

INDEFINITE INTEGRAL

INTRODUCTION

A function $\phi(x)$ is called a primitive or an antiderivative of a function $f(x)$ if $\phi'(x) = f(x)$.

For example, $\frac{x^5}{5}$ is a primitive of x^4 because $\frac{d}{dx}\left(\frac{x^5}{5}\right) = x^4$

Let $\phi(x)$ be a primitive of a function $f(x)$ and let c be any constant.

Then $\frac{d}{dx}[\phi(x) + c] = \phi'(x) = f(x)$ $[\ominus \phi'(x) = f(x)]$

$\Rightarrow \phi(x) + c$ is also a primitive of $f(x)$.

Thus, if a function $f(x)$ possesses a primitive, then it possesses infinitely many primitives which are contained in the expression $\phi(x) + c$, where c is a constant.

For example $\frac{x^5}{5}, \frac{x^5}{5} + 2, \frac{x^5}{5} - 1$ etc. are primitives of x^4 .

Note : $\square$ If $F_1(x)$ and $F_2(x)$ are two antiderivatives of a function $f(x)$ on an interval $[a, b]$, then the difference between them is a constant.

5.1 DEFINITION

Let $f(x)$ be a function. Then the collection of all its primitives is called the indefinite integral of $f(x)$ and is denoted by $\int f(x) dx$

Thus, $\frac{d}{dx}(\phi(x) + c) = f(x) \Rightarrow \int f(x) dx = \phi(x) + c$

where $\phi(x)$ is primitive of $f(x)$ and c is an arbitrary constant known as the constant of integration.

Here $\int$ is the integral sign, $f(x)$ is the integrand, x is the variable of integration and dx is the element of integration.

The process of finding an indefinite integral of a given function is called integration of the function.

It follows from the above discussion that integrating a function $f(x)$ means finding a function $\phi(x)$ such that $\frac{d}{dx}(\phi(x)) = f(x)$.

5.2 COMPARISON BETWEEN DIFFERENTIATION AND INTEGRATION

- (1) Differentiation and integration both are operations on functions and each gives rise to a function.
- (2) Each function is not differentiable or integrable.
- (3) The derivative of a function, if it exists, is unique. The integral of a function, if it exists, is not unique.

(4) The derivative of a polynomial function decreases its degree by 1, but the integral of a polynomial function increases its degree by 1.

(5) The derivative has a geometrical meaning, namely, the slope of the tangent to a curve at a point on it. The integral has also a geometrical meaning, namely, the area of some region.

(6) The derivative is used in obtaining some physical quantities like velocity, acceleration etc. of a particle. The integral is used in obtaining some physical quantities like centre of mass, momentum etc.

(7) Differentiation and integration are inverse of each other.

5.3 PROPERTIES OF INTEGRALS

(1) The differentiation of an integral is the integrand itself or the process of differentiation and integration neutralize each other, i.e., $\frac{d}{dx} \left[\int f(x) dx \right] = f(x)$.

(2) The integral of the product of a constant and a function is equal to the product of the constant and the integral of the function, i.e., $\int cf(x) dx = c \int f(x) dx$

(3) Integral of the sum or difference of two functions is equal to the sum or difference of their integrals, i.e., $\int \{f_1(x) \pm f_2(x)\} dx = \int f_1(x) dx \pm \int f_2(x) dx$

In the general form,
 $\int \{k_1 \cdot f_1(x) \pm k_2 \cdot f_2(x) \pm k_3 \cdot f_3(x) \pm \dots\} dx = k_1 \int f_1(x) dx \pm k_2 \int f_2(x) dx \pm k_3 \int f_3(x) dx \pm \dots$

5.4 FUNDAMENTAL INTEGRATION FORMULAE

$$(1) (i) \int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1 \quad \ominus \quad \frac{d}{dx} \left(\frac{x^{n+1}}{n+1} \right) = x^n \quad (ii) \int dx = x + c$$

$$(iii) \int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + c \quad (iv) \int (ax+b)^n dx = \frac{1}{a} \cdot \frac{(ax+b)^{n+1}}{n+1} + c$$

$$(2) (i) \int \frac{1}{x} dx = \log|x| + c \quad \ominus \quad \frac{d}{dx} (\log|x|) = \frac{1}{x} \quad (ii) \int \frac{1}{ax+b} dx = \frac{1}{a} (\log|ax+b|) + c$$

$$(3) \int e^x dx = e^x + c \quad \ominus \quad \frac{d}{dx} (e^x) = e^x$$

$$(4) \int a^x dx = \frac{a^x}{\log_e a} + c \quad \ominus \quad \frac{d}{dx} \left(\frac{a^x}{\log_e a} \right) = a^x$$

$$(5) \int \sin x dx = -\cos x + c \quad \ominus \quad \frac{d}{dx} (-\cos x) = \sin x$$

$$(6) \int \cos x dx = \sin x + c \quad \ominus \quad \frac{d}{dx} (\sin x) = \cos x$$

$$(7) \int \sec^2 x dx = \tan x + c \quad \ominus \quad \frac{d}{dx} (\tan x) = \sec^2 x$$

$$(8) \int \operatorname{cosec}^2 x dx = -\cot x + c \quad \ominus \quad \frac{d}{dx} (-\cot x) = \operatorname{cosec}^2 x$$

$$(9) \int \sec x \tan x \, dx = \sec x + c$$

$$\ominus \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$(10) \int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$$

$$\ominus \frac{d}{dx}(-\operatorname{cosec} x) = \operatorname{cosec} x \cot x$$

$$(11) \int \tan x \, dx = -\log |\cos x| + c = \log |\sec x| + c$$

$$\ominus \frac{d}{dx}(\log \cos x) = -\tan x$$

$$(12) \int \cot x \, dx = \log |\sin x| + c = -\log |\operatorname{cosec} x| + c$$

$$\ominus \frac{d}{dx}(\log \sin x) = \cot x$$

$$(13) \int \sec x \, dx = \log |\sec x + \tan x| + c = \log \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) + c \quad \ominus \frac{d}{dx} \log(\sec x + \tan x) = \sec x$$

$$(14) \int \operatorname{cosec} x \, dx = \log |\operatorname{cosec} x - \cot x| + c = \log \tan \frac{x}{2} + c \quad \ominus \frac{d}{dx}(\log |\operatorname{cosec} x - \cot x|) = \operatorname{cosec} x$$

$$(15) \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c = -\cos^{-1} x + c$$

$$\ominus \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

$$(16) \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} + c = -\cos^{-1} \frac{x}{a} + c$$

$$\ominus \frac{d}{dx} \left(\sin^{-1} \frac{x}{a} \right) = \frac{1}{\sqrt{a^2-x^2}}, \quad \frac{d}{dx} \left(\cos^{-1} \frac{x}{a} \right) = \frac{-1}{\sqrt{a^2-x^2}}$$

$$(17) \int \frac{dx}{1+x^2} = \tan^{-1} x + c = -\cot^{-1} x + c$$

$$\ominus \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$, \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

$$(18) \int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c = \frac{-1}{a} \cot^{-1} \frac{x}{a} + c$$

$$\ominus \frac{d}{dx} \left(\tan^{-1} \frac{x}{a} \right) = \frac{a}{a^2+x^2} \quad \frac{d}{dx} \left(\cot^{-1} \frac{x}{a} \right) = \frac{-a}{a^2+x^2}$$

$$(19) \int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + c = -\operatorname{cosec}^{-1} x + c$$

$$\ominus \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}} \quad \frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$$

$$(20) \int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a} + c = \frac{-1}{a} \operatorname{cosec}^{-1} \frac{x}{a} + c$$

$$\ominus \frac{d}{dx} \left(\sec^{-1} \frac{x}{a} \right) = \frac{a}{x\sqrt{x^2-a^2}} \quad \frac{d}{dx} \left(\operatorname{cosec}^{-1} \frac{x}{a} \right) = \frac{-a}{x\sqrt{x^2-a^2}}$$

Note : $\square$ In any of the fundamental integration formulae, if x is replaced by $ax + b$, then the same formulae is applicable but we must divide by coefficient of x or derivative of $(ax + b)$ i.e., a .

In general, if $\int f(x) dx = \phi(x) + c$, then $\int f(ax+b) dx = \frac{1}{a} \phi(ax+b) + c$

$$\int \sin(ax+b) dx = \frac{-1}{a} \cos(ax+b) + c, \quad \int \sec(ax+b) dx = \frac{1}{a} \log |\sec(ax+b) + \tan(ax+b)| + c \text{ etc.}$$

Some more results:

$$(i) \int \frac{1}{x^2-a^2} = \frac{-1}{a} \coth^{-1} \frac{x}{a} + c = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c, \quad \text{when } x > a$$

$$(ii) \int \frac{1}{a^2-x^2} dx = \frac{1}{a} \tanh^{-1} \frac{x}{a} + c = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c, \quad \text{when } x < a$$

$$(iii) \int \frac{dx}{\sqrt{x^2-a^2}} = \log \{ |x + \sqrt{x^2-a^2}| \} + c = \cosh^{-1} \left(\frac{x}{a} \right) + c$$

$$(iv) \int \frac{dx}{\sqrt{x^2 + a^2}} = \log\{ |x + \sqrt{x^2 + a^2}| \} + c = \sinh^{-1}\left(\frac{x}{a}\right) + c$$

$$(v) \int \sqrt{a^2 - x^2} dx = \frac{1}{2} x \sqrt{a^2 - x^2} + \frac{1}{2} a^2 \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$(vi) \int \sqrt{x^2 - a^2} dx = \frac{1}{2} x \sqrt{x^2 - a^2} - \frac{1}{2} a^2 \log\{x + \sqrt{x^2 - a^2}\} + c = \frac{1}{2} x \sqrt{x^2 - a^2} - \frac{1}{2} a^2 \cosh^{-1}\left(\frac{x}{a}\right) + c$$

$$(vii) \int \sqrt{x^2 + a^2} dx = \frac{1}{2} x \sqrt{x^2 + a^2} + \frac{1}{2} a^2 \log\{x + \sqrt{x^2 + a^2}\} + c = \frac{1}{2} x \sqrt{x^2 + a^2} + \frac{1}{2} a^2 \sinh^{-1}\left(\frac{x}{a}\right) + c$$

Important Tips

- ☞ The signum function has an antiderivative on any interval which doesn't contain the point $x = 0$, and does not possess an anti-derivative on any interval which contains the point.
- ☞ The antiderivative of every odd function is an even function and vice-versa

Example: 1 $\int \frac{(x+1)^2}{x(x^2+1)} dx$ is equal to

- (a) $\log_e x$ (b) $\log_e x + 2 \tan^{-1} x$ (c) $\log_e \frac{1}{x^2+1}$ (d) $\log_e \{x(x^2+1)\}$

Solution: (b) $\int \frac{(x+1)^2}{x(x^2+1)} dx = \int \frac{x^2+1+2x}{x(x^2+1)} dx = \int \frac{x^2+1}{x(x^2+1)} dx + 2 \int \frac{x}{x(x^2+1)} dx = \int \frac{dx}{x} + 2 \int \frac{dx}{x^2+1} = \log_e x + 2 \tan^{-1} x$

Example: 2 $\int \frac{ax^3 + bx^2 + c}{x^4} dx$ equals to

- (a) $a \log x + \frac{b}{x^2} + \frac{c}{3x^3} + k$ (b) $a \log x + \frac{b}{x} - \frac{c}{3x^3} + k$ (c) $a \log x - \frac{b}{x} - \frac{c}{3x^3} + k$ (d) None of these

Solution: (c) $I = \int \frac{ax^3 + bx^2 + c}{x^4} dx = \int \left[\frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^4} \right] dx = a \log x - \frac{b}{x} - \frac{c}{3x^3} + k$

Example: 3 $\int \frac{1+x+\sqrt{x+x^2}}{\sqrt{x}+\sqrt{1+x}} dx =$

- (a) $\frac{1}{2} \sqrt{1+x} + c$ (b) $\frac{2}{3} (1+x)^{3/2} + c$ (c) $\sqrt{1+x} + c$ (d) $2(1+x)^{3/2} + c$

Solution: (b) $\int \frac{1+x+\sqrt{x+x^2}}{\sqrt{x}+\sqrt{1+x}} dx = \int \frac{\sqrt{1+x}[\sqrt{1+x}+\sqrt{x}]}{(\sqrt{x}+\sqrt{1+x})} dx = \int \sqrt{1+x} dx = \frac{2}{3} (1+x)^{3/2} + c$

Example: 4 $\int (\sin^4 x - \cos^4 x) dx =$

- (a) $-\frac{\cos 2x}{2} + c$ (b) $-\frac{\sin 2x}{2} + c$ (c) $\frac{\sin 2x}{2} + c$ (d) $\frac{\cos 2x}{2} + c$

Solution: (b) $\int (\sin^4 x - \cos^4 x) dx = \int (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x) dx$
 $= \int (\sin^2 x - \cos^2 x) dx = - \int (\cos^2 x - \sin^2 x) dx = - \int \cos 2x dx = -\frac{\sin 2x}{2} + c$

Example: 5 $\int \sqrt{1 + \sin\left(\frac{x}{4}\right)} dx =$

(a) $8\left(\sin\frac{x}{8} - \cos\frac{x}{8}\right) + c$ (b) $\left(\sin\frac{x}{8} + \cos\frac{x}{8}\right) + c$ (c) $\frac{1}{8}\left(\sin\frac{x}{8} - \cos\frac{x}{8}\right) + c$ (d) $8\left(\cos\frac{x}{8} - \sin\frac{x}{8}\right) + c$

Solution: (a) $\int \sqrt{1 + \sin\left(\frac{x}{4}\right)} dx = \int \sqrt{\left[\sin^2\frac{x}{8} + \cos^2\frac{x}{8}\right] + \left[2\sin\frac{x}{8} \cdot \cos\frac{x}{8}\right]} dx = \int \sqrt{\left(\sin\frac{x}{8} + \cos\frac{x}{8}\right)^2}$
 $= \int \left(\sin\frac{x}{8} + \cos\frac{x}{8}\right) dx = \frac{-\cos x/8}{1/8} + \frac{\sin x/8}{1/8}$
 $= 8\left(\sin\frac{x}{8} - \cos\frac{x}{8}\right) + c$

5.5 INTEGRATION BY SUBSTITUTION

(1) When integrand is a function i.e., $\int f(\phi(x))\phi'(x) dx$:

Here, we put $\phi(x) = t$ so that $\phi'(x)dx = dt$ and in that case the integrand is reduced to $\int f(t)dt$. In this method, the integrand is broken into two factors so that one factor can be expressed in terms of the function whose differential coefficient is the second factor.

Example: 6 $\int \frac{dx}{2\sqrt{x}(1+x)} =$

(a) $\frac{1}{2}\tan^{-1}(\sqrt{x}) + c$ (b) $\tan^{-1}(\sqrt{x}) + c$ (c) $2\tan^{-1}(\sqrt{x}) + c$ (d) None of these

Solution: (b) $I = \int \frac{dx}{2\sqrt{x}(1+x)}$, put $\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}} dx = dt$

$\therefore I = \int \frac{dt}{1+t^2} = \tan^{-1} t + c = \tan^{-1} \sqrt{x} + c$

(2) When integrand is the product of two factors such that one is the derivative of the others i.e., $I = \int f(x) \cdot f(x) \cdot dx$

In this case we put $f(x) = t$ and convert it into a standard integral.

Example: 7 For any natural number m $\int (x^{3m} + x^{2m} + x^m)(2x^{2m} + 3x^m + 6)^{1/m} dx \quad (x > 0) =$

(a) $\frac{(2x^{3m} + 3x^{2m} + 6x^m)^{\frac{m+1}{m}}}{6(m+1)}$ (b) $\frac{(2x^{3m} + 3x^{2m} + 6x^m)^{\frac{m}{m+1}}}{6(m+1)}$
(c) $-\frac{(2x^{3m} + 3x^{2m} + 6x^m)^{\frac{m+1}{m}}}{6(m+1)}$ (d) $-\frac{(2x^{3m} + 3x^{2m} + 6x^m)^{\frac{m}{m+1}}}{6(m+1)}$

Solution: (a) $I = \int (x^{3m-1} + x^{2m-1} + x^{m-1})(2x^{2m} + 3x^m + 6)^{1/m} dx$

Put $2x^{3m} + 3x^{2m} + 6x^m = t \Rightarrow 6m(x^{3m-1} + x^{2m-1} + x^{m-1})dx = dt$

$$I = \int \frac{1}{6m} t^{1/m} dt = \frac{1}{6m} \frac{t^{1/m+1}}{1/m+1} + c = \frac{1}{6(m+1)} t^{(m+1)/m} + c$$

Again put the value of t , then we get option (a).

Example: 8 $\int \frac{x^2 \tan^{-1} x^3}{1+x^6} dx$ is equal to

- (a) $\tan^{-1}(x^3) + c$ (b) $\frac{1}{6}(\tan^{-1} x^3)^2 + c$ (c) $-\frac{1}{2}(\tan^{-1} x^3)^2 + c$ (d) $\frac{1}{2}(\tan^{-1} x^3)^2 + c$

Solution: (b) Put $\tan^{-1} x^3 = t$

$$\frac{1}{1+x^6} \cdot 3x^2 dx = dt \Rightarrow \frac{x^2}{1+x^6} dx = \frac{dt}{3} \Rightarrow \frac{1}{3} \int t dt = \frac{1}{3} \cdot \frac{t^2}{2} = \frac{(\tan^{-1} x^3)^2}{6} + c$$

Example: 9 $\int x^{-3} 5^{1/x^2} dx = k 5^{1/x^2} + c$, then k is

- (a) $-\frac{1}{2 \log 5}$ (b) $-2 \log 5$ (c) $\frac{2}{\log 5}$ (d) $\frac{-2}{\log 5}$

Solution: (a) Put $x^{-2} = t \Rightarrow -2x^{-3} dx = dt \Rightarrow x^{-3} dx = -\frac{dt}{2}$

$$\int x^{-3} 5^{1/x^2} dx = -\frac{1}{2} \int 5^t dt = -\frac{1}{2} \frac{5^t}{\log 5} + c = \frac{-1}{2 \log 5} \cdot 5^{1/x^2} + c. \text{ On comparing, } k = \frac{-1}{2 \log 5}$$

(3) **Integral of a function of the form $f(ax+b)$:** Here we put $ax+b = t$ and convert it into standard integral. Obviously if $\int f(x) dx = \phi(x)$, then, $\int f(ax+b) dx = \frac{1}{a} \phi(ax+b) + c$

Example: 10 $\int \tan(3x-5) \sec(3x-5) dx =$

- (a) $\sec(3x-5) + c$ (b) $\frac{1}{3} \sec(3x-5) + c$ (c) $\tan(3x-5) + c$ (d) None of these

Solution: (b) $\ominus \int \sec x \tan x dx = \sec x + c$

$$\therefore \int \sec(3x-5) \tan(3x-5) dx = \frac{\sec(3x-5)}{3} + c$$

(4) **If integral of a function of the form $\frac{f(x)}{f'(x)} dx = \log[f(x)] + c$**

Example: 11 $\int \frac{\sin 2x}{a^2 \sin^2 x + b^2 \cos^2 x} dx$ is equal to

- (a) $\frac{1}{b^2 - a^2} \log(a^2 \sin^2 x + b^2 \cos^2 x) + c$ (b) $\frac{1}{a^2 - b^2} \log(a^2 \sin^2 x + b^2 \cos^2 x) + c$
(c) $\log(a^2 \sin^2 x - b^2 \cos^2 x) + c$ (d) None of these

Solution: (b) Put $a^2 \sin^2 x + b^2 \cos^2 x = t \Rightarrow (a^2 \cdot 2 \sin x \cos x - b^2 \cdot 2 \cos x \sin x) dx = dt \Rightarrow \sin 2x (a^2 - b^2) dx = dt$

$$\int \frac{\sin 2x}{a^2 \sin^2 x + b^2 \cos^2 x} dx = \frac{1}{(a^2 - b^2)} \int \frac{dt}{t} = \frac{1}{a^2 - b^2} \log t + c = \frac{1}{a^2 - b^2} \log(a^2 \sin^2 x + b^2 \cos^2 x) + c$$

Example: 12 $\int \frac{dx}{1+e^x} =$

- (a) $\log|1+e^x|$ (b) $-\log|1+e^{-x}|$ (c) $-\log|1-e^{-x}|$ (d) $\log|e^{-x}+e^{-2x}|$

Solution: (b) $I = \int \frac{dx}{1+e^x} = \int \frac{e^{-x}}{e^{-x}+1} dx$, Put $e^{-x}+1 = t \Rightarrow -e^{-x}dx = dt \Rightarrow I = -\int \frac{dt}{t} = -\log t = -\log|1+e^{-x}|$

Example: 13 The value of the integral $\int \frac{x}{1+x \tan x} dx$ is equal to

- (a) $\log|xcosx+sinx|+c$ (b) $\log|cosx+x|+c$ (c) $\log|cosx+xsinx|+c$ (d) None of these

Solution: (c) $I = \int \frac{x}{1+x \tan x} dx = \int \frac{x \cos x}{\cos x + x \sin x} dx$

Put $(\cos x + x \sin x) = t \Rightarrow (-\sin x + x \cos x + \sin x) dx = dt \Rightarrow x \cos x dx = dt$

$I = \int \frac{dt}{t} = \log|t| + c = \log|\cos x + x \sin x| + c$

(5) If integral of a function of the form, $\int [f(x)]^n f(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c$ $[n \neq -1]$

Example: 14 $\int x\sqrt{1+x^2} dx =$

- (a) $\frac{1+2x^2}{\sqrt{1+x^2}} + c$ (b) $\sqrt{1+x^2} + c$ (c) $3(1+x^2)^{3/2} + c$ (d) $\frac{1}{3}(1+x^2)^{3/2} + c$

Solution: (d) Put $1+x^2 = t \Rightarrow 2x dx = dt$, $\int x\sqrt{1+x^2} dx = \frac{1}{2} \int t^{1/2} dt = \frac{1}{2} \cdot \frac{t^{3/2}}{3/2} + c = \frac{1}{3}(1+x^2)^{3/2} + c$

Example: 15 $\int \sqrt{2+\sin 3x} \cdot \cos 3x dx =$

- (a) $\frac{2}{9}\sqrt{2+\sin 3x} + c$ (b) $\frac{2}{3}(2+\sin 3x)^{2/3} + c$ (c) $\frac{2}{3}(2+\sin 3x)^{3/2} + c$ (d) $\frac{2}{9}(2+\sin 3x)^{3/2} + c$

Solution: (d) Put $(2+\sin 3x) = t \Rightarrow 3 \cos 3x dx = dt$

$\int \sqrt{2+\sin 3x} \cdot \cos 3x dx = \frac{1}{3} \int \sqrt{t} dt = \frac{1}{3} \cdot \frac{t^{3/2}}{3/2} + c = \frac{2}{9} \cdot (2+\sin 3x)^{3/2} + c$

(6) If the integral of a function of the form, $\int \frac{f(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c$

Example: 16 $\int \frac{\sqrt{\tan x}}{\sin x \cdot \cos x} dx$

- (a) $2\sqrt{\sec x} + c$ (b) $2\sqrt{\tan x} + c$ (c) $\frac{2}{\sqrt{\tan x}} + c$ (d) $\frac{2}{\sqrt{\sec x}} + c$

Solution: (b) $\int \frac{\sqrt{\tan x}}{\sin x \cdot \cos x} dx = \int \frac{\tan x}{\sqrt{\tan x} \cdot \sin x \cos x} dx = \int \frac{\sin x \cdot \sec x}{\sqrt{\tan x} \sin x \cos x} dx = \int \frac{\sec^2 x}{\sqrt{\tan x}} dx$

Put $t = \tan x \Rightarrow dt = \sec^2 x dx$ then it reduces to, $\int \frac{1}{\sqrt{t}} dt = 2t^{1/2} + c = 2\sqrt{\tan x} + c$

Example: 17 $\int \frac{x^5}{\sqrt{1+x^3}} dx$

(a) $\frac{2}{9}(1+x^3)^{3/2} + c$

(b) $\frac{2}{9}(1+x^3)^{3/2} + \frac{2}{3}(1+x^3)^{1/2} + c$

(c) $\frac{2}{9}(1+x^3)^{3/2} - \frac{2}{3}(1+x^3)^{1/2} + c$

(d) None of these

Solution: (c) $\int \frac{x^5}{\sqrt{1+x^3}} dx = \int \frac{x^3 \cdot x^2}{\sqrt{1+x^3}} dx$. Put $1+x^3 = t \Rightarrow 3x^2 dx = dt \Rightarrow x^2 dx = \frac{dt}{3}$
 $= \frac{1}{3} \int \frac{(t-1)}{\sqrt{t}} dt = \frac{1}{3} \int \left(\sqrt{t} - \frac{1}{\sqrt{t}} \right) dt = \frac{1}{3} \left[\frac{t^{3/2}}{3/2} - 2\sqrt{t} \right] + c = \frac{2}{9}(1+x^3)^{3/2} - \frac{2}{3}(1+x^3)^{1/2} + c$

(7) Standard substitutions

	Integrand form	Substitution
(i)	$\sqrt{a^2 - x^2}$, $\frac{1}{\sqrt{a^2 - x^2}}$, $a^2 - x^2$	$x = a \sin \theta$, $x = a \cos \theta$
(ii)	$\sqrt{x^2 + a^2}$, $\frac{1}{\sqrt{x^2 + a^2}}$, $x^2 + a^2$	$x = a \tan \theta$ or $x = a \sinh \theta$
(iii)	$\sqrt{x^2 - a^2}$, $\frac{1}{\sqrt{x^2 - a^2}}$, $x^2 - a^2$	$x = a \sec \theta$ or $x = a \cosh \theta$
(iv)	$\frac{x}{\sqrt{a+x}}$, $\sqrt{\frac{a+x}{x}}$, $\sqrt{x(a+x)}$, $\frac{1}{\sqrt{x(a+x)}}$	$x = a \tan^2 \theta$
(v)	$\frac{x}{\sqrt{a-x}}$, $\sqrt{\frac{a-x}{x}}$, $\sqrt{x(a-x)}$, $\frac{1}{\sqrt{x(a-x)}}$	$x = a \sin^2 \theta$
(vi)	$\frac{x}{\sqrt{x-a}}$, $\sqrt{\frac{x-a}{x}}$, $\sqrt{x(x-a)}$, $\frac{1}{\sqrt{x(x-a)}}$	$x = a \sec^2 \theta$
(vii)	$\frac{a-x}{\sqrt{a+x}}$, $\sqrt{\frac{a+x}{a-x}}$	$x = a \cos 2\theta$
(viii)	$\frac{x-\alpha}{\sqrt{\beta-x}}$, $\sqrt{(x-\alpha)(\beta-x)}$, $(\beta > \alpha)$	$x = \alpha \cos^2 \theta + \beta \sin^2 \theta$

Example: 18 $\int \frac{dx}{(a^2 + x^2)^{3/2}}$ is equal to

(a) $\frac{x}{(a^2 + x^2)^{1/2}} + c$

(b) $\frac{x}{a^2(a^2 + x^2)^{1/2}} + c$

(c) $\frac{1}{a^2(a^2 + x^2)^{1/2}} + c$

(d) None of these

Solution: (b) $I = \int \frac{dx}{(a^2 + x^2)^{3/2}}$

Put $x = a \tan \theta \Rightarrow dx = a \sec^2 \theta d\theta$

$$\therefore I = \int \frac{a \sec^2 \theta d\theta}{(a^2 + a^2 \tan^2 \theta)^{3/2}} = \int \frac{a \sec^2 \theta d\theta}{a^3 (\sec^2 \theta)^{3/2}} = \frac{1}{a^2} \int \frac{\sec^2 \theta d\theta}{\sec^3 \theta}$$

$$\Rightarrow I = \frac{1}{a^2} \int \frac{d\theta}{\sec \theta} = \frac{1}{a^2} \int \cos \theta d\theta \Rightarrow I = \frac{1}{a^2} \sin \theta + c \Rightarrow \frac{x}{a^2(x^2 + a^2)^{1/2}} + c$$

Example: 19 $\int \sqrt{\frac{1-x}{1+x}} dx =$

(a) $\sin^{-1} x - \frac{1}{2}\sqrt{1-x^2} + C$ (b) $\sin^{-1} x + \frac{1}{2}\sqrt{1-x^2} + C$ (c) $\sin^{-1} x - \sqrt{1-x^2} + C$ (d) $\sin^{-1} x + \sqrt{1-x^2} + C$

Solution: (d) Put $x = \cos 2\theta$, then $\theta = \frac{1}{2} \cos^{-1} x \Rightarrow dx = -2 \sin 2\theta d\theta$

$$\therefore I = -2 \int \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} \cdot \sin 2\theta d\theta \Rightarrow I = -2 \int \sqrt{\frac{2\sin^2 \theta}{2\cos^2 \theta}} \cdot \sin 2\theta d\theta$$

$$\Rightarrow I = -2 \int \tan \theta \cdot 2 \sin \theta \cos \theta d\theta \Rightarrow I = -2 \cdot 2 \int \sin^2 \theta d\theta \Rightarrow I = -2 \int (1 - \cos 2\theta) d\theta \Rightarrow I = -2 \left[\theta - \frac{\sin 2\theta}{2} \right] + C_1$$

$$\Rightarrow I = -2\theta + \sin 2\theta + C_1 \Rightarrow I = -\cos^{-1} x + \sqrt{1-x^2} + C_1 \Rightarrow I = -\frac{\pi}{2} + \sin^{-1} x + \sqrt{1-x^2} + C_1$$

$$\Rightarrow I = \sin^{-1} x + \sqrt{1-x^2} + C \quad \text{[where } C = C_1 - \frac{\pi}{2}]$$

Trick : Rationalization of denominator and put $1-x^2 = t^2$.

5.6 INTEGRATION BY PARTS

(1) **When integrand involves more than one type of functions :** We may solve such integrals by a rule which is known as integration by parts. We know that,

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx} \Rightarrow d(uv) = u dv + v du \Rightarrow \int d(uv) = \int u dv + \int v du$$

If u and v are two functions of x , then $\int \frac{u v dx}{\cancel{dx}} = u \int \frac{v dx}{\cancel{dx}} - \int \left\{ \frac{du}{dx} \cdot \int v dx \right\} dx$ i.e., the integral of the product of two functions = (First function) $\times$ (Integral of second function) – Integral of {(Differentiation of first function) $\times$ (Integral of second function)}

Integration with the help of above rule is called integration by parts. Before applying this rule proper choice of first and second function is necessary. Normally we use the following methods :

(i) In the product of two functions, one of the function is not directly integrable (i.e., $\log|x|, \sin^{-1} x, \cos^{-1} x, \tan^{-1} x$etc), then we take it as the first function and the remaining function is taken as the second function.

(ii) If there is no other function, then unity is taken as the second function e.g. In the integration of $\int \sin^{-1} x dx, \int \log x dx, 1$ is taken as the second function.

(iii) If both of the function are directly integrable then the first function is chosen in such a way that the derivative of the function thus obtained under integral sign is easily integrable.

Usually, we use the following preference order for the first function. (Inverse, Logarithmic, Algebraic, Trigonometric, exponential). This rule is simply called as “I LATE”.

Important Tips

- ☞ If $I_n = \int x^n \cdot e^{ax} dx$ then $I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$
- ☞ If $I_n = \int (\log x) dx$ then $I_n = x \log x - x$
- ☞ If $I_n = \int \frac{1}{\log x} dx$ then $I_n = \log(\log x) + \log x + \frac{(\log x)^2}{2 \cdot (2!)} + \frac{(\log x)^3}{3(3!)} + \dots$
- ☞ If $I_n = \int (\log x)^n dx$; then $I_n = x(\log x)^n - n I_{n-1}$
- ☞ Successive integration by parts can be performed when one of the functions is x^n (n is positive integer) which will be successively differentiated and the other is either of the following $\sin ax, \cos ax, e^{ax}, e^{-ax}, (x+a)^m$ which will be successively integrated.
- ☞ **Chain rule :** $\int u \cdot v dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots + (-1)^{n-1} u^{n-1}v_n + (-1)^n \int u^n \cdot v_n dx$ Where u^n stands for n^{th} differential coefficient of u and v_n stands for n^{th} integral of v.

Example: 20 If $I_n = \int (\log x)^n dx$ then $I_n + n I_{n-1} =$

- (a) $x(\log x)^n$ (b) $(x \log x)^n$ (c) $(\log x)^{n-1}$ (d) $n(\log x)^n$

Solution: (a) $I_n = \int (\log x)^n dx, \therefore I_{n-1} = \int (\log x)^{n-1} dx$

$$\text{Now } I_n = \int (\log x)^n \cdot 1 dx = (\log x)^n \cdot x - n \int (\log x)^{n-1} \cdot \frac{1}{x} \cdot x dx = (\log x)^n \cdot x - n \int (\log x)^{n-1} dx$$

$$I_n = x(\log x)^n - n I_{n-1} \therefore I_n + n I_{n-1} = x(\log x)^n$$

Example: 21 If $\int x \sin x dx = -x \cos x + A$, then $A =$

- (a) $\sin x + \text{constant}$ (b) $\cos x + \text{constant}$ (c) Constant (d) None of these

Solution: (a) Since $\int x \sin x dx = -x \cos x + A \Rightarrow -x \cos x + \sin x + \text{constant} = -x \cos x + A$

Equating it, we get $A = \sin x + \text{constant}$

Example: 22 $\int x \sec^2 x dx =$

- (a) $\tan x + \log \cos x + c$ (b) $\frac{x^2}{2} \sec^2 x + \log \cos x + c$ (c) $x \tan x + \log \sec x + c$ (d) $x \tan x + \log \cos x + c$

Solution: (d) $\int x \sec^2 x dx = x \tan x - \int \tan x dx = x \tan x + \log(\cos x) + c$

Example: 23 $\int [f(x)g'(x) - f'(x)g(x)] dx$ is equal to

- (a) $\frac{f(x)}{g'(x)}$ (b) $f(x)g(x) - f(x)g'(x)$ (c) $f(x)g'(x) - f'(x)g(x)$ (d) $f(x)g'(x) + f'(x)g(x)$

Solution: (c) $\int [f(x)g'(x) - f'(x)g(x)] dx = \int f(x)g'(x) dx - \int f'(x)g(x) dx$

$$= [f(x)g'(x) - \int f(x)g'(x) dx] - [g(x)f'(x) - \int g'(x)f(x) dx] = f(x)g'(x) - f'(x)g(x)$$

Example: 24 $\int \frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}} dx$ is equal to

- (a) $\frac{2}{\pi} [(2x-1) \sin^{-1} \sqrt{x} + \sqrt{x(1-x)}] - x + c$ (b) $\frac{2}{\pi} [(2x-1) \sin^{-1} \sqrt{x} - \sqrt{x(1-x)}] + x + c$
 (c) $\frac{\pi}{2} [(2x-1) \sin^{-1} \sqrt{x} + \sqrt{x(1-x)}] - x + c$ (d) None of these

Solution: (a) $\int \frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}} dx = \frac{2}{\pi} \left[\int \sin^{-1} \sqrt{x} dx - \int \cos^{-1} \sqrt{x} dx \right] \left[\ominus \sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x} = \frac{\pi}{2} \right]$

Now we solve first and second expressions separately. For first expression, $\int \sin^{-1} \sqrt{x} dx$

Put $x = \sin^2 \theta \Rightarrow \cos 2\theta = 1 - 2x \Rightarrow dx = \sin 2\theta d\theta$

$$\int \theta \cdot \sin 2\theta d\theta = \frac{-\theta \cos 2\theta}{2} + \frac{1}{2} \int \cos 2\theta d\theta = \frac{-\theta \cos 2\theta}{2} + \frac{\sin 2\theta}{4} + c_1 = \frac{(2x-1)}{2} \sin^{-1} \sqrt{x} + \frac{1}{2} \sqrt{x(1-x)} + c_1$$

For second expression, $\int \cos^{-1} \sqrt{x} dx$

Put $x = \cos^2 \theta \Rightarrow dx = -2\cos\theta \sin\theta d\theta = -\sin 2\theta d\theta$

$$\int \cos^{-1} \sqrt{x} dx = -\int \theta \cdot \sin 2\theta d\theta = \frac{\theta \cos 2\theta}{2} - \frac{\sin 2\theta}{4} + c_2$$

Therefore, $I = \frac{2}{\pi} \left[\frac{(2x-1)}{2} \{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}\} + \sqrt{x(1-x)} \right] + c$

$$I = \frac{2}{\pi} (2x-1) \sin^{-1} \sqrt{x} - \frac{(2x-1)}{2} \cdot \frac{\pi}{2} + \sqrt{x(1-x)} + c \Rightarrow I = \frac{2}{\pi} [(2x-1) \sin^{-1} \sqrt{x} + \sqrt{x(1-x)}] - x + c$$

Example: 25 $\int \frac{x^2 dx}{(x \sin x + \cos x)^2} =$

- (a) $\frac{\sin x + \cos x}{x \sin x + \cos x}$ (b) $\frac{x \sin x - \cos x}{x \sin x + \cos x}$ (c) $\frac{\sin x - x \cos x}{x \sin x + \cos x}$ (d) None of these

Solution: (c) Differentiation of $x \sin x + \cos x$ is $x \cos x$. Then, $I = \int \frac{x^2 dx}{(x \sin x + \cos x)^2} = \int \frac{x \cos x}{(x \sin x + \cos x)^2} \cdot \frac{x}{\cos x} dx$

Integrate by parts, $\left[\ominus \int \frac{1}{t^2} dt = -\frac{1}{t} \right]$

$$\therefore I = \frac{-1}{(x \sin x + \cos x)} \cdot \left(\frac{x}{\cos x} \right) + \int \frac{1}{(x \sin x + \cos x)} \cdot \frac{\cos x \cdot 1 - x(-\sin x)}{\cos^2 x} dx \Rightarrow I = \frac{-1}{x \sin x + \cos x} \cdot \frac{x}{\cos x} + \int \sec^2 x dx$$

$$I = \frac{-1}{x \sin x + \cos x} \cdot \frac{x}{\cos x} + \frac{\sin x}{\cos x} \Rightarrow I = \frac{-x + x \sin^2 x + \sin x \cos x}{(x \sin x + \cos x) \cos x} \Rightarrow I = \frac{\sin x \cos x - x(1 - \sin^2 x)}{(x \sin x + \cos x) \cos x} = \frac{\sin x - x \cos x}{x \sin x + \cos x}$$

(2) **Integral is of the form** $\int e^x \{f(x) + f'(x)\} dx$: If the integral is of the form $\int e^x \{f(x) + f'(x)\} dx$, then by breaking this integral into two integrals integrate one integral by parts and keeping other integral as it is, by doing so, we get

$$(i) \int e^x [f(x) + f'(x)] dx = e^x f(x) + c$$

$$(ii) \int e^{mx} [mf(x) + f'(x)] dx = e^{mx} f(x) + c$$

$$(iii) \int e^{mx} \left[f(x) + \frac{f'(x)}{m} \right] dx = \frac{e^{mx} f(x)}{m} + c$$

Example: 26 $\int e^x(1 - \cot x + \csc^2 x) dx$

- (a) $e^x \cot x + c$ (b) $e^x \csc x + c$ (c) $-e^x \cot x + c$ (d) $e^x \csc x + c$

Solution: (c) $\int e^x(\csc^2 x - \cot x) dx = \int e^x[-\cot x + \csc^2 x] = -e^x \cot x + c$

Example: 27 $\int e^x \left(\frac{1 - \sin x}{1 - \cos x} \right) dx$

- (a) $-e^x \tan \frac{x}{2} + c$ (b) $-e^x \cot \frac{x}{2} + c$ (c) $-\frac{1}{2} e^x \tan \frac{x}{2} + c$ (d) $\frac{1}{2} e^x \cot \frac{x}{2} + c$

Solution: (b) $\ominus \frac{1 - \sin x}{1 - \cos x} = \frac{1}{2 \sin^2 \frac{x}{2}} - \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \sin^2 \frac{x}{2}} = \frac{1}{2} \csc^2 \frac{x}{2} - \cot \frac{x}{2}$

$$\therefore \int e^x \left(\frac{1 - \sin x}{1 - \cos x} \right) dx = \int e^x \left(\frac{1}{2} \csc^2 \frac{x}{2} - \cot \frac{x}{2} \right) dx = -e^x \cot \frac{x}{2} + c$$

Example: 28 $\int \frac{e^{2 \tan^{-1} x} (1 + x)^2}{1 + x^2} dx =$

- (a) $x e^{\tan^{-1} x} + c$ (b) $x e^{2 \tan^{-1} x} + c$ (c) $2x e^{\tan^{-1} x} + c$ (d) None of these

Solution: (b) Put $\tan^{-1} x = t \Rightarrow \frac{1}{1 + x^2} dx = dt$

$$\therefore I = \int \frac{e^{2 \tan^{-1} x} (1 + x)^2}{1 + x^2} dx = \int e^{2t} (1 + \tan t)^2 dt = \int e^{2t} (\sec^2 t + 2 \tan t) dt$$

$$I = \int e^{2t} \sec^2 t dt + 2 \left[\tan t \frac{e^{2t}}{2} - \int \sec^2 t \frac{e^{2t}}{2} dt \right] \Rightarrow I = \int e^{2t} \sec^2 t dt + e^{2t} \tan t - \int e^{2t} \sec^2 t dt + c$$

$$I = e^{2t} \tan t + c \Rightarrow I = x e^{2 \tan^{-1} x} + c$$

Example: 29 $\int \frac{(x+3)e^x}{(x+4)^2} dx$ is equal to

- (a) $\frac{e^x}{x+4} + c$ (b) $\frac{e^x}{x+3} + c$ (c) $\frac{1}{(x+4)^2} + c$ (d) $\frac{e^x}{(x+4)^2} + c$

Solution: (a) $\int \frac{(x+3)e^x}{(x+4)^2} dx = \int \frac{(x+4-1)e^x}{(x+4)^2} dx = \int \frac{(x+4)e^x}{(x+4)^2} dx - \int \frac{e^x}{(x+4)^2} dx = \int \frac{e^x}{(x+4)} dx - \int \frac{e^x}{(x+4)^2} dx$

$$= \left[\frac{e^x}{(x+4)} + \int \frac{e^x}{(x+4)^2} dx \right] - \int \frac{e^x}{(x+4)^2} dx + c = \frac{e^x}{x+4} + c$$

(3) **Integral is of the form $[x f(x) + f(x)] dx$:** If the integral is of the form $\int [x f(x) + f(x)] dx$ then by breaking this integral into two integrals, integrate one integral by parts and keeping other integral as it is, by doing so, we get, $\int [x f(x) + f(x)] dx = x f(x) + c$

Example: 30 If $\frac{d}{dx} f(x) = x \cos x + \sin x$ and $f(0) = 2$, then $f(x) =$

- (a) $x \sin x$ (b) $x \cos x + \sin x + 2$ (c) $x \sin x + 2$ (d) $x \cos x + 2$

Solution: (c) $\frac{d}{dx} f(x) = x \cos x + \sin x \Rightarrow f(x) = \int (x \cos x + \sin x) dx = x \sin x + c$

Since $f(0) = 2 \Rightarrow c = 2$, $\therefore f(x) = x \sin x + 2$

Example: 31 $\int \frac{x + \sin x}{1 + \cos x} dx =$

- (a) $-x \tan x / 2 + c$ (b) $x \tan x / 2 + c$ (c) $x \tan x + c$ (d) $\frac{1}{2} x \tan x + c$

Solution: (b) $\int \frac{x + \sin x}{1 + \cos x} dx = \frac{1}{2} \int x \sec^2 \frac{x}{2} dx + \int \tan \frac{x}{2} dx = \frac{1}{2} \frac{x \tan x / 2}{1/2} - \int \tan \frac{x}{2} dx + \int \tan \frac{x}{2} dx = x \tan \frac{x}{2} + c$

Trick : By inspection, $\frac{d}{dx} \left\{ x \tan \frac{x}{2} + c \right\} = x \sec^2 \frac{x}{2} + \tan \frac{x}{2} = \frac{1}{2} \left[\frac{x}{\cos^2 x/2} + \frac{2 \sin x / 2}{\cos x / 2} \right] = \frac{x + \sin x}{1 + \cos x}$

Hence the result.

(4) Integrals of the form $\int e^{ax} \sin bx dx, \int e^{ax} \cos bx dx$:

Working rule : To evaluate $\int e^{ax} \sin bx dx$ or $\int e^{ax} \cos bx dx$ proceed as follows

- Put the given integral equal to I .
- Integrate by parts, taking e^{ax} as the first function.
- Again, integrate by parts taking e^{ax} as the first function. This will involve I .
- Transpose and collect terms involving I and then obtain the value of I .

$$\begin{aligned} \text{Let } I &= \int e^{ax} \sin bx dx. \text{ Then } I = \int e^{ax} \cdot \sin bx dx = -e^{ax} \cdot \frac{\cos bx}{b} - \int a e^{ax} \cdot \left(\frac{-\cos bx}{b} \right) dx \\ &= -\frac{1}{b} e^{ax} \cdot \cos bx + \frac{a}{b} \int e^{ax} \cdot \cos bx dx = -\frac{1}{b} e^{ax} \cdot \cos bx + \frac{a}{b} \left[\frac{e^{ax} \cdot \sin bx}{b} - \int a e^{ax} \cdot \frac{\sin bx}{b} dx \right] \\ &= -\frac{1}{b} e^{ax} \cdot \cos bx + \frac{a}{b^2} e^{ax} \cdot \sin bx - \frac{a^2}{b^2} \int e^{ax} \cdot \sin bx dx = -\frac{1}{b} e^{ax} \cdot \cos bx + \frac{a}{b^2} e^{ax} \cdot \sin bx - \frac{a^2}{b^2} I \\ I + I \cdot \frac{a^2}{b^2} &= \frac{e^{ax}}{b^2} (a \sin bx - b \cos bx) \Rightarrow I = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + c \end{aligned}$$

$$\text{Thus, } \int e^{ax} \sin bx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + c = \frac{e^{ax}}{\sqrt{a^2 + b^2}} \sin \left(bx - \tan^{-1} \frac{b}{a} \right) + c$$

$$\text{Similarly } \int e^{ax} \cdot \cos bx dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + c = \frac{e^{ax}}{\sqrt{a^2 + b^2}} \cos \left(bx - \tan^{-1} \frac{b}{a} \right) + c$$

$$\begin{aligned} \text{Note : } \square \int e^{ax} \cdot \sin (bx + c) dx &= \frac{e^{ax}}{a^2 + b^2} [a \sin (bx + c) - b \cos (bx + c)] + k = \frac{e^{ax}}{\sqrt{a^2 + b^2}} \sin \left[(bx + c) - \tan^{-1} \left(\frac{b}{a} \right) \right] + k \\ \int e^{ax} \cdot \cos (bx + c) dx &= \frac{e^{ax}}{a^2 + b^2} [a \cos (bx + c) + b \sin (bx + c)] + k = \frac{e^{ax}}{\sqrt{a^2 + b^2}} \cos \left[(bx + c) - \tan^{-1} \left(\frac{b}{a} \right) \right] + k \end{aligned}$$

Important Tips

$$\oint \int x e^{ax} \sin (bx + c) dx = \frac{x e^{ax}}{a^2 + b^2} [a \sin (bx + c) - b \cos (bx + c)] - \frac{e^{ax}}{(a^2 + b^2)^2} [(a^2 - b^2) \sin bx (bx + c) - 2ab \cos bx (bx + c)] + k$$

$$\oint \int x \cdot e^{ax} \cos (bx + c) dx = \frac{x \cdot e^{ax}}{a^2 + b^2} [a \cos (bx + c) + b \sin (bx + c)] - \frac{e^{ax}}{(a^2 + b^2)^2} [(a^2 - b^2) \cos bx (bx + c) + 2ab \sin bx (bx + c)] + k$$

$$\int a^x \cdot \sin(bx + c) dx = \frac{a^x}{(\log a)^2 + b^2} [(\log a) \sin(bx + c) - b \cos(bx + c)] + k$$

$$\int a^x \cdot \cos(bx + c) dx = \frac{a^x}{(\log a)^2 + b^2} [(\log a) \cos(bx + c) + b \sin(bx + c)] + k$$

Example: 32 If $\int e^x \sin x dx = \frac{1}{2} e^x \cdot a + c$, then $a =$

(a) $\sin x - \cos x$

(b) $\cos x - \sin x$

(c) $\tan x + c$

(d) None of these

Solution: (a) $\int e^x \sin x dx = \frac{e^x}{1^2 + 1^2} (1 \cdot \sin x - 1 \cdot \cos x) + c = \frac{e^x}{2} (\sin x - \cos x) + c$

Clearly $a = (\sin x - \cos x)$

Example: 33 If $u = \int e^{ax} \cos bx dx$ and $v = \int e^{ax} \sin bx dx$, then $(a^2 + b^2)(u^2 + v^2) =$

(a) $2e^{ax}$

(b) $(a^2 + b^2)e^{2ax}$

(c) e^{2ax}

(d) $(a^2 - b^2)e^{2ax}$

Solution: (c) $u = \int e^{ax} \cos bx dx = e^{ax} \frac{\sin bx}{b} - \frac{a}{b} \int e^{ax} \cdot bx dx = \frac{e^{ax} \sin bx}{b} - \frac{a}{b} v$

$\Rightarrow bu + av = e^{ax} \sin bx$ (i)

Similarly, $bv - au = -e^{ax} \cos bx$ (ii)

Squaring (i) and (ii) and adding. We get, $(a^2 + b^2)(u^2 + v^2) = e^{2ax}$.

5.7 EVALUATION OF THE VARIOUS FORMS OF INTEGRALS BY USE OF STANDARD RESULTS

(1) Integral of the form $\int \frac{dx}{ax^2 + bx + c}$, where $ax^2 + bx + c$ can not be resolved into factors.

(2) Integral of the form $\int \frac{px + q}{ax^2 + bx + c} dx$

(3) Integral of the form $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$.

(4) Integral of the form $\int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx$

(5) Integral of the form $\int \frac{f(x)}{ax^2 + bx + c} dx$, where $f(x)$ is a polynomial of degree 2 or greater than 2.

(6) Integral of the form (i) $\int \frac{x^2 + 1}{x^4 + kx^2 + 1} dx$, (ii) $\int \frac{x^2 - 1}{x^4 + kx^2 + 1} dx$, where k is any constant

(7) Integral of the form $\int \sqrt{ax^2 + bx + c} dx$

(8) Integral of the form $\int (px + q)\sqrt{ax^2 + bx + c} dx$

(9) Integral of the form $\int \frac{dx}{P\sqrt{Q}}$

(1) Integrals of the form $\int \frac{dx}{ax^2 + bx + c}$, where $ax^2 + bx + c$ can not be resolved into factors.

$$\text{We have, } ax^2 + bx + c = a \left(x^2 + \frac{b}{a}x + \frac{c}{a} \right) = a \left[\left(x + \frac{b}{2a} \right)^2 - \left(\frac{b^2}{4a^2} - \frac{c}{a} \right) \right] = a \left[\left(x + \frac{b}{2a} \right)^2 - \left(\frac{b^2 - 4ac}{4a^2} \right) \right]$$

Case (i) : When $b^2 - 4ac > 0$

$$\begin{aligned} \therefore \int \frac{dx}{ax^2 + bx + c} &= \frac{1}{a} \int \frac{dx}{\left(x + \frac{b}{2a} \right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a} \right)^2}, & \left[\text{form} \int \frac{dx}{x^2 - a^2} \right] \\ &= \frac{1}{a} \cdot \frac{1}{2 \cdot \frac{\sqrt{b^2 - 4ac}}{2a}} \log \left| \frac{x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}}{x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}} \right| + c = \frac{1}{\sqrt{b^2 - 4ac}} \log \left| \frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}} \right| + c \end{aligned}$$

Case (ii) : When $b^2 - 4ac < 0$

$$\begin{aligned} \int \frac{dx}{ax^2 + bx + c} &= \frac{1}{a} \int \frac{dx}{\left(x + \frac{b}{2a} \right)^2 + \left(\frac{\sqrt{4ac - b^2}}{2a} \right)^2}, & \left[\text{form} \int \frac{dx}{x^2 + a^2} \right] \\ &= \frac{1}{a} \cdot \frac{1}{\frac{\sqrt{4ac - b^2}}{2a}} \tan^{-1} \left[\frac{x + \frac{b}{2a}}{\frac{\sqrt{4ac - b^2}}{2a}} \right] + c = \frac{2}{\sqrt{4ac - b^2}} \tan^{-1} \left[\frac{2ax + b}{\sqrt{4ac - b^2}} \right] + c \end{aligned}$$

Working rule for evaluating $\int \frac{dx}{ax^2 + bx + c}$: To evaluate this form of integrals proceed as follows :

- (i) Make the coefficient of x^2 unity by taking 'a' common from $ax^2 + bx + c$
- (ii) Express the terms containing x^2 and x in the form of a perfect square by adding and subtracting the square of half of the coefficient of x .
- (iii) Put the linear expression in x equal to t and express the integrals in terms of t .
- (iv) The resultant integrand will be either in $\int \frac{dx}{x^2 + a^2}$ or $\int \frac{dx}{x^2 - a^2}$ or $\int \frac{dx}{a^2 - x^2}$ standard form. After using the standard formulae, express the results in terms of x .

Example: 34 $\int \frac{dx}{2x^2 + x + 1} =$

(a) $\frac{1}{\sqrt{7}} \tan^{-1} \left(\frac{4x+1}{\sqrt{7}} \right) + c$ (b) $\frac{1}{2\sqrt{7}} \tan^{-1} \left(\frac{4x+1}{\sqrt{7}} \right) + c$ (c) $\frac{1}{2} \tan^{-1} \left(\frac{4x+1}{\sqrt{7}} \right) + c$ (d) None of these

Solution: (d) $I = \frac{1}{2} \int \frac{dx}{x^2 + \frac{x}{2} + \frac{1}{2}} \Rightarrow I = \frac{1}{2} \int \frac{dx}{\left(x + \frac{1}{4} \right)^2 + \left(\frac{\sqrt{7}}{4} \right)^2}$

$$I = \frac{1}{2} \cdot \frac{4}{\sqrt{7}} \cdot \tan^{-1} \left[\frac{x + \frac{1}{4}}{(\sqrt{7}/4)} \right] + c \Rightarrow I = \frac{2}{\sqrt{7}} \tan^{-1} \left[\frac{4x+1}{\sqrt{7}} \right] + c$$

Example: 35 $\int \frac{\cos x}{\sin^2 x + 4 \sin x + 5} dx$ equals

- (a) $\tan^{-1}(\sin x) + c$ (b) $\tan^{-1}(\sin x + 2) + c$ (c) $\tan^{-1}(\sin x + 1) + c$ (d) None of these

Solution: (b) $I = \int \frac{\cos x}{\sin^2 x + 4 \sin x + 5} dx = \int \frac{\cos x}{(\sin x + 2)^2 + 1} dx$

Put $\sin x + 2 = t \Rightarrow \cos x dx = dt$

$$I = \int \frac{dt}{t^2 + 1} = \tan^{-1} t + c = \tan^{-1}(\sin x + 2) + c.$$

(2) **Integral of the form** $\int \frac{px + q}{ax^2 + bx + c} dx$: The integration of the function $\frac{px + q}{ax^2 + bx + c}$ is effected by breaking $px + q$ into two parts such that one part is the differential coefficient of the denominator and the other part is a constant.

If M and N are two constants, then we express $px + q$ as

$$px + q = M \frac{d}{dx}(ax^2 + bx + c) + N = M(2ax + b) + N = (2aM)x + Mb + N.$$

Comparing the coefficients of x and constant terms on both sides, we have, $p = 2aM \Rightarrow M = \frac{p}{2a}$ and

$$q = Mb + N \Rightarrow N = q - Mb = q - \frac{p}{2a} b.$$

$$\begin{aligned} \text{Thus, } M \text{ and } N \text{ are known. Hence, the given integral is } & \int \frac{px + q}{ax^2 + bx + c} dx = \int \frac{\frac{p}{2a}(2ax + b) + \left(q - \frac{p}{2a}b\right)}{ax^2 + bx + c} dx \\ & = \frac{p}{2a} \int \frac{2ax + b}{ax^2 + bx + c} dx + \left(q - \frac{p}{2a}b\right) \int \frac{dx}{ax^2 + bx + c} = \frac{p}{2a} \log|ax^2 + bx + c| + \left(q - \frac{p}{2a}b\right) \int \frac{dx}{ax^2 + bx + c} + C \end{aligned}$$

The integral on R.H.S. can be evaluated by the method discussed in previous section.

$$(i) \text{ If } b^2 - 4ac < 0, \text{ then } \int \frac{px + q}{ax^2 + bx + c} dx = \frac{p}{2a} \log|ax^2 + bx + c| + \frac{(2aq - bp)}{a\sqrt{4ac - b^2}} \tan^{-1} \frac{2ax + b}{\sqrt{4ac - b^2}} + k$$

$$(ii) \text{ If } b^2 - 4ac > 0, \text{ then } \int \frac{px + q}{ax^2 + bx + c} dx = \frac{p}{2a} \log|ax^2 + bx + c| + \frac{(2aq - bp)}{2a\sqrt{b^2 - 4ac}} \log \left| \frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}} \right| + k$$

Example: 36 $\int \frac{2\sin 2\theta - \cos \theta}{6 - \cos^2 \theta - 4\sin \theta} d\theta =$

- (a) $2\log|\sin^2 \theta - 4\sin \theta + 5| + 7\tan^{-1}(\sin \theta - 2) + c$ (b) $2\log|\sin^2 \theta - 4\sin \theta + 5| - 7\tan^{-1}(\sin \theta - 2) + c$
 (c) $-2\log|\sin^2 \theta - 4\sin \theta + 5| + 7\tan^{-1}(\sin \theta - 2) + c$ (d) $-2\log|\sin^2 \theta - 4\sin \theta + 5| - 7\tan^{-1}(\sin \theta - 2) + c$

Solution: (a) $I = \int \frac{2(2\sin \theta \cos \theta) - \cos \theta}{6 - (1 - \sin^2 \theta) - 4\sin \theta} d\theta \Rightarrow I = \int \frac{(4\sin \theta - 1)\cos \theta}{\sin^2 \theta - 4\sin \theta + 5} d\theta$

$$\text{Put } \sin \theta = t \Rightarrow \cos \theta d\theta = dt, \therefore I = \int \frac{4t - 1}{t^2 - 4t + 5} dt$$

$$\text{Let } 4t - 1 = M \frac{d}{dt}(t^2 - 4t + 5) + N \Rightarrow 4t - 1 = M(2t - 4) + N$$

Comparing the coefficient of t and constant terms on both side, then $M=2, N=7$

$$\therefore I = \int \frac{2(2t-4)+7}{t^2-4t+5} dt$$

$$\Rightarrow I = 2 \int \frac{2t-4}{t^2-4t+5} dt + \int \frac{7dt}{t^2-4t+5} \Rightarrow I = 2 \log|t^2-4t+5| + 7 \int \frac{dt}{(t-2)^2+1}$$

$$\Rightarrow I = 2 \log|t^2-4t+5| + 7 \tan^{-1}(t-2) + c = 2 \log|\sin^2 \theta - 4 \sin \theta + 5| + 7 \tan^{-1}(\sin \theta - 2) + c$$

(3) **Integral of the form** $\int \frac{dx}{\sqrt{ax^2+bx+c}}$: To evaluate this form of integrals proceed as follows :

(i) Make the coefficient of x^2 unity by taking $\sqrt{a}$ common from $\sqrt{ax^2+bx+c}$.

$$\text{Then, } \int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}} \int \frac{dx}{\sqrt{x^2 + \frac{b}{a}x + \frac{c}{a}}}.$$

(ii) Put $x^2 + \frac{b}{a}x + \frac{c}{a}$, by the method of completing the square in the form, $\sqrt{A^2 - X^2}$ or $\sqrt{X^2 + A^2}$ or $\sqrt{X^2 - A^2}$ where, X is a linear function of x and A is a constant.

(iii) After this, use any of the following standard formulae according to the case under consideration

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c \Rightarrow \int \frac{dx}{\sqrt{x^2 + a^2}} = \log|x + \sqrt{x^2 + a^2}| + c \text{ and } \int \frac{dx}{\sqrt{x^2 - a^2}} = \log|x + \sqrt{x^2 - a^2}| + c$$

$$\text{Note : } \square \text{ If } a < 0, b^2 - 4ac > 0, \text{ then } \int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{-a}} \sin^{-1}\left(\frac{2ax+b}{\sqrt{b^2-4ac}}\right) + k$$

$$\square \text{ If } a > 0, b^2 - 4ac < 0, \text{ then } \int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}} \sinh^{-1}\left[\frac{2ax+b}{\sqrt{4ac-b^2}}\right] + k$$

$$\square \text{ If } a > 0, b^2 - 4ac > 0 \int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}} \cosh^{-1} \frac{2ax+b}{\sqrt{b^2-4ac}} + k$$

Example: 37 $\int \frac{dx}{\sqrt{2-3x-x^2}}$ equals

$$(a) \tan^{-1}\left(\frac{2x+3}{\sqrt{17}}\right) + c \quad (b) \sec^{-1}\left(\frac{2x+3}{\sqrt{17}}\right) + c \quad (c) \sin^{-1}\left(\frac{2x+3}{\sqrt{17}}\right) + c \quad (d) \cos^{-1}\left(\frac{2x+3}{\sqrt{17}}\right) + c$$

$$\text{Solution: (c) } I = \int \frac{dx}{\sqrt{2-3x-x^2}} = \int \frac{dx}{\sqrt{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x + \frac{3}{2}\right)^2}}$$

$$\text{Put } x + 3/2 = t \Rightarrow dx = dt$$

$$I = \int \frac{dx}{\sqrt{\left(\frac{\sqrt{17}}{2}\right)^2 - t^2}} = \sin^{-1}\left(\frac{t}{\sqrt{17}/2}\right) + c = \sin^{-1}\left(\frac{2x+3}{\sqrt{17}}\right) + c$$

Example: 38 $\int \frac{dx}{\sqrt{x^2 - 4x + 2}} =$

(a) $\log|x - 2 + \sqrt{x^2 + 2 - 4x}| + c$

(b) $\log|x - 2 - \sqrt{x^2 + 2 - 4x}| + c$

(c) $\log|x - 2 + \sqrt{x^2 + 2 + 4x}| + c$

(d) $\log|x - 2 - \sqrt{x^2 + 2 + 4x}| + c$

Solution: (a) $I = \int \frac{dx}{\sqrt{(x-2)^2 - 2}} \Rightarrow I = \int \frac{dt}{\sqrt{t^2 - (\sqrt{2})^2}}$

Put $x - 2 = t \Rightarrow dx = dt \Rightarrow I = \log|t + \sqrt{t^2 - 2}| + c \Rightarrow I = \log|x - 2 + \sqrt{x^2 - 4x + 2}| + c$

(4) **Integral of the form** $\int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx$: To evaluate this form of integrals, first we write,

$$px + q = M \frac{d}{dx}(ax^2 + bx + c) + N \Rightarrow px + q = M(2ax + b) + N$$

Where M and N are constants.

By equating the coefficients of x and constant terms on both sides, we get

$$p = 2aM \Rightarrow M = \frac{p}{2a} \text{ and } q = bM + N \Rightarrow N = q - \frac{bp}{2a}$$

In this way, the integral breaks up into two parts given by

$$\int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx = \frac{p}{2a} \int \frac{2ax + b}{\sqrt{ax^2 + bx + c}} dx + \left(q - \frac{bp}{2a} \right) \int \frac{dx}{\sqrt{ax^2 + bx + c}} = I_1 + I_2, \text{ (say)}$$

$$\text{Now, } I_1 = \frac{p}{2a} \int \frac{2ax + b}{\sqrt{ax^2 + bx + c}} dx$$

$$\text{Putting } ax^2 + bx + c \Rightarrow (2ax + b)dx = dt \text{ we have, } I_1 = \frac{p}{2a} \int t^{1/2} dt = \frac{p}{2a} \cdot \frac{t^{3/2}}{3/2} + C_1 = \frac{p}{a} \sqrt{ax^2 + bx + c} + C_1$$

and I_2 is calculated as in the previous section.

$$\text{Note : } \square \int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx = \frac{p}{a} \sqrt{ax^2 + bx + c} + \frac{2aq - bp}{2a} \int \frac{dx}{\sqrt{ax^2 + bx + c}}.$$

Example: 39 $\int \frac{5 - 2x}{\sqrt{6 + x - x^2}} dx =$

(a) $2\sqrt{6 + x - x^2} - 4 \sin^{-1}\left(\frac{2x-1}{5}\right) + c$

(b) $2\sqrt{6 + x - x^2} + 4 \sin^{-1}\left(\frac{2x-1}{5}\right) + c$

(c) $-2\sqrt{6 + x - x^2} - 4 \sin^{-1}\left(\frac{2x-1}{5}\right) + c$

(d) $-2\sqrt{6 + x - x^2} + 4 \sin^{-1}\left(\frac{2x-1}{5}\right) + c$

Solution: (b) $I = \int \frac{5 - 2x}{\sqrt{6 + x - x^2}} dx$

$$\text{Let } 5 - 2x = M \frac{d}{dx}(6 + x - x^2) + N \Rightarrow 5 - 2x = M(1 - 2x) + N$$

Equating the coefficients of x and constant terms on both sides, we get

$$-2 = -2M \Rightarrow M = 1 \text{ and } 5 = M + N \Rightarrow N = 5 - 1 = 4 \therefore 5 - 2x = (1 - 2x) + 4$$

$$\text{Hence, } I = \int \frac{(1 - 2x) + 4}{\sqrt{6 + x - x^2}} dx = \int \frac{1 - 2x}{\sqrt{6 + x - x^2}} dx + 4 \int \frac{dx}{\sqrt{6 + x - x^2}} = I_1 + 4I_2, \text{ (say)}$$

$$\text{Now, } I_1 = \int \frac{1-2x}{\sqrt{6+x-x^2}} dx$$

Putting $6+x-x^2 = t \Rightarrow (1-2x)dx = dt$ we have,

$$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} + C_1 = 2\sqrt{6+x-x^2} + C_1 \text{ and } I_2 = \int \frac{dx}{\sqrt{6+x-x^2}} = \int \frac{dx}{\sqrt{6-(x^2-x)}}$$

$$I = \int \frac{dx}{\sqrt{6-\left(x^2-x+\frac{1}{4}\right)+\frac{1}{4}}} = \int \frac{dx}{\sqrt{\frac{25}{4}-\left(x-\frac{1}{2}\right)^2}} = \int \frac{du}{\sqrt{\left(\frac{5}{2}\right)^2-u^2}} \quad \left(\text{where, } u = x - \frac{1}{2}\right)$$

$$= \sin^{-1}\left(\frac{u}{5/2}\right) + C_2 = \sin^{-1}\left[\frac{2}{5}\left(x - \frac{1}{2}\right)\right] + C_2 = \sin^{-1}\left(\frac{2x-1}{5}\right) + C_2$$

$$\therefore I = I_1 + 4I_2 = 2\sqrt{6+x-x^2} + 4\sin^{-1}\left(\frac{2x-1}{5}\right) + C \text{ (where, } C = C_1 + 4C_2)$$

Example: 40 $\int \frac{x+2}{\sqrt{x^2-2x+4}}$ equals

$$(a) \sqrt{x^2-2x+4} + 3\sinh^{-1}\left[\frac{(x-1)}{\sqrt{3}}\right] + c$$

$$(b) \sqrt{x^2-2x+4} - 3\sinh^{-1}\left[\frac{(x-1)}{\sqrt{3}}\right] + c$$

$$(c) \sqrt{x^2-2x+4} + 3\cosh^{-1}\left[\frac{(x-1)}{\sqrt{3}}\right] + c$$

$$(d) \sqrt{x^2-2x+4} - 3\cosh^{-1}\left[\frac{(x-1)}{\sqrt{3}}\right] + c$$

Solution: (a) $\int \frac{x+2}{\sqrt{x^2-2x+4}} dx = \int \frac{x+2}{\sqrt{(x-1)^2+(\sqrt{3})^2}} dx = \int \frac{x-1+3}{\sqrt{(x-1)^2+(\sqrt{3})^2}} dx = \int \frac{x-1}{\sqrt{x^2-2x+4}} dx + \int \frac{3 dx}{\sqrt{(x-1)^2+(\sqrt{3})^2}}$

Put $x^2-2x+4 = t^2$ in the first expression $\Rightarrow 2(x-1)dx = 2tdt$

$$\Rightarrow (x-1)dx = tdt$$

$$\int \frac{x+2}{\sqrt{x^2-2x+4}} dx = \int \frac{tdt}{t} + 3 \int \frac{dx}{\sqrt{(x-1)^2+(\sqrt{3})^2}}$$

$$\int \frac{x+2}{\sqrt{x^2-2x+4}} dx = \sqrt{x^2-2x+4} + 3\sinh^{-1}\left[\frac{x-1}{\sqrt{3}}\right] + c$$

(5) Integrals of the form $\int \frac{f(x)}{ax^2+bx+c} dx$, where $f(x)$ is a polynomial of degree 2 or greater than

2:

To evaluate the integrals of the above form, divide the numerator by the denominator. Then, the integrals take the form given by $\frac{f(x)}{ax^2+bx+c} = Q(x) + \frac{R(x)}{ax^2+bx+c} dx$

where, $Q(x)$ is a polynomial and $R(x)$ is a linear polynomial in x .

$$\text{Then, we have } \int \frac{f(x)}{ax^2+bx+c} dx = \int Q(x) dx + \int \frac{R(x)}{ax^2+bx+c} dx$$

The integrals on R.H.S. can be obtained by the methods discussed earlier.

Example: 41 $\frac{(x^3+8)(x-1)}{x^2-2x+4} dx$ equals.

- (a) $\left(\frac{x^3}{3}\right) + \left(\frac{x^2}{2}\right) - 2x + c$ (b) $x^3 + x^2 - 2x + c$ (c) $\frac{(x^3+x^2-x)}{3} + c$ (d) None of these

Solution: (a) $\int \frac{(x^3+8)(x-1)}{x^2-2x+4} dx = \int \frac{(x+2)(x^2-2x+4)(x-1)}{x^2-2x+4} = \int (x+2)(x-1) dx = \int (x^2+x-2) dx = \frac{x^3}{3} + \frac{x^2}{2} - 2x + c$

Example: 42 The value of $\int \frac{2x^3-3x^2+5x+6}{x^2+3x+2} dx$ is

- (a) $x^2+3x+4\ln|x^2+3x+2|+12\ln\frac{x+1}{x+2}+c$ (b) $x+3x^2+4\ln|x+1|-12\ln(x+2)+c$
 (c) $(x^2+3x)+8\ln\frac{|x+1|}{(x+2)^2}+c$ (d) None of these

Solution: (c) $\int \frac{2x^3-3x^2+5x+6}{x^2+3x+2} dx$
 $= \int \left((2x+3) - \frac{8x}{x^2+3x+2} \right) dx = \int (2x+3) - 4 \int \frac{(2x+3) dx}{x^2+3x+2} + 12 \int \frac{dx}{(x+1)(x+2)}$
 $= (x^2+3x) - 4 \ln|x^2+3x+2| + 12 \int \frac{dx}{x+1} - 12 \int \frac{dx}{x+2}$
 $= (x^2+3x) - 4 \ln(x+1) - 4 \ln(x+2) + 12 \ln\frac{x+1}{x+2} + c$
 $= x^2-3x+8\ln(x+1)-16\ln(x+2)+c = (x^2-3x)+8\ln\frac{|x+1|}{(x+2)^2}+c.$

(6) Integrals of the form $\int \frac{x^2+1}{x^4+kx^2+1} dx$ and $\int \frac{x^2-1}{x^4+kx^2+1} dx$:

To evaluate the integral of the form $I = \int \frac{x^2+1}{x^4+kx^2+1} dx$, proceed as follows

(i) Divide the numerator and denominator by x^2 to get $I = \int \frac{1+\frac{1}{x^2}}{x^2+k+\frac{1}{x^2}} dx$

(ii) Put $x - \frac{1}{x} = t \Rightarrow \left(1 + \frac{1}{x^2}\right) dx = dt$ and $x^2 + \frac{1}{x^2} - 2 = t^2 \Rightarrow x^2 + \frac{1}{x^2} = t^2 + 2$.

Then, the given integral reduces to the form $I = \int \frac{dt}{t^2+2+k}$, which can be integrand as usual.

(iii) To evaluate $I = \int \frac{x^2-1}{x^4+kx^2+1} dx$ we divide the numerator and denominator by x^2 and get

$$I = \int \frac{1 - \frac{1}{x^2}}{x^2 + k + \frac{1}{x^2}} dx$$

Then, we put $x + \frac{1}{x} = t \Rightarrow \left(1 - \frac{1}{x^2}\right) dx = dt$ and $x^2 + \frac{1}{x^2} + 2 = t^2 \Rightarrow x^2 + \frac{1}{x^2} = t^2 - 2$.

Thus, we have $t = \int \frac{dt}{t^2 - 2 + k}$, which can be evaluated as usual.

Important Tips

☞ **Algebraic twins:** $\int \frac{2x^2}{x^4 + 1} dx = \int \frac{x^2 + 1}{x^4 + 1} dx + \int \frac{x^2 - 1}{x^4 + 1} dx$

$$\int \frac{2}{x^4 + 1} dx = \int \frac{x^2 + 1}{x^4 + 1} dx - \int \frac{x^2 - 1}{x^4 + 1} dx, \int \frac{2x^2}{x^4 + 1 + kx^2} dx = \int \frac{2}{(x^4 + 1 + kx^2)} dx$$

We know the result of $I_1 = \int \frac{x^2 + 1}{x^4 + 1} dx$ and $I_2 = \int \frac{x^2 - 1}{x^4 + 1} dx$ so for $\int \frac{x^2}{x^4 + 1} dx$ and for $\int \frac{dx}{x^4 + 1}$, we can use the result of $\frac{I_1 + I_2}{2}$ and $\frac{I_1 - I_2}{2}$.

☞ **Trigonometric twins:** $\int \sqrt{\tan x} dx, \int \sqrt{\cot x} dx, \int \frac{dx}{(\sin^4 x + \cos^4 x)}, \int \frac{dx}{\sin^6 x + \cos^6 x}, \int \frac{\pm \sin x \pm \cos x}{a + b \sin x \cos x} dx$

Example: 43 $\int \frac{x^2 - 1}{x^4 + x^2 + 1} dx =$

$$(a) \frac{1}{2} \log \left[\frac{x^2 + x + 1}{x^2 - x + 1} \right] + c \quad (b) \frac{1}{2} \log \left[\frac{x^2 - x - 1}{x^2 + x + 1} \right] + c \quad (c) \log \left[\frac{x^2 - x + 1}{x^2 + x + 1} \right] + c \quad (d) \frac{1}{2} \log \left[\frac{x^2 - x + 1}{x^2 + x + 1} \right] + c$$

Solution: (d) $\int \frac{1 - \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx = \int \frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)^2 - 1} dx = \frac{1}{2} \log \left| \frac{x + \frac{1}{x} - 1}{x + \frac{1}{x} + 1} \right| + c = \frac{1}{2} \log \left| \frac{x^2 + 1 - x}{x^2 + 1 + x} \right| + c = \frac{1}{2} \log \left| \frac{x^2 - x + 1}{x^2 + x + 1} \right| + c$

Example: 44 $\int \sqrt{\tan x} dx =$

$$(a) \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\tan x - 1}{\sqrt{2 \tan x}} \right) + \frac{1}{2\sqrt{2}} \log \left| \frac{\tan x - \sqrt{2 \tan x} + 1}{\tan x + \sqrt{2 \tan x} + 1} \right| + c$$

$$(b) \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\tan x - 1}{\sqrt{2 \tan x}} \right) - \frac{1}{2\sqrt{2}} \log \left| \frac{\tan x - \sqrt{2 \tan x} + 1}{\tan x + \sqrt{2 \tan x} + 1} \right| + c$$

$$(c) \log \left(\frac{\tan x - 1}{\sqrt{2 \tan x}} \right) + c$$

(d) None of these

Solution: (a) Let $I = \int \sqrt{\tan x} dx$

Putting $\sqrt{\tan x} = t \Rightarrow \tan x = t^2$, we have $\sec^2 x dx = 2t dt \Rightarrow (1 + \tan^2 x) dx = 2t dt$

$$\text{i.e., } (1 + t^4) dx = 2t dt \Rightarrow dx = \frac{2t}{1 + t^4} dt$$

$$\therefore I = \int \sqrt{\tan x} dx = \int t \frac{2t}{1 + t^4} dt = \int \frac{2t^2}{1 + t^4} dt = \int \frac{(t^2 + 1) + (t^2 - 1)}{t^4 + 1} dt = \int \frac{t^2 + 1}{t^4 + 1} dt + \int \frac{t^2 - 1}{t^4 + 1} dt$$

$$= \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} dt + \int \frac{1 - \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} dt = \int \frac{1 + \frac{1}{t^2}}{\left(t - \frac{1}{t}\right)^2 + 2} dt + \int \frac{1 - \frac{1}{t^2}}{\left(t + \frac{1}{t}\right)^2 - 2} dt.$$

Again putting $t - \frac{1}{t} = u \Rightarrow \left(1 + \frac{1}{t^2}\right) dt = du$ in the first integral and $t + \frac{1}{t} = v \Rightarrow \left(1 - \frac{1}{t^2}\right) dt = dv$ in the second integral, we have,

$$\begin{aligned} I &= \int \frac{du}{u^2 + (\sqrt{2})^2} + \int \frac{dv}{v^2 - (\sqrt{2})^2} = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{u}{\sqrt{2}} \right) + \frac{1}{2\sqrt{2}} \log \left| \frac{v - \sqrt{2}}{v + \sqrt{2}} \right| + c \\ &= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{t - \frac{1}{t}}{\sqrt{2}} \right) + \frac{1}{2\sqrt{2}} \log \left| \frac{t + \frac{1}{t} - \sqrt{2}}{t + \frac{1}{t} + \sqrt{2}} \right| + c = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{t^2 - 1}{t\sqrt{2}} \right) + \frac{1}{2\sqrt{2}} \log \left| \frac{t^2 - t\sqrt{2} + 1}{t^2 + t\sqrt{2} + 1} \right| + c \\ &= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\tan x - 1}{\sqrt{2} \tan x} \right) + \frac{1}{2\sqrt{2}} \log \left| \frac{\tan x - \sqrt{2} \tan x + 1}{\tan x + \sqrt{2} \tan x + 1} \right| + c \end{aligned}$$

Example: 45 $\int (\sqrt{\tan x} + \sqrt{\cot x}) dx$

(a) $\sqrt{2} \tan^{-1} \left(\frac{\sqrt{\tan x} - \sqrt{\cot x}}{\sqrt{2}} \right) + c$

(b) $\sqrt{2} \tan^{-1} \left(\frac{\sqrt{\tan x} + \sqrt{\cot x}}{\sqrt{2}} \right) + c$

(c) $\tan^{-1} \left(\frac{\sqrt{\tan x} + \sqrt{\cot x}}{\sqrt{2}} \right) + c$

(d) None of these

Solution: (a) $I = \int (\sqrt{\tan x} + \sqrt{\cot x}) dx$

Put $\tan x = t^2 \Rightarrow \sec^2 x dx = 2t dt \Rightarrow dx = \frac{2t}{1+t^4} dt$

$$I = \int \left(t + \frac{1}{t} \right) \cdot \frac{2t}{1+t^4} dt = 2 \int \frac{\left(1 + \frac{1}{t^2}\right)}{t^2 + \frac{1}{t^2} + 2} dt = 2 \int \frac{\left(1 + \frac{1}{t^2}\right) dt}{\left(t - \frac{1}{t}\right)^2 + (\sqrt{2})^2}$$

$$\begin{aligned} \text{Put } t - \frac{1}{t} = p &\Rightarrow \left(1 + \frac{1}{t^2}\right) dt = dp = \int \frac{dp}{p^2 + (\sqrt{2})^2} = \frac{2}{\sqrt{2}} \tan^{-1} \frac{p}{\sqrt{2}} + c = \sqrt{2} \tan^{-1} \left(\frac{t - \frac{1}{t}}{\sqrt{2}} \right) + c \\ &= \sqrt{2} \tan^{-1} \left(\frac{\sqrt{\tan x} - \sqrt{\cot x}}{\sqrt{2}} \right) + c \end{aligned}$$

(7) Integrals of the forms $\int \sqrt{ax^2 + bx + c} dx$: To evaluate this form of integrals, express $ax^2 + bx + c$ in the form $a[(x + \alpha)^2 + \beta^2]$ by the method of completing the square and apply the standard result discussed in the above section according to the case as may be.

Note : $\square \int \sqrt{ax^2 + bx + c} dx = \frac{(2ax + b)\sqrt{ax^2 + bx + c}}{4a} + \frac{4ac - b^2}{8a} \int \frac{dx}{\sqrt{ax^2 + bx + c}}$

Example: 46 $\int \sqrt{x^2 + 8x + 12} dx$

(a) $\frac{1}{2}(x+4)\sqrt{x^2 + 8x + 12} + 2\log|x+4 + \sqrt{x^2 + 8x + 12}| + c$

(b) $\frac{1}{2}(x+4)\sqrt{x^2 + 8x + 12} - 2\log|x+4 + \sqrt{x^2 + 8x + 12}| + c$

(c) $(x+4)\sqrt{x^2 + 8x + 12} + \log|x+4 + \sqrt{x^2 + 8x + 12}| + c$

(d) $(x+4)\sqrt{x^2 + 8x + 12} - \log|x+4 + \sqrt{x^2 + 8x + 12}| + c$

Solution: (b) Let $I = \int \sqrt{x^2 + 8x + 12} dx = \int \sqrt{(x^2 + 8x + 16) - 4} dx$
 $= \int \sqrt{(x+4)^2 - 2^2} dx = \int \sqrt{t^2 - 2^2} dx = \int \sqrt{t^2 - 2^2} dt$ (putting $x+4 = t \Rightarrow dx = dt$)
 $= \frac{1}{2} t \sqrt{t^2 - 2^2} - \frac{1}{2} \cdot 2^2 \log|t + \sqrt{t^2 - 2^2}| + c = \frac{1}{2} (x+4) \sqrt{(x+4)^2 - 4} - 2\log|x+4 + \sqrt{(x+4)^2 - 4}| + c$
 $= \frac{1}{2} (x+4) \sqrt{x^2 + 8x + 12} - 2\log|x+4 + \sqrt{x^2 + 8x + 12}| + c.$

Example: 47 $\int \sqrt{2ax - x^2} dx =$

(a) $\frac{1}{2}(x-a)\sqrt{2ax - x^2} + \frac{1}{2}a^2 \sin^{-1}\left(\frac{x-a}{a}\right) + c$

(b) $\frac{1}{2}(x-a)\sqrt{2ax - x^2} - \frac{1}{2}a^2 \sin^{-1}\left(\frac{x-a}{a}\right) + c$

(c) $\frac{1}{2}(x-a)\sqrt{2ax - x^2} + \frac{1}{2}a^2 \cos^{-1}\left(\frac{x-a}{a}\right) + c$

(d) $\frac{1}{2}(x-a)\sqrt{2ax - x^2} - \frac{1}{2}a^2 \cos^{-1}\left(\frac{x-a}{a}\right) + c$

Solution: (a) $\int \sqrt{2ax - x^2} dx = \int \sqrt{a^2 - a^2 + 2ax - x^2} dx = \int \sqrt{a^2 - (x^2 - 2ax + a^2)} dx = \int \sqrt{a^2 - (x-a)^2} dx$
 $= \frac{1}{2}(x-a)\sqrt{2ax - x^2} + \frac{1}{2}a^2 \sin^{-1}\left(\frac{x-a}{a}\right) + c$

(8) Integrals of the form $\int (px + q)\sqrt{ax^2 + bx + c} dx$: To evaluate this form of integral, proceed as follows:

(i) First express $(px + q)$ as $px + q = M \frac{d}{dx}(ax^2 + bx + c) + N \Rightarrow px + q = M(2ax + b) + N$

Where, M and N are constant.

(ii) Compare the coefficients of x and constant terms on both sides, will get

$$p = 2aM \Rightarrow M = \frac{p}{2a} \text{ and } q = Mb + N \Rightarrow N = q - Mb = q - \frac{p}{2a}b$$

(iii) Now, write the given integral as

$$\begin{aligned} \int (px + q)\sqrt{ax^2 + bx + c} dx &= \frac{p}{2a} \int (2ax + b)\sqrt{ax^2 + bx + c} dx + \left(q - \frac{p}{2a}b\right) \int \sqrt{ax^2 + bx + c} dx \\ &= \frac{p}{2a} I_1 + \left(q - \frac{p}{2a}b\right) I_2, \text{ (say).} \end{aligned}$$

(iv) To evaluate I_1 , put $ax^2 + bx + c = t$ and to evaluate I_2 , follows the method discussed in (7)

Example: 48 $\int (2x+3)\sqrt{x^2 + 4x + 3} dx =$

(a) $\log|(x+2) + \sqrt{x^2 + 4x + 3}| + c$

(b) $\log|(x+2) + \sqrt{x^2 + 4x + 3}| + c$

(c) $\log|(x-2) + \sqrt{x^2 + 4x + 3}| + c$

(d) None of these

Solution: (d) Let $2x+3 = M \frac{d}{dx}(x^2+4x+3) + N \Rightarrow 2x+3 = M(2x+4) + N$

Equating the coefficients of x and constant terms on both sides, we get

$$2 = 2M \Rightarrow M = 1 \text{ and } 3 = 4M + N \Rightarrow N = 3 - 4 \times 1 = -1$$

$$\therefore 2x+3 = (2x+4) - 1$$

$$\text{Hence, } I = \int [(2x+4) - 1] \sqrt{x^2+4x+3} dx = \int (2x+4) \sqrt{x^2+4x+3} dx - \int \sqrt{x^2+4x+3} dx = I_1 - I_2, (\text{say})$$

$$\text{Now, } I_1 = \int (2x+4) \sqrt{x^2+4x+3} dx$$

Putting $x^2+4x+3 = t \Rightarrow (2x+4)dx = dt$ we have

$$I_1 = \int t^{1/2} dt = \frac{t^{3/2}}{3/2} + c_1 = \frac{2}{3}(x^2+4x+3)^{3/2} + c_1 \quad I_2 = \int \sqrt{x^2+4x+3} dx = \int \sqrt{(x+2)^2 - 1^2} dx$$

$$= \frac{1}{2}(x+2)\sqrt{(x+2)^2 - 1^2} - \frac{1}{2} \cdot 1^2 \log \left| x+2 + \sqrt{(x+2)^2 - 1^2} \right| + c_2 = \frac{1}{2}(x+2)\sqrt{x^2+4x+3} - \frac{1}{2} \log \left| x+2 + \sqrt{x^2+4x+3} \right| + c_2$$

$$\therefore I = I_1 - I_2 = \frac{2}{3}(x^2+4x+3)^{3/2} - \left[\frac{1}{2}(x+2)\sqrt{x^2+4x+3} - \frac{1}{2} \log \left| x+2 + \sqrt{x^2+4x+3} \right| \right] + c \quad (\text{where, } c = c_1 - c_2),$$

Example: 49 $\int (2x-5) \sqrt{x^2-4x+3} dx =$

(a) $\log |x-2 + \sqrt{x^2-4x+3}| + c$

(b) $\log |x-2 - \sqrt{x^2-4x+3}| + c$

(c) $\log |x+2 + \sqrt{x^2-4x+3}| + c$

(d) None of these

Solution: (d) Let $2x-5 = M \frac{d}{dx}(x^2-4x+3) + N \Rightarrow 2x-5 = M(2x-4) + N$

Equating the coefficients of x and constant terms on both sides, we get

$$2 = 2M \Rightarrow M = 1$$

$$-5 = -4M + N \Rightarrow N = 4M - 5 = -1$$

$$\text{Hence, } I = \int \{(2x-4) - 1\} \sqrt{x^2-4x+3} dx \Rightarrow I = \int (2x-4) \sqrt{x^2-4x+3} dx - \int \sqrt{x^2-4x+3} dx \quad I = I_1 - I_2 (\text{say})$$

$$I_1 = \int (2x-4) \sqrt{x^2-4x+3} dx = \frac{2}{3}(x^2-4x+3)^{3/2} + c_1 \quad I_2 = \int \sqrt{x^2-4x+3} dx = \int \sqrt{x^2-4x+4-4+3} dx$$

$$= \int \sqrt{(x-2)^2 - 1^2} \Rightarrow I_2 = \frac{1}{2}(x-2)\sqrt{x^2-4x+3} - \frac{1}{2} \cdot 1^2 \log \left\{ (x-2) + \sqrt{x^2-4x+3} \right\}$$

$$= \frac{1}{2}(x-2)\sqrt{x^2-4x+3} - \frac{1}{2} \log \left[(x-2) + \sqrt{x^2-4x+3} \right] + c_2, \text{ Therefore } I = I_1 - I_2$$

$$I = \frac{2}{3}(x^2-4x+3)^{3/2} + \frac{1}{2}(-2)\sqrt{x^2-4x+3} - \frac{1}{2} \log \left[(x-2) + \sqrt{x^2-4x+3} \right] + c \quad (\text{Where } c = c_1 + c_2)$$

(9) Integrals of the form $\int \frac{dx}{P\sqrt{Q}}$, (where P and Q are linear or quadratic expressions in x): To

evaluate such types of integrals, we have following substitutions according to the nature of expressions of P and Q in x :

(i) When Q is linear and P is linear or quadratic, we put $Q = t^2$.

(ii) When P is linear and Q is quadratic, we put $P = \frac{1}{t}$.

(iii) When both P and Q are quadratic, we put $x = \frac{1}{t}$.

Example: 50 $\int \frac{dx}{(x-3)\sqrt{x+1}} =$

(a) $\frac{1}{2} \log \left(\frac{\sqrt{x+1}+2}{\sqrt{x+1}-2} \right) + c$ (b) $\frac{1}{2} \log \left(\frac{\sqrt{x+1}+2}{\sqrt{x+1}-2} \right) + c$ (c) $\frac{1}{2} \log \left(\frac{\sqrt{x+1}-2}{\sqrt{x+1}+2} \right) + c$ (d) None of these

Solution: (c) Put $x+1 = t^2 \Rightarrow dx = 2t dt$

$$\int \frac{dx}{(x-3)\sqrt{x+1}} = \int \frac{2t dt}{(t^2-4)t} \quad (\because x+1 = t^2 \Rightarrow x = t^2-1 \Rightarrow x-3 = t^2-4)$$

$$= 2 \int \frac{dt}{t^2-2^2} = 2 \cdot \frac{1}{2 \cdot 2} \log \left(\frac{t-2}{t+2} \right) + c = \frac{1}{2} \log \left(\frac{\sqrt{x+1}-2}{\sqrt{x+1}+2} \right) + c.$$

5.8 INTEGRALS OF THE FORM $\int \frac{dx}{a+b\cos x}$ and $\int \frac{dx}{a+b\sin x}$

To evaluate such form of integrals, proceed as follows:

(1) Put $\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$ and $\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$.

(2) Replace $1 + \tan^2 \frac{x}{2}$ in the numerator by $\sec^2 \frac{x}{2}$.

(3) Put $\tan \frac{x}{2} = t$ so that $\frac{1}{2} \sec^2 \frac{x}{2} dx = dt$

(4) Now, evaluate the integral obtained which will be of the form $\int \frac{dt}{at^2 + bt + c}$ by the method discussed earlier.

(i) $\int \frac{dx}{a+b\cos x}$

Case I: When $a > b$, then $\int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}} \tan^{-1} \left(\sqrt{\frac{a-b}{a+b}} \tan \frac{x}{2} \right) + c$

Case II: When $a < b$, then $\int \frac{dx}{a+b\cos x} = \frac{1}{\sqrt{b^2-a^2}} \log \left| \frac{\sqrt{b-a} \tan \frac{x}{2} + \sqrt{b+a}}{\sqrt{b-a} \tan \frac{x}{2} - \sqrt{b+a}} \right| + c$

Case III: When $a = b$, then $\int \frac{dx}{a+b\cos x} = \frac{1}{a} \tan \frac{x}{2} + c.$

(ii) $\int \frac{dx}{a+b\sin x}$

Case I: When $a^2 > b^2$ or $a > 0$ and $a > b$, then $\int \frac{dx}{a+b\sin x} = \frac{2}{\sqrt{a^2-b^2}} \tan^{-1} \left[\frac{a \tan \frac{x}{2} + b}{\sqrt{a^2-b^2}} \right] + c$

Case II: When $a^2 < b^2$, then $\int \frac{dx}{a+b\sin x} = \frac{1}{\sqrt{b^2-a^2}} \log \left| \frac{(a \tan \frac{x}{2} + b) - (\sqrt{b^2-a^2})}{(a \tan \frac{x}{2} + b) + \sqrt{b^2-a^2}} \right| + c$

Case III: When $a^2 = b^2$

In this case, either $b = a$ or $b = -a$

(a) When $b = a$, then $\int \frac{dx}{a + b \sin x} = \frac{-1}{a} \cot\left(\frac{\pi}{4} + \frac{x}{2}\right) + c = \frac{1}{a} [\tan x - \sec x] + c$

(b) When $b = -a$, then $\int \frac{dx}{a + b \sin x} = \frac{1}{a} \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c.$

Example: 51 $\int \frac{dx}{3 + 4 \cos x} =$

(a) $\frac{1}{\sqrt{7}} \log\left(\frac{\tan(x/2) - \sqrt{7}}{\tan(x/2) + \sqrt{7}}\right) + c$

(b) $\frac{1}{\sqrt{7}} \log\left(\frac{\tan(x/2) + \sqrt{7}}{\tan(x/2) - \sqrt{7}}\right) + c$

(c) $\frac{1}{\sqrt{7}} \log\left(\frac{\sqrt{7} + \tan(x/2)}{\sqrt{7} - \tan(x/2)}\right) + c$

(d) $\frac{1}{\sqrt{7}} \log\left(\frac{\sqrt{7} - \tan(x/2)}{\sqrt{7} + \tan(x/2)}\right) + c$

Solution: (b) If $b > a$ then, $\int \frac{dx}{a + b \cos x} = \frac{1}{\sqrt{b^2 - a^2}} \log \left[\frac{\sqrt{b-a} \tan \frac{x}{2} + \sqrt{b+a}}{\sqrt{b-a} \tan \frac{x}{2} - \sqrt{b+a}} \right] + c \Rightarrow I = \frac{1}{\sqrt{7}} \log \left(\frac{\tan \frac{x}{2} + \sqrt{7}}{\tan \frac{x}{2} - \sqrt{7}} \right) + c$

Example: 52 If $\int \frac{dx}{1 + \sin x} = \tan\left(\frac{x}{2} + a\right) + b$, then

(a) $a = \frac{\pi}{4}, b = 3$

(b) $a = \frac{-\pi}{4}, b = 3$

(c) $a = \frac{\pi}{4}, b = \text{arbitrary constant}$

(d) $a = \frac{-\pi}{4}, b = \text{arbitrary constant}$

Solution: (d) If $a = b$, then $\int \frac{dx}{a + b \sin x} = -\frac{1}{a} \cot\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$

$\therefore \int \frac{dx}{1 + \sin x} = -\cot\left(\frac{\pi}{4} + \frac{x}{2}\right) = -\tan\left(\frac{\pi}{2} - \frac{\pi}{4} - \frac{x}{2}\right) + c = -\tan\left(\frac{\pi}{4} - \frac{x}{2}\right) + c = \tan\left(\frac{x}{2} - \frac{\pi}{4}\right) + c$

Hence $a = \frac{-\pi}{4}$ and $b = \text{arbitrary constant}$.

Example: 53 $\int \frac{dx}{5 + 4 \cos x} =$

(a) $\frac{2}{3} \tan^{-1}\left(\frac{1}{3} \tan x\right) + c$ (b) $\frac{1}{3} \tan^{-1}\left(\frac{1}{3} \tan x\right) + c$ (c) $\frac{2}{3} \tan^{-1}\left(\frac{1}{3} \tan \frac{x}{2}\right) + c$ (d) $\frac{1}{3} \tan^{-1}\left(\frac{1}{3} \tan \frac{x}{2}\right) + c$

Solution: (c) If $a > b$ then $\int \frac{dx}{a + b \cos x} = \frac{2}{\sqrt{a^2 - b^2}} \cdot \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}} \tan \frac{x}{2}\right) + c$

$\therefore \int \frac{dx}{5 + 4 \cos x} = \frac{2}{3} \tan^{-1}\left(\frac{1}{3} \tan \frac{x}{2}\right) + c.$

5.9 INTEGRALS OF THE FORM $\int \frac{dx}{a + b \cos x + c \sin x}, \int \frac{dx}{a \sin x + b \cos x}$

(1) **Integral of the form** $\int \frac{dx}{a + b \cos x + c \sin x}$: To evaluate such integrals, we put $b = r \cos \alpha$ and $c = r \sin \alpha$.

So that, $r^2 = b^2 + c^2$ and $\alpha = \tan^{-1} \frac{c}{b}$. $\therefore I = \int \frac{dx}{a + r(\cos \alpha \cos x + \sin \alpha \sin x)} = \int \frac{dx}{a + r \cos(x - \alpha)}$

Again, Put $x - \alpha = t \Rightarrow dx = dt$ we have $I = \int \frac{dt}{a + r \cos t}$

Which can be evaluated by the method discussed earlier.

(2) **Integral of the form** $\int \frac{dx}{a \sin x + b \cos x}$: To evaluate this type of integrals we substitute

$$a = r \cos \theta, \quad b = r \sin \theta \quad \text{and so} \quad r = \sqrt{a^2 + b^2}, \quad \alpha = \tan^{-1} \frac{b}{a}$$

$$\begin{aligned} \text{So, } \int \frac{dx}{a \sin x + b \cos x} &= \frac{1}{r} \int \frac{dx}{\sin(x + \alpha)} = \frac{1}{r} \int \operatorname{cosec}(x + \alpha) dx \\ &= \frac{1}{r} \log \left| \tan \left(\frac{x}{2} + \frac{\alpha}{2} \right) \right| = \frac{1}{\sqrt{a^2 + b^2}} \log \left| \tan \left(\frac{x}{2} + \frac{1}{2} \tan^{-1} \frac{b}{a} \right) \right| + c \end{aligned}$$

Note : $\square$ The integral of the above form can be evaluated by using $\cos x = \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}$ and

$$\sin x = \frac{2 \tan x/2}{1 + \tan^2 x/2}.$$

Important Tips

$$\text{If } a > b, \quad a^2 > b^2 + c^2, \text{ then } \int \frac{dx}{a + b \cos x + c \sin x} = \frac{2}{\sqrt{a^2 - b^2 - c^2}} \tan^{-1} \left[\frac{(a - b) \tan x/2 + c}{\sqrt{a^2 - b^2 - c^2}} \right] + k$$

$$\text{If } a > b, \quad a^2 < b^2 + c^2, \text{ then } \int \frac{dx}{a + b \cos x + c \sin x} = \frac{1}{\sqrt{b^2 + c^2 - a^2}} \log \left[\frac{(a - b) \tan x/2 + c - \sqrt{b^2 + c^2 - a^2}}{(a - b) \tan x/2 + c + \sqrt{b^2 + c^2 - a^2}} \right] + k$$

$$\text{If } a < b, \quad \int \frac{dx}{a + b \cos x + c \sin x} = \frac{-1}{\sqrt{b^2 + c^2 - a^2}} \log \left[\frac{(b - a) \tan x/2 - c - \sqrt{b^2 + c^2 - a^2}}{(b - a) \tan x/2 - c + \sqrt{b^2 + c^2 - a^2}} \right] + k$$

Example: 54 $\int \frac{dx}{\sin x + \cos x} =$

(a) $\log \tan(\pi/8 + x/2) + c$

(b) $\log \tan(\pi/8 - x/2) + c$

(c) $\frac{1}{\sqrt{2}} \log \tan(\pi/8 + x/2) + c$

(d) None of these

Solution: (c) $\int \frac{dx}{\frac{2 \tan x/2}{1 + \tan^2 x/2} + \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}} \Rightarrow \int \frac{\sec^2 x/2}{2 \tan x/2 + 1 - \tan^2 x/2} dx$

Put $\tan x/2 = t \Rightarrow \frac{1}{2} \sec^2 x/2 dx = dt$

$$\therefore I = 2 \int \frac{dt}{2t + 1 - t^2} = 2 \int \frac{dt}{2 - (t^2 - 2t + 1)}$$

$$\Rightarrow I = 2 \int \frac{dt}{(\sqrt{2})^2 - (t - 1)^2} = \frac{2}{2\sqrt{2}} \log \left| \frac{\sqrt{2} + t - 1}{\sqrt{2} - t + 1} \right| + c \Rightarrow I = \frac{1}{\sqrt{2}} \log \left[\frac{(\sqrt{2} - 1) + \tan x/2}{(\sqrt{2} + 1) - \tan x/2} \right] + c$$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \log \frac{\tan \pi/8 + \tan x/2}{1 - (\sqrt{2} - 1) \tan x/2} + \frac{1}{\sqrt{2}} \log(\sqrt{2} - 1) + c \Rightarrow I = \frac{1}{\sqrt{2}} \log \tan(\pi/8 + x/2) + c_1$$

where $c_1 = \frac{1}{\sqrt{2}} \log(\sqrt{2} - 1) + c$

$$\begin{aligned} \text{Trick: } I &= \frac{1}{\sqrt{2}} \int \frac{dx}{\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sin(x/4 + x)} \\ &= \frac{1}{\sqrt{2}} \int \operatorname{cosec}(\pi/4 + x) dx = \frac{1}{\sqrt{2}} \log \tan(\pi/8 + x/2) + c. \end{aligned}$$

Example: 55 $\int \frac{dx}{1 - \cos x - \sin x} =$

(a) $\log \left| 1 + \cot \frac{x}{2} \right| + c$ (b) $\log \left| 1 - \tan \frac{x}{2} \right| + c$ (c) $\log \left| 1 - \cot \frac{x}{2} \right| + c$ (d) $\log \left| 1 + \tan \frac{x}{2} \right| + c$

Solution: (c) Given $I = \int \frac{dx}{1 - \cos x - \sin x}$

$$I = \int \frac{dx}{1 - \frac{(1 - \tan^2 x/2)}{(1 + \tan^2 x/2)} - \frac{2 \tan x/2}{1 + \tan^2 x/2}} \Rightarrow I = \int \frac{\sec^2 x/2 \cdot dx}{1 + \tan^2 x/2 - 1 + \tan^2 x/2 - 2 \tan x/2}$$

$$I = \int \frac{\sec^2 x/2 \cdot dx}{2 \tan^2 x/2 - 2 \tan x/2} \Rightarrow \int \frac{1/2 \cdot \sec^2 x/2 \cdot dx}{\tan^2 x/2 - \tan x/2}$$

Put $\tan x/2 = t \Rightarrow \frac{1}{2} \sec^2 x/2 \cdot dx = dt$ therefore $I = \int \frac{dt}{t^2 - t} \Rightarrow I = \int \frac{dt}{t(t-1)}$

$$\Rightarrow \int \left[-\frac{1}{t} + \frac{1}{t-1} \right] dt \Rightarrow I = \int \frac{dt}{t-1} - \int \frac{dt}{t}$$

$$I = \log(t-1) - \log t + c \Rightarrow I = \log \left| \frac{t-1}{t} \right| + c \Rightarrow I = \log \left| \frac{\tan x/2 - 1}{\tan x/2} \right| + c \Rightarrow I = \log |1 - \cot x/2| + c$$

Example: 56 The antiderivative of $f(x) = \frac{1}{3 + 5 \sin x + 3 \cos x}$ whose graph passes through the point (0,0) is

(a) $\frac{1}{5} \left(\log \left| 1 - \frac{5}{3} \tan x/2 \right| \right)$ (b) $\frac{1}{5} \left(\log \left| 1 + \frac{5}{3} \tan x/2 \right| \right)$ (c) $\frac{1}{5} \left(\log \left| 1 + \frac{5}{3} \cot x/2 \right| \right)$ (d) None of these

Solution: (b) $y = \int \frac{dx}{(3 + 5 \sin x + 3 \cos x)} = \int \frac{\sec^2 x/2 dx}{10 \tan x/2 + 6} = \frac{1}{5} \log |5 \tan x/2 + 3| + c = \frac{1}{5} \log \left| \frac{5}{3} \tan x/2 + 1 \right| + c$

Passes through (0,0)

$$\therefore c = 0 \text{ then } y = \frac{1}{5} \log \left| 1 + \frac{5}{3} \tan x/2 \right|.$$

5.10 INTEGRALS OF THE

FORM $\int \frac{dx}{a + b \cos^2 x}, \int \frac{dx}{a + b \sin^2 x}, \int \frac{dx}{a \sin^2 x + b \cos^2 x}, \int \frac{dx}{(a \sin x + b \cos x)^2}, \int \frac{dx}{a + b \sin^2 x + c \cos^2 x}$

To evaluate the above forms of integrals proceed as follows:

(1) Divide both the numerator and denominator by $\cos^2 x$.

(2) Replace $\sec^2 x$ in the denominator, if any by $(1 + \tan^2 x)$.

(3) Put $\tan x = t \Rightarrow \sec^2 x dx = dt$

(4) Now, evaluate the integral thus obtained, by the method discussed earlier.

Example: 57 $\int \frac{dx}{1 + 3 \sin^2 x} =$

(a) $\frac{1}{3} \tan^{-1}(3 \tan^2 x) + c$ (b) $\frac{1}{2} \tan^{-1}(2 \tan x) + c$ (c) $\tan^{-1}(\tan x) + c$ (d) None of these

Solution: (b) $\int \frac{\sec^2 x dx}{\sec^2 x + 3 \tan^2 x} = \int \frac{\sec^2 x dx}{1 + 4 \tan^2 x}$

Put $2 \tan x = t$

$$\sec^2 x dx = \frac{dt}{2}$$

$$\therefore I = \frac{1}{2} \int \frac{dt}{1+t^2} = \frac{1}{2} \tan^{-1} t + c \Rightarrow I = \frac{1}{2} \tan^{-1}(2 \tan x) + c.$$

Example: 58 $\int \frac{dx}{4 \sin^2 x + 5 \cos^2 x}$

(a) $\frac{1}{\sqrt{5}} \tan^{-1} \left(\frac{2 \tan x}{\sqrt{5}} \right) + c$ (b) $\frac{1}{\sqrt{5}} \tan^{-1} \left(\frac{\tan x}{\sqrt{5}} \right) + c$ (c) $\frac{1}{2\sqrt{5}} \tan^{-1} \left(\frac{2 \tan x}{\sqrt{5}} \right) + c$ (d) None of these

Solution: (c) $\int \frac{\sec^2 x dx}{4 \tan^2 x + 5}$ (Put $2 \tan x = t$)
 $2 \sec^2 x dx = dt$)

$$\Rightarrow \frac{1}{2} \int \frac{dt}{t^2 + (\sqrt{5})^2} = \frac{1}{2\sqrt{5}} \tan^{-1} \left(\frac{2 \tan x}{\sqrt{5}} \right) + c$$

Example: 59 $\int \frac{dx}{(2 \sin x + \cos x)^2}$

(a) $\frac{1}{2} \left(\frac{1}{2 \tan x + 1} \right)$ (b) $\frac{1}{2} \log |2 \tan x + 1| + c$ (c) $\frac{1}{2 + \cot x} + c$ (d) $-\frac{1}{2} \left(\frac{1}{2 \tan x - 1} \right) + c$

Solution: (c) $\int \frac{dx}{(2 \sin x + \cos x)^2} = \int \frac{dx}{\sin^2 x (2 + \cot x)^2} = \int \frac{\csc^2 x dx}{(2 + \cot x)^2}$

Put $(2 + \cot x) = t \Rightarrow -\csc^2 x dx = dt = \int \frac{-dt}{t^2} = \frac{1}{t} + c = \frac{1}{2 + \cot x} + c$

Example: 60 $\int \frac{\cos x dx}{\cos 3x} =$

(a) $\frac{1}{2\sqrt{3}} \log \left| \frac{1 + \sqrt{3} \tan x}{1 - \sqrt{3} \tan x} \right| + c$ (b) $\frac{-1}{2\sqrt{3}} \log \left| \frac{1 + \sqrt{3} \tan x}{1 - \sqrt{3} \tan x} \right| + c$

(c) $\frac{1}{\sqrt{6}} \tan^{-1}(\sqrt{3} \tan x) + c$ (d) None of these

Solution: (a) Let $I = \int \frac{\cos x}{\cos 3x} dx = \int \frac{\cos x}{4 \cos^3 x - 3 \cos x} dx$

Dividing the numerator and denominator by $\cos^2 x$, we have $I = \int \frac{dx}{4 \cos^2 x - 3}$

$$I = \int \frac{dt}{1-3t^2} = \frac{1}{3} \int \frac{dt}{\frac{1}{3} - t^2} = \frac{1}{3} \int \frac{dt}{\left(\frac{1}{\sqrt{3}} \right)^2 - t^2} = \frac{1}{3} \cdot \frac{1}{2 \cdot \frac{1}{\sqrt{3}}} \log \left| \frac{\frac{1}{\sqrt{3}} + t}{\frac{1}{\sqrt{3}} - t} \right| + c = \frac{1}{2\sqrt{3}} \log \left| \frac{1 + \sqrt{3} t}{1 - \sqrt{3} t} \right| + c$$

$$I = \frac{1}{2\sqrt{3}} \log \left| \frac{1 + \sqrt{3} \tan x}{1 - \sqrt{3} \tan x} \right| + c$$

Example: 61 $\int \frac{dx}{4 \sin^2 x + 4 \sin x \cos x + 5 \cos^2 x}$ equals

(a) $\tan^{-1} \left(\tan x + \frac{1}{2} \right) + c$ (b) $\frac{1}{4} \tan^{-1} \left(\tan x + \frac{1}{2} \right) + c$ (c) $4 \tan^{-1} \left(\tan x + \frac{1}{2} \right) + c$ (d) None of these

Solution: (b) $I = \int \frac{dx}{4 \sin^2 x + 4 \sin x \cos x + 5 \cos^2 x} = \int \frac{\sec^2 x dx}{4 \tan^2 x + 5 + 4 \tan x} = \frac{1}{4} \int \frac{\sec^2 x dx}{\left(\tan x + \frac{1}{2} \right)^2 + 1}$

Put $\tan x + \frac{1}{2} = t \Rightarrow \sec^2 x dx = dt = \frac{1}{4} \int \frac{dt}{t^2 + 1} = \frac{1}{4} \tan^{-1} t + c \Rightarrow I = \frac{1}{4} \tan^{-1} \left(\tan x + \frac{1}{2} \right) + c.$

5.11 INTEGRALS OF THE FORM $\int \frac{a \sin x + b \cos x}{c \sin x + d \cos x}$ and $\int \frac{a \sin x + b \cos x + q}{c \sin x + d \cos x + r}$

(1) Integrals of the form $\int \frac{a \sin x + b \cos x}{c \sin x + d \cos x} dx$: Such rational functions of $\sin x$ and $\cos x$ may be integrated by expressing the numerator of the integrand as follows:

Numerator = M (Diff. of denominator) + N (Denominator)

$$\text{i.e., } a \sin x + b \cos x = M \frac{d}{dx}(c \sin x + d \cos x) + N(c \sin x + d \cos x)$$

The arbitrary constants M and N are determined by comparing the coefficients of $\sin x$ and $\cos x$ from two sides of the above identity. Then, the given integral is

$$\begin{aligned} I &= \int \frac{a \sin x + b \cos x}{c \sin x + d \cos x} dx = \int \frac{M(cc \cos x - d \sin x) + N(c \sin x + d \cos x)}{c \sin x + d \cos x} dx = M \int \frac{cc \cos x - d \sin x}{c \sin x + d \cos x} dx + N \int 1 dx \\ &= M \log |c \sin x + d \cos x| + Nx + c \end{aligned}$$

(2) Integrals of the form $\int \frac{a \sin x + b \cos x + q}{c \sin x + d \cos x + r} dx$: To evaluate this type of integrals, we express the numerator as follows: Numerator = M (Denominator) + N (Differentiation of denominator) + P

$$\text{i.e., } (c \sin x + b \cos x + q) = M(c \sin x + d \cos x + r) + N(cc \cos x - d \sin x) + P.$$

where M, N, P are constants to be determined by comparing the coefficients of $\sin x, \cos x$ and constant term on both sides.

$$\begin{aligned} \therefore \int \frac{a \sin x + b \cos x + q}{c \sin x + d \cos x + r} dx &= \int M dx + N \int \frac{\text{Diff. of denominator}}{\text{Denominator}} dx + \int \frac{dx}{c \sin x - d \cos x + r} \\ &= Mx + M \log |\text{Denominator}| + P \int \frac{dx}{c \sin x + d \cos x + r}. \end{aligned}$$

Important Tips

$$\int \frac{a \cos x + b \sin x}{c \cos x + d \sin x} dx = \frac{ac + bd}{c^2 + d^2} x + \frac{ad - bc}{c^2 + d^2} \log |c \cos x + d \sin x| + c.$$

Example: 62 $\int \frac{3 \cos x + 3 \sin x}{4 \sin x + 5 \cos x} dx =$

(a) $\frac{27}{41} x - \frac{3}{41} \log |4 \sin x + 5 \cos x| + c$

(b) $\frac{27}{41} x + \frac{3}{41} \log |4 \sin x + 5 \cos x| + c$

(c) $\frac{27}{41} x - \frac{3}{41} \log |4 \sin x - 5 \cos x| + c$

(d) None of these

Solution: (a) $3 \cos x + 3 \sin x = M \frac{d}{dx}(4 \sin x + 5 \cos x) + N(4 \sin x + 5 \cos x)$

$$\Rightarrow 3 \cos x + 3 \sin x = M(4 \cos x - 5 \sin x) + N(4 \sin x + 5 \cos x)$$

$\Rightarrow$ Comparing the coefficient of $\sin x$ and $\cos x$ on both sides.

$$\Rightarrow -5M + 4N = 3 \text{ and } 4M + 5N = 3 \Rightarrow M = \frac{-3}{41}, N = \frac{27}{41}$$

$$\therefore I = \int \frac{\frac{-3}{41}(4 \cos x - 5 \sin x) + \frac{27}{41}(4 \sin x + 5 \cos x)}{4 \sin x + 5 \cos x} dx \Rightarrow I = \int \frac{27}{41} dx + \left(\frac{-3}{41}\right) \int \frac{4 \cos x - 5 \sin x}{4 \sin x + 5 \cos x} dx$$

$$\Rightarrow I = \frac{27}{41} x - \frac{3}{41} \log |4 \sin x + 5 \cos x| + c.$$

Example: 63 $\int \frac{dx}{1+\cot x} =$

(a) $\frac{1}{2}x + \frac{1}{2}\log|\sin x + \cos x| + c$

(b) $\frac{1}{2}x - \frac{1}{2}\log|\sin x + \cos x| + c$

(c) $-\frac{1}{2}x + \frac{1}{2}\log|\sin x + \cos x| + c$

(d) None of these

Solution: (b) Here, $I = \int \frac{dx}{1+\cot x} = \int \frac{dx}{1+\frac{\cos x}{\sin x}} = \int \frac{\sin x}{\sin x + \cos x} dx$

Let $\sin x = M \frac{d}{dx}(\sin x + \cos x) + N(\sin x + \cos x) \Rightarrow \sin x = M(\cos x - \sin x) + N(\sin x + \cos x)$

Comparing the coefficients of $\sin x$ and $\cos x$ of both the sides, we have $1 = -M + N$ and $0 = M + N$

Solving these equations, we have $M = -\frac{1}{2}$ and $N = \frac{1}{2}$

$\therefore \sin x = -\frac{1}{2}(\cos x - \sin x) + \frac{1}{2}(\sin x + \cos x)$

Hence, $I = \int \frac{-\frac{1}{2}(\cos x - \sin x) + \frac{1}{2}(\sin x + \cos x)}{\sin x + \cos x} dx = -\frac{1}{2} \int \frac{\cos x - \sin x}{\sin x + \cos x} dx + \frac{1}{2} \int 1 dx$
 $= -\frac{1}{2} \log|\sin x + \cos x| + \frac{1}{2}x + c$

5.12 INTEGRATION OF RATIONAL FUNCTIONS BY USING PARTIAL FRACTIONS

(1) **Proper rational functions:** Functions of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial and $g(x) \neq 0$, are called rational functions of x .

If degree of $f(x)$ is less than degree of $g(x)$, then $\frac{f(x)}{g(x)}$ is called a proper rational function.

(2) **Improper rational function :** If degree of $f(x)$ is greater than or equal to degree of $g(x)$, then $\frac{f(x)}{g(x)}$, is called an improper rational function and every improper rational function can be transformed to a proper rational function by dividing the numerator by the denominator.

For example, $\frac{x^3}{x^2 - 5x + 6}$ is an improper rational function and can be expressed as $(x+5) + \frac{19x-30}{x^2-5x+6}$, which is the sum of a polynomial $(x+5)$ and a proper function $\frac{19x-30}{x^2-5x+6}$.

(3) **Partial fractions:** Any proper rational function can be broken up into a group of different rational fractions, each having a simple factor of the denominator of the original rational function. Each such fraction is called a partial fraction.

If by some process, we can break a given rational function $\frac{f(x)}{g(x)}$ into different fractions, whose denominators are the factors of $g(x)$, then the process of obtaining them is called the resolution or decomposition of $\frac{f(x)}{g(x)}$ into its partial fractions.

Depending on the nature of the factors of the denominator, the following cases arise.

Case I: When the denominator consists of non-repeated linear factors: To each linear factor $(x - a)$ occurring once in the denominator of a proper fraction, there corresponds a single partial fraction of the form $\frac{A}{x - a}$, where A is a constant to be determined.

Case II: When the denominator consists of linear factors, some repeated: To each linear factor $(x - a)$ occurring r times in the denominator of a proper rational function, there corresponds a sum of r partial fractions of the form.

$$\frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \dots + \frac{A_r}{(x - a)^r}$$

Where A_s are constants to be determined. Of course, A_r is not equal to zero.

Case III: When the denominator consists of quadratic factors: To each irreducible non repeated quadratic factor $ax^2 + bx + c$, there corresponds a partial fraction of the form $\frac{Ax + B}{ax^2 + bx + c}$, where A and B are constants to be determined.

To each irreducible quadratic factor $ax^2 + bx + c$ occurring r times in the denominator of a proper rational fraction there corresponds a sum of r partial fractions of the form

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \dots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

Where, A 's and B 's are constants to be determined.

(4) General methods of finding the constants

(i) In the given proper fraction, first of all factorize the denominator.

(ii) Express the given proper fraction into its partial fractions according to rules given above and multiply both the sides by the denominator of the given fraction.

(iii) Equate the coefficients of like powers of x in the resulting identity and solve the equations so obtained simultaneously to find the various constant is short method. Sometimes, we substitute particular values of the variable x in the identity obtained after clearing of fractions to find some or all the constants. For non-repeated linear factors, the values of x used as those for which the denominator of the corresponding partial fractions become zero.

Note : $\square$ If the given fraction is improper, then before finding partial fractions, the given fraction must be expressed as sum of a polynomial and a proper fraction by division.

(5) **Special cases:** Some times a suitable substitution transform the given function to a rational fraction which can be integrated by breaking it into partial fractions.

Example: 64 $\int \frac{\cos x}{(1 + \sin x)(2 + \sin x)} dx =$

(a) $\log[(1 + \sin x)(2 + \sin x)] + c$

(b) $\log \left[\frac{2 + \sin x}{1 + \sin x} \right] + c$

(c) $\log \left[\frac{1 + \sin x}{2 + \sin x} \right] + c$

(d) None of these

Solution: (c) Put $\sin x = t \Rightarrow \cos x dx = dt$

$$\int \frac{\cos x dx}{(1 + \sin x)(2 + \sin x)} = \int \frac{dt}{(1 + t)(2 + t)} = \int \left[\frac{1}{t + 1} - \frac{1}{t + 2} \right] dt = \log(t + 1) - \log(t + 2) + c = \log \left[\frac{t + 1}{t + 2} \right] + c$$

$$= \log \left[\frac{1 + \sin x}{2 + \sin x} \right] + c$$

Example: 65 $\int \frac{3x+1}{(x-2)^2(x+2)} dx =$

(a) $\frac{5}{16} \log \left| \frac{x+2}{x-2} \right| + \frac{7}{4(x-2)} + c$

(b) $\frac{5}{16} \log \left| \frac{x-2}{x+2} \right| + \frac{7}{4(x-2)} + c$

(c) $\frac{16}{5} \log \left| \frac{x-2}{x+2} \right| - \frac{7}{4(x-2)} + c$

(d) None of these

Solution: (b) We have, $\frac{3x+1}{(x-2)^2(x+2)} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x+2)}$

$$3x+1 = A(x-2)(x+2) + B(x+2) + C(x-2)^2 \quad \dots\dots(i)$$

Putting $x=2$ and $x=-2$ successively in equation (i), we get $B = \frac{7}{4}$, $C = \frac{-5}{16}$

Now, we put $x=0$ and get $A = \frac{5}{16}$

$$\begin{aligned} \int \frac{3x+1}{(x-2)^2(x+2)} dx &= \frac{5}{16} \int \frac{dx}{x-2} + \frac{7}{4} \int \frac{dx}{(x-2)^2} + \frac{-5}{16} \int \frac{dx}{x+2} \\ &= \frac{5}{16} \log(x-2) - \frac{7}{4} \frac{1}{(x-2)} - \frac{5}{16} \log(x+2) + c \\ &= \frac{5}{16} \log \left| \frac{x-2}{x+2} \right| - \frac{7}{4(x-2)} + c \end{aligned}$$

Example: 66 $\int \frac{x^2+1}{x(x^2-1)} dx =$

(a) $\log \frac{x^2-1}{x} + c$

(b) $-\log \frac{x^2-1}{x} + c$

(c) $\log \frac{x}{x^2+1} + c$

(d) $-\log \frac{x}{x^2+1} + c$

Solution: (a) $I = \int \left(\frac{2x}{x^2-1} - \frac{1}{x} \right) dx \Rightarrow I = \log(x^2-1) - \log x + c \Rightarrow I = \log \frac{x^2-1}{x} + c.$

Example: 67 If $\int \frac{2x^2+3}{(x^2-1)(x^2+4)} dx = a \log \left(\frac{x-1}{x+1} \right) + b \tan^{-1} \left(\frac{x}{2} \right) + c$ then values of a and b are

(a) 1, -1

(b) -1, 1

(c) 1/2, -1/2

(d) 1/2, 1/2

Solution: (d) Put $x^2 = y$

$$\therefore \frac{2x^2+3}{(x^2-1)(x^2+4)} = \frac{2y+3}{(y-1)(y+4)} \Rightarrow \frac{2y+3}{(y-1)(y+4)} = \frac{A}{(y-1)} + \frac{B}{(y+4)}$$

$$\therefore 2y+3 = A(y+4) + B(y-1)$$

Comparing the coefficient of y and constant terms

$$\Rightarrow A+B=2 \text{ and } 4A-B=3$$

$$\therefore A=1 \text{ and } B=1$$

$$\therefore I = \int \frac{1}{y-1} dx + \int \frac{1}{y+4} dx \Rightarrow I = \int \frac{1}{x^2-1} dx + \int \frac{1}{x^2+4} dx \Rightarrow I = \frac{1}{2} \log \left| \frac{x-1}{x+1} \right| + \frac{1}{2} \tan^{-1} \frac{x}{2} + c$$

$$\therefore a = \frac{1}{2} \text{ and } b = \frac{1}{2}.$$

Example: 68 $\int \frac{x^4}{(x-1)(x^2+1)} dx =$

(a) $\frac{x(x+2)}{2} + \frac{\log(x-1)}{2} - \frac{\log(x^2+1)}{4} - \frac{\tan^{-1} x}{2} + c$

(b) $\frac{x(x+2)}{2} + \frac{\log(x-1)}{2} + \frac{\log(x^2+1)}{4} - \frac{\tan^{-1} x}{2} + c$

(c) $\frac{x(x+2)}{2} + \frac{\log(x-1)}{2} + \frac{\log(x^2+1)}{4} + \frac{\tan^{-1} x}{2} + c$

(d) None of these

Solution: (a) $\int \frac{x^4}{(x-1)(x^2+1)} dx = \int \frac{x^4-1}{(x-1)(x^2+1)} dx + \int \frac{1}{(x-1)(x^2+1)} dx = \int \frac{(x+1)(x-1)(x^2+1)}{(x-1)(x^2+1)} dx + \int \frac{dx}{(x-1)(x^2+1)}$

$$= \int (x+1) dx + \int \frac{dx}{(x-1)(x^2+1)} = \frac{x^2}{2} + x + \left[\frac{1}{2} \log(x-1) - \frac{1}{4} \log(x^2+1) - \frac{1}{2} \tan^{-1} x \right] + c$$

$$= \frac{x(x+2)}{2} + \frac{\log(x-1)}{2} - \frac{\log(x^2+1)}{4} - \frac{\tan^{-1} x}{2} + c$$

Example: 69 $\int \frac{\tan \phi + \tan^3 \phi}{1 + \tan^3 \phi} d\phi$

(a) $\frac{1}{3} \log|1 + \tan \phi| + \frac{1}{6} \log|\tan^2 \phi - \tan \phi + 1| + \frac{1}{3} \tan^{-1} \left(\frac{2 \tan \phi - 1}{\sqrt{3}} \right) + c$

(b) $-\frac{1}{3} \log|1 + \tan \phi| + \frac{1}{6} \log|\tan^2 \phi - \tan \phi + 1| + \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2 \tan \phi - 1}{\sqrt{3}} \right) + c$

(c) $\frac{1}{3} \log \left(\frac{1 + \tan^2 \phi}{\tan^3 \phi} \right) + c$

(d) None of these

Solution: (b) Let $I = \int \frac{\tan \phi + \tan^3 \phi}{1 + \tan^3 \phi} d\phi = \int \frac{\tan \phi (1 + \tan^2 \phi)}{1 + \tan^3 \phi} d\phi \Rightarrow I = \int \frac{\tan \phi \sec^2 \phi}{1 + \tan^3 \phi} d\phi$

Putting $\tan \phi = t \Rightarrow \sec^2 \phi d\phi = dt$ we have, $I = \int \frac{t}{1+t^3} dt = \int \frac{t dt}{(1+t)(1-t+t^2)}$

Let $\frac{t}{(1+t)(1-t+t^2)} = \frac{A}{1+t} + \frac{Bt+C}{1-t+t^2}$

$\Rightarrow t = A(1-t+t^2) + (Bt+C)(1+t)$ (i)

Putting $t = -1$ in equation (i), we get, $-1 = A(1+1+1) \Rightarrow A = -\frac{1}{3}$

Comparing the coefficients of t^2 and constant terms on both the sides of equation (i)

we get

$0 = A + B \Rightarrow B = -A = \frac{1}{3}$ and $0 = A + C \Rightarrow C = -A = \frac{1}{3}$

$\therefore \frac{t}{(1+t)(1-t+t^2)} = -\frac{1}{3} \cdot \frac{1}{1+t} + \frac{1}{3} \cdot \frac{t+1}{1-t+t^2}$

Hence, $I = -\frac{1}{3} \int \frac{dt}{1+t} + \frac{1}{3} \int \frac{t+1}{t^2-t+1} dt = -\frac{1}{3} \int \frac{dt}{1+t} + \frac{1}{6} \int \frac{2t+2}{t^2-t+1} dt = -\frac{1}{3} \int \frac{dt}{1+t} + \frac{1}{6} \int \frac{(2t-1)+3}{t^2-t+1} dt$

$$= -\frac{1}{3} \int \frac{dt}{1+t} + \frac{1}{6} \int \frac{2t-1}{t^2-t+1} dt + \frac{3}{6} \int \frac{dt}{t^2-t+1} = -\frac{1}{3} \int \frac{dt}{1+t} + \frac{1}{6} \int \frac{2t-1}{t^2-t+1} dt + \frac{1}{2} \int \frac{dt}{\left(t-\frac{1}{2}\right)^2 + \frac{3}{4}}$$

$= -\frac{1}{3} \log|1+t| + \frac{1}{6} \log|t^2-t+1| + \frac{1}{2} \cdot \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) + c = -\frac{1}{3} \log|1+t| + \frac{1}{6} \log|t^2-t+1| + \frac{1}{3} \tan^{-1} \left(\frac{2t-1}{\sqrt{3}} \right) + c$

$= -\frac{1}{3} \log|1 + \tan \phi| + \frac{1}{6} \log|\tan^2 \phi - \tan \phi + 1| + \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{2 \tan \phi - 1}{\sqrt{3}} \right) + c$

Example: 70 $\int \frac{dx}{x(x^n+1)}$

(a) $\frac{1}{n} \log \frac{x^n}{x^n+1} + c$

(b) $n \log \frac{x^n+1}{x^n} + c$

(c) $-\frac{1}{n} \log \frac{x^n}{x^n+1} + c$

(d) $-n \log \frac{x^n+1}{x^n} + c$

Solution: (a) Let $I = \int \frac{dx}{x(x^n+1)} = \int \frac{x^{n-1}}{x^n(x^n+1)} dx$

Putting $x^n = t \Rightarrow nx^{n-1} dx = dt$ we have

$I = \frac{1}{n} \int \frac{dt}{t(t+1)} = \frac{1}{n} \int \left[\frac{1}{t} - \frac{1}{t+1} \right] dt$

(by resolving into partial fractions)

$$= \frac{1}{n} [\log t - \log(t+1)] + c = \frac{1}{n} \log \left| \frac{t}{t+1} \right| + c = \frac{1}{n} \log \left| \frac{x^n}{x^n+1} \right| + c$$

5.13 INTEGRATION OF TRIGONOMETRIC FUNCTIONS

(1) **Integral of the form $\int \sin^m x \cos^n x dx$:** (i) To evaluate the integrals of the form $I = \int \sin^m x \cos^n x dx$ where m and n are rational numbers.

(a) Substitute $\sin x = t$, if n is odd;

(b) Substitute $\cos x = t$, if m is odd;

(c) Substitute $\tan x = t$, if $m+n$ is a negative even integer; and

(d) Substitute $\cot x = t$, if $\frac{1}{2}(n-1)$ is an integer.

(e) If m and n are rational numbers and $\left(\frac{m+n-2}{2}\right)$ is a negative integer, then substitution $\cos x = t$ or $\tan x = t$ is found suitable.

(ii) Integrals of the form $\int R(\sin x, \cos x) dx$, where R is a rational function of $\sin x$ and $\cos x$, are transformed into integrals of a rational function by the substitution $\tan \frac{x}{2} = t$, where $-\pi < x < \pi$. This is the so called universal substitution. Sometimes it is more convenient to make the substitution $\cot \frac{x}{2} = t$ for $0 < x < 2\pi$.

The above substitution enables us to integrate any function of the form $R(\sin x, \cos x)$. However, in practice, it sometimes leads to extremely complex rational function. In some cases, the integral can be simplified by:

(a) Substituting $\sin x = t$, if the integral is of the form $\int R(\sin x) \cos x dx$

(b) Substituting $\cos x = t$, if the integral is of the form $\int R(\cos x) \sin x dx$

(c) Substituting $\tan x = t$, i.e., $dx = \frac{dt}{1+t^2}$, if the integral is dependent only on $\tan x$

(d) Substituting $\cos x = t$, if $R(-\sin x, \cos x) = -R(\sin x, \cos x)$

(e) Substituting $\sin x = t$, if $R(\sin x, -\cos x) = -R(\sin x, \cos x)$

(f) Substituting $\tan x = t$, if $R(-\sin x, -\cos x) = -R(\sin x, \cos x)$

Important Tips

☞ To evaluate integrals of the form $\int \sin m x \cos n x dx$, $\int \sin m x \sin n x dx$, $\int \cos m x \cos n x dx$ and $\int \cos m x \sin n x dx$ we use the following trigonometrical identities.

$$\sin m x \cos n x = \frac{1}{2} [\sin(m-n)x + \sin(m+n)x] \Rightarrow \cos m x \sin n x = \frac{1}{2} [\sin(m+n)x - \sin(m-n)x]$$

$$\sin m x \sin n x = \frac{1}{2} [\cos(m-n)x - \cos(m+n)x] \Rightarrow \cos m x \cos n x = \frac{1}{2} [\cos(m-n)x + \cos(m+n)x]$$

Example: 71 $\int \sin^3 x \cos^2 x dx$

- (a) $\frac{\cos^2 x}{5} - \frac{\cos^3 x}{3} + c$ (b) $\frac{\cos^5}{5} + \frac{\cos^3}{3} + c$ (c) $\frac{\sin^5}{5} - \frac{\sin^3}{3} + c$ (d) $\frac{\sin^5}{5} + \frac{\sin^3}{3} + c$

Solution: (a) $I = \int \sin x (1 - \cos^2 x) \cos^2 x dx$

Put $\cos x = t \Rightarrow -\sin x dx = dt \Rightarrow I = -\int (t^2 - t^4) dt = \frac{t^5}{5} - \frac{t^3}{3} + c = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + c$

Example: 72 $\int \frac{d\theta}{\sin \theta \cdot \cos^3 \theta} =$

- (a) $\log \tan \theta + \tan^2 \theta + c$ (b) $\log \tan \theta - \frac{1}{2} \tan^2 \theta + c$ (c) $\log \tan \theta + \frac{1}{2} \tan^2 \theta + c$ (d) None of these

Solution: (c) $\int \frac{d\theta}{\sin \theta \cos^3 \theta} = \int \frac{\sec^2 \theta d\theta}{\sin \theta \cos \theta} = \int \frac{\sec^2 \theta (1 + \tan^2 \theta) d\theta}{\tan \theta d\theta}$

Put $t = \tan \theta \Rightarrow dt = \sec^2 \theta d\theta$ then it reduces to $\Rightarrow \int \frac{1+t^2}{t} dt = \int \left(\frac{1}{t} + t \right) dt$

$= \log t + \frac{t^2}{2} + c = \log \tan \theta + \frac{1}{2} \tan^2 \theta + c$

Example: 73 $\int \frac{\sin^3 2x}{\cos^5 2x} dx =$

- (a) $\tan^4 x + c$ (b) $\tan 4x + c$ (c) $\tan^4 2x + x + c$ (d) $\frac{1}{8} \tan^4 2x + c$

Solution: (d) Given, $I = \int \frac{\sin^3 2x}{\cos^5 2x} dx$. The given equation may be written as

$\int \frac{\sin^3 2x}{\cos^3 2x} \cdot \frac{1}{\cos^2 2x} dx = \int \tan^3 2x \cdot \sec^2 2x dx$

Put $\tan 2x = t$ and $2 \sec^2 2x dx = dt$

$I = \frac{1}{2} \int t^3 dt = \frac{t^4}{8} + c = \frac{\tan^4 2x}{8} + c$

Example: 74 If $I_n = \int \frac{\sin nx}{\sin x} dx$ where $n > 1$, then $I_n - I_{n-2} =$

- (a) $\frac{2}{(n-1)} \cos(n-1)x$ (b) $\frac{2}{n-1} \sin(n-1)x$ (c) $\frac{2}{n} \cos nx$ (d) $\frac{2}{n} \sin nx$

Solution: (b) $I_n = \int \frac{\sin nx}{\sin x} dx$

$I_{n-2} = \int \frac{\sin(n-2)x}{\sin x} dx \Rightarrow I_n - I_{n-2} = \int \frac{\sin nx - \sin(n-2)x}{\sin x} dx = \int \frac{2 \cos(n-1)x \cdot \sin x}{\sin x} dx$

$I_n - I_{n-2} = \frac{2 \sin(n-1)x}{(n-1)}$

(2) Reduction formulae for special cases

(i) $\int \sin^n x dx = \frac{-\cos x \cdot \sin^{n-1} x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx$

(ii) $\int \cos^n x dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx$

(iii) $\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx$

$$(iv) \int \cot^n x dx = \frac{-1}{n-1} \cot^{n-1} x - \int \cot^{n-2} x dx$$

$$(v) \int \sec^n x dx = \frac{1}{(n-1)} \left[\sec^{n-2} x \tan x + (n-2) \int \sec^{n-2} x dx \right]$$

$$(vi) \int \operatorname{cosec}^n x dx = \frac{1}{(n-1)} \left[-\operatorname{cosec}^{n-2} x \cot x + (n-2) \int \operatorname{cosec}^{n-2} x dx \right]$$

$$(vii) \int \sin^p x \cos^q x dx = -\frac{\sin^{q+1} x \cos^{p-1} x}{p+q} + \frac{p-1}{p+q} \int \sin^{p-2} x \cos^q x dx$$

$$(viii) \int \sin^p x \cos^q x dx = \frac{\sin^{p+1} x \cos^{q-1} x}{p+q} + \frac{p-1}{p+q} \int \sin^p x \cos^{q-2} x dx$$

$$(ix) \int \frac{dx}{(x^2+k)^n} = \frac{x}{k(2n-2)(x^2+k)^{n-1}} + \frac{(2n-3)}{k(2n-2)} \int \frac{dx}{(x^2+k)^{n-1}}$$

Important Tips

☞ Reduction formulae for $I_{(n,m)} = \int \frac{\sin^n x}{\cos^m x} dx$ is $I_{(n,m)} = \frac{1}{m-1} \cdot \frac{\sin^{n-1} x}{\cos^{m-1} x} - \frac{(n-1)}{(m-1)} \cdot I_{(n-2,m-2)}$

Example: 75 $\int \tan^5 \theta d\theta =$

$$(a) \frac{\tan^4 \theta}{4} - \frac{\tan^2 \theta}{2} + \log \sec \theta + c$$

$$(b) \frac{\tan^4 \theta}{4} - \frac{\tan^2 \theta}{2} - \log \sec \theta + c$$

$$(c) \frac{\tan^4 \theta}{4} - \frac{\tan^2 \theta}{2} + \log |\sec \theta| + c$$

(d) None of these

Solution: (c) $\int \tan^5 \theta d\theta = I_5 = \frac{\tan^4 \theta}{4} - I_3 = \frac{\tan^4 \theta}{4} - \frac{\tan^2 \theta}{2} + I_1 = \frac{\tan^4 \theta}{4} - \frac{\tan^2 \theta}{2} + \log |\sec \theta| + c$

Example: 76 $\int \operatorname{cosec}^4 x dx$

$$(a) \cot x + \frac{\cot^3 x}{3} + c$$

$$(b) \tan x + \frac{\tan^3 x}{3} + c$$

$$(c) \cot x - \frac{\cot^3 x}{3} + c$$

$$(d) -\tan x - \frac{\tan^3 x}{3} + c$$

Solution: (c) $\int \operatorname{cosec}^4 x dx = \int \operatorname{cosec}^2 x (1 + \cot^2 x) dx = \int \operatorname{cosec}^2 x dx + \int \cot^2 x \operatorname{cosec}^2 x dx = -\cot x + \frac{(\cot x)^3}{3} + c$

Example: 77 Integrate $\int \sin^8 x dx$

$$(a) \frac{1}{2^7} \left[\frac{\sin 8x}{8} - 8 \frac{\sin 6x}{6} + 28 \frac{\sin 4x}{4} - 56 \frac{\sin 2x}{2} + 35x + c \right]$$

$$(b) \frac{1}{2^7} \left[-\frac{\sin 8x}{8} + 8 \frac{\sin 6x}{6} + 28 \frac{\sin 4x}{4} + 56 \frac{\sin 2x}{2} - 35x + c \right]$$

$$(c) \frac{\sin 8x}{8} - 8 \frac{\sin 6x}{6} + 28 \frac{\sin 4x}{4} - 56 \frac{\sin 2x}{2} + 35x + c$$

(d) None of these

Solution: (a) Let $\cos x + i \sin x = y$; then

$$2 \cos x = y + \frac{1}{y}, 2 \cos nx = y^n + \frac{1}{y^n} \Rightarrow 2 i \sin x = y - \frac{1}{y}, 2 i \sin nx = y^n - \frac{1}{y^n} \quad (\text{Remember as the standard results})$$

$$\begin{aligned}\text{Thus } 2^8 \sin^8 x &= \left(y - \frac{1}{y}\right)^8 = \left(y^8 + \frac{1}{y^8}\right) - 8\left(y^6 + \frac{1}{y^6}\right) + 28\left(y^4 + \frac{1}{y^4}\right) - 56\left(y^2 + \frac{1}{y^2}\right) + 70 \\ &= 2\cos 8x - 16\cos 6x + 56\cos 4x - 112\cos 2x + 70\end{aligned}$$

$$\text{Thus } \sin^8 x = \frac{1}{2^7}(\cos 8x - 8\cos 6x + 28\cos 4x - 56\cos 2x + 35),$$

$$\text{and } \int \sin^8 x dx = \frac{1}{2^7} \left[\frac{\sin 8x}{8} - 8 \frac{\sin 6x}{6} + 28 \frac{\sin 4x}{4} - 56 \frac{\sin 2x}{2} + 35x \right] + c$$

5.14 INTEGRATION OF HYPERBOLIC FUNCTIONS

$$(1) \int \sinh x dx = \cosh x + c$$

$$(2) \int \cosh x dx = \sinh x + c$$

$$(3) \int \operatorname{sech}^2 x dx = \tanh x + c$$

$$(4) \int \operatorname{cosech}^2 x dx = -\coth x + c$$

$$(5) \int \operatorname{sech} x \tanh x dx = -\operatorname{sech} x + c$$

$$(6) \int \operatorname{cosech} x \coth x dx = -\operatorname{cosech} x + c$$

Example: 78 $\int \cos^{-1}\left(\frac{1}{x}\right) dx =$

$$(a) x \sec^{-1} x + \cosh^{-1} x + c$$

$$(b) x \sec^{-1} x - \cosh^{-1} x + c$$

$$(c) \cosh^{-1} x - x \sec^{-1} x + c$$

$$(d) \text{None of these}$$

Solution: (b) $\int \cos^{-1}\left(\frac{1}{x}\right) dx = \int \sec^{-1} x dx = x \sec^{-1} x - \int \frac{1}{x\sqrt{x^2-1}} x dx = x \sec^{-1} x - \cosh^{-1} x + c$

Example: 79 $\int \frac{\cos x - \sin x}{\sqrt{\sin 2x}} dx$ equals

$$(a) \cosh^{-1}(\sin x + \cos x) + c$$

$$(b) \sinh^{-1}(\sin x + \cos x) + c$$

$$(c) -\cosh^{-1}(\sin x + \cos x) + c$$

$$(d) -\sinh^{-1}(\sin x + \cos x) + c$$

Solution: (a) $I = \int \frac{\cos x - \sin x}{\sqrt{\sin 2x}} dx = \int \frac{\cos x - \sin x}{\sqrt{1 + \sin 2x - 1}} dx = \int \frac{\cos x - \sin x}{\sqrt{(\cos x + \sin x)^2 - 1}} dx$

Put $\cos x + \sin x = t \Rightarrow (\cos x - \sin x) dx = dt \Rightarrow \int \frac{dt}{\sqrt{t^2 - 1}} = \cosh^{-1} t + c = \cosh^{-1}(\sin x + \cos x) + c$

Example: 80 $\int \sinh^{-1} x dx$ equals

$$(a) x \cosh^{-1} x - \sqrt{x^2 + 1} + c$$

$$(b) x \cosh^{-1} x + \sqrt{x^2 + 1} + c$$

$$(c) x \sinh^{-1} x + \sqrt{x^2 + 1} + c$$

$$(d) x \sinh^{-1} x - \sqrt{x^2 + 1} + c$$

Solution: (d) $\int \sinh^{-1} x dx = x \sinh^{-1} x - \int \frac{1}{\sqrt{1+x^2}} \cdot x dx = x \sinh^{-1} x - \sqrt{1+x^2} + c$

Example: 81 $\int \sqrt{\frac{e^x + a}{e^x - a}} dx =$

$$(a) \cosh^{-1}(e^x/a) + \sec^{-1}(e^x/a) + c$$

$$(b) \left(\frac{1}{a}\right) [\cosh^{-1}(e^x/a) + \sec^{-1}(e^x/a)] + c$$

$$(c) \left(\frac{1}{a}\right) [\cosh^{-1}(e^x/a) + \sec^{-1}(e^x/a)] + c$$

$$(d) \text{None of these}$$

Solution: (a) $\int \sqrt{\frac{e^x + a}{e^x - a}} dx = \int \frac{e^x + a}{\sqrt{e^{2x} - a^2}} dx = \int \frac{e^x}{\sqrt{e^{2x} - a^2}} dx + \int \frac{a}{\sqrt{e^{2x} - a^2}} dx = \int \frac{e^x/a}{\sqrt{(e^x/a)^2 - 1^2}} dx + \int \frac{1}{\sqrt{(e^x/a)^2 - 1^2}} dx$

$$\text{Put } \frac{e^x}{a} = t \Rightarrow \frac{e^x}{a} dx = dt = \int \frac{dt}{\sqrt{t^2 - 1}} + \int \frac{dt}{t\sqrt{t^2 - 1}} = \cosh^{-1}(e^x/a) + \sec^{-1}(e^x/a) + c$$

Example: 82 $\int \frac{dx}{(e^x + e^{-x})^2} =$

- (a) $\tanh x + c$ (b) $\frac{1}{2} \tanh x + c$ (c) $\frac{1}{4} \tanh x + c$ (d) None of these

Solution: (c) $\int \frac{dx}{(e^x + e^{-x})^2} = \frac{1}{4} \int \frac{4 \cdot dx}{(e^x + e^{-x})^2} = \frac{1}{4} \int \text{sech}^2 x dx = \frac{1}{4} \tanh x + c$

Example: 83 $\int \frac{\tan x}{\sqrt{1 - \tan^2 x}} dx =$

- (a) $\cosh^{-1}(\sqrt{2} \cos x) + c$ (b) $-\cosh^{-1}(\sqrt{2} \cos x) + c$ (c) $\frac{1}{\sqrt{2}} \cos^{-1}(\sqrt{2} \cos x) + c$ (d) $\frac{-1}{\sqrt{2}} \cosh^{-1}(\sqrt{2} \cos x) + c$

Solution: (d) $\int \frac{\tan x dx}{\sqrt{1 - \tan^2 x}} = \int \frac{\sin x dx}{\sqrt{\cos^2 x - \sin^2 x}} = \int \frac{\sin x dx}{\sqrt{2 \cos^2 x - 1}}$

Put $\sqrt{2} \cos x = t \Rightarrow -\sqrt{2} \sin x dx = dt$

$$I = \frac{-1}{\sqrt{2}} \int \frac{dt}{\sqrt{t^2 - 1}} = \frac{-1}{\sqrt{2}} \cosh^{-1}(\sqrt{2} \cos x + c).$$

Example: 84 $\int \sin x \sinh x dx =$

- (a) $\frac{1}{2}(e^x - e^{-x}) + c$ (b) $\frac{1}{2}[\sin x \cosh x - \cos x \sinh x] + c$
(c) $\frac{1}{2}[\sin x \cosh x + \cos x \sinh x] + c$ (d) None of these

Solution: (b) $I = \int \sin x \sinh x dx = -\sinh x \cos x + \int \cosh x \cos x dx = \sinh x \cos x + [\cosh x \sin x - \int \sinh x \sin x dx]$

$$I = \frac{1}{2}[\sin x \cosh x - \cos x \sinh x] + c$$

5.15 INTEGRAL OF THE TYPE $f[x, (ax + b)^{m_1/n_1}, (ax + b)^{m_2/n_2} \dots]$ WHERE F IS A RATIONAL FUNCTION AND m_1, n_1, m_2, n_2 ARE INTEGERS

To evaluate such type of integral, we transform it into an integral of rational function by putting $(ax + b) = t^s$, where s is the least common multiple (L.C.M.) of the numbers n_1, n_2 .

Integrals of the form $\int x^m(a + bx^n)^p dx$

Case I : If $p \in \mathbf{N}$ (Natural number). We expand the integral with the help of binomial theorem and integrate.

Example: 85 Evaluate $\int x^{1/3}(2 + x^{1/2})^2 dx$

- (a) $3x^{4/3} + \frac{7}{3}x^{7/3} + c$ (b) $x^{4/3} + \frac{5}{3}x^{5/3} + c$ (c) $3x^{4/3} + \frac{3}{5}x^{5/3} + c$ (d) $3x^{4/3} + \frac{3}{7}x^{7/3} + \frac{24}{11}x^{11/6} + c$

Solution: (d) $I = \int x^{1/3}(2 + x^{1/2})^2 dx$

Since P is natural number,

$$I = \int x^{1/3}(4 + x + 4x^{1/2}) dx = \int (4x^{1/3} + x^{4/3} + 4x^{5/6}) dx = \frac{4x^{4/3}}{4/3} + \frac{x^{7/3}}{7/3} + \frac{4x^{11/6}}{11/6} + c = 3x^{4/3} + \frac{3}{7}x^{7/3} + \frac{24}{11}x^{11/6} + c$$

Case II: If $p \in I$ (i.e. negative integer). Write $x = t^k$, where k is the L.C.M. of denominator of m and n .

Example: 86 Evaluate $\int x^{-2/3}(1+x^{2/3})^{-1} dx$

- (a) $3 \tan^{-1}(x^{1/3}) + c$ (b) $3 \tan^{-1} x + c$ (c) $3 \tan^{-1}(x^{2/3}) + c$ (d) None of these

Solution: (a) If we substitute $x = t^3$ (as we know $p \in$ negative integer)

$\therefore$ Let $x = t^k$, where k is L.C.M. of denominator m and n .

$$\therefore x = t^3 \Rightarrow dx = 3t^2 dt \text{ OR } I = \int \frac{3t^2 dt}{t^2(1+t^2)} = 3 \int \frac{dt}{t^2+1} = 3 \tan^{-1}(t) + c \Rightarrow I = 3 \tan^{-1}(x^{1/3}) + c.$$

Case III: If $\frac{m+1}{n}$ is an integer and $p \in$ fraction put $(a+bx^n) = t^k$, where k is the denominator of the fraction p .

Example: 87 Evaluate $\int x^{-2/3}(1+x^{1/3})^{1/2} dx$

- (a) $2(1+x^{1/3})^{2/3} + c$ (b) $2(1+x^{1/3})^{3/2} + c$ (c) $2(1+x^{2/3})^{3/2} + c$ (d) $2(1+x^{2/3})^{2/3} + c$

Solution: (b) If we substitute $1+x^{1/3} = t^2$ then, $\frac{1}{3x^{2/3}} dx = dt$

$$\therefore I = \int \frac{t \cdot 6t dt}{1} = 6 \int t^2 dt = 2t^3 + c \text{ OR } I = 2(1+x^{1/3})^{3/2} + c$$

Case IV: If $\left(\frac{m+1}{n} + P\right)$ is an integer and $P \in$ fraction. We put $(a+bx^n) = t^k \cdot x^n$, where k is the denominator of the fraction P .

Example: 88 Evaluate $\int x^{-11}(1+x^4)^{-1/2} dx$

- (a) $\frac{1}{2}(t^5 + t^3 + t) + c$ (b) $-\frac{1}{2}\left[\frac{t^5}{5} - \frac{2t^3}{3} + t\right] + c$ (c) $\frac{1}{2}\left[\frac{t^5}{4} + \frac{2t^3}{3} + t\right] + c$ (d) None of these

$$\text{where } t = \sqrt{1 + \frac{1}{x^4}}$$

Solution: (b) Here $\left(\frac{m+1}{n} + P\right) = \left[\frac{-11+1}{4} + \frac{1}{2}\right] = -3$

If we substitute then

$$1 + \frac{1}{x^4} = t^2 \text{ and } \frac{-4}{x^5} dx = 2t dt \Rightarrow I = \int \frac{dx}{x^{11}(1+x^4)^{1/2}} = \int \frac{dx}{x^{11} \cdot x^2(1+1/x^4)^{1/2}}$$

$$I = \int \frac{dx}{x^{13}(1+1/x^4)^{1/2}} = \frac{1}{4} \int \frac{2t dt}{x^8 t} = -\frac{1}{2} \int (t^2 - 1)^2 dt = -\frac{1}{2} \int (t^4 - 2t^2 + 1) dt = -\frac{1}{2} \left[\frac{t^5}{5} - \frac{2t^3}{3} + t \right] + c$$

$$\text{where } t = \sqrt{1 + \frac{1}{x^4}}$$

5.16 INTEGRALS USING EULER'S SUBSTITUTION

Integrals of the form $\int f(x) \sqrt{ax^2 + bx + c} dx$ are calculated with the aid of one of the three Euler substitutions:

(1) $\sqrt{ax^2 + bx + c} = t \pm x\sqrt{a}$, if $a > 0$.

(2) $\sqrt{ax^2 + bx + c} = tx \pm \sqrt{c}$, if $c > 0$.

(3) $\sqrt{ax^2 + bx + c} = (x - \alpha)t$, if $ax^2 + bx + c = a(x - \alpha)(x - \beta)$, i.e., if x is real root of $(ax^2 + bx + c)$.

Note : $\square$ The Euler substitution often leads to rather some calculations, therefore they should be applied only when it is difficult to find another method for calculating the given integral.

Example: 89 Evaluate $\int \frac{dx}{x + \sqrt{x^2 - x + 1}}$

(a) $2\log_e |t| - \frac{1}{2}\log_e |t-1| - \frac{3}{2}\log_e |t+1| + \frac{3}{(t+1)} + c$ (b) $2\log_e |t| + \frac{1}{2}\log_e |t-1| + \frac{3}{2}\log_e |t+1| + \frac{3}{(t+1)} + c$

(c) $2\log_e |t| - \frac{1}{2}\log_e |t-1| + \frac{3}{2}\log_e |t+1| + \frac{3}{(t+1)} + c$ (d) None of these

(Where $t = \frac{\sqrt{x^2 - x + 1} + 1}{x}$)

Solution: (a) Since here $c=1$, we can apply the second Euler substitution.

$$\sqrt{x^2 - x + 1} = tx - 1$$

Hence $(2t-1)x = (t^2-1)x^2$; $x = \frac{2t-1}{t^2-1}$

$$\therefore dx = \frac{2(t^2-t+1)dt}{(t^2-1)^2} \text{ and } x + \sqrt{x^2 - x + 1} = \frac{1}{t-1}$$

$$\therefore I = \int \frac{dx}{x + \sqrt{x^2 - x + 1}} = \int \frac{-2t^2 + 2t - 2}{t(t-1)(t+1)^2} dt$$

Using partial fractions, we have,

$$\frac{-2t^2 + 2t - 2}{t(t-1)(t+1)^2} = \frac{A}{t} + \frac{B}{t-1} + \frac{C}{(t+1)} + \frac{D}{(t+1)^2} \text{ or } (-2t + 2t - 2) = A(t-1)(t+1)^2 + B(t+1)^2 + C(t-1)(t+1)t + Dt$$

We get $A=2, B=1/2, C=-3/2, D=-3$.

$$\text{Hence } I = 2 \int \frac{dt}{t} - \frac{1}{2} \int \frac{dt}{t-1} - \frac{3}{2} \int \frac{dt}{(t+1)} - 3 \int \frac{dt}{(t+1)^2} = 2\log_e |t| - \frac{1}{2}\log_e |t-1| - \frac{3}{2}\log_e |t+1| + \frac{3}{(t+1)} + c$$

(Where $t = \frac{\sqrt{x^2 - x + 1} + 1}{x}$)

Example: 90 Evaluate $\int \frac{x dx}{\sqrt{(7x-10-x^2)^3}}$

(a) $\frac{-2}{9} \left(\frac{-5}{t} + 2t \right) + c$

(b) $\frac{2}{9} \left(\frac{-5}{t} - 2t \right) + c$

(c) $\frac{-1}{9} \left(\frac{5}{t} + 2t \right) + c$

(d) None of these

where $t = \frac{\sqrt{7x-10-x^2}}{x-2}$

Solution: (a) In this case $a < 0$ and $c < 0$. Therefore neither (I) nor (II) Euler substitution is applicable. But the quadratic $7x-10-x^2$ has real roots $\alpha=2, \beta=5$.

$$\therefore \text{We use the (III) i.e., } \sqrt{7x-10-x^2} = \sqrt{(x-2)(5-x)} = (x-2)t$$

where $(5-x) = (x-2)t^2$ or $5+2t^2 = x(1+t^2)$

$$\therefore x = \frac{5+2t^2}{1+t^2} \Rightarrow (x-2)t = \left(\frac{5+2t^2}{1+t^2} - 2 \right) t = \frac{3t}{1+t^2}, \quad \therefore dx = \frac{-6t}{(1+t^2)^2} dt$$

$$\text{Hence, } I = \int \frac{x dx}{(\sqrt{7x-10-x^2})^3} = \int \frac{\left(\frac{5+2t^2}{1+t^2} \right) \cdot \frac{-6t}{(1+t^2)^2} dt}{\left(\frac{3t}{1+t^2} \right)^3} = \frac{-6}{27} \int \frac{5+2t^2}{t^2} dt = \frac{-2}{9} \left(\frac{5}{t^2} + 2 \right) dt = \frac{-2}{9} \left[\frac{-5}{t} + 2t \right] + c$$

$$\therefore \int \frac{x dx}{(\sqrt{7x-10-x^2})^3} = \frac{-2}{9} \left(\frac{-5}{t} + 2t \right) + c, \quad \text{where } t = \frac{\sqrt{7x-10-x^2}}{x-2}$$

-----*****-----