

HYPERBOLIC FNCTIONS

6.1 DEFINITION

We know that parametric co-ordinates of any point on the unit circle $x^2 + y^2 = 1$ is $(\cos\theta, \sin\theta)$; so that these functions are called circular functions and co-ordinates of any point on unit hyperbola $x^2 - y^2 = 1$ is $\left(\frac{e^\theta + e^{-\theta}}{2}, \frac{e^\theta - e^{-\theta}}{2}\right)$ i.e., $(\cosh\theta, \sinh\theta)$. It means that the relation which exists amongst $\cos\theta, \sin\theta$ and unit circle, that relation also exist amongst $\cosh\theta, \sinh\theta$ and unit hyperbola. Because of this reason these functions are called as Hyperbolic functions.

For any (real or complex) variable quantity x ,

$$(1) \sinh x = \frac{e^x - e^{-x}}{2} \text{ [Read as 'hyperbolic sine x']} \quad (2) \cosh x = \frac{e^x + e^{-x}}{2} \text{ [Read as 'hyperbolic cosine x']}$$

$$(3) \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$(4) \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$(5) \operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

$$(6) \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

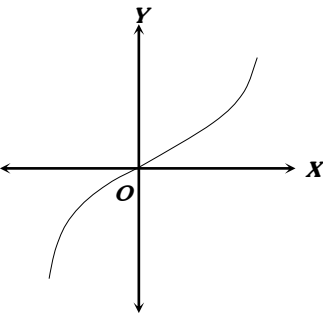
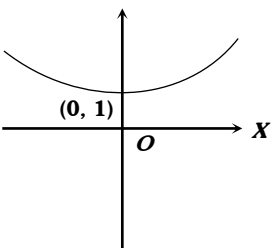
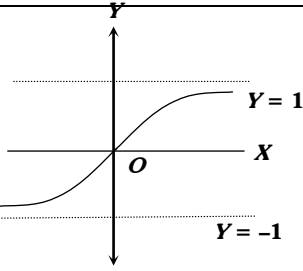
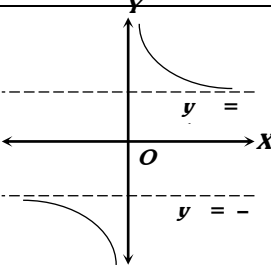
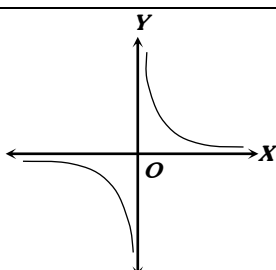
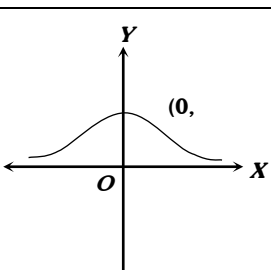
Note : $\square \sinh 0 = 0, \cosh 0 = 1, \tanh 0 = 0$

6.2 DOMAIN AND RANGE OF HYPERBOLIC FUNCTIONS

Let x is any real number

Function	Domain	Range
$\sinh x$	R	R
$\cosh x$	R	$[1, \infty)$
$\tanh x$	R	$(-1, 1)$
$\coth x$	R_0	$R - [-1, 1]$
$\operatorname{sech} x$	R	$(0, 1]$
$\operatorname{cosech} x$	R_0	R_0

6.3 GRAPH OF REAL HYPERBOLIC FUNCTIONS

(1) $\sinh x$ 	(2) $\cosh x$ 
(3) $\tanh x$ 	(4) $\coth x$ 
(5) $\operatorname{cosech} x$ 	(6) $\operatorname{sech} x$ 

6.4 FORMULAE FOR HYPERBOLIC FUNCTIONS

The following formulae can easily be established directly from above definitions

(1) Reciprocal formulae

$$(i) \operatorname{cosech} x = \frac{1}{\sinh x}$$

$$(ii) \operatorname{sech} x = \frac{1}{\cosh x}$$

$$(iii) \coth x = \frac{1}{\tanh x}$$

$$(iv) \tanh x = \frac{\sinh x}{\cosh x}$$

$$(v) \coth x = \frac{\cosh x}{\sinh x}$$

(2) Square formulae

$$(i) \cosh^2 x - \sinh^2 x = 1$$

$$(ii) \operatorname{sech}^2 x + \tanh^2 x = 1$$

$$(iii) \coth^2 x - \operatorname{cosech}^2 x = 1$$

$$(iv) \cosh^2 x + \sinh^2 x = \cosh 2x$$

(3) Expansion or Sum and difference formulae

$$(i) \sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$(ii) \cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$(iii) \tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

(4) Formulae to transform the product into sum or difference

- (i) $\sinh x + \sinh y = 2 \sinh \frac{x+y}{2} \cosh \frac{x-y}{2}$ (ii) $\sinh x - \sinh y = 2 \cosh \frac{x+y}{2} \sinh \frac{x-y}{2}$
 (iii) $\cosh x + \cosh y = 2 \cosh \frac{x+y}{2} \cosh \frac{x-y}{2}$ (iv) $\cosh x - \cosh y = 2 \sinh \frac{x+y}{2} \sinh \frac{x-y}{2}$
 (v) $2 \sinh x \cosh y = \sinh(x+y) + \sinh(x-y)$ (vi) $2 \cosh x \sinh y = \sinh(x+y) - \sinh(x-y)$
 (vii) $2 \cosh x \cosh y = \cosh(x+y) + \cosh(x-y)$ (viii) $2 \sinh x \sinh y = \cosh(x+y) - \cosh(x-y)$
 (ix) $\cosh x + \sinh x = e^x$ (x) $\cosh x - \sinh x = e^{-x}$ (xi) $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$

(5) Trigonometric ratio of multiple of an angle

- (i) $\sinh 2x = 2 \sinh x \cosh x = \frac{2 \tanh x}{1 - \tanh^2 x}$
 (ii) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2 \cosh^2 x - 1 = 1 + 2 \sinh^2 x = \frac{1 + \tanh^2 x}{1 - \tanh^2 x}$
 (iii) $2 \cosh^2 x = \cosh 2x + 1$ (iv) $2 \sinh^2 x = \cosh 2x - 1$ (v) $\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$
 (vi) $\sinh 3x = 3 \sinh x + 4 \sinh^3 x$ (vii) $\cosh 3x = 4 \cosh^3 x - 3 \cosh x$ (viii) $\tanh 3x = \frac{3 \tanh x + \tanh^3 x}{1 + 3 \tanh^2 x}$
 (6) (i) $\cosh x + \sinh x = e^x$ (ii) $\cosh x - \sinh x = e^{-x}$
 (iii) $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$

Example: 1 $\frac{\sinh x - \sinh y}{\cosh x - \cosh y}$ is equal to

- (a) $2 \coth(x+y)$ (b) $\tanh\left(\frac{x+y}{2}\right)$ (c) $\coth\left(\frac{x+y}{2}\right)$ (d) $\coth\left(\frac{x-y}{2}\right)$

Solution: (c) $\frac{\sinh x - \sinh y}{\cosh x - \cosh y} = \frac{2 \cosh \frac{x+y}{2} \sinh \frac{x-y}{2}}{2 \sinh \frac{x+y}{2} \cosh \frac{x-y}{2}} = \coth\left(\frac{x+y}{2}\right).$

Example: 2 If $\tanh^2 x = \tan^2 \theta$, then $\cosh 2x$ is equal to

- (a) $-\sin 2\theta$ (b) $\sec 2\theta$ (c) $\cos 3\theta$ (d) $\cos 2\theta$

Solution: (b) $\cosh 2x = \frac{1 + \tanh^2 x}{1 - \tanh^2 x} = \frac{1 + \tan^2 \theta}{1 - \tan^2 \theta} = \frac{1}{\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}} = \frac{1}{\cos 2\theta} = \sec 2\theta.$

Example: 3 If $u = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$, then $\cosh u$ is equal to

- (a) $\sec x$ (b) $\operatorname{cosec} x$ (c) $\tan x$ (d) $\sin x$

Solution: (a) $u = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) \Rightarrow \therefore \frac{e^u}{1} = \frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}}$

$$\cosh u = \frac{e^u + e^{-u}}{2} = \frac{e^{2u} + 1}{2e^u} = \frac{\left(\frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}}\right)^2 + 1}{2 \cdot \left(\frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}}\right)} = \frac{2 \left[\frac{1 + \tan^2 \frac{x}{2}}{1 - \tan^2 \frac{x}{2}} \right]}{2 \left(\frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}} \right) \left(1 + \tan \frac{x}{2} \right)} = \frac{1 + \tan^2 \frac{\pi}{2}}{1 - \tan^2 \frac{\pi}{2}} = \frac{1}{\frac{1 - \tan^2 \frac{\pi}{2}}{1 + \tan^2 \frac{\pi}{2}}} = \frac{1}{\cos x} = \sec x.$$

Example: 4 $\cosh 2 + \sinh 2 =$

(a) $\frac{1}{e}$

(b) e

(c) $\frac{1}{e^2}$

(d) e^2

Solution: (d) $\cosh 2 + \sinh 2 = \frac{e^2 + e^{-2}}{2} + \frac{e^2 - e^{-2}}{2} = e^2.$

Example: 5 If $\cosh x = \sec x$, then $\tan^2 \frac{x}{2} =$

(a) $\cos^2 \frac{\alpha}{2}$

(b) $\sin^2 \frac{\alpha}{2}$

(c) $\cot^2 \frac{\alpha}{2}$

(d) $\tan^2 \frac{\alpha}{2}$

Solution: (d) $\cosh x = \sec x = \frac{1}{\cos x} = \frac{1}{\frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}}$

$$\frac{\cosh x}{1} = \frac{1 + \tan^2 \frac{x}{2}}{1 - \tan^2 \frac{x}{2}} \Rightarrow \frac{\cosh x - 1}{\cosh x + 1} = \frac{2 \tan^2 \frac{x}{2}}{2} = \tan^2 \frac{x}{2} \Rightarrow \frac{2 \sinh^2 \frac{\alpha}{2}}{2 \cosh^2 \frac{\alpha}{2}} = \tan^2 \frac{x}{2} \Rightarrow \tan^2 \frac{\alpha}{2} = \tan^2 \frac{x}{2}.$$

6.5 TRANSFORMATION OF A HYPERBOLIC FUNCTIONS

Since, $\cosh^2 x - \sinh^2 x = 1$

$$\Rightarrow \sinh x = \sqrt{\cosh^2 x - 1} \quad \Rightarrow \sinh x = \frac{\sqrt{1 - \operatorname{sech}^2 x}}{\operatorname{sech} x}$$

$$\Rightarrow \sinh x = \frac{\tanh x}{\sqrt{1 - \tanh^2 x}} \quad \Rightarrow \sinh x = \frac{1}{\sqrt{\coth^2 x - 1}}$$

Also, $\sinh x = \frac{1}{\operatorname{cosech} x}$

In a similar manner we can express $\cosh x$, $\tanh x$, $\coth x$,..... in terms of other hyperbolic functions.

6.6 EXPANSION OF HYPERBOLIC FUNCTIONS

$$(1) \sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$$

$$(2) \cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$

$$(3) \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = x - \frac{x^3}{3} + 2x^5 - \frac{17}{315}x^7 + \dots$$

The expansion of $\coth x$, $\operatorname{cosech} x$ does not exist because $\coth(0) = \infty$, $\operatorname{cosech}(0) = \infty$.

6.7 RELATION BETWEEN HYPERBOLIC AND CIRCULAR FUNCTIONS

We have from Euler formulae,

$$e^{ix} = \cos x + i \sin x \quad \dots\dots(i) \quad \text{and} \quad e^{-ix} = \cos x - i \sin x \quad \dots\dots(ii)$$

$$\text{Adding (i) and (ii)} \Rightarrow \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\text{Subtracting (ii) from (i)} \Rightarrow \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\text{Replacing } x \text{ by } ix \text{ in these values, we get } \cos(ix) = \frac{e^{-x} + e^x}{2} = \cosh x$$

$$\therefore \cos(ix) = \cosh x$$

$$\sin(ix) = \frac{e^{-x} - e^x}{2i} = i \left(\frac{e^x - e^{-x}}{2} \right)$$

$$\therefore \sin(ix) = i \sinh x$$

$$\text{Also } \tan(ix) = \frac{\sin(ix)}{\cos(ix)} = \frac{i \sinh x}{\cosh x}$$

$$\tan(ix) = i \tanh x$$

$$\text{Similarly replacing } x \text{ by } ix \text{ in the definitions of } \sinh x \text{ and } \cosh x, \text{ we get } \cosh(ix) = \frac{e^{ix} + e^{-ix}}{2} = \cos x$$

$$\text{Also, } \tanh(ix) = \frac{\sinh(ix)}{\cosh(ix)} = \frac{i \sin x}{\cos x} = i \tan x$$

Thus, we obtain the following relations between hyperbolic and trigonometrical functions.

(1) $\sin(ix) = i \sinh x$ $\sinh(ix) = i \sin x$ $\sinh x = -i \sin(ix)$ $\sin x = -i \sinh(ix)$	(2) $\cos(ix) = \cosh x$ $\cosh(ix) = \cos x$ $\cosh x = \cos(ix)$ $\cos x = \cosh(ix)$
(3) $\tan(ix) = i \tanh x$ $\tanh(ix) = i \tan x$ $\tanh x = -i \tan(ix)$ $\tan x = -i \tanh(ix)$	(4) $\cot(ix) = -i \coth x$ $\coth(ix) = -i \cot x$ $\coth x = i \cot(ix)$ $\cot x = i \coth(ix)$
(5) $\sec(ix) = \operatorname{sech} x$ $\operatorname{sech}(ix) = \sec x$ $\operatorname{sech} x = \sec(ix)$ $\sec x = \operatorname{sech}(ix)$	(6) $\operatorname{cosec}(ix) = -i \operatorname{cosech} x$ $\operatorname{cosech}(ix) = i \operatorname{cosec} x$ $\operatorname{cosech} x = i \operatorname{cosec}(ix)$ $\operatorname{cosec} x = i \operatorname{cosech}(ix)$

Important Tips

☞ For obtaining any formula given in (5)th article, use the following substitutions in the corresponding formula for trigonometric functions.

$$\begin{array}{lll} \sin x \longrightarrow i \sinh x & \cos x \longrightarrow \cosh x & \tan x \longrightarrow i \tanh x \\ \sin^2 x \longrightarrow -\sinh^2 x & \cos^2 x \longrightarrow \cosh^2 x & \tan^2 x \longrightarrow -\tanh^2 x \end{array}$$

For example,

For finding out the formula for $\cosh 2x$ in terms of $\tanh x$, replace $\tan x$ by $i \tanh x$ and $\tan^2 x$ by $-\tanh^2 x$ in the following formula of trigonometric function of $\cos 2x$:

$$\cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x} \text{ we get, } \cosh 2x = \frac{1 + \tanh^2 x}{1 - \tanh^2 x}$$

6.8 PERIOD OF HYPERBOLIC FUNCTIONS

If for any function $f(x)$, $f(x + T) = f(x)$, then $f(x)$ is called the **Periodic function** and least positive value of T is called the **Period** of the function.

$$\ominus \sinh x = \sinh(2\pi i + x)$$

$$\cosh x = \cosh(2\pi i + x)$$

$$\text{and } \tanh x = \tanh(\pi i + x)$$

Therefore the period of these functions are respectively $2\pi i$, $2\pi i$ and πi . Also period of cosech x , sech x and $\coth x$ are respectively $2\pi i$, $2\pi i$ and πi .

Note : $\square$ Remember that if the period of $f(x)$ is T , then period of $f(nx)$ will be $\left(\frac{T}{n}\right)$.

$\square$ Hyperbolic function are neither periodic functions nor their curves are periodic but they show the algebraic properties of periodic functions and having imaginary period.

Example: 6 If $\cos(x + iy) = A + iB$, then A equals

$$(a) \cos x \cosh y \quad (b) \sin x \sinh y \quad (c) -\sin x \sinh y \quad (d) \cos x \sinh y$$

Solution: (a) $\cos(x + iy) = A + iB$

$$\Rightarrow \cos x \cos(iy) - \sin x \sin(iy) = A + iB \Rightarrow \cos x \cosh y - i \sin x \sinh y = A + iB$$

$$\therefore A = \cos x \cosh y$$

Example: 7 If $\cos(u + iv) = x + iy$, then $x^2 + y^2 + 1$ is equal to

$$(a) \cos^2 u + \sin^2 v \quad (b) \sin^2 u + \cosh^2 v \quad (c) \cos^2 u + \cosh^2 v \quad (d) \sin^2 u + \sinh^2 v$$

Solution: (c) $\cos(u + iv) = x + iy$

$$\Rightarrow \cos u \cos(iv) - \sin u \sin(iv) = x + iy \Rightarrow \cos u \cosh v - i \sin u \sinh v = x + iy$$

$$\therefore x = \cos u \cosh v$$

$$y = -\sin u \sinh v$$

$$x^2 + y^2 = \cos^2 u \cosh^2 v + \sin^2 u \sinh^2 v$$

$$= (1 - \sin^2 u) \cosh^2 v + \sin^2 u [\cosh^2 v - 1] = \cosh^2 v - \sin^2 v$$

$$\therefore x^2 + y^2 + 1 = \cosh^2 v + 1 - \sin^2 u = \cosh^2 v + \cos^2 u.$$

Example: 8 The value of $\operatorname{sech}(i\pi)$ is

- (a) -1 (b) i (c) 0 (d) 1

Solution: (a) $\operatorname{sech}(i\pi) = \frac{2}{e^{i\pi} + e^{-i\pi}} = \sec\pi = -1.$

Example: 9 $\cos ix + i \sin ix$ equals

- (a) e^{ix} (b) e^{-ix} (c) e^x (d) e^{-x}

Solution: (d) $\cos ix + i \sin ix$

$$= \cosh x + i i \sinh x = \cosh x - \sinh x = \frac{e^x + e^{-x} - e^x + e^{-x}}{2} = e^{-x}.$$

Example: 10 $\sinh\left(\frac{\pi}{6}i\right)$ is equal to

- (a) $-\frac{i}{2}$ (b) $\frac{i}{2}$ (c) $i\frac{\sqrt{3}}{2}$ (d) $-i\frac{\sqrt{3}}{2}$

Solution: (b) $\sinh\left(\frac{\pi}{6}i\right) = i \sin \frac{\pi}{6} = i \frac{1}{2} = \frac{i}{2}.$

Example: 11 $\operatorname{sech}(\pi i) + \operatorname{cosech}\left(\frac{\pi}{2}i\right)$ equals

- (a) $1-i$ (b) $-1+i$ (c) $-1-i$ (d) $1+i$

Solution: (c) $\operatorname{sech}(\pi i) + \operatorname{cosech}\left(\frac{\pi}{2}i\right) = \sec\pi - i \operatorname{cosec}\frac{\pi}{2} = -1 - i.$

Example: 12 The period of $\cosh \frac{\theta}{3}$ is

- (a) $6\pi i$ (b) $2\pi i$ (c) πi (d) $9\pi i$

Solution: (a) Since the period of $\cosh \theta$ is $2\pi i$, so the period of $\cosh \frac{\theta}{3}$ is $3 \cdot 2\pi i = 6\pi i.$

Example: 13 The period of $\sinh\left(\frac{x}{2}\right)$ is

- (a) $2\pi i$ (b) 2π (c) $4\pi i$ (d) 4π

Solution: (c) Since period of $\sinh x$ is $2\pi i$, therefore period of $\sinh\left(\frac{x}{2}\right)$ will be $4\pi i.$

6.9 INVERSE HYPERBOLIC FUNCTIONS

If $\sinh y = x$, then y is called the inverse hyperbolic sine of x and it is written as $y = \sinh^{-1} x$. Similarly $\operatorname{cosech}^{-1} x, \cosh^{-1} x, \tanh^{-1} x$ etc. can be defined.

(1) Domain and range of Inverse hyperbolic function

Function	Domain	Range
$\sinh^{-1} x$	R	R
$\cosh^{-1} x$	$[1, \infty)$	R
$\tanh^{-1} x$	$(-1, 1)$	R
$\coth^{-1} x$	$R - [-1, 1]$	R_0
$\operatorname{sech}^{-1} x$	$(0, 1]$	R
$\operatorname{cosech}^{-1} x$	R_0	R_0

(2) Relation between inverse hyperbolic function and inverse circular function

Method : Let $\sinh^{-1} x = y$

$$\Rightarrow x = \sinh y = -i \sin(iy) \Rightarrow ix = \sin(iy) \Rightarrow iy = \sin^{-1}(ix)$$

$$\Rightarrow y = -i \sin^{-1}(ix) \Rightarrow \sinh^{-1} x = -i \sin^{-1}(ix)$$

Therefore we get the following relations

$$(i) \sinh^{-1} x = -i \sin^{-1}(ix) \quad (ii) \cosh^{-1} x = -i \cos^{-1} x \quad (iii) \tanh^{-1} x = -i \tan^{-1}(ix)$$

$$(iv) \operatorname{sech}^{-1} x = -i \sec^{-1} x \quad (v) \operatorname{cosech}^{-1} x = i \operatorname{cosec}^{-1}(ix)$$

(3) To express any one inverse hyperbolic function in terms of the other inverse hyperbolic functions

To express $\sinh^{-1} x$ in terms of the others

$$(i) \text{ Let } \sinh^{-1} x = y \Rightarrow x = \sinh y \Rightarrow \operatorname{cosech} y = \frac{1}{x} \Rightarrow y = \operatorname{cosech}^{-1}\left(\frac{1}{x}\right)$$

$$(ii) \ominus \cosh y = \sqrt{1 + \sinh^2 y} = \sqrt{1 + x^2}$$

$$\ominus y = \cosh^{-1} \sqrt{1 + x^2} \Rightarrow \sinh^{-1} x = \cosh^{-1} \sqrt{1 + x^2}$$

$$(iii) \ominus \tanh y = \frac{\sinh y}{\cosh y} = \frac{\sinh y}{\sqrt{1 + \sinh^2 y}} = \frac{x}{\sqrt{1 + x^2}}$$

$$\therefore y = \tanh^{-1} \frac{x}{\sqrt{1 + x^2}} \Rightarrow \sinh^{-1} x = \tanh^{-1} \frac{x}{\sqrt{1 + x^2}}$$

$$(iv) \ominus \coth y = \frac{\sqrt{1 + \sinh^2 y}}{\sinh y} = \frac{\sqrt{1 + x^2}}{x}$$

$$\therefore y = \coth^{-1} \frac{\sqrt{1 + x^2}}{x} \Rightarrow \sinh^{-1} x = \coth^{-1} \frac{\sqrt{1 + x^2}}{x}$$

$$(v) \ominus \operatorname{sech} y = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}$$

$$y = \operatorname{sech}^{-1} \frac{1}{\sqrt{1 + x^2}} \Rightarrow \sinh^{-1} x = \operatorname{sech}^{-1} \frac{1}{\sqrt{1 + x^2}}$$

$$(vi) \text{ Also, } \sinh^{-1} x = \operatorname{cosech}^{-1}\left(\frac{1}{x}\right)$$

From the above, it is clear that

$$\coth^{-1} x = \tanh^{-1}\left(\frac{1}{x}\right)$$

$$\operatorname{sech}^{-1} x = \cosh^{-1}\left(\frac{1}{x}\right)$$

$$\operatorname{cosech}^{-1} = \sinh^{-1}\left(\frac{1}{x}\right)$$

Note : $\square$ If x is real then all the above six inverse functions are single valued.

(4) Relation between inverse hyperbolic functions and logarithmic functions

Method : Let $\sinh^{-1} x = y$

$$\Rightarrow x = \sinh y = \frac{e^y - e^{-y}}{2} \Rightarrow e^{2y} - 2xe^y - 1 = 0 \Rightarrow e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}$$

But $e^y > 0, \forall y$ and $x < \sqrt{x^2 + 1}$

$$\therefore e^y = x + \sqrt{x^2 + 1} \Rightarrow y = \log(x + \sqrt{x^2 + 1})$$

$$\therefore \sinh^{-1} x = \log(x + \sqrt{x^2 + 1})$$

By the above method we can obtain the following relations between inverse hyperbolic functions and principal values of logarithmic functions.

$$(i) \sinh^{-1} x = \log(x + \sqrt{x^2 + 1}) \quad (-\infty < x < \infty) \quad (ii) \cosh^{-1} x = \log(x + \sqrt{x^2 - 1}) \quad (x \geq 1)$$

$$(iii) \tanh^{-1} x = \frac{1}{2} \log\left(\frac{1+x}{1-x}\right) \quad |x| < 1 \quad (iv) \coth^{-1} x = \frac{1}{2} \log\left(\frac{x+1}{x-1}\right) \quad |x| > 1$$

$$(v) \operatorname{sech}^{-1} x = \log\left(\frac{1 + \sqrt{1 - x^2}}{x}\right) \quad 0 < x \leq 1 \quad (vi) \operatorname{cosech}^{-1} x = \log\left(\frac{1 + \sqrt{1 + x^2}}{x}\right) \quad (x \neq 0)$$

Note : $\square$ Formulae for values of $\operatorname{cosech}^{-1} x$, $\operatorname{sech}^{-1} x$ and $\coth^{-1} x$ may be obtained by replacing x by $\frac{1}{x}$ in the values of $\sinh^{-1} x$, $\cosh^{-1} x$ and $\tanh^{-1} x$ respectively.

6.10 SEPARATION OF INVERSE TRIGONOMETRIC AND INVERSE HYPERBOLIC FUNCTIONS

If $\sin(\alpha + i\beta) = x + iy$ then $(\alpha + i\beta)$, is called the inverse sine of $(x + iy)$. We can write it as,

$$\sin^{-1}(x + iy) = \alpha + i\beta$$

Here the following results for inverse functions may be easily established.

$$(1) \cos^{-1}(x + iy) = \frac{1}{2} \cos^{-1}\left[(x^2 + y^2) - \sqrt{(1 - x^2 + y^2)^2 + 4x^2 y^2}\right] + \frac{i}{2} \cosh^{-1}\left[(x^2 + y^2) + \sqrt{(1 - x^2 + y^2)^2 + 4x^2 y^2}\right]$$

$$(2) \sin^{-1}(x + iy) = \frac{\pi}{2} - \cos^{-1}(x + iy) \\ = \frac{\pi}{2} - \frac{1}{2} \cos^{-1}\left[(x^2 + y^2) - \sqrt{(1 - x^2 + y^2)^2 + 4x^2 y^2}\right] - \frac{i}{2} \cosh^{-1}\left[(x^2 + y^2) + \sqrt{(1 - x^2 + y^2)^2 + 4x^2 y^2}\right]$$

$$(3) \tan^{-1}(x + iy) = \frac{1}{2} \tan^{-1}\left(\frac{2x}{1 - x^2 - y^2}\right) + \frac{i}{2} \tanh^{-1}\left(\frac{2y}{1 + x^2 + y^2}\right) = \frac{1}{2} \tan^{-1}\left(\frac{2x}{1 - x^2 - y^2}\right) + \frac{i}{4} \log\left[\frac{x^2 + (1 + y)^2}{x^2 + (1 - y)^2}\right]$$

$$(4) \sin^{-1}(\cos\theta + i\sin\theta) = \cos^{-1}(\sqrt{\sin\theta}) + i\sinh^{-1}(\sqrt{\sin\theta}) \text{ or } \cos^{-1}(\sqrt{\sin\theta}) + i\log(\sqrt{\sin\theta} + \sqrt{1 + \sin\theta})$$

$$(5) \cos^{-1}(\cos\theta + i\sin\theta) = \sin^{-1}(\sqrt{\sin\theta}) - i\sinh^{-1}(\sqrt{\sin\theta}) \text{ or } \sin^{-1}(\sqrt{\sin\theta}) - i\log(\sqrt{\sin\theta} + \sqrt{1 + \sin\theta})$$

$$(6) \tan^{-1}(\cos\theta + i\sin\theta) = \frac{\pi}{4} + \frac{i}{4} \log\left(\frac{1+\sin\theta}{1-\sin\theta}\right), (\cos\theta) > 0$$

$$\text{and } \tan^{-1}(\cos\theta + i\sin\theta) = \left(-\frac{\pi}{4}\right) + \frac{1}{4} \log\left(\frac{1+\sin\theta}{1-\sin\theta}\right), (\cos\theta) < 0$$

Since each inverse hyperbolic function can be expressed in terms of logarithmic function, therefore for separation into real and imaginary parts of inverse hyperbolic function of complex quantities use the appropriate method.

Note : $\square$ Both inverse circular and inverse hyperbolic functions are many valued.

Example: 14 If $x = \log(y + \sqrt{y^2 + 1})$, then $y =$

(a) $\tanh x$

(b) $\coth x$

(c) $\sinh x$

(d) $\cosh x$

Solution: (c) $x = \sinh^{-1} y \Rightarrow y = \sinh x$.

Example: 15 If $\cosh^{-1} x = \log 2 + \sqrt{3}$, then $x =$

(a) 2

(b) 1

(c) 3

(d) 5

Solution: (a) $\cosh^{-1} x = \log(x + \sqrt{x^2 - 1}) = \log 2 + \sqrt{3}$

$$\therefore x = 2.$$

Example: 16 $\log 3 + 2\sqrt{2} =$

(a) $\sinh^{-1} 3$

(b) $\cosh^{-1} 3$

(c) $\tanh^{-1} 3$

(d) $\cosh^{-1} 3$

Solution: (b) $\log 3 + 2\sqrt{2} = \log 3 + \sqrt{8} = \log 3 + \sqrt{9-1} = \log 3 + \sqrt{3^2-1}$

$$\log 3 + 2\sqrt{2} = \cosh^{-1} 3.$$

Example: 17 $\operatorname{sech}^{-1}\left(\frac{1}{2}\right)$ is

(a) $\log(\sqrt{3} + \sqrt{2})$

(b) $\log(\sqrt{3} + 1)$

(c) $\log 2 + \sqrt{3}$

(d) None of these

Solution: (c) $\operatorname{sech}^{-1}\left(\frac{1}{2}\right) = \cosh^{-1}(2) = \log 2 + \sqrt{2^2 - 1} = \log 2 + \sqrt{3}$.

Example: 18 $\sinh^{-1}(2^{3/2})$ is

(a) $\log 2 + \sqrt{18}$

(b) $\log 3 + \sqrt{8}$

(c) $\log 3 - \sqrt{8}$

(d) $\log(\sqrt{8} + \sqrt{27})$

Solution: (b) $\sinh^{-1}(2^{3/2}) = \log 2^{3/2} + \sqrt{(2^{3/2})^2 + 1} = \log 3 + \sqrt{8}$.

Example: 19 If $\tan^{-1}(\alpha + i\beta) = x + iy$, then $x =$

(a) $\frac{1}{2} \tan^{-1}\left(\frac{2\alpha}{1-\alpha^2-\beta^2}\right)$

(b) $\frac{1}{2} \tan^{-1}\left(\frac{2\alpha}{1+\alpha^2+\beta^2}\right)$

(c) $\tan^{-1}\left(\frac{2\alpha}{1-\alpha^2-\beta^2}\right)$

(d) None of these

Solution: (a) $\tan^{-1}(\alpha + i\beta) = x + iy$

$$\tan^{-1}(\alpha - i\beta) = x - iy$$

$$2x = x + iy + x - iy = \tan^{-1}(\alpha + i\beta) + \tan^{-1}(\alpha - i\beta)$$

$$\therefore x = \frac{1}{2} \tan^{-1} \frac{2\alpha}{1-\alpha^2-\beta^2} = \frac{1}{2} \tan^{-1} \frac{\alpha + i\beta + \alpha - i\beta}{1 - (\alpha + i\beta)(\alpha - i\beta)} = \frac{1}{2} \tan^{-1}\left(\frac{2\alpha}{1-\alpha^2-\beta^2}\right).$$

Example: 20 If $-\frac{\pi}{2} < x < \frac{\pi}{2}$, then the value of $\log \sec x$ is

(a) $2 \coth^{-1}\left(\operatorname{cosec}^2 \frac{x}{2} - 1\right)$

(b) $2 \coth^{-1}\left(\operatorname{cosec}^2 \frac{x}{2} + 1\right)$

(c) $2 \operatorname{cosech}^{-1}\left(\cot^2 \frac{x}{2} - 1\right)$

(d) $2 \operatorname{cosech}^{-1}\left(\cot^2 \frac{x}{2} + 1\right)$

Solution: (a) Let $\log \sec x = y$; $\therefore \frac{1}{\cos x} = \frac{e^{y/2}}{e^{-y/2}}$

By componendo and Dividendo rule, $\frac{1+\cos x}{1-\cos x} = \frac{e^{y/2} + e^{-y/2}}{e^{y/2} - e^{-y/2}} \Rightarrow \coth^2\left(\frac{x}{2}\right) = \coth^2\left(\frac{y}{2}\right)$

$$\Rightarrow y = 2\coth^{-1}\left(\csc^2 \frac{x}{2} - 1\right).$$

Example: 21 The value of $\cosh^{-1}(\sec x)$ is

- (a) $\log\left(\frac{1+\sin x}{\cos x}\right)$ (b) $\log\left(\frac{1-\sin x}{\cos x}\right)$ (c) $\log\left(\frac{1+\cos x}{\sin x}\right)$ (d) $\log\left(\frac{1-\cos x}{\sin x}\right)$

Solution: (a) Here $\cosh^{-1}(\sec x) = \log(\sec x + \sqrt{\sec^2 x - 1}) = \log(\sec x + \tan x) = \log\left(\frac{1+\sin x}{\cos x}\right).$

Example: 22 $2\sinh^{-1}(\theta)$ is equal to

- (a) $\sinh^{-1}(2\theta\sqrt{1+\theta^2})$ (b) $\sinh^{-1}(2\theta\sqrt{1-\theta^2})$ (c) $\sinh^{-1}(\theta\sqrt{1+\theta^2})$ (d) None of these

Solution: (a) We know that, $2\sin^{-1} x = \sin^{-1}(2x\sqrt{1-x^2})$

Putting the value of $x = i\theta$

$$2\sin^{-1}(i\theta) = \sin^{-1}(2i\theta\sqrt{1-\theta^2})$$

$$2i\sinh^{-1}(\theta) = \sin^{-1}(2i\theta\sqrt{1+\theta^2}) \text{ OR } 2i\sinh^{-1}(\theta) = i\sinh^{-1}(2\theta\sqrt{1+\theta^2}) \quad (\ominus \sin^{-1}(ix) = i\sinh^{-1} x)$$

$$\Rightarrow 2\sinh^{-1}(\theta) = \sinh^{-1}(2\theta\sqrt{1+\theta^2}).$$

Example: 23 The value of $\sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ is

- (a) $\tanh^{-1} x$ (b) $\coth^{-1} x$ (c) $\sinh^{-1}(2x)$ (d) $\cosh^{-1}(2x)$

Solution: (a) Let $x = \tanh y$, then $\frac{x}{\sqrt{1-x^2}} = \frac{\tanh y}{\operatorname{sech} y} = \sinh y$

$$\therefore \sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sinh^{-1}(\sinh y) \Rightarrow y = \tanh^{-1}(x).$$

Example: 24 If $\cos \alpha \cosh \beta = 1$, then β is equal to

- (a) $\log \sec\left(\frac{\alpha}{2}\right)$ (b) $\log \tan \alpha$ (c) $\log(\sec \alpha + \tan \alpha)$ (d) $\log \sin\left(\frac{\alpha}{2}\right)$

Solution: (c) $\cos \alpha \cosh \beta = 1 \Rightarrow \cosh \beta = \sec \alpha \Rightarrow \beta = \cosh^{-1}(\sec \alpha) = \log(\sec \alpha + \sqrt{\sec^2 \alpha - 1}) = \log(\sec \alpha + \tan \alpha)$

Example: 25 $\sinh^{-1}(\sinh^{-1} \theta)$ is equal to

- (a) $i\theta$ (b) θ (c) $-i\theta$ (d) $\pi + i\theta$

Solution: (b) $\sinh^{-1}(\sinh^{-1} \theta) = -i\sin^{-1}(\sinh^{-1} \theta) = -i\sin^{-1}[-i\sin^{-1}(i\theta)] = -\sin[\sin^{-1}(i\theta)] = -i\theta = -i^2\theta = \theta.$

Example: 26 If $\sinh^{-1} x = \operatorname{cosech}^{-1} y$, then the correct statement is

- (a) $x = y$ (b) $xy = -1$ (c) $xy = 1$ (d) $x + y = 0$

Solution: (c) Given that, $\sinh^{-1} x = \operatorname{cosech}^{-1} y$ OR $\sinh^{-1} x = \sinh^{-1}\left(\frac{1}{y}\right)$ OR $x = \sinh\left\{\sinh^{-1}\left(\frac{1}{y}\right)\right\}$ OR $x = \frac{1}{y} \Rightarrow xy = 1.$

Example: 27 Find real part of $\tan^{-1}(1+i)$

- (a) $-\frac{1}{2}\tan^{-1}(2)$ (b) $\frac{1}{2}\tan^{-1}(2)$ (c) $-\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)$ (d) 0

Solution: (a) Real part = $\frac{1}{2}\tan^{-1}\frac{2(1)}{1-1-1} = -\frac{1}{2}\tan^{-1}(2)$.

Example: 28 Find real part of $\cosh^{-1}(1)$

- (a) -1 (b) 1 (c) 0 (d) None of these

Solution: (c) We know that $\cosh^{-1}x = \log(x + \sqrt{x^2 - 1})$

$$\therefore \cosh^{-1}(1) = \log(1 + \sqrt{1^2 - 1}) = \log 1 = 0.$$

Example: 29 Find imaginary part of $\sin^{-1}\left(\frac{5\sqrt{7}-9i}{16}\right)$

- (a) $\log 2$ (b) $-\log 2$ (c) 0 (d) None of these

Solution: (b) $\left[\sin^{-1}\left(\frac{5\sqrt{7}}{16} - \frac{9i}{16}\right)\right] = -\log\left[\sqrt{\frac{9}{16}} + \sqrt{1 + \frac{9}{16}}\right] = -\log 2$.

Example: 30 Find real part of $\cos^{-1}\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$

- (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\log\left(\frac{\sqrt{3}-1}{2}\right)$ (d) None of these

Solution: (b) Expression $\cos^{-1}(\cos\theta + i\sin\theta) = \sin^{-1}\sqrt{\sin\theta} - i\log(\sqrt{\sin\theta} + \sqrt{1 + \sin\theta})$

$$\text{Where } \theta = \frac{\pi}{6}$$

$$\therefore \cos^{-1}\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = \sin^{-1}\sqrt{\frac{1}{2}} - i\log\left(\sqrt{\frac{1}{2}} + \sqrt{1 + \frac{1}{2}}\right) = \frac{\pi}{4} - i\log\left(\frac{1 + \sqrt{3}}{2}\right) = \frac{\pi}{4} + i\log\left(\frac{\sqrt{3}-1}{2}\right)$$

$$\text{Real part} = \frac{\pi}{4}, \text{ Imaginary part} = \log\left(\frac{\sqrt{3}-1}{2}\right).$$

Example: 31 Find imaginary part of $\sin^{-1}(\csc\theta)$

- (a) $\log\left(\cot\frac{\theta}{2}\right)$ (b) $\frac{\pi}{2}$ (c) $\frac{1}{2}\log\left(\cot\frac{\theta}{2}\right)$ (d) None of these

Solution: (a) Let $\sin^{-1}(\csc\theta) = x + iy$

$$\therefore \csc\theta = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$$

By comparing we get, $\sin x \cosh y = \csc\theta$ (i) and $\cos x \sinh y = 0$ (ii)

$$\text{From (ii), } \cos x = 0 \Rightarrow x = \frac{\pi}{2}$$

$$\therefore \text{ from (i) } \sin\frac{\pi}{2} \cdot \cosh y = \csc\theta \text{ OR } y = \cosh^{-1}(\csc\theta) = \log[\csc\theta]$$

$$\Rightarrow y = \log[\csc\theta + \cot\theta] = \log\left(\cot\frac{\theta}{2}\right)$$

$$\therefore \sin^{-1}(\csc\theta) = \frac{\pi}{2} + i\log\left(\cot\frac{\theta}{2}\right)$$

$$\text{Real part} = \frac{\pi}{2}, \text{ Imaginary part} = \log\left(\cot\frac{\theta}{2}\right).$$

ASSIGNMENT

FORMULAE AND TRANSFORMATION OF HYPERBOLIC FUNCTION

Basic Level

1. The value of $\cosh 2x$ is
 (a) $\cosh^2 x - \sinh^2 x$ (b) $1 + 2\cosh^2 x$ (c) $1 + 2\sinh^2 x$ (d) None of these
2. $\sinh 3z$ equals
 (a) $3\sinh z - 4\sinh^3 z$ (b) $4\sinh^3 z - 3\sinh z$ (c) $3\sinh z + 4\sinh^3 z$ (d) None of these
3. Which of the following statement is true
 (a) $\sinh^2 x - \cosh^2 x = 1$ (b) $\sinh^2 x + \cosh^2 x = 1$ (c) $\operatorname{sech}^2 x - \tanh^2 x = 1$ (d) $\coth^2 x - \operatorname{cosech}^2 x = 1$
4. $\tan(x + y)$ equals
 (a) $\frac{\tanh x + \tanh y}{1 - \tanh x \tanh y}$ (b) $\frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$ (c) $\frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$ (d) $\frac{\tanh x - \tanh y}{1 + \tanh x \tanh y}$
5. $\sinh^2 x$ equals
 (a) $\cosh 2x - 1$ (b) $\cosh^2 x + 1$ (c) $\frac{1}{2}(\cosh 2x - 1)$ (d) $\frac{1}{2}(\cosh 2x + 1)$
6. Which of the following functions is not defined at $x = 0$
 (a) $\tanh x$ (b) $\operatorname{cosech} x$ (c) $\sin x$ (d) $\operatorname{sech} x$
7. The value of $(\cosh \theta + \sinh \theta)^n$ is
 (a) $e^{n\theta}$ (b) $e^{-n\theta}$ (c) $e^{in\theta}$ (d) $e^{-in\theta}$
8. The value of $\frac{e^{2\theta} - 1}{e^{2\theta} + 1}$ is
 (a) $\coth \theta$ (b) $\coth 2\theta$ (c) $\tanh \theta$ (d) $\tanh 2\theta$
9. $\left(\frac{1 + \tanh \theta}{1 - \tanh \theta}\right)^5$ is equal to
 (a) $e^{10\theta}$ (b) $e^{5\theta}$ (c) 1 (d) -1
10. $\frac{1 + \tanh x}{1 - \tanh x}$ is equal to
 (a) e^{2x} (b) e^{-2x} (c) i (d) -1
11. If $\cosh z = \sec \theta$, then $\sinh z$ equals
 (a) $\operatorname{cosec} \theta$ (b) $\cot \theta$ (c) $\tan \frac{\theta}{2}$ (d) $\tan \theta$

12. If $\operatorname{cosec} \theta = \coth x$, then the value of $\tanh \theta$ is
 (a) $\cosh x$ (b) $\sinh x$ (c) $\tanh x$ (d) $\operatorname{cosech} x$
13. If $\cosh y = \sec x$, then the value of $\tanh^2\left(\frac{y}{2}\right)$ is
 (a) $\tan^2\left(\frac{x}{2}\right)$ (b) $\cot^2 \frac{x}{2}$ (c) $\sin^2\left(\frac{x}{2}\right)$ (d) $\cos^2 \frac{x}{2}$
14. $u = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$, then the value of $\tanh \frac{u}{2}$ is
 (a) $\cot \frac{x}{2}$ (b) $-\cot \frac{x}{2}$ (c) $-\tan \frac{x}{2}$ (d) $\tan \frac{x}{2}$
15. If $\tan\left(\frac{x}{2}\right) \coth\left(\frac{x}{2}\right) = 1$, then the value of $\cos x \cosh x$ is
 (a) 1 (b) -1 (c) $\cos^2 x$ (d) $\sin^2 x$
16. $\frac{\operatorname{cosech} x}{\sqrt{\operatorname{cosech}^2 x + 1}}$ equals
 (a) $\tanh x$ (b) $\coth x$ (c) $\operatorname{sech} x$ (d) $\cosh x$

Advance Level

17. If $f(x) = \cosh x + \sinh x$ and $f(p) = f(x) \cdot f(y)$, then the value of p is
 (a) xy (b) $x - y$ (c) $x + y$ (d) None of these
18. If $f(x) = \cosh x - \sinh x$, then $f(x_1 + x_2 + \dots + x_n)$ is equal to
 (a) $f(x_1) \cdot f(x_2) \dots f(x_n)$ (b) $f(x_1) + f(x_2) + \dots + f(x_n)$ (c) 0 (d) 1
19. If $\sin x \cosh y = \cos \theta$ and $\cos x \sinh y = \sin \theta$, then $\sin^2 y$ equals
 (a) $\sin^2 x$ (b) $\cos^2 x$ (c) $\cos^2 x$ (d) 1
20. If $\tan \theta = \tanh x \cot y$ and $\tan \phi = \tanh x \tan y$, then $\frac{\sin 2\theta}{\sin 2\phi}$ equals
 (a) $\frac{\cosh 2x + \cos 2y}{\cosh 2x - \cos 2y}$ (b) $\frac{\cosh 2x - \cos 2y}{\cosh 2x + \cos 2y}$
 (c) $\frac{\cos 2x + \cosh 2y}{\cos 2x - \cosh 2y}$ (d) None of these

RELATION BETWEEN HYPERBOLIC AND CIRCULAR FUNCTIONS

21. The value of $\operatorname{cosech}\left(\frac{\pi i}{6}\right)$ is
 (a) -2 (b) 2 (c) $-2i$ (d) $2i$
22. $\sinh\left(x + \frac{\pi i}{2}\right)$ equals
 (a) $i \cosh x$ (b) $-i \cosh x$ (c) $\cos x$ (d) $-\cos x$
23. $\tanh\left(\frac{\pi i}{4}\right) - \coth\left(\frac{\pi i}{4}\right)$ is equal to
 (a) 0 (b) 2 (c) $\sqrt{2}$ (d) None of these
24. $\cos(\pi x)$ equals
 (a) $i \cosh x$ (b) $-i \cosh x$ (c) $\cosh x$ (d) $-\cosh x$
25. If $\sin(x + iy) = A + iB$ then A equals
 (a) $\sinh x \cos y$ (b) $\sin x \cosh y$ (c) $\cos x \sinh y$ (d) $\cosh x \sin y$
26. The imaginary part of $\sin^2(x + iy)$ is
 (a) $\frac{1}{2} \cosh 2x \cos 2y$ (b) $\frac{1}{2} \cos 2x \cosh 2y$ (c) $\frac{1}{2} \sinh 2x \sin 2y$ (d) $\frac{1}{2} \sin 2x \sinh 2y$
27. Imaginary part of $\cosh(\alpha + i\beta) - \cosh(\alpha - i\beta) =$
 (a) $2 \sinh \alpha \sinh \beta$ (b) $2 \sinh \alpha \sin \beta$ (c) $\cosh \alpha \cos \beta$ (d) $2 \cos \alpha \cosh \beta$
28. Real part of $\cosh(\alpha + i\beta)$ is
 (a) $\cosh \alpha \cos \beta$ (b) $\cos \alpha \cos \beta$ (c) $\cos \alpha \cosh \beta$ (d) $\sin \alpha \sinh \beta$
29. The value of $\sinh(x + 2\pi i)$ is
 (a) $\frac{e^x + e^{-x}}{2}$ (b) $\frac{e^x - e^{-x}}{2}$ (c) $\frac{e^x - e^{-x}}{2i}$ (d) $\frac{e^x + e^{-x}}{2i}$
30. $\sin^2(ix) + \cosh^2 x$ is equal to
 (a) 1 (b) -1 (c) $2 \cosh^2 x$ (d) $\cosh 2x$
31. The period of $\coth\left(\frac{nx}{4}\right)$ is
 (a) $\frac{\pi i}{n}$ (b) $\frac{4\pi i}{n}$ (c) $\frac{n\pi i}{4}$ (d) πi
32. The period of e^z is
 (a) 2π (b) π (c) $2\pi i$ (d) πi

33. The period of $\cosh 4x$ is

- (a) $2\pi i$ (b) πi (c) $\frac{\pi i}{2}$ (d) 2π

Advance Level

34. If $\tan(\theta + i\phi) = \sin(x + iy)$, then the value of $\coth x \sinh 2\phi$ is

- (a) $\tan x \cot 2\theta$ (b) $\tan x \sin 2\theta$ (c) $\cot x \sin 2\theta$ (d) None of these

35. If $\cos(x + i\beta) = \rho(\cos\alpha + i\sin\alpha)$, then value of $\tan\alpha$ is

- (a) $\tanh\alpha \tan\beta$ (b) $\frac{-\tanh\alpha \tan\beta}{\rho}$ (c) $-\cot\alpha \coth\beta$ (d) $-\tan\alpha \tanh\beta$

INVERSE HYPERBOLIC FUNCTIONS

Basic Level

36. $\sinh^{-1} x =$

- (a) $\log(x + \sqrt{1 - x^2})$ (b) $\log(x + \sqrt{x^2 + 1})$ (c) $\log(x + \sqrt{x^2 - 1})$ (d) None of these

37. $\cosh^{-1} x =$

- (a) $\log(x + \sqrt{x^2 + 1})$ (b) $\log(x - \sqrt{x^2 + 1})$ (c) $\log(x - \sqrt{x^2 - 1})$ (d) $\log(x + \sqrt{x^2 - 1})$

38. $\tanh^{-1} x =$

- (a) $\frac{1}{2} \log\left(\frac{x+1}{x-1}\right)$ (b) $\frac{1}{2} \log\left(\frac{x-1}{x+1}\right)$ (c) $\frac{1}{2} \log\left(\frac{1-x}{1+x}\right)$ (d) $\frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$

39. The value of $\sinh^{-1}(1)$ is

- (a) 0 (b) $\log(\sqrt{2} + 1)$ (c) $\log(1 - \sqrt{2})$ (d) None of these

40. $\coth^{-1} x$ equals

- (a) $\frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$ (b) $\frac{1}{2} \log\left(\frac{x+1}{x-1}\right)$ (c) $\frac{1}{2} \log\left(\frac{x-1}{x+1}\right)$ (d) None of these

41. $\operatorname{cosech}^{-1} x$ equals

- (a) $\log\left(\frac{1 + \sqrt{1 + x^2}}{x}\right)$ (b) $\log\left(\frac{1 + \sqrt{1 - x^2}}{x}\right)$ (c) $\log\left(\frac{1 - \sqrt{1 - x^2}}{x}\right)$ (d) $\log\left(\frac{1 - \sqrt{1 + x^2}}{x}\right)$

42. The value of $2\coth^{-1}\left(\frac{Z}{2}\right)$ is

- (a) $\log\left(\frac{Z-2}{Z+2}\right)$ (b) $\frac{1}{2} \log\left(\frac{Z-1}{Z+1}\right)$ (c) $\frac{1}{2} \log\left(\frac{Z+1}{Z-1}\right)$ (d) $-\log\left(\frac{Z-2}{Z+2}\right)$

43. $\operatorname{sech}^{-1}(\sin x)$ equals

- (a) $\log \cot \frac{x}{2}$ (b) $\log \tan \frac{x}{2}$ (c) $\log \cot x$ (d) None of these

44. If $\operatorname{cosech}^{-1}(1) = x + iy$ then the value of y is

- (a) 1 (b) 0 (c) $\log(1 + \sqrt{2})$ (d) -1

45. The value of $\tanh^{-1}(2^{-1})$ is

- (a) $\log 2$ (b) $\log 2^{-1}$
(c) $\log \sqrt{3}$ (d) None of these

46. If $\log(2 + \sqrt{3}) = \cosh^{-1} K$ then K equals

- (a) 1 (b) 0 (c) 2 (d) None of these

47. $-i \tan^{-1}(ix)$ equals

- (a) $\tanh^{-1} x$ (b) $-\tanh^{-1} x$ (c) $\tanh^{-1}(ix)$ (d) None of these

48. $2 \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is equal to

- (a) $\cosh^{-1}\left(\frac{1}{3}\right)$ (b) $\cosh^{-1}(\sqrt{3})$ (c) $\cosh^{-1}(3)$ (d) $\cosh^{-1}\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$

49. If $\tanh x = \frac{3}{4}$, then the value of x is

- (a) $\sqrt{7}$ (b) $-\sqrt{7}$ (c) $\log \sqrt{7}$ (d) $-\log \sqrt{7}$

50. $\tanh^{-1}\left(\frac{1}{2}\right) + \tanh^{-1}\left(\frac{1}{3}\right)$ is equal to

- (a) $\tanh^{-1}\left(\frac{5}{7}\right)$ (b) $\tanh^{-1}\left(\frac{7}{5}\right)$ (c) $\tanh^{-1}\left(\frac{1}{6}\right)$ (d) $\tanh^{-1}\left(\frac{5}{6}\right)$

51. $\sinh^{-1}\left(\frac{1}{2}\right)$ is equal to

- (a) $\tanh^{-1}(\sqrt{5})$ (b) $\tanh^{-1}\left(\frac{1}{\sqrt{5}}\right)$ (c) $\tanh^{-1}(\sqrt{3})$ (d) $\tanh^{-1}\left(\frac{2}{\sqrt{5}}\right)$

52. $\log \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$ is equal to

- (a) $\tanh^{-1}\left(\tan \frac{\theta}{2}\right)$ (b) $\tanh^{-1}\left(\tanh \frac{\theta}{2}\right)$ (c) $2 \tanh^{-1}\left(\tan \frac{\theta}{2}\right)$ (d) $2 \tanh^{-1}\left(\tanh \frac{\theta}{2}\right)$

53. If $\sin^{-1}(A + iB) = x + iy$ then $\frac{A}{B}$ equals

- (a) $\frac{\tan x}{\tanh y}$ (b) $\frac{\tanh x}{\tan y}$
(c) $\frac{\tanh x}{\tanh y}$ (d) $\frac{\cos x}{\cosh y}$

Advance Level

54. The general value of $\cosh^{-1} x$ is

- (a) $2\pi ri + \log(x + \sqrt{x^2 + 1})$ (b) $2\pi ri + \log(x + \sqrt{x^2 - 1})$ (c) $\pi ri + (-1)^r \log(x + \sqrt{x^2 + 1})$ (d) $2\pi ri + (-1)^r \log(x + \sqrt{x^2 - 1})$

55. If $x = \log\left(\frac{1}{y} + \sqrt{\frac{1}{y^2} + 1}\right)$, then y is equal to

- (a) $\tanh x$ (b) $\cosh x$ (c) $\sinh x$ (d) $\operatorname{cosech} x$

56. The imaginary part of $\tan^{-1}(\cos\theta + i\sin\theta)$ is

- (a) $\tanh^{-1}(\sin\theta)$ (b) $\tanh^{-1}(\cos\theta)$ (c) $\frac{1}{2}\tanh^{-1}(\sin\theta)$ (d) None of these

57. If $\cosh^{-1}(p + iq) = u + iv$ then the equation with roots $\cos^2 u$ and $\cos^2 v$

- (a) $x^2 - x(p^2 + q^2) + p^2 = 0$ (b) $x^2 - x(p^2 + q^2 + 1) + 1 = 0$ (c) $x^2 + x(p^2 + q^2 + 1) + 1 = 0$ (d) $x^2 - x(p^2 + q^2 + 1) + p^2 = 0$

58. The value of $\log \tan\left(\frac{\pi}{4} + \frac{ix}{2}\right)$ is

- (a) $i \tan^{-1}(\sinh x)$ (b) $-i \tan^{-1}(\sinh x)$ (c) $i \tan^{-1}(\cosh x)$ (d) None of these

ANSWER

HYPERBOLIC FUNCTIONS

ASSIGNMENT (BASIC & ADVANCE LEVEL)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
c	c	d	b	c	b	a	c	a	a	d	b	a	d	a	c	c	a	c	a
21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
c	a	b	c	b	d	b	a	b	a	b	c	c	c	d	b	d	d	b	b
41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58		
a	d	a	b	c	c	a	c	c	a	b	c	a	b	d	c	d	a		