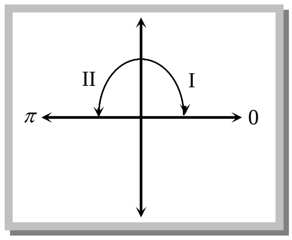
$\cot^{-1}(-x) = \pi - \cot^{-1} x$ $\sec^{-1}(-x) = \pi - \sec^{-1} x$ $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x$

(6) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, for all $x \in [-1, 1]$ $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$, for all $x \in R$

$\sec^{-1} x + \text{cosec}^{-1} x = \frac{\pi}{2}$, for all $x \in (-\infty, -1] \cup [1, \infty)$

## Important Tips




- Here; $\sin^{-1} x, \text{cosec}^{-1} x, \tan^{-1} x$ belong to I and IV Quadrant.
- Here; $\cos^{-1} x, \sec^{-1} x, \cot^{-1} x$ belong to I and II Quadrant.
- I Quadrant is common to all the inverse functions.
- III Quadrant is not used in inverse function.
- IV Quadrant is used in the clockwise direction i.e., $-\frac{\pi}{2} \le y \le 0$

(7) **Principal values for inverse circular functions**

| Principal values for $x \ge 0$ | Principal values for $x < 0$ |
|---|---|
| $0 \le \sin^{-1} x \le \frac{\pi}{2}$ | $-\frac{\pi}{2} \le \sin^{-1} x < 0$ |
| $0 \le \cos^{-1} x \le \frac{\pi}{2}$ | $\frac{\pi}{2} < \cos^{-1} x \le \pi$ |
| $0 \le \tan^{-1} x < \frac{\pi}{2}$ | $-\frac{\pi}{2} < \tan^{-1} x < 0$ |
| $0 < \cot^{-1} x \le \frac{\pi}{2}$ | $\frac{\pi}{2} < \cot^{-1} x < \pi$ |
| $0 \le \sec^{-1} x < \frac{\pi}{2}$ | $\frac{\pi}{2} < \sec^{-1} x \le \pi$ |
| $0 < \text{cosec}^{-1} x \le \frac{\pi}{2}$ | $-\frac{\pi}{2} \le \text{cosec}^{-1} x < 0$ |

Thus $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$, not $\frac{5\pi}{6}$; $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$ not $\frac{4\pi}{3}$; $\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$ not $\frac{2\pi}{3}$; $\cot^{-1}(-1) = \frac{3\pi}{4}$ not $-\frac{\pi}{4}$ etc.

*Note* : ❑ $\sin^{-1} x, \cos^{-1} x, \tan^{-1} x$ are also written as arc $\sin x$, arc $\cos x$ and arc $\tan x$ respectively.

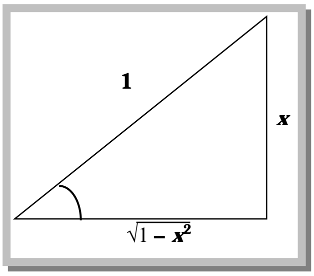
❑ It should be noted that if not otherwise stated only principal values of inverse circular functions are to be considered.

(8) **Conversion property :** Let, $\sin^{-1} x = y \Rightarrow x = \sin y \Rightarrow \text{cosec}\, y = \left(\frac{1}{x}\right) \Rightarrow y = \text{cosec}^{-1}\left(\frac{1}{x}\right)$

$$\sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x} = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \text{cosec}^{-1}\left(\frac{1}{x}\right)$$

$$\cos^{-1} x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \sec^{-1}\frac{1}{x} = \text{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$\tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \cot^{-1}\left(\frac{1}{x}\right) = \sec^{-1}\sqrt{1+x^2} = \text{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$$




*Note* : ❑ $\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1} x$, for all $x \in (-\infty, 1] \cup [1, \infty)$

❑ $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x$, for all $x \in (-\infty, 1] \cup [1, \infty)$

❑ $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x, & \text{for } x > 0 \\ -\pi + \cot^{-1} x, & \text{for } x < 0 \end{cases}$

(9) **General values of inverse circular functions:** We know that if □□□is the smallest angle whose sine is *x*, then all the angles whose sine is *x* can be written as $n\pi + (-1)^n\alpha$, where $n = 0, 1, 2, \ldots$. Therefore, the general value of $\sin^{-1} x$ can be taken as $n\pi + (-1)^n\alpha$. The general value of $\sin^{-1} x$ is denoted by $\sin^{-1} x$.

Thus, we have $\boxed{\sin^{-1} x = n\pi + (-1)^n\alpha, -1 \le x \le 1, \text{if } \sin\alpha = x \text{ and } -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}}$

Similarly, general values of other inverse circular functions are given as follows:

| | |
|---|---|
| $\cos^{-1} x = 2n\pi \pm \alpha, -1 \le x \le 1$; | If $\cos\alpha = x$, $0 \le \alpha \le \pi$ |
| $\tan^{-1} x = n\pi + \alpha$, $x \in R$; | If $\tan\alpha = x$, $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ |
| $\cot^{-1} x = n\pi + \alpha$, $x \in R$; | If $\cot\alpha = x$, $0 < \alpha < \pi$ |
| $\sec^{-1} x = 2n\pi \pm \alpha$, $x \ge 1$ or $x \le -1$; | If $\sec\alpha = x, 0 \le \alpha \le \pi$ and $\ne \frac{\pi}{2}$ |
| $\text{cosec}^{-1} x = n\pi + (-1)^n\alpha$, $x \ge 1$ or $x \le -1$; | If $\text{cosec}\,\alpha = x$, $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$ and $x \ne 0$ |

**Example: 1** The principal value of $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$ is

(a) $-\frac{2\pi}{3}$ (b) $-\frac{\pi}{3}$ (c) $\frac{4\pi}{3}$ (d) $\frac{5\pi}{8}$

**Solution:** (b) $\sin^{-1}\left[\sin\left(-\frac{\pi}{3}\right)\right] = -\frac{\pi}{3}$ $\left(\because -\frac{\pi}{2} < \sin^{-1} x < \frac{\pi}{2}\right)$

**Example: 2** $\sec^{-1}[\sec(-30^o)] =$

(a) $-60^o$ (b) $-30^o$ (c) $30^o$ (d) $150^o$

**Solution:** (c) $\sec^{-1}[\sec(-30^o)] = \sec^{-1}(\sec 30^o) = 30^o$.

**Example: 3** The principal value of $\sin^{-1}\left(\sin\frac{5\pi}{3}\right)$ is

(a) $\frac{5\pi}{3}$ (b) $-\frac{5\pi}{3}$ (c) $-\frac{\pi}{3}$ (d) $\frac{4\pi}{3}$

**Solution:** (c) $\sin^{-1}\left(\sin\frac{5\pi}{3}\right) = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$.

**Example: 4** The principal value of $\sin^{-1}\left[\sin\left(\frac{2\pi}{3}\right)\right]$ is

(a) $-\frac{2\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $\frac{4\pi}{3}$ (d) None of these

**Solution:** (d) The principal vlaue of $\sin^{-1}[\sin(\pi - \frac{2\pi}{3})]$ = $\sin^{-1}\sin\left(\frac{\pi}{3}\right) = \frac{\pi}{3}$.

**Example: 5** Considering only the principal values, if $\tan(\cos^{-1} x) = \sin\left[\cot^{-1}\left(\frac{1}{2}\right)\right]$, then *x* is equal to

(a) $\frac{1}{\sqrt{5}}$ (b) $\frac{2}{\sqrt{5}}$ (c) $\frac{3}{\sqrt{5}}$ (d) $\frac{\sqrt{5}}{3}$

**Solution:** (d) Put $\cot^{-1}\left(\frac{1}{2}\right) = \theta \Rightarrow \cot\theta = \frac{1}{2}$

$\therefore\ \sin\theta = \frac{2}{\sqrt{5}}$. Put $\cos^{-1} x = \phi$ then $x = \cos\phi$

Also □ $\tan\phi = \frac{2}{\sqrt{5}}$, $\therefore x = \cos\phi = \frac{\sqrt{5}}{3}$.

**Example: 6** If $\theta = \sin^{-1}[\sin(-600^o)]$, then one of the possible value of $\theta$ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{2\pi}{3}$ (d) $-\frac{2\pi}{3}$

**Solution:** (a) $\theta = \sin^{-1}[\sin(-600^o)]$ = $\sin^{-1}[-\sin(360 + 240)]$

$\Rightarrow \theta = \sin^{-1}[-\sin 240^o]$ = $\sin^{-1}[-\sin(180^o + 60^o)]$ $\Rightarrow \theta = \sin^{-1}\sin 60^o$ $= \sin^{-1}\left[\sin\left(\frac{\pi}{3}\right)\right] = \frac{\pi}{3}$.

**Example: 7** Value of $\cos^{-1}\left(\cos\frac{5\pi}{3}\right) + \sin^{-1}\left(\sin\frac{5\pi}{3}\right)$ is

(a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{2\pi}{3}$ (d) $\frac{10\pi}{3}$

**Solution:** (a) $\cos^{-1}\left(\cos\frac{5\pi}{3}\right) + \sin^{-1}\left(\sin\frac{5\pi}{3}\right)$ $= \cos^{-1}\left[\cos\left(2\pi - \frac{\pi}{3}\right)\right] + \sin^{-1}\left[\sin\left(2\pi - \frac{\pi}{3}\right)\right]$ $= \frac{\pi}{3} - \frac{\pi}{3} = 0$.

**Example: 8** The equation $2\cos^{-1} x + \sin^{-1} x = \frac{11\pi}{6}$ has

(a) No solution (b) Only one solution (c) Two solutions (d) Three solutions

**Solution:** (a) Given equation is $2\cos^{-1} x + \sin^{-1} x = \frac{11\pi}{6} \Rightarrow \cos^{-1} x + (\cos^{-1} x + \sin^{-1} x) = \frac{11\pi}{6}$

$\Rightarrow \cos^{-1} x = \frac{11\pi}{6} - \frac{\pi}{2}$ $\Rightarrow \cos^{-1} x = \frac{4\pi}{3}$, which is not possible as $\cos^{-1} x \in [0, \pi]$.

**Example: 9** If $\sin^{-1} x + \sin^{-1} y = \frac{2\pi}{3}$, then $\cos^{-1} x + \cos^{-1} y =$

(a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\pi$

**Solution:** (b) $\sin^{-1} x+\sin^{-1} y=\frac{2\pi}{3}$

$\Rightarrow \ \frac{\pi}{2}-\cos^{-1} x+\frac{\pi}{2}-\cos^{-1} y=\frac{2\pi}{3} \ \Rightarrow \ \cos^{-1} x+\cos^{-1} y=\frac{\pi}{3}$.

**Example: 10** If $\theta=\sin^{-1} x+\cos^{-1} x-\tan^{-1} x, \ x\geq 0$ then the smallest interval in which $\theta$ lies is

(a) $\frac{\pi}{2}\leq\theta\leq\frac{3\pi}{4}$ (b) $0\leq\theta\leq\frac{\pi}{4}$ (c) $-\frac{\pi}{4}\leq\theta\leq 0$ (d) $\frac{\pi}{4}\leq\theta\leq\frac{\pi}{2}$

**Solution:** (b) $\theta=\sin^{-1} x+\cos^{-1} x-\tan^{-1} x=\frac{\pi}{2}-\tan^{-1} x$

We know $-\frac{\pi}{2}<\tan^{-1} x<\frac{\pi}{2}\Rightarrow\frac{\pi}{2}>-\tan^{-1} x>-\frac{\pi}{2} \quad \therefore 0<\frac{\pi}{2}-\tan^{-1} x<\frac{\pi}{4}$.

**Example: 11** If $\sin^{-1}\left(x-\frac{x^2}{2}+\frac{x^3}{4}-\ldots\right)+\cos^{-1}\left(x^2-\frac{x^4}{2}+\frac{x^6}{4}-\ldots\right)=\frac{\pi}{2}$ for $0<|x|<\sqrt{2}$, then $x$ equals

(a) $\frac{1}{2}$ (b) 1 (c) $-\frac{1}{2}$ (d) – 1

**Solution:** (b) We know that $\sin^{-1} y+\cos^{-1} y=\frac{\pi}{2}, \ |y|\leq 1$

$\therefore$ According to question, $x-\frac{x^2}{2}+\frac{x^3}{4}-\ldots\ldots\ldots=x^2-\frac{x^4}{2}+\frac{x^6}{4}-\ldots\ldots\ldots$

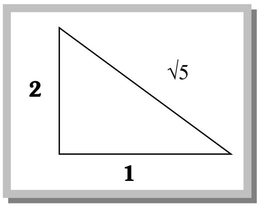
$\Rightarrow \ \frac{x}{1+\frac{x}{2}}=\frac{x^2}{1+\frac{x^2}{2}}$, $(\because 0<|x|<\sqrt{2})$ $\Rightarrow \ \frac{x}{2+x}=\frac{x^2}{2+x^2}\Rightarrow 2x+x^3=2x^2+x^3\Rightarrow \ x=x^2$

$\therefore x-x^2=0\Rightarrow x(1-x)=0\Rightarrow x=0$ and $x=1$, but $x\neq 0$. So, $x=1$.

**Example: 12** If $\sin^{-1} x+\cot^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{2}$, then $x$ is

(a) 0 (b) $\frac{1}{\sqrt{5}}$ (c) $\frac{2}{\sqrt{5}}$ (d) $\frac{\sqrt{3}}{2}$

**Solution:** (b) $\sin^{-1} x+\cot^{-1}\left(\frac{x}{2}\right)=\frac{\pi}{2}$ $\left(\because \cot^{-1}\frac{1}{2}=\cos^{-1}\frac{1}{\sqrt{5}}\right)$




$\sin^{-1} x+\cos^{-1}\frac{1}{\sqrt{5}}=\frac{\pi}{2}$; Clearly, $x=\frac{1}{\sqrt{5}}$.

**Example: 13** The value of $\sin(\cot^{-1} x)$ is

(a) $(1+x^2)^{3/2}$ (b) $(1+x^2)^{-3/2}$ (c) $(1+x^2)^{1/2}$ (d) $(1+x^2)^{-1/2}$

**Solution:** (d) $\sin(\cot^{-1} x)=\sin\left(\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right)=\frac{1}{\sqrt{1+x^2}}$.

**Example: 14** The number of real solutions of $\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2}$ is

(a) Zero (b) One (c) Two (d) Infinite

**Solution:** (c) $\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2}$

$\tan^{-1}\sqrt{x(x+1)}$ is defined, when $x(x+1)\geq 0$ ........(i)

$\sin^{-1}\sqrt{x^2+x+1}$ is defined, when $0\leq x(x+1)+1\leq 1$ or $0\leq x(x+1)\leq 0$ ........(ii)

From (i) and (ii), $x(x+1)=0$ or $x=0$ and –1.

Hence, number of solutions is 2.

## 5.2 Formulae for Sum and Difference of Inverse Trigonometric Function

(1) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$; If $x > 0, y > 0$ and $xy < 1$

(2) $\tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$; If $x > 0, y > 0$ and $xy > 1$

(3) $\tan^{-1} x + \tan^{-1} y = -\pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$; If $x < 0, y < 0$ and $xy > 1$

(4) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$; If $xy > -1$

(5) $\tan^{-1} - \tan^{-1} y = \pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ ; If $x > 0, y < 0$ and $xy < -1$

(6) $\tan^{-1} x - \tan^{-1} y = -\pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right)$; If $x < 0, y > 0$ and $xy < -1$

(7) $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$

(8) $\tan^{-1} x_1 + \tan^{-1} x_2 + \text{..........} + \tan^{-1} x_n = \tan^{-1}\left[\frac{S_1 - S_3 + S_5 - \text{..........}}{1 - S_2 + S_4 - S_6 + \text{........}}\right]$ ,

where $S_k$ denotes the sum of the products of $x_1, x_2, \text{........}, x_n$ taken $k$ at a time.

(9) $\cot^{-1} x + \cot^{-1} y = \cot^{-1} \frac{xy-1}{y+x}$

(10) $\cot^{-1} x - \cot^{-1} y = \cot^{-1} \frac{xy+1}{y-x}$

(11) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$;

If $-1 \le x, y \le 1$ and $x^2 + y^2 \le 1$ or if $xy < 0$ and $x^2 + y^2 > 1$

(12) $\sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$, If $0 < x, y \le 1$ and $x^2 + y^2 > 1$

(13) $\sin^{-1} x + \sin^{-1} y = -\pi - \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$, If $-1 \le x; y < 0$ and $x^2 + y^2 > 1$

(14) $\sin^{-1} x - \sin^{-1} y = \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}$, If $-1 \le x; y \le 1$ and $x^2 + y^2 \le 1$ if or $xy > 0$ and $x^2 + y^2 > 1$.

(15) $\sin^{-1} x - \sin^{-1} y = \pi - \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}$, If $0 < x \le 1, -1 \le y < 0$ and $x^2 + y^2 > 1$.

(16) $\sin^{-1} x - \sin^{-1} y = -\pi - \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}$, If $-1 \le x < 0, 0 < y \le 1$ and $x^2 + y^2 > 1$.

(17) $\cos^{-1} x + \cos^{-1} y = \cos^{-1}\{xy - \sqrt{1-x^2}.\sqrt{1-y^2}\}$, If $-1 \le x, y \le 1$ and $x + y \ge 0$.

(18) $\cos^{-1}x+\cos^{-1}y=2\pi-\cos^{-1}\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\}$, If $-1\le x,y\le 1$ and $x+y\le 0$

(19) $\cos^{-1}x-\cos^{-1}y=\cos^{-1}\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\}$, If $-1\le x,y\le 1$, and $x\le y$.

(20) $\cos^{-1}x-\cos^{-1}y=-\cos^{-1}\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\}$, If $-1\le y\le 0$, $0<x\le 1$ and $x\ge y$.

## Important Tips

- If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2}$, then $xy+yz+zx=1$.
- If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, then $x+y+z=xyz$.
- If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{\pi}{2}$, then $x^2+y^2+z^2+2xyz=1$.
- If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\pi$, then $x\sqrt{1-x^2}+y\sqrt{1-y^2}+z\sqrt{1-z^2}=2xyz$.
- If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=3\pi$, then $xy+yz+zx=3$.
- If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, then $x^2+y^2+z^2+2xyz=1$.
- If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$, then $xy+yz+zx=3$.
- If $\sin^{-1}x+\sin^{-1}y=\theta$, then $\cos^{-1}x+\cos^{-1}y=\pi-\theta$.
- If $\cos^{-1}x+\cos^{-1}y=\theta$, then $\sin^{-1}x+\sin^{-1}y=\pi-\theta$.
- If $\tan^{-1}x+\tan^{-1}y=\frac{\pi}{2}$, then $xy=1$.
- If $\cot^{-1}x+\cot^{-1}y=\frac{\pi}{2}$, then $xy=1$.
- If $\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b}=\theta$, then $\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\theta+\frac{y^2}{b^2}=\sin^2\theta$.

**Example: 15** The value of $\tan\left[\sin^{-1}\left(\frac{3}{5}\right)+\cos^{-1}\left(\frac{3}{\sqrt{13}}\right)\right]$ is

(a) $\frac{6}{17}$ (b) $\frac{6}{\sqrt{13}}$ (c) $\frac{\sqrt{13}}{5}$ (d) $\frac{17}{6}$

**Solution:** (d) $\tan\left[\sin^{-1}\left(\frac{3}{5}\right)+\cos^{-1}\left(\frac{3}{\sqrt{13}}\right)\right]=\tan\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{2}{3}\right)$

$$=\tan\left(\tan^{-1}\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}.\frac{2}{3}}\right)=\tan\left[\tan^{-1}\frac{17}{12}\times\frac{12}{6}\right]=\frac{17}{6}.$$

**Example: 16** $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=$

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\pi$

**Solution:** (b) $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}.\frac{1}{3}}=\tan^{-1}1=\frac{\pi}{4}$.

**Example: 17** If $\sin^{-1}\frac{1}{3}+\sin^{-1}\frac{2}{3}=\sin^{-1}x,$ then $x$ is equal to

(a) 0 (b) $\frac{\sqrt{5}-4\sqrt{2}}{9}$ (c) $\frac{\sqrt{5}+4\sqrt{2}}{9}$ (d) $\frac{\pi}{2}$

**Solution:** (c) $\sin^{-1}\frac{1}{3}+\sin^{-1}\frac{2}{3}=\sin^{-1}\left[\frac{1}{3}\sqrt{1-\frac{4}{9}}+\frac{2}{3}\sqrt{1-\frac{1}{9}}\right]=\sin^{-1}\left[\frac{\sqrt{5}+4\sqrt{2}}{9}\right]$

Therefore, $x=\frac{\sqrt{5}+4\sqrt{2}}{9}$.

**Example: 18** $\sin^{-1}\frac{1}{\sqrt{5}}+\cot^{-1}3$ is equal to

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

**Solution:** (b) $\sin^{-1}\frac{1}{\sqrt{5}}+\cot^{-1}3=\cot^{-1}\left(\frac{\sqrt{1-\frac{1}{5}}}{\frac{1}{\sqrt{5}}}\right)+\cot^{-1}3=\cot^{-1}(2)+\cot^{-1}(3)=\cot^{-1}\left(\frac{2\times 3-1}{3+2}\right)=\cot^{-1}(1)=\frac{\pi}{4}$.

**Example: 19** If $\sin^{-1}\frac{3}{5}+\cos^{-1}\left(\frac{12}{13}\right)=\sin^{-1}C,$ then $C=$

(a) $\frac{65}{56}$ (b) $\frac{24}{65}$ (c) $\frac{16}{65}$ (d) $\frac{56}{65}$

**Solution:** (d) Given, $\sin^{-1}C=\sin^{-1}\frac{3}{5}+\cos^{-1}\frac{12}{13}$

$\therefore\ \sin^{-1}C=\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}=\sin^{-1}\left\{\frac{3}{5}\sqrt{1-\frac{25}{169}}+\frac{5}{13}\sqrt{1-\frac{9}{25}}\right\}=\sin^{-1}\left(\frac{56}{65}\right)\Rightarrow C=\frac{56}{65}$.

**Example: 20** If $f(x)=\cos^{-1}x+\cos^{-1}\left\{\frac{x}{2}+\frac{1}{2}\sqrt{3-3x^2}\right\},$ then

(a) $f\left(\frac{2}{3}\right)=\frac{\pi}{3}$ (b) $f\left(\frac{2}{3}\right)=2\cos^{-1}\frac{2}{3}-\frac{\pi}{3}$ (c) $f\left(\frac{1}{3}\right)=\frac{\pi}{3}$ (d) $f\left(\frac{1}{3}\right)=2\cos^{-1}\frac{1}{3}-\frac{\pi}{3}$

**Soltuion:** (a,d) $f(x)=\cos^{-1}x+\cos^{-1}\left\{\frac{1}{2}x+\frac{\sqrt{3}}{2}.\sqrt{1-x^2}\right\}$

$=\cos^{-1}x\pm(\cos^{-1}\frac{1}{2}-\cos^{-1}x)$, according as $\cos^{-1}\frac{1}{2}>$ or $<\cos^{-1}x$

$=\cos^{-1}\frac{1}{2}$ if $\cos^{-1}\frac{1}{2}>\cos^{-1}x,$ which holds for $x=\frac{2}{3}$

$=2\cos^{-1}x-\cos^{-1}\frac{1}{2}$ if $\cos^{-1}\frac{1}{2}<\cos^{-1}x,$ which holds for $x=\frac{1}{3}$.

**Example: 21** $\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{63}{16}=$

(a) 0 (b) $\frac{\pi}{2}$ (c) $\pi$ (d) $\frac{3\pi}{2}$

**Solution:** (c) $\tan^{-1}\frac{12}{5}+\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{63}{16}=\pi+\tan^{-1}\frac{48+15}{20-36}+\tan^{-1}\frac{63}{16}\ (xy>1)=\pi-\tan^{-1}\frac{63}{16}+\tan^{-1}\frac{63}{16}=\pi$.

**Example: 22** If $\alpha=\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{1}{3}$ and $\beta=\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{1}{3}$, then

(a) $\alpha<\beta$ (b) $\alpha=\beta$ (c) $\alpha>\beta$ (d) None of these

**Solution:** (a) $\alpha=\sin^{-1}\left[\frac{4}{5}\sqrt{1-\frac{1}{9}}+\frac{1}{3}\sqrt{1-\frac{16}{25}}\right]=\sin^{-1}\left[\frac{8\sqrt{2}}{15}+\frac{3}{15}\right]=\sin^{-1}\left(\frac{8\sqrt{2}+3}{15}\right)$

Since $\frac{8\sqrt{2}+3}{15} < 1, \therefore \alpha < \frac{\pi}{2}$

$\beta = \left(\frac{\pi}{2} - \sin^{-1}\frac{4}{5} + \frac{\pi}{2} - \sin^{-1}\frac{1}{3}\right) = (\pi - \alpha) > \frac{\pi}{2} \Rightarrow \alpha < \beta$ .

**Example: 23** If $\cos^{-1}\frac{x}{2} + \cos^{-1}\frac{y}{3} = \theta$, then $9x^2 - 12xy\cos\theta + 4y^2 =$

(a) $36\sin^2\theta$ (b) $36\cos^2\theta$ (c) $36\tan^2\theta$ (d) None of these

**Solution:** (a) $\cos^{-1}\frac{x}{2} + \cos^{-1}\frac{y}{3} = \theta$

$\Rightarrow \frac{x}{2}.\frac{y}{3} - \sqrt{\left(1-\frac{x^2}{4}\right)}\sqrt{\left(1-\frac{y^2}{9}\right)} = \cos\theta$

$\therefore (xy - 6\cos\theta)^2 = (4-x^2)(9-y^2) \Rightarrow 9x^2 - 12xy\cos\theta + 4y^2 = 36(1-\cos^2\theta) = 36\sin^2\theta$ .

**Example: 24** The number of solutions of $\sin^{-1}x + \sin^{-1}2x = \frac{\pi}{3}$ is

(a) 0 (b) 1 (c) 2 (d) Infinitie

**Solution:** (b) $\sin^{-1}2x = \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}x = \sin^{-1}\left[\frac{\sqrt{3}}{2}.\sqrt{1-x^2} - x\sqrt{1-\frac{3}{4}}\right]$

$\therefore 2x = \frac{\sqrt{3}}{2}\sqrt{1-x^2} - \frac{x}{2}$

$\therefore \left(\frac{5x}{2}\right)^2 = \frac{3}{4}(1-x^2)$ or $28x^2 = 3 \Rightarrow x = \sqrt{\frac{3}{28}} = \frac{1}{2}\sqrt{\frac{3}{7}}$ , (not $-\frac{1}{2}\sqrt{\frac{3}{7}}$ ) .

**Example: 25** If $\cos^{-1}\left(\frac{x}{a}\right) + \cos^{-1}\left(\frac{y}{b}\right) = \alpha$, then $\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2} =$

(a) $\sin^2\alpha$ (b) $\cos^2\alpha$ (c) $\tan^2\alpha$ (d) $\cot^2\alpha$

**Solution:** (a) We have $\cos^{-1}\left[\frac{x}{a}.\frac{y}{b} - \sqrt{\left(1-\frac{x^2}{a^2}\right)}\sqrt{\left(1-\frac{y^2}{b^2}\right)}\right] = \alpha \Rightarrow \frac{xy}{ab} - \sqrt{\left(1-\frac{x^2}{a^2}\right)}\sqrt{\left(1-\frac{y^2}{b^2}\right)} = \cos\alpha$

$\therefore \left(\frac{xy}{ab} - \cos\alpha\right)^2 = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{x^2y^2}{a^2b^2}$

$\frac{x^2y^2}{a^2b^2} + \cos^2\alpha - \frac{2xy}{ab}\cos\alpha = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{x^2y^2}{a^2b^2} \Rightarrow \frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2} = 1 - \cos^2\alpha = \sin^2\alpha$ .

**Example: 26** If $a$, $b$, $c$ be positive real numbers and the value of $\theta = \tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}}$ $+ \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$, then $\tan\theta$ is

(a) 0 (b) 1 (c) $a+b+c$ (d) None of these

**Solution:** (a) $\theta = \tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}} + \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$

Let $s^2 = \frac{a+b+c}{abc}$ $\therefore \theta = \tan^{-1}\sqrt{a^2s^2} + \tan^{-1}\sqrt{b^2s^2} + \tan^{-1}\sqrt{c^2s^2}$

$\Rightarrow \theta = \tan^{-1}(as) + \tan^{-1}(bs) + \tan^{-1}(cs) \Rightarrow \theta = \tan^{-1}\left[\frac{as+bs+cs-abcs^3}{1-abs^2-bcs^2-cas^2}\right]$

$\Rightarrow \tan\theta = s\left[\dfrac{(a+b+c)-abcs^2}{1-(ab+bc+ca)s^2}\right]=0$   $[\because abcs^2=(a+b+c)]$

**Trick :** Since it is an identity so it will be true for any value of *a*,*b*,c. Let $a=b=c=1$ then

$\theta=\tan^{-1}\sqrt{3}+\tan^{-1}\sqrt{3}+\tan^{-1}\sqrt{3}=\pi,\quad \tan\theta=0.$

**Example: 27** All possible values of *p* and *q* for which $\cos^{-1}\sqrt{p}+\cos^{-1}\sqrt{1-p}+\cos^{-1}\sqrt{1-q}=\dfrac{3\pi}{4}$ holds, is

(a) $p=-1, q=\dfrac{1}{2}$   (b) $q>1, p=\dfrac{1}{2}$   (c) $0\le p\le 1, q=\dfrac{1}{2}$   (d) None of these

**Solution:** (c) $\cos^{-1}\sqrt{p}+\cos^{-1}\sqrt{1-p}=\dfrac{3\pi}{4}-\cos^{-1}\sqrt{1-q} \Rightarrow \cos^{-1}\sqrt{p}+\cos^{-1}\sqrt{1-p}=\cos^{-1}\left(\dfrac{-1}{\sqrt{2}}\right)-\cos^{-1}\sqrt{1-q}$

$\Rightarrow \sqrt{p}\sqrt{1-p}-\sqrt{1-p}\sqrt{p}=-\left[\dfrac{1}{\sqrt{2}}.\sqrt{1-q}-\dfrac{1}{\sqrt{2}}.\sqrt{q}\right]\Rightarrow 0=\sqrt{1-q}-\sqrt{q} \Rightarrow q=\dfrac{1}{2}.$

## 5.3 Inverse Trigonometric Ratios of Multiple Angles

(1) $2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$, If $-\dfrac{1}{\sqrt{2}}\le x\le\dfrac{1}{\sqrt{2}}$

(2) $2\sin^{-1}x=\pi-\sin^{-1}(2x\sqrt{1-x^2})$, If $\dfrac{1}{\sqrt{2}}\le x\le 1$

(3) $2\sin^{-1}x=-\pi-\sin^{-1}(2x\sqrt{1-x^2})$, If $-1\le x\le\dfrac{-1}{\sqrt{2}}$

(4) $3\sin^{-1}x=\sin^{-1}(3x-4x^3)$, If $\dfrac{-1}{2}\le x\le\dfrac{1}{2}$

(5) $3\sin^{-1}x=\pi-\sin^{-1}(3x-4x^3)$, If $\dfrac{1}{2}<x\le 1$

(6) $3\sin^{-1}x=-\pi-\sin^{-1}(3x-4x^3)$, If $-1\le x<-\dfrac{1}{2}$

(7) $2\cos^{-1}x=\cos^{-1}(2x^2-1)$, If $0\le x\le 1$

(8) $2\cos^{-1}x=2\pi-\cos^{-1}(2x^2-1)$, if $-1\le x\le 0$

(9) $3\cos^{-1}x=\cos^{-1}(4x^3-3x)$ If $\dfrac{1}{2}\le x\le 1$

(10) $3\cos^{-1}x=2\pi-\cos^{-1}(4x^3-3x)$, If $-\dfrac{1}{2}\le x\le\dfrac{1}{2}$

(11) $3\cos^{-1}x=2\pi+\cos^{-1}(4x^3-3x)$, If $-1\le x\le-\dfrac{1}{2}$

(12) $2\tan^{-1}x=\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$, if $-1<x\le 1$

(13) $2\tan^{-1}x=\pi+\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$, If $x>1$

(14) $2\tan^{-1}x=-\pi+\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$, If $x<-1$

(15) $2\tan^{-1}x=\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$, If $-1\le x\le 1$

(16) $2\tan^{-1}x=\pi-\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$, If $x>1$

(17) $2\tan^{-1}x=-\pi-\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$, If $x<-1$

(18) $2\tan^{-1}x=\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$, If $0\le x<\infty$

(19) $2\tan^{-1}x=-\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$, If $-\infty<x\le 0$

(20) $3\tan^{-1}x=\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$, If $\dfrac{-1}{\sqrt{3}}<x<\dfrac{1}{\sqrt{3}}$

(21) $3\tan^{-1}x=\pi+\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$, If $x>\dfrac{1}{\sqrt{3}}$

(22) $3\tan^{-1}x=-\pi+\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$, If $x<-\dfrac{1}{\sqrt{3}}$

(23) $\tan^{-1}\left[\dfrac{x}{\sqrt{a^2-x^2}}\right]=\sin^{-1}\dfrac{x}{a}$

(24) $\tan^{-1}\left[\dfrac{3a^2x-x^3}{a(a^2-3x^2)}\right]=3\tan^{-1}\dfrac{x}{a}$

(25) $\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$ (26) $\tan^{-1}\sqrt{\frac{1-x}{1+x}}=\frac{1}{2}\cos^{-1}x$

**Example: 28** $2\tan^{-1}(\cos x)=\tan^{-1}(\operatorname{cosec}^2 x)$, then $x=$

(a) $\frac{\pi}{2}$ (b) $\pi$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{3}$

**Solution:** (d) $2\tan^{-1}(\cos x)=\tan^{-1}(\operatorname{cosec}^2 x)$

$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right)=\tan^{-1}\left(\frac{1}{\sin^2 x}\right)\Rightarrow\frac{2\cos x}{\sin^2 x}=\frac{1}{\sin^2 x}\Rightarrow 2\cos x=1\Rightarrow x=\frac{\pi}{3}$.

**Example: 29** The solution set of the equation $\sin^{-1}x=2\tan^{-1}x$ is

(a) {1, 2} (b) $\{-1,2\}$ (c) $\{-1,1,0\}$ (d) $\{1,\frac{1}{2},0\}$

**Soltuion:** (c) $\sin^{-1}x=2\tan^{-1}x \Rightarrow \sin^{-1}x=\sin^{-1}\frac{2x}{1+x^2}\Rightarrow\frac{2x}{1+x^2}=x\Rightarrow x^3-x=0\Rightarrow x(x+1)(x-1)=0\Rightarrow x=\{-1,1,0\}$.

**Exmaple: 30** $\sin\left\{\tan^{-1}\left(\frac{1-x^2}{2x}\right)+\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right\}$ is equal to

(a) 0 (b) 1 (c) $\sqrt{2}$ (d) $\frac{1}{\sqrt{2}}$

**Solution:** (b) $\sin\left\{\left(\frac{\pi}{2}-2\tan^{-1}x\right)+2\tan^{-1}x\right\}=\sin\frac{\pi}{2}=1$.

**Example: 31** If $\sin^{-1}\frac{2a}{1+a^2}-\cos^{-1}\frac{1-b^2}{1+b^2}=\tan^{-1}\frac{2x}{1-x^2}$, then $x=$

(a) $a$ (b) $b$ (c) $\frac{a+b}{1-ab}$ (d) $\frac{a-b}{1+ab}$

**Solution:** (d) Put $a=\tan\theta, b=\tan\phi$ and $x=\tan\psi$, then reduced form is

$\sin^{-1}(\sin 2\theta)-\cos^{-1}(\cos 2\phi)=\tan^{-1}(\tan 2\psi)\Rightarrow 2\theta-2\phi=2\psi\Rightarrow\theta-\phi=\psi$

Taking tan on both sides, we get $\tan(\theta-\phi)=\tan\psi\Rightarrow\frac{\tan\theta-\tan\phi}{1+\tan\theta.\tan\phi}=\tan\psi$

Substituting these values, we get $\frac{a-b}{1+ab}=x$

**Example: 32** $\tan\left[2\tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right]=$

(a) $\frac{17}{7}$ (b) $-\frac{17}{7}$ (c) $\frac{7}{17}$ (d) $-\frac{7}{17}$

**Solution:** (d) $\tan\left[2\tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right]=\tan\left[\tan^{-1}\frac{\frac{2}{5}}{1-\frac{1}{25}}-\tan^{-1}(1)\right]=\tan\left[\tan^{-1}\frac{5}{12}-\tan^{-1}(1)\right]=\tan.\tan^{-1}\left(\frac{\frac{5}{12}-1}{1+\frac{5}{12}}\right)=\frac{-7}{17}$.

**Example: 33** $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) None of these

**Solution:** (c) $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=2\tan^{-1}\left[\frac{\frac{2}{5}}{1-\frac{1}{25}}\right]-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}$

$$= 2\tan^{-1}\left(\frac{5}{12}\right) - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99} = \tan^{-1}\left[\frac{\frac{5}{6}}{1-\frac{25}{144}}\right] - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$$

$$= \tan^{-1}\left(\frac{120}{119}\right) - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99} = \tan^{-1}\frac{120}{119} + \tan^{-1}\left[\frac{\frac{1}{99}-\frac{1}{70}}{1+\frac{1}{99}.\frac{1}{70}}\right] = \tan^{-1}\frac{120}{119} + \tan^{-1}\left(-\frac{29}{6931}\right)$$

$$= \tan^{-1}\frac{120}{119} - \tan^{-1}\frac{29}{6931} = \tan^{-1}\frac{120}{119} - \tan^{-1}\frac{1}{239} = \tan^{-1}\left[\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\times\frac{1}{239}}\right] = \tan^{-1}(1) = \frac{\pi}{4}.$$

**Example: 34** The value of $\sin\left(2\tan^{-1}\left(\frac{1}{3}\right)\right) + \cos(\tan^{-1} 2\sqrt{2}) =$

(a) $\frac{16}{15}$ (b) $\frac{14}{15}$ (c) $\frac{12}{15}$ (d) $\frac{11}{15}$

**Solution:** (b) $\sin\left[2\tan^{-1}\left(\frac{1}{3}\right)\right] + \cos[\tan^{-1}(2\sqrt{2})] = \sin\left[\tan^{-1}\frac{\frac{2}{3}}{1-\frac{1}{9}}\right] + \cos[\tan^{-1}(2\sqrt{2})]$

$$= \sin\left[\tan^{-1}\frac{3}{4}\right] + \cos[\tan^{-1} 2\sqrt{2}] = \sin\left[\sin^{-1}\frac{3}{5}\right] + \cos\left[\cos^{-1}\frac{1}{3}\right] = \frac{3}{5} + \frac{1}{3} = \frac{14}{15}.$$

**Example: 35** $\tan\left[\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right] + \tan\left[\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right]$ equal to

(a) $\frac{2a}{b}$ (b) $\frac{2b}{a}$ (c) $\frac{a}{b}$ (d) $\frac{b}{a}$

**Solution:** (b) Let $\cos^{-1}\frac{a}{b} = \theta \Rightarrow \cos\theta = \frac{a}{b}$

$\tan\left[\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right] + \tan\left[\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right] = \frac{1+t}{1-t} + \frac{1-t}{1+t}$, where $t = \tan\frac{\theta}{2}$ $= 2\frac{(1+t^2)}{1-t^2} = \frac{2}{\cos\theta} = \frac{2b}{a}$.

**Example: 36** $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$ is equal to

(a) $\pi$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

**Solution:** (d) Since, $2\tan^{-1} x = \tan^{-1}\frac{2x}{1-x^2}$

$$\therefore\ 4\tan^{-1}\frac{1}{5} = 2\left[2\tan^{-1}\frac{1}{5}\right] = 2\tan^{-1}\frac{\frac{2}{5}}{1-\frac{1}{25}} = 2\tan^{-1}\frac{10}{24} = \tan^{-1}\frac{\frac{20}{24}}{1-\frac{100}{576}} = \tan^{-1}\frac{120}{119}$$

So, $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = \tan^{-1}\frac{120}{119} - \tan^{-1}\frac{1}{239} = \tan^{-1}\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}.\frac{1}{239}} = \tan^{-1}\frac{(120\times 239)-119}{(119\times 239)+120} = \tan^{-1} 1 = \frac{\pi}{4}$.

**Example: 37** The formula $\cos^{-1}\frac{1-x^2}{1+x^2} = 2\tan^{-1} x$ holds only for

(a) $x \in R$ (b) $|x| \le 1$ (c) $x \in (-1, 1]$ (d) $x \in [1, +\infty)$

**Solution:** (d) If $x = -1$, LHS = $\frac{\pi}{2}$, RHS = $2 \times \left(-\frac{\pi}{2}\right)$. So, the fomula does not hold.

If $x < -1$, the angle on the LHS is in the second quadrant while the angle on the RHS is $2\times$ (angle in the fourth quadrant), which cannot be equal.

If $x > 1$, the angle on the LHS is in the second quadrant while the angle on the RHS is $2\times$ (angle in the first quadrant) and these two may be equal.

If $-1 < x < 0$, the angle on the LHS is positive and that on the RHS is negative and the two cannot be equal.

**Example: 38** $2\tan^{-1} x + \sin^{-1}\frac{2x}{1+x^2}$ is independent of $x$, then

(a) $x \in [1, +\infty)$ (b) $x \in [-1, 1]$ (c) $x \in (-\infty, -1]$ (d) None of these

**Solution:** (a) Let $x = \tan\theta$. Then $\sin^{-1}\frac{2x}{1+x^2} = \sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta} = \sin^{-1}(\sin 2\theta)$

$\therefore\ 2\tan^{-1} x + \sin^{-1}\frac{2x}{1+x^2} = 2\theta + \sin^{-1}(\sin 2\theta)$

If $-\frac{\pi}{2} \le 2\theta \le \frac{\pi}{2}$, $2\tan^{-1} x + \sin^{-1}\frac{2x}{1+x^2}$ = $2\theta + 2\theta = 4\tan^{-1} x \ne$ independent of $x$.

If $-\frac{\pi}{2} \le \pi - 2\theta \le \frac{\pi}{2}$, $2\tan^{-1} x + \sin^{-1}\frac{2x}{1+x^2}$ = $2\theta + \sin^{-1}[\sin(\pi - 2\theta)] = 2\theta + \pi - 2\theta =$ $\pi$ = independent of $x$.

$\therefore\ \theta \notin \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ but $\theta \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ and from the principal value of $\tan^{-1} x$.

$\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Hence, $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$\therefore\ \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \Rightarrow \tan^{-1} x + \sin^{-1}\frac{2x}{1+x^2} = \pi$.

Also at $\theta = \frac{\pi}{4}, 2\tan^{-1} x + \sin^{-1}\frac{2x}{1+x^2} = 2.\frac{\pi}{4} + \sin^{-1} 1 = \frac{\pi}{2} + \frac{\pi}{2} = \pi$.

$\therefore$ The given function = $\pi$ = constant if $\theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$. *i.e.*, $x \in [1, +\infty)$.

**Example: 39** The number of positive integral solutions of the equation $\tan^{-1} x + \cos^{-1}\frac{y}{\sqrt{1+y^2}} = \sin^{-1}\frac{3}{\sqrt{10}}$ or $\tan^{-1} x + \cot^{-1} y = \tan^{-1} 3$ is

(a) One (b) Two (c) Zero (d) None of these

**Solution:** (b) $\tan^{-1} x + \tan^{-1}\frac{1}{y} = \tan^{-1} 3$ or $\tan^{-1}\frac{1}{y} =$ $\tan^{-1} 3 - \tan^{-1} x$ or $\tan^{-1}\frac{1}{y} = \tan^{-1}\frac{3-x}{1+3x} \Rightarrow y = \frac{1+3x}{3-x}$

As $x$, $y$ are positive integers, $x = 1$, 2 and corresponding $y = 2$, 7

$\therefore$ Solutions are $(x, y) = (1, 2), (2, 7)$.

**Example: 40** $\alpha, \beta$ and $\gamma$ are three angles given by $\alpha = 2\tan^{-1}(\sqrt{2} - 1), \beta = 3\sin^{-1}\frac{1}{\sqrt{2}} + \sin^{-1}\left(-\frac{1}{2}\right)$ and $\gamma = \cos^{-1}\left(\frac{1}{3}\right)$. Then

(a) $\alpha > \beta$ (b) $\beta > \gamma$ (c) $\alpha < \gamma$ (d) None of these

**Solution:** (b,c) $\alpha = 2\tan^{-1}(\sqrt{2} - 1) = 2\tan^{-1}\tan\frac{\pi}{8} = 2 \times \frac{\pi}{8} = \frac{\pi}{4} = \cos^{-1}\frac{1}{\sqrt{2}}$

$\beta = 3.\frac{\pi}{4} - \frac{\pi}{6} = \frac{7\pi}{12}$

$\therefore\ \beta > \alpha$. Also, $\frac{1}{3} < \frac{1}{\sqrt{2}} \Rightarrow \cos^{-1}\frac{1}{3} > \cos^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$

So, $\gamma > \alpha$

Again $\cos^{-1}\left(\frac{1}{3}\right)$ belongs to the first quadrant and $\beta$ is in the second quadrant.

$\therefore \beta > \gamma$.

**Example: 41** $\frac{a^3}{2}\,cosec^2\left(\frac{1}{2}\tan^{-1}\frac{a}{b}\right)+\frac{b^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{b}{a}\right)$ is equal to

(a) $(a-b)(a^2+b^2)$ (b) $(a+b)(a^2-b^2)$ (c) $(a+b)(a^2+b^2)$ (d) None of these

**Soltuion:** (c) Let $\tan^{-1}\frac{a}{b}=\theta, \tan^{-1}\frac{b}{a}=\phi \Rightarrow \therefore \tan\theta=\frac{a}{b}, \tan\phi=\frac{b}{a}$

$$\frac{a^3}{2}\,cosec^2\left(\frac{1}{2}\tan^{-1}\frac{a}{b}\right)+\frac{b^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{b}{a}\right)$$

$$=\frac{a^3}{2\sin^2\left(\frac{\theta}{2}\right)}+\frac{b^3}{2\cos^2\left(\frac{\phi}{2}\right)}=\frac{a^3}{1-\cos\theta}+\frac{b^3}{1+\cos\phi}=\frac{a^3}{1-\frac{b}{\sqrt{a^2+b^2}}}+\frac{b^3}{1+\frac{a}{\sqrt{a^2+b^2}}}$$

$$=\sqrt{a^2+b^2}\left[\frac{a^3[\sqrt{a^2+b^2}+b]}{(a^2+b^2)-b^2}+\frac{b^3[\sqrt{a^2+b^2}-a]}{(a^2+b^2)-a^2}\right]$$ (rationalized)

$$=\sqrt{a^2+b^2}\,[a\{\sqrt{a^2+b^2}+b\}+b\{\sqrt{a^2+b^2}-a\}]=\sqrt{a^2+b^2}\,[\sqrt{a^2+b^2}\,(a+b)]=(a^2+b^2)(a+b).$$

# ASSIGNMENT

## PROPERTIES OF INVERSE TRIGNOMETRICAL FUNCTION

***Basic Level***

**1.** The domain of $\sin^{-1} x$ is

(a) $(-\pi, \pi)$ (b) [– 1, 1] (c) $(0, 2\pi)$ (d) $(-\infty, \infty)$

**2.** The range of $\tan^{-1} x$ is

(a) $\left(\pi, \frac{\pi}{2}\right)$ (b) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (c) $(-\pi, \pi)$ (d) $(0, \pi)$

**3.** $\sin^{-1} x + \cos^{-1} x$ is equal to

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) – 1 (d) 1

**4.** $\sin\left\{\sin^{-1}\frac{1}{2} + \cos^{-1}\frac{1}{2}\right\} =$

(a) 0 (b) – 1 (c) 2 (d) 1

**5.** The value of $\cos^{-1}\left(\cos\frac{5\pi}{3}\right) + \sin^{-1}\left(\cos\frac{5\pi}{3}\right)$ is

(a) $\frac{\pi}{2}$ (b) $\frac{5\pi}{3}$ (c) $\frac{10\pi}{3}$ (d) 0

**6.** $\cos\left[\cos^{-1}\left(\frac{-1}{7}\right) + \sin^{-1}\left(\frac{-1}{7}\right)\right] =$

(a) $-\frac{1}{3}$ (b) 0 (c) $\frac{1}{3}$ (d) $\frac{4}{9}$

**7.** The value of $\tan^{-1} x + 2\cot^{-1} x$ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$ (d) $2\pi$

**8.** If $\tan^{-1} x + 2\cot^{-1} x = \frac{2\pi}{3}$, then $x =$

(a) $\sqrt{2}$ (b) 3 (c) $\sqrt{3}$ (d) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$

**9.** If $4\sin^{-1} x + \cos^{-1} x = \pi$, then *x* is equal to

(a) 0 (b) $\frac{1}{2}$ (c) $-\frac{\sqrt{3}}{2}$ (d) $\frac{1}{\sqrt{2}}$

**10.** $\cos\left[2\cos^{-1}\frac{1}{5} + \sin^{-1}\frac{1}{5}\right] =$

(a) $\frac{2\sqrt{6}}{5}$ (b) $-\frac{2\sqrt{6}}{5}$ (c) $\frac{1}{5}$ (d) $-\frac{1}{5}$

**11.** The value of $\sin^{-1}\left(\cos\frac{33\pi}{5}\right)$ is

(a) $\frac{3\pi}{5}$ (b) $\frac{7\pi}{5}$ (c) $\frac{\pi}{10}$ (d) $-\frac{\pi}{10}$

12. If $\sec^{-1}\left(\frac{1}{x}\right)+2\sin^{-1}(1)=\pi,$ then $x$ equals

(a) $\frac{1}{2}$ (b) 1 (c) $\frac{\pi}{2}$ (d) None of these

13. The value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)-\sin^{-1}\left(\frac{1}{2}\right)$ is

(a) $45^o$ (b) $90^o$ (c) $15^o$ (d) $30^o$

14. The value of $\cos(\tan^{-1}(\tan 2))$ is

(a) $\frac{1}{\sqrt{5}}$ (b) $-\frac{1}{\sqrt{5}}$ (c) $\cos 2$ (d) $-\cos 2$

15. The value of $x$ which satisfies the equation $\tan^{-1} x=\sin^{-1}\left(\frac{3}{\sqrt{10}}\right)$ is

(a) 3 (b) – 3 (c) $\frac{1}{3}$ (d) $-\frac{1}{3}$

16. If $\sec^{-1} x=\operatorname{cosec}^{-1} y,$ then $\cos^{-1}\frac{1}{x}+\cos^{-1}\frac{1}{y}=$

(a) $\pi$ (b) $\frac{\pi}{4}$ (c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{2}$

17. If $\cos^{-1}\left(\frac{1}{x}\right)=\theta,$ then $\tan\theta=$

(a) $\frac{1}{\sqrt{x^2-1}}$ (b) $\sqrt{x^2+1}$ (c) $\sqrt{1-x^2}$ (d) $\sqrt{x^2-1}$

18. If $\sin^{-1} x=\frac{\pi}{5}$ for some $x\in(-1,1)$, then the value of $\cos^{-1} x$ is

(a) $\frac{3\pi}{10}$ (b) $\frac{5\pi}{10}$ (c) $\frac{7\pi}{10}$ (d) $\frac{9\pi}{10}$

19. $\sec(\operatorname{cosec}^{-1} x)$ is equal to

(a) $\operatorname{cosec}(\sec^{-1} x)$ (b) $\cot x$ (c) $\pi$ (d) None of these

20. $\tan(\cos^{-1} x)$ is equal to

(a) $\frac{\sqrt{1-x^2}}{x}$ (b) $\frac{x}{1+x^2}$ (c) $\frac{\sqrt{1+x^2}}{x}$ (d) $\sqrt{1-x^2}$

21. $\sin(\cot^{-1} x)=$

(a) $\sqrt{1+x^2}$ (b) $x$ (c) $(1+x^2)^{-3/2}$ (d) $(1+x^2)^{-\frac{1}{2}}$

22. $\cos(\tan^{-1} x)=$

(a) $\sqrt{1+x^2}$ (b) $\frac{1}{\sqrt{1+x^2}}$ (c) $1+x^2$ (d) None of these

23. $\cot\left[\cos^{-1}\left(\frac{7}{25}\right)\right]=$

(a) $\frac{25}{24}$ (b) $\frac{25}{7}$ (c) $\frac{24}{25}$ (d) None of these

**24.** The value of $\sin\cot^{-1}\tan\cos^{-1}x$ is equal to

(a) $x$ (b) $\frac{\pi}{2}$ (c) 1 (d) None of these

**25.** $\left[\sin\left(\tan^{-1}\frac{3}{4}\right)\right]^2 =$

(a) $\frac{3}{5}$ (b) $\frac{5}{3}$ (c) $\frac{9}{25}$ (d) $\frac{25}{9}$

**26.** $\sec^2(\tan^{-1}2)+\operatorname{cosec}^2(\cot^{-1}3)=$

(a) 5 (b) 13 (c) 15 (d) 6

***Advance Level***

**27.** If $\cos^{-1}x>\sin^{-1}x$, then

(a) $x<0$ (b) $-1<x<0$ (c) $0\le x<\frac{1}{\sqrt{2}}$ (d) $-1\le x<\frac{1}{\sqrt{2}}$

**28.** If $(\cos^{-1}x)^2-(\sin^{-1}x)^2>0$, then

(a) $x<\frac{1}{2}$ (b) $-1<x<\sqrt{2}$ (c) $0\le x<\frac{1}{\sqrt{2}}$ (d) $-1\le x<\frac{1}{\sqrt{2}}$

**29.** The greatest and the least values of $(\sin^{-1}x)^3+(\cos^{-1}x)^3$ are

(a) $\frac{-\pi}{2},\frac{\pi}{2}$ (b) $\frac{-\pi^3}{8},\frac{\pi^3}{8}$ (c) $\frac{\pi^3}{32},\frac{7\pi^3}{8}$ (d) None of these

**30.** If $x$ satisfies the equation $t^2-t-2>0$, then there exists a value for

(a) $\sin^{-1}x$ (b) $\cos^{-1}x$ (c) $\sec^{-1}x$ (d) None of these

**31.** If $f(x)=\sec^{-1}x+\tan^{-1}x$, then $f(x)$ is real for

(a) $x\in[-1,1]$ (b) $x\in R$ (c) $x\in(-\infty,1]\cup[1,+\infty)$ (d) None of these

**32.** If $\sum_{r=1}^{2n}\sin^{-1}x_r=n\pi$, then $\sum_{r=1}^{2n}x_r=$

(a) $n$ (b) $2n$ (c) $\frac{n(n+1)}{2}$ (d) None of these

**33.** $\frac{-2\pi}{5}$ is the principal value of

(a) $\cos^{-1}(\cos\frac{7\pi}{5})$ (b) $\sin^{-1}(\sin\frac{7\pi}{5})$ (c) $\sec^{-1}(\sec\frac{7\pi}{5})$ (d) None of these

**34.** The number of real solutions of $(x,y)$; where $|y|=\sin x, y=\cos^{-1}(\cos x), -2\pi\le x\le 2\pi$ is

(a) 2 (b) 1 (c) 3 (d) 4

**35.** The set of values of $k$ for which $x^2-kx+\sin^{-1}(\sin 4)>0$ for all real $x$ is

(a) $\phi$ (b) $(-2,2)$ (c) $R$ (d) None of these

**36.** $\cos^{-1}\left\{\frac{1}{2}x^2+\sqrt{1-x^2}.\sqrt{1-\frac{x^2}{4}}\right\}=\cos^{-1}\frac{x}{2}-\cos^{-1}x$ holds for

(a) $|x|\le 1$ (b) $x\in R$ (c) $0\le x\le 1$ (d) $-1\le x\le 0$

## SUM & DIFFERENCE OF INVERSE TRIGNOETRICAL

### *Basic Level*

**37.** If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = 3\pi$, then $xy + yz + zx =$

(a) 0 (b) 1 (c) 3 (d) – 3

**38.** The value of $\tan\left(\tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{3}\right)$ is

(a) $\frac{5}{6}$ (b) $\frac{7}{6}$ (c) $\frac{1}{6}$ (d) $\frac{1}{7}$

**39.** $\tan^{-1}\left(\frac{1}{11}\right) + \tan^{-1}\left(\frac{2}{12}\right) =$

(a) $\tan^{-1}\left(\frac{33}{132}\right)$ (b) $\tan^{-1}\left(\frac{1}{2}\right)$ (c) $\tan^{-1}\left(\frac{132}{33}\right)$ (d) None of these

**40.** If $\sin^{-1} a + \sin^{-1} b + \sin^{-1} c = \pi$, then the value of $a\sqrt{(1-a^2)} + b\sqrt{(1-b^2)} + c\sqrt{(1-c^2)}$ will be

(a) $2abc$ (b) $abc$ (c) $\frac{1}{2}abc$ (d) $\frac{1}{3}abc$

**41.** If $\tan^{-1} x - \tan^{-1} y = \tan^{-1} A$, then $A =$

(a) $x - y$ (b) $x + y$ (c) $\frac{x-y}{1+xy}$ (d) $\frac{x+y}{1-xy}$

**42.** If $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$ then $x$ =

(a) –1 (b) $\frac{1}{6}$ (c) $-1, \frac{1}{6}$ (d) None of these

**43.** If $\cos^{-1}\frac{3}{5} - \sin^{-1}\frac{4}{5} = \cos^{-1} x$, then $x =$

(a) 0 (b) 1 (c) –1 (d) 2

**44.** If $\cot^{-1}\alpha + \cot^{-1}\beta = \cot^{-1} x$, then $x =$

(a) $\alpha + \beta$ (b) $\alpha - \beta$ (c) $\frac{1+\alpha\beta}{\alpha+\beta}$ (d) $\frac{\alpha\beta - 1}{\alpha+\beta}$

**45.** $\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5} - \tan^{-1}\frac{8}{19} =$

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) None of these

**46.** $\cos\left[\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2}\right] =$

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) $\frac{\pi}{4}$

**47.** $\tan^{-1}\frac{x}{y} - \tan^{-1}\frac{x-y}{x+y} =$ (where $x > y > 0$ )

(a) $-\frac{\pi}{4}$ (b) $\frac{\pi}{4}$ (c) $\frac{3\pi}{4}$ (d) None of these

**48.** $\tan\left[\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right]=$

(a) $\frac{6}{17}$ (b) $\frac{17}{6}$ (c) $\frac{7}{16}$ (d) $\frac{16}{7}$

**49.** $\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{2}{9}\right)=$

(a) $\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$ (b) $\frac{1}{2}\sin^{-1}\left(\frac{3}{5}\right)$ (c) $\tan^{-1}\left(\frac{1}{2}\right)$ (d) Both (a) and (c)

**50.** $\sin^{-1}\frac{1}{\sqrt{5}}+\cot^{-1}3$ is equal to

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

**51.** If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$, then $x=$

(a) 1 (b) 0 (c) $\frac{4}{5}$ (d) $\frac{1}{5}$

**52.** A solution of the equation $\tan^{-1}(1+x)+\tan^{-1}(1-x)=\frac{\pi}{2}$ is

(a) $x = 1$ (b) $x = -1$ (c) $x = 0$ (d) $x=\pi$

**53.** If $\tan^{-1}\frac{x-1}{x+1}+\tan^{-1}\frac{2x-1}{2x+1}=\tan^{-1}\frac{23}{36}$, then $x=$

(a) $\frac{3}{4},\frac{-3}{8}$ (b) $\frac{3}{4},\frac{3}{8}$ (c) $\frac{4}{3},\frac{3}{8}$ (d) None of these

**54.** If $\sin^{-1}\frac{x}{5}+cosec^{-1}\left(\frac{5}{4}\right)=\frac{\pi}{2}$, then $x=$

(a) 4 (b) 5 (c) 1 (d 3

**55.** $\sin^{-1}\left(\frac{3}{5}\right)+\tan^{-1}\left(\frac{1}{7}\right)=$

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\cos^{-1}\left(\frac{4}{5}\right)$ (d) $\pi$

**56.** If $\tan^{-1}2, \tan^{-1}3$ are two angles of a triangle, then the third angle is

(a) $\frac{\pi}{4}$ (b) $\frac{3\pi}{4}$ (c) $\frac{\pi}{4}$ (d) None of these

**57.** If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$ then

(a) $x^2+y^2+z^2+xyz=0$ (b) $x^2+y^2+z^2+2xyz=0$ (c) $x^2+y^2+z^2+xyz=1$ (d) $x^2+y^2+z^2+2xyz=1$

**58.** If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2}$, then

(a) $x+y+z-xyz=0$ (b) $x+y+z+xyz=0$ (c) $xy+yz+zx+1=0$ (d) $xy+yz+zx-1=0$

**59.** If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, then $\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}=$

(a) 0 (b) 1 (c) $\frac{1}{xyz}$ (d) $xyz$

*Advance Level*

**60.** If we consider only the principal values of the inverse trigonometric functions, then the value of $\tan\left(\cos^{-1}\frac{1}{5\sqrt{2}}-\sin^{-1}\frac{4}{\sqrt{(17)}}\right)$ is

(a) $\sqrt{\frac{29}{3}}$ (b) $\frac{29}{3}$ (c) $\sqrt{\frac{3}{29}}$ (d) $\frac{3}{29}$

**61.** The sum of first 10 terms of the series $\cot^{-1}3+\cot^{-1}7+\cot^{-1}13+\cot^{-1}21+.......$ is

(a) $\tan^{-1}\left(\frac{5}{6}\right)$ (b) $\tan^{-1}(100)$ (c) $\tan^{-1}\left(\frac{6}{5}\right)$ (d) $\tan^{-1}\left(\frac{1}{100}\right)$

**62.** Sum of infinite terms of the series $\cot^{-1}\left[1^2+\frac{3}{4}\right]+\cot^{-1}\left[2^2+\frac{3}{4}\right]+\cot^{-1}\left[3^2+\frac{3}{4}\right]+$ …….is

(a) $\frac{\pi}{4}$ (b) $\tan^{-1}2$ (c) $\tan^{-1}3$ (d) None of these

**63.** $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{18}+.........+\tan^{-1}\left(\frac{1}{n^2+n+1}\right)+.......$ to $\infty$ is equal

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{2\pi}{3}$ (d) 0

**64.** If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\pi$, then $x^4+y^4+z^4+4x^2y^2z^2=K(x^2y^2+y^2z^2+z^2x^2)$, where $K$ =

(a) 1 (b) 2 (c) 4 (d) None of these

**65.** The sum of the infinite series $\sin^{-1}\frac{1}{\sqrt{2}}+\sin^{-1}\left(\frac{\sqrt{2}-1}{\sqrt{6}}\right)+\sin^{-1}\left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{12}}\right)+.............+\sin^{-1}\left(\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}}\right)$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\pi$

**66.** If sum of the infinite series $\cot^{-1}(2.1^2)+\cot^{-1}(2.2^2)+\cot^{-1}(2.2^3)+\cot(2.2^4)+.....$ is equal to

(a) $\frac{\pi}{5}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

**67.** If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$, then the value of $x^{100}+y^{100}+z^{100}-\frac{9}{x^{101}+y^{101}+z^{101}}$ is equal to

(a) 0 (b) 3 (c) – 3 (d) 9

**68.** If $x_1,x_2,x_3,x_4$ are roots of the equation $x^4-x^3\sin 2\beta+x^2\cos 2\beta-x\cos\beta-\sin\beta=0$,

then $\tan^{-1}x_1+\tan^{-1}x_2+\tan^{-1}x_3+\tan^{-1}x_4=$

(a) $\beta$ (b) $\frac{\pi}{2}-\beta$ (c) $\pi-\beta$ (d) $-\beta$

**69.** If $a_1,a_2,a_3,........,a_n$ is an A.P. with common difference $d$, then

$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+..........+\tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)\right]=$

(a) $\frac{(n-1)d}{a_1+a_n}$ (b) $\frac{(n-1)d}{1+a_1a_n}$ (c) $\frac{nd}{1+a_1a_n}$ (d) $\frac{a_n-a_1}{a_n+a_1}$

# INVERSE TRIGNOMETRIC RATIOS OF MULTIPLE ANGLES

## *Basic Level*

**70.** $3\tan^{-1} a$ is equal to

(a) $\tan^{-1}\frac{3a+a^3}{1+3a^2}$ (b) $\tan^{-1}\frac{3a-a^3}{1+3a^2}$ (c) $\tan^{-1}\frac{3a+a^3}{1-3a^2}$ (d) $\tan^{-1}\frac{3a-a^3}{1-3a^2}$

**71.** If $A=\tan^{-1} x$, then $\sin 2A=$

(a) $\frac{2x}{\sqrt{1-x^2}}$ (b) $\frac{2x}{1-x^2}$ (c) $\frac{2x}{1+x^2}$ (d) None of these

**72.** $\sin(2\sin^{-1} 0.8)=$

(a) 0.96 (b) 0.48 (c) 0.64 (d) None of these

**73.** If $\cos(2\sin^{-1} x)=\frac{1}{9}$, then $x=$

(a) Only $\frac{2}{3}$ (b) Only $-\frac{2}{3}$ (c) $\frac{2}{3},-\frac{2}{3}$ (d) Neither $\frac{2}{3}$ nor $-\frac{2}{3}$

**74.** $\cos^{-1}\left(\frac{15}{17}\right)+2\tan^{-1}\left(\frac{1}{5}\right)=$

(a) $\frac{\pi}{2}$ (b) $\cos^{-1}\left(\frac{171}{221}\right)$ (c) $\frac{\pi}{4}$ (d) None of these

**75.** $2\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)=$

(a) $\tan^{-1}\left(\frac{49}{29}\right)$ (b) $\frac{\pi}{2}$ (c) 0 (d) $\frac{\pi}{4}$

**76.** $\sin^{-1}\frac{4}{5}+2\tan^{-1}\frac{1}{3}=$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) None of these

**77.** If $\sin^{-1}\frac{2a}{1+a^2}+\sin^{-1}\frac{2b}{1+b^2}=2\tan^{-1} x$, then $x$ =

(a) $\frac{a-b}{1+ab}$ (b) $\frac{b}{1+ab}$ (c) $\frac{b}{1-ab}$ (d) $\frac{a+b}{1-ab}$

**78.** If $3\tan^{-1}\left(\frac{1}{2+\sqrt{3}}\right)-\tan^{-1}\frac{1}{x}=\tan^{-1}\frac{1}{3}$, then $x$ equals

(a) 1 (b) 2 (c) 3 (d) $\sqrt{3}$

**79.** $\cot^{-1}\left[\frac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{\sqrt{1-\sin x}-\sqrt{1+\sin x}}\right]$ =

(a) $\pi-x$ (b) $2\pi-x$ (c) $\frac{\pi}{2}$ (d) $\pi-\frac{x}{2}$

**80.** $\sin\left(\frac{1}{2}\cos^{-1}\frac{4}{5}\right) =$

(a) $\frac{1}{\sqrt{10}}$ (b) $-\frac{1}{\sqrt{10}}$ (c) $\frac{1}{10}$ (d) $-\frac{1}{10}$

## *Advance Level*

**81.** $2\tan^{-1}\left[\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right] =$

(a) $\cos^{-1}\left(\frac{a\cos\theta + b}{a + b\cos\theta}\right)$ (b) $\cos^{-1}\left(\frac{a + b\cos\theta}{a\cos\theta + b}\right)$ (c) $\cos^{-1}\left(\frac{a\cos\theta}{a + b\cos\theta}\right)$ (d) $\cos^{-1}\left(\frac{b\cos\theta}{a\cos\theta + b}\right)$

**82.** If $\cot^{-1}[(\cos\alpha)^{1/2}] - \tan^{-1}[(\cos\alpha)^{1/2}] = x$. Then $\sin x =$

(a) $\tan^2\left(\frac{\alpha}{2}\right)$ (b) $\cot^2\left(\frac{\alpha}{2}\right)$ (c) $\tan\alpha$ (d) $\cot\left(\frac{\alpha}{2}\right)$

**83.** The value of $\sin^{-1}\left\{\left(\sin\frac{\pi}{3}\right)\frac{x}{\sqrt{x^2 + k^2 - kx}}\right\} - \cos^{-1}\left\{\cos\frac{\pi}{6}\frac{x}{\sqrt{x^2 + k^2 - kx}}\right\}$, $\left(\text{where } \frac{k}{2} < x < 2k, k > 0\right)$ is

(a) $\tan^{-1}\left(\frac{2x^2 + xk - k^2}{x^2 - 2xk + k^2}\right)$ (b) $\tan^{-1}\left(\frac{x^2 + 2xk - k^2}{x^2 - 2xk + k^2}\right)$ (c) $\tan^{-1}\left(\frac{x^2 + 2xk - 2k^2}{2x^2 - 2xk + 2k^2}\right)$ (d) None of these

**84.** Solution of equation $\sin[2\cos^{-1}\{\cot(2\tan^{-1} x)\}] = 0$ is

(a) $x = \pm 1$ only (b) $x = 1 \pm \sqrt{2}$ only (c) $x = (-1 \pm \sqrt{2})$ only (d) All of these

**85.** The greater of the two angles $A = 2\tan^{-1}(2\sqrt{2} - 1)$ and $B = 3\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{3}{5}\right)$ is

(a) *B* (b) *A* (c) *C* (d) None of these

**86.** $\tan\left[\frac{1}{2}\cos^{-1}\left(\frac{\sqrt{5}}{3}\right)\right] =$

(a) $\frac{3-\sqrt{5}}{2}$ (b) $\frac{3+\sqrt{5}}{2}$ (c) $\frac{2}{3-\sqrt{5}}$ (d) $\frac{2}{3+\sqrt{5}}$

***

# ANSWER

# INVERSE TRIGNOMETRICAL FUNCTIONS

## ASSIGNMENT ( BASIC & ADVANCE LEVEL )

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| b | b | b | d | a | b | c | c | b | b | d | b | d | c | a | d | d | a | a | a |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| d | b | d | a | c | c | d | d | c | c | c | b | b | c | a | c | c | d | d | a |
| 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 |
| c | b | b | d | a | a | b | b | d | b | d | c | d | d | a | b | d | d | b | d |
| 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 |
| a | b | b | b | c | b | a | b | b | d | c | a | c | d | d | a | d | b | d | a |
| 81 | 82 | 83 | 84 | 85 | 86 | | | | | | | | | | | | | | |
| a | a | c | d | b | d | | | | | | | | | | | | | | |