COMPLEX NUMBERS

Abraham de Moivre

De Moivre was a competent mathematician with a good knowledge of many of the standard texts. De Moivre had hoped for a chair of mathematics, but foreigners were at a disadvantage in England so although he now was free from religious discrimination, he still suffered discrimination as a Frenchman in England. His first mathematics paper arose from his study of fluxions in the Principia and in March 1695 In 1697 he was elected a fellow of the Royal Society. In 1710 de Moivre was appointed to the Commission set up by the Royal Society to review the rival claims of Newton and Leibniz to be the discovers of the calculus.

De Moivre pioneered the development of analytic geometry and the theory of probability. De Moivre's most significant contribution to this area, namely the approximation to the binomial distribution by the normal distribution in the case of a large number of trials. De Moivre also investigated mortality statistics and the foundation of the theory of annuities.

De Moivre is also remembered for his formula for $(\cos x + i \sin x)^n$ which took trigonometry into analysis, and was important in the early development of the theory of complex numbers. It appears in this form in a paper which de Moivre published in 1722,

De Moivre, like Cardan, is famed for predicting the day of his own death. He found that he was sleeping 15 minutes longer each night and summing the arithmetic progression, calculated that he would die on the day that he slept for 24 hours. He was right!

Introduction

We know that value of $\sqrt{x}$ is real if and only if $x \geq 0$. In other words in the set of real numbers value of $\sqrt{x}$ does not exist for $x < 0$. To make this possible another type of number is introduced, called as imaginary numbers.

Let us consider equation $x^2 - 2x + 2 = 0$ whose solution will be $\frac{2 \pm \sqrt{4-8}}{2}$ i.e., $1 \pm 2\sqrt{-1}$ which is meaningless in the set of real numbers.

To make the roots meaningful the symbol i is introduced such that $i^2 = -1$ or $i = \sqrt{-1}$. Hence the roots will be $1 \pm 2i$.

1.1 COMPLEX NUMBER

A number in the form of $a + ib$, where a, b are real numbers and $i = \sqrt{-1}$ is called a complex number. A complex number can also be defined as an ordered pair of real numbers a and b and may be written as (a, b), where the first number denotes the real part and the second number denotes the imaginary part of the given complex number.

- **Properties of Complex Numbers**

(i) If $z = a + ib$, then the real part of z is denoted by Re (z) and the imaginary part by Im (z).

(ii) A complex number is said to be purely imaginary if Re $(z) = 0$.

(iii) A complex number is said to be purely real if $\text{Im}(z) = 0$.

(iv) The complex number $0 = 0 + i0$ is both purely real and purely imaginary.

(v) Two complex numbers are said to be equal if and only if their real parts and imaginary parts are separately equal i.e., $a + ib = c + id$ implies $a = c$ and $b = d$.

(vi) However, there is no order relation between complex and the expressions of the type $a + ib <$ (or $>$) $c + id$ are meaningless.

Note:

Clearly $i^2 = -1$, $i^3 = i^2 . i = -i$, $i^4 = 1$. In general, $i^{4n} = 1$, $i^{4n+1} = i$, $i^{4n+2} = -1$, $i^{4n+3} = -i$ for an integer n.

ILLUSTRATIONS

***Illustration* 1**

If $x = -5 + 2\sqrt{-4}$ find the value of $x^4 + 9x^3 + 35x^2 - x + 4$.

Solution

$x = -5 + 4i \left(i = \sqrt{-1}\right)$

$x + 5 = 4i$

Squaring, $x^2 + 10x + 25 = -16$ □ $x^2 + 10x + 41 = 0$

Now, $x^4 + 9x^3 + 35x^2 - x + 4 = (x^2 + 10x + 41)(x^2 - x + 4) - 160$ and $x^2 + 10x + 41 = 0$

Hence given expression $= 0 - 160 = -160$

***Illustration* 2**

Express $\dfrac{1}{(1 - \cos\theta + i\sin\theta)}$ in the form $a + ib$.

Solution

$$\frac{1}{(1-\cos\theta+i\sin\theta)}$$

$$= \frac{(1-\cos\theta)-i\sin\theta}{(1-\cos\theta+i\sin\theta)(1-\cos\theta-i\sin\theta)}$$

$$= \frac{\{(1-\cos\theta)-i\sin\theta\}}{\{(1-\cos\theta)^2+\sin^2\theta\}}$$

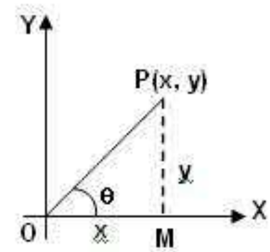
$$= \frac{(1-\cos\theta)-i\sin\theta}{2-2\cos\theta}$$

$$= \frac{1-\cos\theta}{2(1-\cos\theta)} - \frac{i.2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$$

$$= \frac{1}{2} - \frac{i}{2}\cot\frac{\theta}{2}$$

1.2 Geometrical Representation of Complex Number

A complex number $z = x + iy$ written as ordered pair (x, y) can be represented by a point P whose Cartesian coordinates are (x, y) referred to axes OX and OY, usually called the real and the imaginary axes. The plane of OX and OY is called the Argand plane or the complex plane. Since the origin O lies on both OX and OY, the corresponding complex number $z = 0$ is both purely real as well as purely imaginary.



1.3 Modulus and Argument of a Complex Number

Modulus of the complex number $z = x + iy$ is defined as distance of point represented by complex number from the origin and is denoted by | z |. Hence $| z | = \sqrt{x^2+y^2}$.

It may be noted that | *z* | □ 0 and thus | *z* | = 0 would imply that *z* is a zero complex number or simply *z* = 0.

If $z = x + iy$, then angle □ given by tan □ = $\frac{y}{x}$ is said to be the argument or amplitude of the complex number *z* and is denoted by arg (*z*) or amp. (*z*). In case of *x* = 0 (where *y* □ 0), arg (*z*) = + □/2 or – □/2 depending upon *y* > 0 or *y* < 0 and the complex number is called purely imaginary. If *y* = 0 (where *x* □ 0), then arg (*z*) = 0 or □ depending upon *x* > 0 or *x* < 0 and the complex number is called purely real. The argument of the complex 0 is not defined. We can define the argument of a complex number also as any value of the □

which satisfies the system of equations cos □ = $\frac{x}{\sqrt{x^2+y^2}}$, sin □ = $\frac{y}{\sqrt{x^2+y^2}}$.

The argument of a complex number is not unique. If □ is a argument of a complex number, then 2*n*□ + □ (*n* integer) is also argument of *z* for various values of *n*. The value of □ satisfying the inequality – □ < □ □ □ is called the **principal value of the argument.**

From figure, we can see that OP = $\sqrt{x^2+y^2} = |z|$ and if □ = □POM, then tan □ = *y*/*x*. Thus in other words | *z* | is the length of OP i.e., the distance of point *z* from the origin and arg *z* is the angle which OP makes with the positive direction of the x-axis.

1.4 Polar and Euler form of a Complex Number

Let OP = *r*, then *x* = *r* cos □, and *y* = *r* sin □

□ *z* = *x* + *iy* = *r* cos □ + *ir* sin □ = *r*(cos □ + *i* sin □). This is known as Trigonometric (or Polar) form of a Complex Number. Here we should take the principal value of □.

For general values of the argument

z = *r* [cos (2*n*□ + □) + *i* sin (2*n*□ + □)] (where *n* is an integer)

Sometimes cos □ + *i* sin □ is, in short, written as cis (□).

z = *r*(cos □ + *i* sin □) is the polar representation of the complex number where as *z* = *r e*$^{i□}$ is the Euler representation of the given complex number.

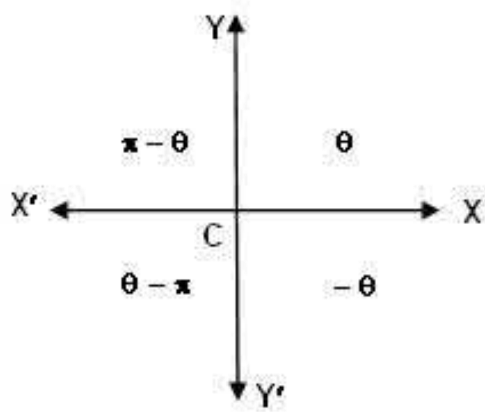
□ **Argument of a Complex Number**

Method of finding the principal value of, the argument of a complex number *z* = *x* + *iy*.

Step I. Find tan □ = $\left|\frac{y}{x}\right|$ and this gives the value of □ in the first quadrant.

Step II. Find the quadrant in which *z* lies, with the help of sign of *x* and *y* co-ordinates.

Step III. Then argument of *z* will be □, □ – □, □ – □ and – □ according as *z* lies in the first second, third or fourth quadrant.



ILLUSTRATIONS

***Illustration* 3**

Represent the given complex numbers in polar form

(i) sin α – *i* cos α (α acute)

(ii) $1+\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$

Solution

(i) Real part > 0; Imaginary part < 0

Argument of sin α – *i* cos α is in the nature of a negative acute angle.

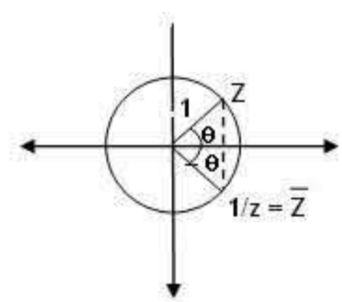
$$\therefore \quad \sin\alpha - i\cos\alpha = \cos\left(\alpha-\frac{\pi}{2}\right)+i\sin\left(\alpha-\frac{\pi}{2}\right)=e^{i\left(\alpha-\frac{\pi}{2}\right)}$$

(ii) $$1+\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}=2\cos^2\frac{\pi}{6}+i.2\sin\frac{\pi}{6}\cos\frac{\pi}{6}$$

$$= 2\cos\frac{\pi}{6}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=2\cos\frac{\pi}{6}e^{i\pi/6}$$

1.5 Uni-modular Complex Number

A Complex Number z such that $|z| = 1$ is said to be uni-modular complex number. Since $|z| = 1$, z lies on a circle of radius 1 unit and centre (0, 0).



$\therefore$ $1/z = (\cos\theta + i\sin\theta)^{-1}$

$= \cos\theta - i\sin\theta$

If $|z| = 1 \Rightarrow z = \cos\theta + i\sin\theta$

1.6 Algebraic Operations with Complex Numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two given complex numbers then the algebraic operations between them is performed as given below

$z_1 + z_2 = (a + ib) + (c + id) = (a + c) + i(b + d)$

$z_1 - z_2 = (a + ib) - (c + id) = (a - c) + i(b - d)$

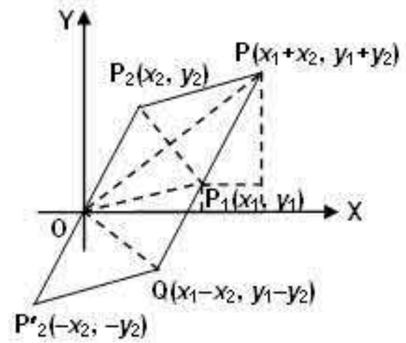
$z_1 \,.\, z_2 = (a + ib)(c + id) = (ac - bd) + i(ad + bc)$

$z_1 / z_2 = \dfrac{a+ib}{c+id}$ (when at least one of c and d is non-zero)

$$= \frac{(ac+bd)}{c^2+d^2}+\frac{i(bc-ad)}{c^2+d^2}$$

- **Geometrical meaning of Addition and Subtraction**

  Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers represented by the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ respectively. By definition $z_1 + z_2$ should be represented by the point $(x_1 + x_2, y_1 + y_2)$. This point is the vertex which completes the parallelogram with the line segments joining the origin with OP_1 and OP_2 as the adjacent sides.

  Also by definition $z_1 - z_2$ should be represented by the point $(x_1 - x_2, y_1 - y_2)$. This point is the vertex which completes the parallelogram with the line segments joining the origin with OP_1 and OP'_2 (where the point P'_2 represents $-z_2$; the point $-z_2$ can be obtained by producing the directed line P_2O by length $|z_2|$) as the adjacent sides.

  $\therefore$ $|z_1 - z_2| = OQ = P_2P_1$.

Note:

- In any triangle, sum of any two sides is greater than the third side and difference of any two sides is less than the third side; we have

(i) $|z_1| + |z_2| \geq |z_1 + z_2|$; here equality holds when arg $(z_1/z_2) = 0$ i.e., z_1 and z_2 are parallel.

(ii) $||z_1| - |z_2|| \leq |z_1 - z_2|$; here equality holds when arg $(z_1/z_2) = \pi$ i.e., z_1 and z_2 are anti-parallel.

- In the parallelogram OP_1PP_2, the sum of the squares of its sides is equal to the sum of the squares of its diagonals; i.e., $OP^2 + P_2P_1^2 = OP_1^2 + P_1P^2 + PP_2^2 + P_2O^2$

  $\Rightarrow$ $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$.

- **Geometrical Meaning of Product and Division**

  Let $z_1 = r_1$ (cos θ_1 + i sin θ_1), $z_2 = r_2$ (cos θ_2 + i sin θ_2) be complex numbers represented by Q_1 and Q_2.

(i) **Construction for the Point Representing the Product z_1z_2:**

Let L be the point on OX which represents unity, so that OL = 1. Draw the triangle OQ_2P directly similar to the triangle OLQ_1. Then point P represents the product z_1z_2.

From similar triangles OPQ_2 and OQ_1L we get

$$\frac{OP}{OQ_1} = \frac{OQ_2}{OL}, \text{ that is } \frac{OP}{r_1} = \frac{r_2}{1}$$

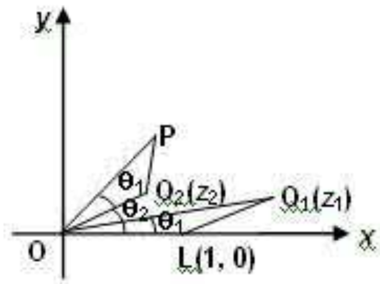
□ OP = r_1r_2

Also, □Q_2OP = □LOQ_1 = $□_1$

□ □LOP = $□_1 + □_2$

Since $z_1z_2 = r_1r_2$ {cos ($□_1 + □_2$) + i sin ($□_1 + □_2$)},

P represents z_1z_2



(ii) **Construction for the Point Representing the Quotient z_1/z_2:**

Draw the triangle OQ_1P directly similar to the triangle OQ_2L

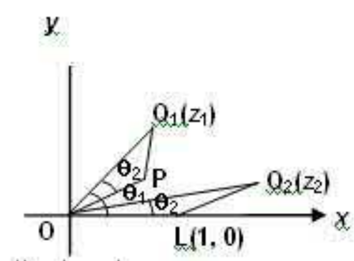
Then P represents the quotient z_1 / z_2.

From the last construction,

$$\frac{OQ_1}{OQ_2} = \frac{OP}{OL} \Rightarrow \frac{r_1}{r_2} = \frac{OP}{1}$$

number represented by P.$z_2 = z_1$

□ number represented by P = $\frac{z_1}{z_2}$



Note:

If $z_1 = r_1$ (cos $□_1$ + i sin $□_1$), and $z_2 = r_2$ (cos $□_2$ + i sin $□_2$)

then $z_1z_2 = r_1r_2\, e^{i(\theta_1+\theta_2)}$ and $\frac{z_1}{z_2} = \frac{r_1}{r_2}.e^{i(\theta_1+\theta_2)}$

Hence $|z_1z_2| = r_1r_2 = |z_1||z_2|$ and $\frac{|z_1|}{|z_2|} = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}, |z_2| \neq 0$

PRACTICE EXERCISE

1. Write the complex number $z = \frac{2+i}{(1+i)(1-i2)}$ in $x + iy$ form.

2. Express $\left(\frac{1}{1-2i} + \frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right)$ in the form A + iB.

3. Express $\frac{1}{1-\cos\theta + 2i\sin\theta}$ in the form $a + ib$.

4. If z_1 and z_2 are $1 - i$, $-2 + 4i$ respectively, find $\text{Im}\left(\dfrac{z_1 z_2}{\bar{z}_1}\right)$.

5. For what real values of x and y are the following numbers equal
$x^2 - 7x + 9yi$ and $y^2 i + 20i - 12$?

6. Find the values of x and y, if $\dfrac{(1+i)x - 2i}{3+i} + \dfrac{(2-3i)y + i}{3-i} = i$.

7. Find real □ such that $\dfrac{3 + 2i\sin\theta}{1 - 2i\sin\theta}$ is purely real.

8. Find the modulus and principal argument of the following complex numbers:

(i) $\dfrac{5}{2}$ (cos 300° + i sin 30°) (ii) cos 70° + i cos 20°

9. Put $\dfrac{1+7i}{(2-i)^2}$ in the polar form.

Answers

1. $\dfrac{1}{2} + i\dfrac{1}{2}$

2. $\dfrac{1}{4} + i\dfrac{9}{4}$

3. $\dfrac{1}{5 + 3\cos\theta} - i\dfrac{2\cot\dfrac{\theta}{2}}{5 + 3\cos\theta}$

4. 2

5. {(4, 5), (4, 4), (3, 5), (3, 4)}

6. $x = 3$, $y = -1$

7. □ = n□, n □ I

8. (i) $\dfrac{\pi}{4}$ (ii) $\dfrac{7\pi}{18}$

9. $\sqrt{2}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$

1.7 Square Root of a Complex Number

Let $a + ib$ be a complex number such that $\sqrt{a+ib} = x + iy$, where x and y are real numbers.

Then,

$\sqrt{a+ib} = x + iy$ □ $(a + ib) = (x + iy)^2$

□ $a + ib = (x^2 - y^2) + 2i\,xy$

By equating real and imaginary parts, we get

$x^2 - y^2 = a$ … (i)

and $2\,xy = b$ … (ii)

Now, $(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4\,x^2y^2$

□ $(x^2 + y^2)^2 = a^2 + b^2$

□ $(x^2 + y^2) = \sqrt{a^2 + b^2}$ … (iii)

Solving the equations (i) and (ii), we get

$x^2 = \left(\dfrac{1}{2}\right)\left\{\sqrt{a^2 + b^2} + a\right\}$ and $y^2 = \left(\dfrac{1}{2}\right)\left\{\sqrt{a^2 + b^2} - a\right\}$

$\therefore$ $x = \pm\sqrt{\left(\frac{1}{2}\right)\left\{\sqrt{a^2+b^2}+a\right\}}$ and $y = \pm\sqrt{\left(\frac{1}{2}\right)\left\{\sqrt{a^2+b^2}-a\right\}}$

If b is positive, then by equation (ii), x and y are of the same sign. Hence,

$$\sqrt{a+ib} = \pm\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}+a\right\}} + i\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}-a\right\}}$$

If b is negative, then by equation (ii), x and y are of different signs. Hence,

$$\sqrt{a+ib} = \pm\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}+a\right\}} - i\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}-a\right\}}$$

Note:

To find the square root of $a - ib$, replace i by $- i$ in the above results.

Alternate Method to find the square root of a complex number

(i) If the imaginary part is not even then multiply and divide the given complex number by 2. e.g. $z = 8 - 15i$, here imaginary part is not even so write $z = \frac{1}{2}(16 - 30i)$ and let $a + ib = 16 - 30i$.

(ii) Now divide the numerical value of imaginary part of $a + ib$ by 2 and let quotient be P and find all possible two factors of the number P thus obtained and take that pair in which difference of squares of the numbers is equal to the real part of $a + ib$. e.g. here numerical value of lm $(16 - 30i)$ is 30. Now $30 = 2 \times 15$. All possible way to express 15 as a product of two are 1×15, 3×5, etc., here $5^3 - 3^2 = 16 = \text{Re}(16 - 30i)$ so we will take 5, 3.

(iii) Take i with the smaller or the greater factor according as the real part of $a + ib$ is positive or negative and if real part is zero then taken equal factors of P and associate i with any one of them. e.g. $\text{Re}(16 - 30i) > 0$, we will take i with 3. Now complete the square and write down the square root of z. e.g.

$$z = \frac{1}{2}[16-30i] = \frac{1}{2}\left[5^5 + (3i)^2 - 2\times5\times3i\right] = \frac{1}{2}[5-3i]^2$$

$$\therefore \quad \sqrt{z} = \pm\frac{1}{\sqrt{2}}(5-3i)$$

PRACTICE EXERCISE

10. Find the square root of $7 - 30\sqrt{-2}$.

11. Find $\sqrt{i} + \sqrt{-i}$.

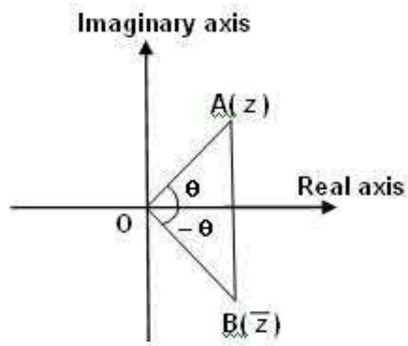
12. Find the square root of $\frac{x^2}{y^2} + \frac{y^2}{x^2} + \frac{1}{2i}\left(\frac{x}{y} + \frac{y}{x}\right) + \frac{31}{16}$.

Answers

10. $\pm\left(5-3\sqrt{2}i\right)$ **11.** $\pm\sqrt{2}$ **12.** $\pm\left(\frac{x}{y}+\frac{y}{x}-\frac{i}{4}\right)$

1.8 Conjugate of a Complex Number

The conjugate of the complex number $z = a + ib$ is defined to be $a - ib$ and is denoted by $\bar{z}$. In other words $\bar{z}$ is the mirror image of z in the real axis.



If $z = a + ib$, $z + \bar{z} = 2a$ (real)

$z - \bar{z} = 2\,ib$ (imaginary)

and $z\bar{z} = (a + ib)(a - ib) = a^2 + b^2$ (real) $= |z|^2 = |\bar{z}|^2$

Also Re(z) = $\frac{z+\bar{z}}{2}$, lm(z) = $\frac{z-\bar{z}}{2i}$

□ **Properties of Conjugate**

(i) $\overline{(\bar{z})} = z$

(ii) $|z| = |\bar{z}|$

(iii) $z + \bar{z} = 2\text{Re}(z)$

(iv) $z - \bar{z} = 2i\,\text{lm}(z)$

(v) If z is purely real $z = \bar{z}$

(vi) If z is purely imaginary $z = -\bar{z}$

(vii) $z\bar{z} = |z|^2 = |\bar{z}|^2$

(viii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (In general, $\overline{z_1 + z_2 + \ldots\ldots + z_n} = \bar{z}_1 + \bar{z}_2 + \ldots\ldots \bar{z}_n$)

(ix) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(x) $\overline{z_1 z_2} = \bar{z}_1.\bar{z}_2$ (In general, $\overline{z_1 z_2 z_3 \ldots\ldots z_n} = \bar{z}_1.\bar{z}_2.\bar{z}_3 \ldots\ldots \bar{z}_n$)

(xi) $\overline{z^n} = (\bar{z})^n$

(xii) $\overline{\left(\frac{z_1}{z_2}\right)} = \left(\frac{\bar{z}_1}{\bar{z}_2}\right)$

□ **Properties of Modulus**

(i) $|z| = 0$ □ $z = 0 + i0$

(ii) $|z_1 - z_2|$ denotes the distance between z_1 and z_2

(iii) $-|z|$ □ Re(z) □ $|z|$

(iv) $-|z|$ □ lm z □ $|z|$

(v) $|z|$ □ $|\text{Re}(z)| + |\text{lm}(z)|$ □ $\sqrt{2}\,|z|$

(vi) $|z|^2 = z\bar{z}$

(vii) $|z_1 z_2| = |z_1||z_2|$

(viii) $|z^n| = |z|^n$, $n \in 1$

(ix) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

(x) $|z_1 \pm z_2| \le |z_1| + |z_2|$

(xi) $|z_1 \pm z_2| \ge ||z_1| - |z_2||$

(xii) $|z_1 + z_2|^2 = (z_1 - z_2)(\bar{z}_1 + \bar{z}_2) = |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + z_2\bar{z}_1 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z}_2)$

(xiii) $|z_1 - z_2|^2 = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) = |z_1|^2 + |z_2|^2 - z_1\bar{z}_2 - z_2\bar{z}_1 = |z_1|^2 + |z_2|^2 - 2\text{Re}(z_1\bar{z}_2)$

- **Properties of Argument**

(i) $\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$

(ii) $\arg(z_1/z_2) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$

(iii) $\arg(z^n) = n\arg(z)$, $n \in I$

In the above result $\theta_1 + \theta_2$ or $\theta_1 - \theta_2$ are not necessarily the principle values of the argument of corresponding complex numbers. e.g., $\arg(z^n) = n\arg(z)$ only shows that one of the argument of z^n is equal to $n\arg(z)$ (if we consider $\arg(z)$ in the principle range) !!

(iv) $\arg(z) = 0$, π $\Rightarrow$ z is a purely real number $\Rightarrow$ $z = -\bar{z}$

(v) $\arg(z) = \pi/2, -\pi/2$ $\Rightarrow$ z is a purely imaginary number $\Rightarrow$ $z = -\bar{z}$

ILLUSTRATIONS

***Illustration* 4**

If $|z - 2 + i| \le 2$, then find the greatest and least value of $|z|$.

Solution

Given that

$$|z - 2 + i| \le 2 \qquad \dots \text{(i)}$$

$$\because \quad |z - 2 + i| \ge ||z| - |2 - i||$$

$$\Rightarrow \quad |z - 2 + i| \ge ||z| - \sqrt{5}| \qquad \dots \text{(ii)}$$

From (i) and (ii)

$$||z| - \sqrt{5}| \le |z - 2 + i| \le 2$$

$$\Rightarrow \quad ||z| - \sqrt{5}| \le 2$$

$$\Rightarrow \quad -2 \le |z| - \sqrt{5} \le 2$$

□ $\sqrt{5} - 2$ □ $| z |$ □ $\sqrt{5} + 2$

Hence greatest value of $| z |$ is $\sqrt{5} + 2$ and least value of $| z |$ is $\sqrt{5} - 2$

***Illustration* 5**

If $\left| Z + \frac{1}{Z} \right| = a$, where Z is a complex number and a is a positive real number, then find the greatest $| Z |$ and least $| Z |$.

Solution

Let us first find greatest $| Z |$

If $| Z |$ is greatest, $\frac{1}{|Z|}$ is least and hence $| Z | > \frac{1}{|Z|}$

Write $a = \left| Z + \frac{1}{Z} \right| = \left| Z - \left(-\frac{1}{Z} \right) \right| \geq |Z| - \frac{1}{|Z|}$

This gives $| Z |^2 - a | Z | - 1$ □ 0; and hence $| Z |$ lies between the roots of the equation $| Z |^2 - a | Z | - 1 = 0$

Roots are $\frac{a \pm \sqrt{a^2 + 4}}{2}$ and hence $\frac{a - \sqrt{a^2 + 4}}{2} \leq |Z| \leq \frac{a + \sqrt{a^2 + 4}}{2}$ … (i)

It is known that $| Z |$ □ 0 while $\frac{a - \sqrt{a^2 + 4}}{2}$ is < 0 and hence (i) gets modified as

$$0 \leq |Z| \leq \frac{a + \sqrt{a^2 + 4}}{2}$$

and thus the greatest value of $| Z |$ is $\frac{a + \sqrt{a^2 + 4}}{2}$

Now for the least $| Z |$

In this case $\frac{1}{|Z|}$ is greatest and hence $\frac{1}{|Z|} - | Z | > 0$

write $a = \left| Z + \frac{1}{Z} \right| = \left| \frac{1}{Z} - (-Z) \right| \geq \frac{1}{|Z|} - |Z|$

This gives $| Z |^2 + a | Z | - 1$ □ 0 and this is possible for all $| Z |$ lying outside the roots of $| Z |^2 + a | Z | - 1 = 0$

Roots are $\frac{-a \pm \sqrt{a^2 + 4}}{2}$; and of these $\frac{-a - \sqrt{a^2 + 4}}{2}$ is negative, hence $| Z |$ cannot be less than this negative value.

Therefore $| Z |$ □ $\frac{-a + \sqrt{a^2 + 4}}{2}$ and this gives the least $| Z |$ value

***Illustration* 6**

Among the complex numbers z which satisfies $| z - 25i |$ □ 15, find the complex numbers z

having

(i) Least positive argument (ii) Maximum positive argument

(iii) Least modulus (iv) Maximum modulus

Solution

The complex numbers z satisfying the condition | z – 25 i | □ 15 are represented by the points inside and on the circle of radius 15 and centre at the point C(0, 25).

The complex number having least positive argument and maximum positive arguments in this region are the points of contact of tangents drawn from origin to the circle.

Here □ = least positive argument

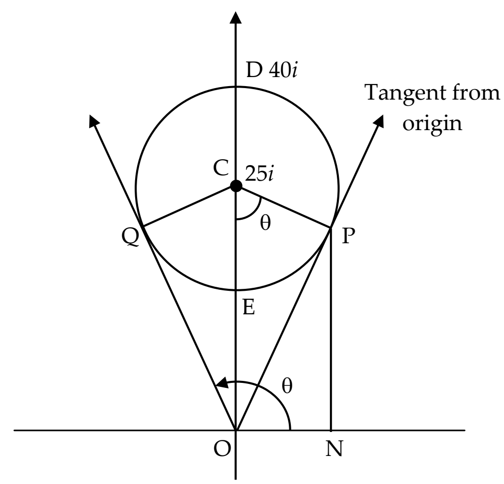
and □ = maximum positive argument

□ In □OCP, OP = $\sqrt{(OC)^2-(CP)^2}=\sqrt{(25)^2-(15)^2}=20$

and sin □ = $\frac{OP}{OC}=\frac{20}{25}=\frac{4}{5}$

□ tan □ = $\frac{4}{3}$ □□□□□□ = $\tan^{-1}\left(\frac{4}{3}\right)$

Thus, complex number at P has modulus 20 and argument □ = $\tan^{-1}\left(\frac{4}{3}\right)$



□ Z_P = 20 (cos □ + i sin □) = 20 $\left(\frac{3}{5}+i\frac{4}{5}\right)$

□ Z_P = 12 + 16i

Similarly Z_Q = – 12 + 16i

From the figure, E is the point with least modulus and D is the point with maximum modulus.

Hence, $Z_E=\overrightarrow{OE}=\overrightarrow{OC}-\overrightarrow{EC}$ = 25i – 15i = 10i

and $Z_D=\overrightarrow{OD}=\overrightarrow{OC}+\overrightarrow{CD}$ = 25i + 15i = 40i

1.9 De Moivre's Theorem

If n is any integer, then $(\cos\theta+i\sin\theta)^n$ = cos n□ + i sin n□. This is known as De Moivre's Theorem.

If n is a rational number, then one of the values of (cos □ + i sin □)^n is cos n□ + i sin n□. Let $n = p/q$, where p and q are integers (q > 0) and p, q have no common factor, then (cos □ + i sin □)^n has q distinct values, one of which is cos n□ + i sin n□.

If z = r (cos □ + i sin □), and n is a positive integer, then $z^{1/n} = r^{1/n}\left[\cos\frac{2k\pi+\theta}{n} + i\sin\frac{2k\pi+\theta}{n}\right]$

where $k = 0, 1, 2, \ldots\ldots n - 1$

ILLUSTRATIONS

***Illustration* 7**

If 2 cos □ = $x + \frac{1}{x}$ and 2 cos □ = $y + \frac{1}{y}$, prove the following

(i) $x^m y^n + \frac{1}{x^m y^n}$ = 2 cos (m□ + n□)

(ii) $\frac{x^m}{y^n} + \frac{y^n}{x^m}$ = 2 cos (m□ – n□)

Solution

(i) Given $x + \frac{1}{x}$ = 2 cos □ □ $x^2 - 2x$ cos □ + 1 = 0. Solving this, x = cos □ ± i sin □

In fact if x = cos □ + i sin □; $\frac{1}{x}$ = cos □ – i sin □. It may be noted also that $x + \frac{1}{x}$ = 2 cos □ is symmetrical w.r.t. $\frac{1}{x}$ and hence if one root is the value for x, the other root is $\frac{1}{x}$ and vice-versa.

Similarly, given that 2 cos □ = $y + \frac{1}{y}$, we have y = cos □ + i sin □

□ x^m = (cos □ + i sin □)^m = cos m□ + i sin m□; and

y^n = (cos □ + i sin □)^n = cos n□ + i sin n□

$x^m y^n$ = (cos m□ + i sin m□) (cos n□ + i sin n□)

= cos (m□ + n□) + i sin (m□ + n□)

and $\frac{1}{x^m y^n}$ = cos (m□ + n□) – i sin (m□ + n□)

Adding we get $x^m y^n + \frac{1}{x^m y^n}$ = 2 cos (m□ + n□)

(ii) By similar reasoning $\frac{x^m}{y^n} + \frac{y^n}{x^m}$ = 2 cos (m□ – n□)

***Illustration* 8**

If *n* be a positive integer, prove that

$$(1+i)^{2n}+(1-i)^{2n}=\begin{cases} 0 & \text{if } \; n \text{ be odd} \\ 2^{n+1} & \text{if } \; \dfrac{n}{2} \text{ be even} \\ -2^{n+1} & \text{if } \; \dfrac{n}{2} \text{ be odd} \end{cases}$$

Solution

$$(1+i)^{2n}=2^n\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^{2n}=2^n\left(\cos\frac{n\pi}{2}+i\sin\frac{n\pi}{2}\right)$$

$$(1-i)^{2n}=2^n\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)^{2n}=2^n\left(\cos\frac{n\pi}{2}-i\sin\frac{n\pi}{2}\right)$$

$$\therefore \quad (1+i)^{2n}+(1-i)^{2n}=2^n\left(\cos\frac{n\pi}{2}+i\sin\frac{n\pi}{2}+\cos\frac{n\pi}{2}-i\sin\frac{n\pi}{2}\right)$$

$$= 2^{n+1}\cos\left(\frac{n\pi}{2}\right)$$

If n be odd = $2m + 1$, then RHS = $2 \cos (2m + 1)\dfrac{\pi}{2}$

$$= 0$$

If n be even and $\dfrac{n}{2}$ also even so that $n = 4k$, then RHS = $2^{n+1} \cos (2k\pi) = 2^{n+1}$

else RHS = $2^{n+1}\cos\left(\dfrac{4k\pi}{2}\right)$

1.10 The *n*th Root of Unity

Let x be nth root of unity. Then

$x^n = 1 = 1 + i \cdot 0 = \cos 0° + i \sin 0° = \cos (2k\pi + 0) + i \sin (2k\pi + 0)$

$= \cos 2k\pi + i \sin 2k\pi$ (where k is an integer)

$\therefore \; x = \cos \dfrac{2k\pi}{n}+i\sin\dfrac{2k\pi}{n}, \quad k = 0, 1, 2, \ldots\ldots n - 1$

Let $\alpha = \cos \dfrac{2\pi}{n}+i\sin\dfrac{2\pi}{n}$. Then the nth roots of unity are α^t

($t = 0, 1, 2, \ldots\ldots n - 1$), i.e., the nth roots of unity are $1, \alpha, \alpha^2, \ldots\ldots \alpha^{n-1}$

□ **Properties of *n* roots of Unity**

(i) **Sum of *n* roots of unity is zero**

$1 + □ + □^2 + \ldots\ldots + □^{n-1} = \frac{1-\alpha^n}{1-\alpha} = 0$

□ $\sum_{k=0}^{n-1} \cos\frac{2k\pi}{n} = 0$ and $\sum_{k=0}^{n-1} \sin\frac{2k\pi}{n} = 0$

Thus the sum of the roots of unity is zero.

(ii) **Sum of *p*th power of *n* roots of unity is zero, if *p* is not a multiple of *n***

$1 + □^p + (\alpha^2)^p + (\alpha^3)^p + \ldots\ldots + (\alpha^{n-1})^p$

$= \frac{1-(\alpha^p)^n}{1-\alpha^p} = \frac{1-\left(e^{i\frac{2\pi p}{n}}\right)^n}{1-\alpha^p}$

$= \frac{1-e^{i2\pi p}}{1-\alpha^p} = 0$

(iii) **Sum of *p*th power of *n* roots of unity is *n*, if *p* is a multiple of *n***

Let $p = □n$, thus $□^p = e^{i\frac{2\pi p}{n}} = e^{i2\pi\lambda}$

$= (\cos 2□□ + i \sin 2□□) = 1$

$1 + □^p + (\alpha^2)^p + (\alpha^3)^p + \ldots\ldots + (\alpha^{n-1})^p$

$= 1 + 1 + 1 + \ldots(n \text{ times}) = n$

(iv) **Product of the roots**

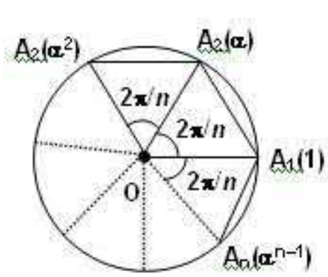
$1.□.□^2. \ldots\ldots □^{n-1} = \alpha^{\frac{n(n-1)}{2}} = \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^{n\left(\frac{n-1}{2}\right)}$

$= \cos\{□(n-1)\} + i\sin\{□(n-1)\}$

If n is even, $\alpha^{\frac{n(n-1)}{2}} = -1$

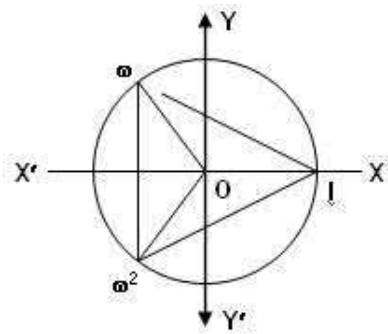
If n is odd, $\alpha^{\frac{n(n-1)}{2}} = 1$

(v) The points represented by the '*n*' *n*th roots of unity are located at the vertices of a regular polygon of *n* sides inscribed in a unit circle having centre at the origin, one vertex being on the positive real axis (Geometrically represented as shown)



1.11 CUBE ROOTS OF UNITY

For $n = 3$, we get the cube roots of unity and they are $1,\ \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ and $\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$ i.e., $1, \frac{-1+i\sqrt{3}}{2}$ and $\frac{-1-i\sqrt{3}}{2}$. They are generally denoted by 1, ω an ω^2 and geometrically represented by the vertices of an equilateral triangle whose circumcentre is the origin and circumradius is unity.



❖ **Properties of Cube Roots of Unity**

(i) $\omega^3 = 1$

(ii) $1 + \omega + \omega^2 = 0$

(iii) $1 + \omega^n + \omega^{2n} = 3$ (n is a multiple of 3)

(iv) $1 + \omega^n + \omega^{2n} = 0$ (n is an integer, not a multiple of 3)

(v) $\omega = 1/\omega^2$ and $\omega^2 = 1/\omega$

(vi) $\omega = (\omega^2)^2$

(vii) $\bar{\omega} = \omega^2$ and $\omega^2 = \bar{\omega}$

ILLUSTRATIONS

***Illustration* 9**

If α, β, γ are roots of $x^3 - 3x^2 + 3x + 7 = 0$ (and ω is cube roots of unity), then find the value of $\frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\gamma-1} + \frac{\gamma-1}{\alpha-1}$.

Solution

We have $x^3 - 3x^2 + 3x + 7 = 0$

$\Rightarrow$ $(x-1)^3 + 8 = 0$

$\Rightarrow$ $(x-1)^3 = (-2)^3$

$\Rightarrow$ $\left(\frac{x-1}{-2}\right)^3 = 1$ $\Rightarrow$ $\frac{x-1}{-2} = (1)^{1/3} = 1, \omega, \omega^2$ (cube roots of unity)

$\Rightarrow$ $x = -1, 1-2\omega, 1-2\omega^2$

Here $\alpha = -1, \beta = 1-2\omega, \gamma = 1-2\omega^2$

$\Rightarrow$ $\alpha - 1 = -2, \beta - 1 = -2\omega, \gamma - 1 = -2\omega^2$

Then $\frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\gamma-1} + \frac{\gamma-1}{\alpha-1} = \left(\frac{-2}{-2\omega}\right) + \left(\frac{-2\omega}{-2\omega^2}\right) + \left(\frac{-2\omega^2}{-2}\right)$

$$= \frac{1}{\omega} + \frac{1}{\omega} + \omega^2$$

$$= \omega^2 + \omega^2 + \omega^2 = 3\omega^2$$

***Illustration* 10**

If 1, α, α^2 ……… α^n are *n*th roots of unity then prove that

(a) $(1 - \alpha)(1 - \alpha^2) \ldots\ldots\ldots (1 - \alpha^{n-1}) = n$

(b) $\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\sin\frac{3\pi}{n}\ldots\ldots\ldots\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}},\ n \geq 2$

Solution

If 1, α, α^2 ……… α^n are roots of $x^n = 1$

$\Rightarrow$ $x^n - 1 = (x - 1)(x - \alpha)(x - \alpha^2) \ldots\ldots\ldots (x - \alpha^{n-1})$

$\Rightarrow$ $(x - \alpha)(x - \alpha^2) \ldots\ldots\ldots (x - \alpha^{n-1}) = \frac{x^n - 1}{x - 1} = 1 + x + x^2 + \ldots\ldots\ldots + x^{n-1}$

Put $x = 1$ $\Rightarrow$ $(1 - \alpha)(1 - \alpha^2) \ldots\ldots\ldots (1 - \alpha^{n-1}) = n$

Also $\alpha^k = e^{\frac{i2k\pi}{n}}$ $\Rightarrow$ $|1 - \alpha^k| = \left| 2\sin\frac{k\pi}{n} \right|$

Taking modulus of the first result

$\Rightarrow$ $|1 - \alpha||1 - \alpha^2| \ldots\ldots\ldots |1 - \alpha^{n-1}| = |n|$

$\Rightarrow$ $\left(2\sin\frac{\pi}{n}\right)\left(2\sin\frac{2\pi}{n}\right)\ldots\ldots\ldots\left(2\sin\frac{(n-1)\pi}{n}\right) = n$

$\Rightarrow$ $\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\ldots\sin(n-1)\frac{\pi}{n} = \frac{n}{2^{n-1}}$

1.12 Logarithm of Complex Number

In order to find log $(x + iy)$, we write log $(x + iy) = a + ib$

$\Rightarrow$ $x + iy = e^{a+ib} = e^a[\cos b + i\sin b]$

$= e^a(\cos(2k\pi + b) + i\sin(2k\pi + b))$

$\Rightarrow$ $e^a\cos(2k\pi + b) = x$ and $e^a\sin(2k\pi + b) = y$

Solve for *a* and *b*

$\Rightarrow$ $e^{2a} = x^2 + y^2$ or $a = \frac{1}{2}\ln\left(x^2 + y^2\right)$, $\tan(2k\pi + b) = (y/x)$

When $k = 0$, corresponding values of *a* and *b* are refered to as principle values

- **Method to Find $(x + iy)^{a+ib}$**

  For evaluating $(x + iy)^{a+ib}$ we write

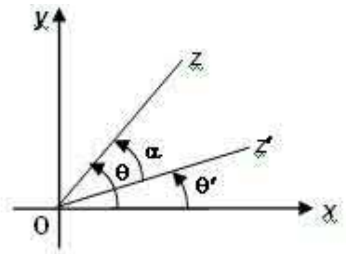
  $c + id = (x + iy)^{a+ib}$

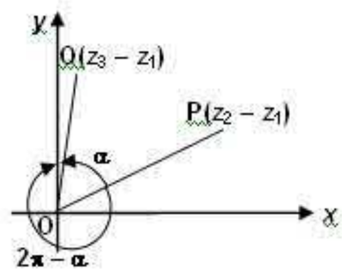
∴ $\log (c + id) = (a + ib) \cdot \log (x + iy)$

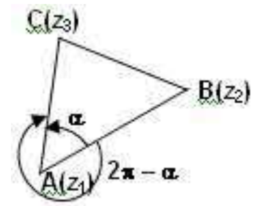
Now evaluate $\log (x + iy)$ and then solve $c + id = e^{(a + ib) \log (x + iy)}$

1.13 Concept of Rotation

If z and z' are two complex numbers then argument of $\frac{z}{z'}$ is the angle through which oz' must be turned in order that it may lie along oz.







$$\frac{z}{z'} = \frac{|z|\, e^{i\theta}}{|z'|\, e^{i\theta'}} = \frac{|z|}{|z'|} e^{i\alpha}$$

In general, let z_1, z_2, z_3, be the three vertices of a triangle ABC described in the counter-clock wise sense. Draw OP and OQ parallel and equal to AB and AC respectively. Then the point P is $z_2 - z_1$ and Q is $z_3 - z_1$ and $\frac{z_3 - z_1}{z_2 - z_1} = \frac{OQ}{OP}$ $(\cos \alpha + i \sin \alpha)$

$$= \frac{CA}{BA} \cdot e^{i\alpha} = \frac{|z_3 - z_1|}{|z_2 - z_1|} \cdot e^{i\alpha}$$

Note that $\arg (z_3 - z_1) - \arg (z_2 - z_1) = \alpha$ is the angle through which OP must be rotated in the anti-clockwise direction so that it becomes parallel to OQ.

Here we can write $\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|} \cdot e^{-i(2\pi - \alpha)}$ also. In this case we are rotating OP in clockwise direction by an angle $(2\pi - \alpha)$. Since the rotation is in clockwise direction, we are taking negative sign with angle $(2\pi - \alpha)$.

ILLUSTRATIONS

***Illustration* 11**

If $\sin (\log i^i) = a + ib$, find a and b. Hence, find $\cos (\log i^i)$.

Solution

$a + ib = \sin (\log i^i) = \sin (i \log i)$

$= \sin (i (\log |i| + i \text{ amp } i))$

$= \sin (i (\log 1 + i\, \pi/2))$

$= \sin (i (0 + i\, \pi/2))$

∴ $a = -1, b = 0$

□ $\sin(\log i^i) = -1$

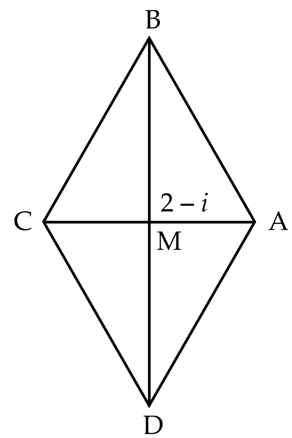
now $\cos(\log i^i) = \sqrt{\{1-\sin^2(\log i^i)\}} = \sqrt{1-1} = 0$

***Illustration* 12**

ABCD is a rhombus. Its diagonals AC and BD intersect at M such that BD = 2AC. If the points D and M represent the complex number 1 + *i* and 2 – *i* respectively, find the complex number(s) representing A.

Solution

Let A be *z*. The position MA can be obtained by rotating MD anticlockwise through an angle $\frac{\pi}{2}$; simultaneously length gets halved.



□ $z-(2-i)=\frac{1}{2}((1+i)-(2-i))e^{i\pi/2}$

$= \frac{1}{2}(-2-i) = -1-\frac{1}{2}i$

$z = -1 - \frac{1}{2}i + 2 - i = 1 - \frac{3i}{2}$

Another position of A corresponds to A and C getting interchanged and in that the complex number of A is $1+\frac{1}{2}i+2-i = 3-\frac{1}{2}i$

□ The complex number of A is either

$1-\frac{3i}{2}$ *or* $3-\frac{1}{2}i$

***Illustration* 13**

Complex numbers Z_1, Z_2, Z_3 are the vertices A, B and C respectively of an isosceles right angled triangle with $\underline{|C} = 90°$. Show that $(Z_1 - Z_2)^2 = 2(Z_1 - Z_3)(Z_3 - Z_2)$ (OR) equivalently $Z_1^2 + Z_2^2 + 2Z_3^2 = 2Z_1Z_3 + 2Z_2Z_3$.

Solution

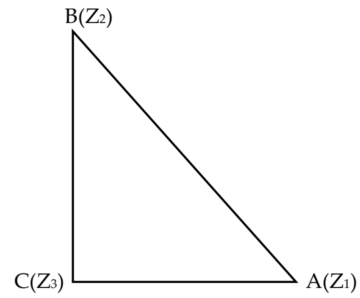
It is seen that when CA is turned anticlockwise through an angle 90°, the position of CB is obtained. Lengthwise CA = CB since the triangle is isosceles.

□ $Z_2 - Z_3 = (Z_1 - Z_3)\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

Squaring $(Z_2 - Z_1)^2 + (Z_1 - Z_3)^2 = 0$

i.e., $Z_1^2 + Z_2^2 + 2Z_3^2 = 2Z_1Z_3 + 2Z_2Z_3$

which is the second result.



To get the first from the second, we have

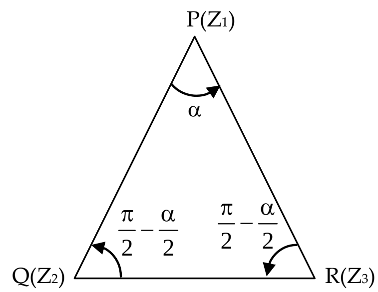
$$Z_1^2 + Z_2^2 = 2Z_1Z_3 + 2Z_2Z_3 - 2Z_3^2$$

$$Z_1^2 + Z_2^2 - 2Z_1Z_2 = 2Z_1Z_3 + 2Z_2Z_3 - 2Z_3^2 - 2Z_1Z_2$$

i.e., $(Z_1 - Z_2)^2 = 2(Z_1 - Z_3)(Z_3 - Z_2)$

which is the desired from of the result.

***Illustration* 14**



The points P, Q and R represent the complex numbers Z_1, Z_2 and Z_3 respectively and the angles of the triangle PQR at Q and R are both $\frac{\pi}{2} - \frac{\alpha}{2}$, prove that $(Z_3 - Z_2)^2 = 4(Z_3 - Z_1)(Z_1 - Z_2)\sin^2\left(\frac{\alpha}{2}\right)$.

Solution

QP is obtained from QR by a rotation counter clockwise through an angle $\frac{\pi}{2} - \frac{\alpha}{2}$; of course the length PQ is different from the length of QR

□ $$Z_1 - Z_2 = \frac{PQ}{QR}(Z_3 - Z_2)\left\{\cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right\}$$

Similarly

□ $$Z_1 - Z_3 = \frac{PR}{QR}(Z_2 - Z_3)\left\{\cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) - i\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right\}$$

Multiplying the two

$$(Z_1 - Z_2)(Z_1 - Z_3) = \frac{PQ \cdot PR}{QR^2}(Z_3 - Z_2)(Z_2 - Z_3)\left\{\cos^2\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) + \sin^2\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right\}$$

Now, $$\frac{QR}{\sin\alpha} = \frac{PQ}{\cos\frac{\alpha}{2}} = \frac{PR}{\cos\frac{\alpha}{2}}$$

□ $$\frac{PQ \cdot PR}{QR^2} = \frac{\cos^2\frac{\alpha}{2}}{4\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}}$$

□ $$(Z_1 - Z_2)(Z_1 - Z_3)\,4\sin^2\frac{\alpha}{2} = (Z_3 - Z_2)(Z_2 - Z_3)$$

i.e., $$(Z_3 - Z_2)^2 = 4(Z_3 - Z_1)(Z_1 - Z_2)\sin^2\frac{\alpha}{2}$$

***Illustration* 15**

Find the value of | Z | from the equation $2Z^3 - 3Z^2 - 18iZ + 27i = 0$

Solution

$2Z^3 - 3Z^2 - 18iZ + 27i = 0$

$Z^2(2Z - 3) - 9i(2Z - 3) = 0$

$(2Z - 3)(Z^2 - 9i) = 0$

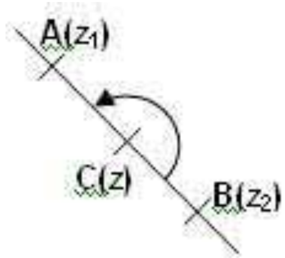
$\Rightarrow$ $2Z - 3 = 0$ $\Rightarrow$ $|Z| = 3/2$

or $Z^2 = 9i$ $\Rightarrow$ $|Z| = 3$

1.14 Application of Complex Numbers in Geometry

- **Section formula**

Let z_1 and z_2 be any two complex numbers representing the points A and B respectively in the argand plane. Let C be the point dividing the line segment AB internally in the ratio $m : n$, i.e., $\frac{AC}{BC} = \frac{m}{n}$ and let the complex number associated with point C be z.



Let us rotate the line BC about the point C so that it becomes parallel to CA. The corresponding equation of rotation will be,

$$\frac{z_1 - z}{z_2 - z} = \frac{|z_1 - z|}{|z_2 - z|} \cdot e^{i\pi} = \frac{m}{n}(-1)$$

$\Rightarrow$ $nz_1 - nz = -mz_2 + mz$

$\Rightarrow$ $z = \frac{nz_1 + mz_2}{m + n}$

Similarly if C(z) divides the segment AB externally in the ratio of $m : n$, then $z = \frac{nz_1 - mz_2}{m - n}$

In the specific case, if C(z) is the mid point of AB then $z = \frac{z_1 + z_2}{2}$.

ILLUSTRATIONS

***Illustration* 16**

If the vertices of a triangle ABC are represented by Z_1, Z_2 and Z_3 respectively; then prove that

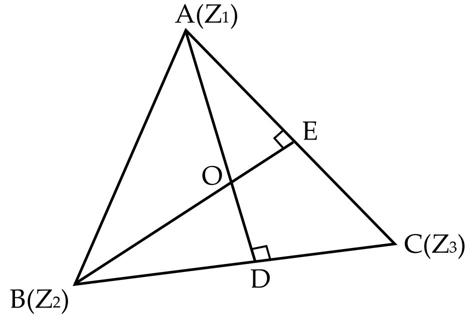
(i) the orthocenter is $\dfrac{(a\sec A)\,Z_1+(b\sec B)Z_2+(c\sec C)Z_3}{a\sec A+b\sec B+c\sec C}$

Or $\dfrac{(\tan A)\,z_1+(\tan B)z_2+(\tan C)z_3}{\tan A+\tan B+\tan C}$

(ii) the circumcentre is $\dfrac{(\sin 2A)\,Z_1+(\sin 2B)Z_2+(\sin 2C)Z_3}{\sin 2A+\sin 2B+\sin 2C}$

Solution

(i) Orthocentre



Let the two altitudes AD and BE intersect at O

Now, $\dfrac{BD}{DC}=\dfrac{c\cos B}{b\cos C}=\dfrac{c\sec C}{b\sec B}$

The point D, dividing BC in the ratio $\dfrac{BD}{DC}$ has a complex number

$$\frac{(c\sec C)Z_3+(b\sec B)Z_2}{b\sec B+c\sec C};$$

Again $\dfrac{AO}{OD}=\dfrac{\text{Area of }\Delta ABO}{\text{Area of }\Delta OBD}$ (triangles of the same altitude)

$$=\frac{\frac{1}{2}AB\,.\,BO\,\sin\angle ABE}{\frac{1}{2}BO\,.\,OD\,\sin\angle DBE}$$

$$=\frac{c\cos A}{(c\cos B\,.\,\cos C)}$$

$$=\frac{a\cos A}{a\cos B\,.\,\cos C}=\frac{b\cos C+c\cos B}{\cos B\cos C}\;\frac{1}{a\sec A}$$

$$=\frac{b\sec B+c\sec C}{a\sec A}$$

$\therefore$ The point O, dividing AD, in the ratio $\dfrac{AO}{OD}$ has a complex number

$$\frac{\text{AO (complex number of D) + OD (complex number of A)}}{\text{AO + OD}}$$

$$= \frac{(b\sec B + c\sec C)\left(\frac{b\sec B.Z_2 + c\sec C.Z_3}{b\sec B + c\sec C}\right) + a\sec A.Z_1}{b\sec B + c\sec C + a\sec A}$$

$$= \frac{(a\sec A)\,Z_1 + (b\sec B)Z_2 + (c\sec C)Z_3}{a\sec A + b\sec B + c\sec C}$$

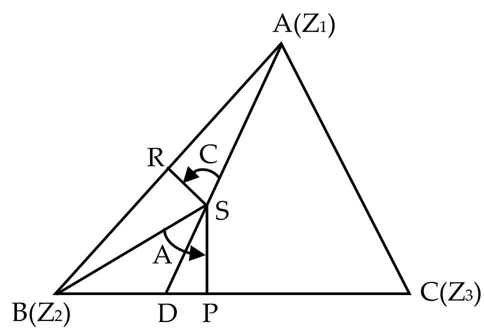
The symmetry of this result in *a*, *b*, *c* and A, B, C indicates that O lies on the third altitude also. Hence O, the orthocenter, is $\frac{Z_1\,a\sec A + Z_2\,b\sec B + Z_3\,c\sec C}{a\sec A + b\sec B + c\sec C}$

To prove the other result substituting a = 2R sin A, b = 2R sin B and c = 2R sin C in the above result.

$$\frac{Z_1 \tan A + Z_2 \tan B + Z_3 \tan C}{\tan A + \tan B + \tan C}$$

(ii) Circum-centre

Let S be the point of intersection of perpendicular bisectors of BC and AB. S lies on the third perpendicular bisector also



Let AS produced meet BC in D. Now,

$$\frac{BD}{DC} = \frac{\text{area of } \Delta ABD}{\text{area of } \Delta ACD}$$

(Triangles of the same altitude)

$$= \frac{AB.AD.\sin\angle BAD}{AC.AD.\sin\angle CAD} = \frac{c\sin(90° - C)}{b\sin(90° - B)}$$

$$= \frac{\sin 2C}{\sin 2B} \qquad \ldots \text{(i)}$$

□ D is represented by the complex number = $\frac{(\sin 2C)\,Z_3 + (\sin 2B)Z_2}{\sin 2B + \sin 2C}$

$$\frac{AS}{SD} = \frac{\text{area of } \Delta ASB}{\text{area of } \Delta BSD} = \frac{AS.BS.\sin 2C}{BS.BD\sin(90° - A)}$$

$$= \frac{R\sin 2C}{BD\cos A} \qquad \ldots \text{(ii)}$$

From (i), $\frac{BD}{\sin 2C} = \frac{DC}{\sin 2B} = \frac{BD + DC}{\sin 2B + \sin 2C} = \frac{a}{\sin 2B + \sin 2C}$

Substituting in (ii)

$$\frac{AS}{SD} = \frac{R\sin 2C}{\frac{a\sin 2C}{\sin 2B + \sin 2C}.\cos A}$$

$$= \frac{R \sin 2C}{\dfrac{2R \sin A \cos A \sin 2C}{\sin 2B + \sin 2C}} = \frac{\sin 2B + \sin 2C}{\sin 2A}$$

- S is represented by

$$\frac{(\sin 2A)\, Z_1 + (\sin 2B + \sin 2C)\left(\dfrac{\sin 2C\,.\,Z_3 + \sin 2B . Z_2}{\sin 2B + \sin 2C}\right)}{\sin 2A + \sin 2B + \sin 2C}$$

i.e., $\dfrac{Z_1 \sin 2A + Z_2 \sin 2B + Z_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}$

- **Condition for Collinearity**

  If there are three real numbers (other than 0) l, m and n such that

  $lz_1 + mz_2 + nz_3 = 0$ and $l + m + n = 0$

  then complex numbers z_1, z_2 and z_3 will be collinear.

1.15 Equation of a straight line

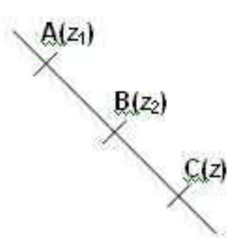
- **Equation of straight line with the help of coordinate geometry**

  Writing $x = \dfrac{z + \bar{z}}{2}$, $y = \dfrac{z - \bar{z}}{2i}$ etc. in $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$ and re-arranging terms, we find that the equation of the line through z_1 and z_2 is given by

  $$\frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1} \quad \text{or} \quad \begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

- **Equation of a straight line with the help of rotation formula**

  Let A(z_1) and B(z_2) be any two points lying on any line and we have to obtain the equation of this line. For this purpose let us take any point C(z) lying on this line. Since arg $\left(\dfrac{z - z_1}{z_2 - z_1}\right) = 0$ or □.



$$\frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1} \qquad \text{… (i)}$$

This is the equation of the line that passes through A(z_1) and B(z_2). After rearranging the terms, it can also be put in the following form $\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$.

- **General equation of the line**

From equation (i) we get, $z(\bar{z}_2-\bar{z}_1)-z_1\bar{z}_2+z_1\bar{z}_1=\bar{z}(z_2-z_1)-\bar{z}_1z_2+z_1\bar{z}_1$

$\Rightarrow \quad z(\bar{z}_2-\bar{z}_1)+\bar{z}(z_1-z_2)+\bar{z}_1z_2-z_1\bar{z}_2=0$

Here $\bar{z}_1z_2-z_1\bar{z}_2$ is a purely imaginary number as $\bar{z}_1z_2-z_1\bar{z}_2=2i\,\text{Im}(\bar{z}_1z_2)$.

Let $\bar{z}_1z_2-z_1\bar{z}_2=ib$, $b \in$ R

$\Rightarrow \quad z(\bar{z}_2-\bar{z}_1)+\bar{z}(z_1-z_2)+ib=0$

$\Rightarrow \quad zi(\bar{z}_1-\bar{z}_2)+\bar{z}i(z_2-z_1)+b=0$

Let $a = i(z_2 - z_1)$

$\Rightarrow \quad \bar{a}=i(\bar{z}_1-\bar{z}_2)$

$\Rightarrow \quad z\bar{a}+\bar{z}a+b=0$

This is the general equation of a line in the complex plane.

- **Slope of a given line**

Let the given line be $z\bar{a}+\bar{z}a+b=0$. Replacing z by $x + iy$, we get

$(x+iy)\bar{a}+(x-iy)a+b=0$

$\Rightarrow \quad (a+\bar{a})x+iy(\bar{a}-a)+b=0$

It's slope is = $\dfrac{a+\bar{a}}{i(a-\bar{a})}=\dfrac{2\,\text{Re}(a)}{2i^2\text{Im}(a)}=-\dfrac{\text{Re}(a)}{\text{Im}(a)}$

- **Equation of a line parallel to given line**

Equation of a line parallel to the line $z\bar{a}+\bar{z}a+b=0$ is $z\bar{a}+\bar{z}a+\lambda=0$ (where λ is a real number)

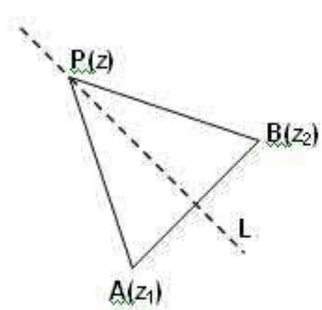
- **Equation of a line perpendicular to given line**

Equation of a line perpendicular to the line $z\bar{a}+\bar{z}a+b=0$ is $z\bar{a}-\bar{z}a+i\lambda=0$ (where λ is a real number)

- **Equation of Perpendicular Bisector**

Consider a line segment joining A(z_1) and B(z_2).

Let the line 'L' be it's perpendicular bisector.



If P(z) be any point on the 'L', we have

PA = PB $\Rightarrow$ $|z-z_1| = |z-z_2|$

$\Rightarrow \quad |z-z_1|^2 = |z-z_2|^2$

$\Rightarrow \quad (z-z_1)(\bar{z}-\bar{z}_1)=(z-z_2)(\bar{z}-\bar{z}_2)$

$\Rightarrow \quad z\bar{z}-z\bar{z}_1-z_1\bar{z}+z_1\bar{z}_1=z\bar{z}-z\bar{z}_2-z_2\bar{z}+z_2\bar{z}_2$

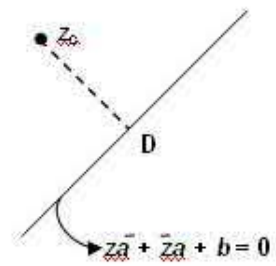
$\Rightarrow \quad z(\bar{z}_2-\bar{z}_1)+\bar{z}(z_2-z_1)+z_1\bar{z}_1-z_2\bar{z}_2=0$

□ **Perpendicular Distance of a given point from a given line**

Let the given line be $z\bar{a}+\bar{z}a+b=0$ and the given point be z_c

Saying $z = x_c + iy_c$

Replacing z by $x + iy$, in the given equation, we get,



$$x\left(a+\bar{a}\right)+iy\left(\bar{a}-a\right)+b=0$$

Distance of (x_c, y_c) from this line is

$$\frac{\left| x_c\left(a+\bar{a}\right)+iy_c\left(\bar{a}-a\right)+b\right|}{\sqrt{\left(a+\bar{a}\right)^2-\left(a-\bar{a}\right)^2}}=\frac{\left| z_c\bar{a}+\bar{z}_c a+b\right|}{\sqrt{4\left(\mathrm{Re}\left(a\right)\right)^2+4\left(im\left(a\right)\right)^2}}$$

$$=\frac{\left| z_c\bar{a}+\bar{z}_c a+b\right|}{2\left| a\right|}$$

Note:

arg $(z - z_0)$ = □ represents a line passing through z_0 with slope tan □. (making angle □ with the positive direction of x-axis)

1.16 Equation of a Circle

Consider a fixed complex number z_0 and let z be any complex number which moves in such a way that it's distance from z_0 is always equals to 'r'. This implies z would lie on a circle whose centre is z_0 and radius r. And it's equation would be $| z - z_0 | = r$.

□ $| z - z_0 |^2 = r^2$

□ $\left(z-z_0\right)\left(\bar{z}-\bar{z}_0\right)=r^2$

□ $z\bar{z}-z\bar{z}_0-\bar{z}z_0+z_0\bar{z}_0-r^2=0$

Let $-a = z_0$ and $z_0\bar{z}_0-r^2=b$

□ $z\bar{z}+a\bar{z}+\bar{a}z+b=0$

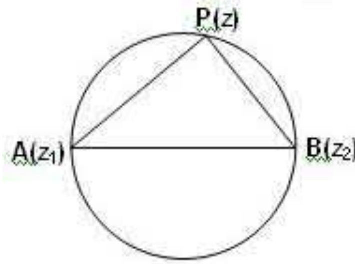
It represent the general equation of a circle in the complex plane.

□ **Properties of Circles**

(i) $z\bar{z}+a\bar{z}+\bar{a}z+b=0$ represents a circle whose centre is $-a$ and radius is $\sqrt{a\bar{a}-b}$. Thus $z\bar{z}+a\bar{z}+\bar{a}z+b=0$, ($b$ □ R) represents a real circle if and only if $a\bar{a}-b\geq 0$.

(ii) Now let us consider a circle described on a line segment AB, (A(z_1), B(z_2)) as diameter.

Let P(z) be any point on the circle. As the angle in the semicircle is □/2, □APB = □/2.



□ $\arg\left(\frac{z_1 - z}{z_2 - z}\right)$ = ± □/2

□ $\frac{z - z_1}{z - z_2}$ is purely imaginary

□ $\frac{z - z_1}{z - z_2} + \frac{\bar{z} - \bar{z}_1}{\bar{z} - \bar{z}_2} = 0$

□ $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$.

(iii) Let z_1 and z_2 be two given complex numbers and z be any complex number.

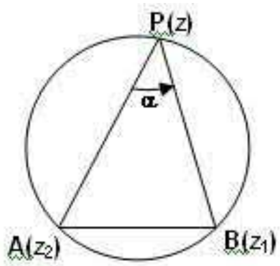


Fig. 1

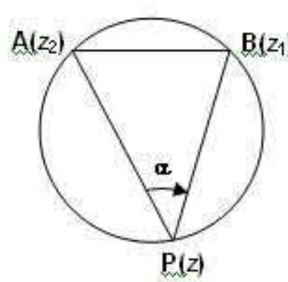


Fig. 2

Such that, $\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha$, where □ □ (0, □)

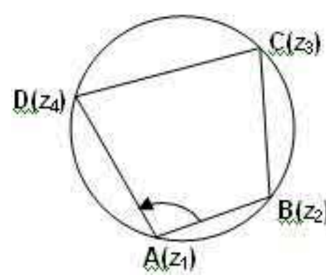
Then 'z' would lie on an arc of segment of a circle on z_1z_2, containing angle □. Clearly if □ □ (□/2), z would lie on the major arc (excluding the points z_1 and z_2) and if □ □ (□/2, □). 'z' would lie on the minor arc (excluding the points z_1 and z_2).

Note:

The sign of □ determines the side of z_1z_2 on which the segment lies. Thus □ is positive in fig. 1 and negative in fig. 2.

(iv) Let ABCD be a cyclic quadrilateral such that A(z_1), B(z_2), C(z_3) and D(z_4) lie on a circle.

Clearly □A + □C = □.



□ $\arg\left(\frac{z_4 - z_1}{z_2 - z_1}\right) + \arg\left(\frac{z_2 - z_3}{z_4 - z_3}\right) = \pi$

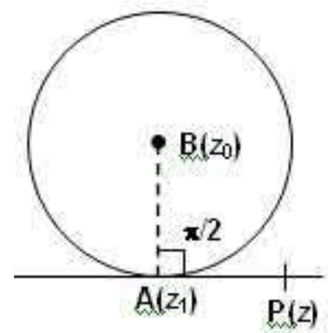
□ $\arg\left(\frac{z_4 - z_1}{z_2 - z_1}\right)\left(\frac{z_2 - z_3}{z_4 - z_3}\right) = \pi$

□ $\frac{(z_4 - z_1)(z_2 - z_3)}{(z_2 - z_1)(z_4 - z_3)}$ is purely real.

1.17 EQUATION OF TANGENT TO A GIVEN CIRCLE

Let $| z - z_0 | = r$ be the given circle and we have to obtain the tangent at A(z_1). Let us take any point P(z) on the tangent line at A(z_1).

Clearly □PAB = □/2



$$\arg\left(\frac{z - z_1}{z_0 - z_1}\right) = \pm\frac{\pi}{2}$$

□ $\dfrac{z - z_1}{z_0 - z_1}$ is purely imaginary

□ $$\frac{z - z_1}{z_0 - z_1} + \frac{\bar{z} - \bar{z}_1}{\bar{z}_0 - \bar{z}_1} = 0$$

□ $$(z - z_1)(\bar{z}_0 - \bar{z}_1) + (\bar{z} - \bar{z}_1)(z_0 - z_1) = 0$$

□ $$z(\bar{z}_0 - \bar{z}_1) + \bar{z}(z_0 - z_1) + z_1\bar{z}_1 - z_1\bar{z}_0 + z_1\bar{z}_1 - \bar{z}_1 z_0 = 0$$

□ $$z(\bar{z}_0 - \bar{z}_1) + \bar{z}(z_0 - z_1) + 2|z_1|^2 - z_1\bar{z}_0 - \bar{z}_1 z_0 = 0$$

In particular if given circle is $| z | = r$, equation of the tangent at $z = z_1$ would be,

$$z\bar{z}_1 + \bar{z}z_1 = 2|z_1|^2 = 2r^2$$

If $\left|\dfrac{z - z_1}{z - z_2}\right| = \lambda$ (□ □ R$^+$, □ □ 1), where z_1 and z_2 are given complex numbers and z is a arbitrary complex number then z would lie on a circle.

□ **Some important results to remember**

(i) The triangle whose vertices are the points represented by complex numbers z_1, z_2, z_3 is equilateral if $\dfrac{1}{z_2 - z_3} + \dfrac{1}{z_3 - z_1} + \dfrac{1}{z_1 - z_2} = 0$, i.e., if $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$.

(ii) $| z - z_1 | + | z - z_2 | =$ □, represents an ellipse if $| z_1 - z_2 | <$ □, having the points z_1 and z_2 as its foci. And if $| z_1 - z_2 | =$ □, then z lies on a line segment connecting z_1 and z_2.

(iii) $| z - z_1 |$ □ $| z - z_2 | =$ □, represents a hyperbola if $| z_1 - z_2 | >$ □, having the points z_1 and z_2 as its foci. And if $| z_1 - z_2 | =$ □, z lies on the line passing z_1 and z_2 excluding the points between z_1 and z_2.

ILLUSTRATIONS

***Illustration* 17**

Examine what locus is represented by $|Z - a|^2 + |Z - b|^2 = k$ (where k is real).

Solution

$$|Z - a|^2 = (Z - a)\,(\bar{Z} - \bar{a}) = Z\bar{Z} + a\bar{a} - \left(Z\bar{a} + \bar{Z}a\right)$$

$$= |Z|^2 + |a|^2 - 2\mathrm{Re}\,(Z\bar{a})$$

Similarly $|Z - b|^2 = |Z|^2 + |b|^2 - 2\mathrm{Re}\,\left(Z\bar{b}\right)$

The given equation becomes

$$2|Z|^2 + |a|^2 + |b|^2 - 2\mathrm{Re}\,\left(Z\left(\bar{a} + \bar{b}\right)\right) = k$$

$$|Z|^2 - 2\mathrm{Re}\left[\frac{Z\left(\bar{a}+\bar{b}\right)}{2}\right] + \frac{|a+b|^2}{4} = \frac{k}{2} + \frac{1}{4}|a+b|^2 - \frac{|a|^2}{2} - \frac{|b|^2}{2}$$

i.e., $$\left|Z - \frac{a+b}{2}\right|^2 = \frac{1}{2}\left\{k - \frac{1}{2}\left[|a|^2 + |b|^2 - 2\mathrm{Re}\,a\bar{b}\right]\right\}$$

i.e., $$\left|Z - \frac{a+b}{2}\right|^2 = \frac{1}{2}\left\{k - \frac{1}{2}|a-b|^2\right\}$$

This will represent a circle with centre at $\frac{a+b}{2}$ and radius $\frac{1}{2}\sqrt{2k - |a-b|^2}$

***Illustration* 18**

If $||z + 2| - |z - 2|| = a^2$, z □ C representing a hyperbola for a □ S, then find the least set C.

Solution

Here foci are at – 2 and 2 at a distance at 4. Hence the given equation represents a hyperbola if $a^2 < 4$ i.e., a □ (– 2, 2).

***Illustration* 19**

Locate the points representing the complex numbers Z in the Argand diagram for which

(i) $|i - 1 - 2Z| > 9$ (ii) 4 □ $|2Z + i|$ □ 6

(iii) $|Z + i| = |Z - 1|$ (iv) $|Z - 1|^2 + |Z + 1|^2 = 4$

Solution

(i) $i - 1 - 2Z = -2\left(Z + \frac{1}{2} - \frac{i}{2}\right)$

$$|i - 1 - 2Z| = \left|-2\left[Z - \left(\frac{-1+i}{2}\right)\right]\right|$$

$$= 2\left|Z - \left(\frac{-1+i}{2}\right)\right|$$

□ The given condition becomes $\left|Z - \left(\frac{-1+i}{2}\right)\right| > \frac{9}{2}$

This represents all points represented by Z and lying outside the circle with centre $\frac{-1+i}{2}$ i.e., $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and radius $\frac{9}{2}$.

(ii) $2Z + i = 2\left(Z + \frac{i}{2}\right)$

□ $|2Z + i| = 2\left|Z + \frac{1}{2}i\right|$

□ 4 □ $|2Z + i|$ □ 6 gives

4 □ 2 $\left|Z + \frac{1}{2}i\right|$ □ 6i.e., 2 □ $\left|Z + \frac{i}{2}\right|$ □ 3

This represents the locations of all points Z on or outside the circle with centre $-\frac{i}{2}$ *i.e.,* $\left(0, -\frac{1}{2}\right)$ and radius 2; and on inside the circle with centre at $\frac{1}{2}i$ *i.e.,* $\left(0, -\frac{1}{2}\right)$ and radius 4. Thus it denotes the circular segment lying between two concentric circles.

(iii) $|Z + i| = |Z - 1|$

$|Z + i| = |Z - (-i)|$ denotes the distance of Z from $-i$ i.e., (0, – 1); and $|Z - 1|$ denotes the distance of Z from 1 i.e., (1, 0). The requirement $|Z + i| = |Z - 1|$ is satisfied for all Z equidistant from (0, – 1) and (1, 0) and thus it is perpendicular bisector of the join of (0, – 1) and (1, 0) whose Cartesian equation is $x + y = 0$.

(iv) $|Z - 1|^2 + |Z + 1|^2 = 4$

$|Z - 1|^2 + |Z + 1|^2 = (Z - 1)\left(\bar{Z} - 1\right) + (Z + 1)\left(\bar{Z} + 1\right)$ $\left(Q\ |\ Z|^2 = 2\bar{Z}\right)$

$= Z\bar{Z} - \left(Z + \bar{Z}\right) + 1 + Z\bar{Z} + \left(Z + \bar{Z}\right) + 1$

$= 2Z\bar{Z} + 2$

The requirement is $2Z\bar{Z} + 2 = 4$ i.e., $|Z|^2 = 1$ i.e., $|Z| = 1$. Thus the location of Z subject to the given condition is the unit circle $|Z| = 1$.

PRACTICE EXERCISE

13. If the vertices of a square are z_1, z_2, z_3 and z_4 taken in the anticlockwise order, prove that

$z_3 = -iz_1 + (1 + i)z_2$ and $z_4 = (1 - i)z_1 + iz_2$.

14. If w_1 and w_2 are the complex slope of two lines on the Argand plane, then prove that the lines are:

(i) perpendicular, if $w_1 + w_2 = 0$

(ii) parallel, if $w_1 = w_2$

15. If z is any non-zero complex number, prove that area of the triangle formed by the complex numbers z, □z and z + □z as its sides is $\frac{\sqrt{3}}{4}|z|^2$.

16. Prove that the complex numbers z_1, z_2 and the origin form an equilateral triangle, if $z_1^2 + z_2^2 - z_1 z_2 = 0$.

17. Find the closest distance of the origin from a curve given as $A\bar{z} + \bar{A}z + A\bar{A} = 0$. ($A$ is a complex number)

18. If $z = x + iy$, and the equation $\left|\frac{2z-i}{z+1}\right| = m$ represents a circle then find m.

19. Let A, B and C represents the complex numbers z_1, z_2, z_3 respectively on the complex plane. If the circumcentre of the triangle ABC lies at the origin, then find the orthocenter.

Answers

17. $\frac{|A|}{2}$ **18.** $\frac{m}{2} \neq 1$ *i.e.*, $m \neq 2$ **19.** $z_1 + z_2 + z_3$

*****

MISCELLANEOUS PROBLEMS

OBJECTIVE TYPE QUESTIONS

Example 1

The vertices of a triangle in the Argand plane are $3 + 4i$, $4 + 3i$ and $2\sqrt{6}+i$, then distance between orthocentre and circumcentre of the triangle is equal to

(a) $\sqrt{137-28\sqrt{6}}$ (b) $\sqrt{137+28\sqrt{6}}$ (c) $\frac{1}{2}\sqrt{137+28\sqrt{6}}$ (d) $\frac{1}{3}\sqrt{137+28\sqrt{6}}$

Solution

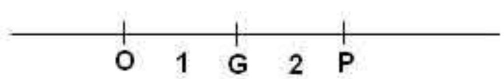
$z_1 = 3 + 4i,\ z_2 = 4 + 3i,\ z_3 = 2\sqrt{6}+i$

Clearly $|z_1| = |z_2| = |z_3| = 5$,

□ Points would lie on the circle centred at origin 'O'

now centroid of the triangle formed by these point

$$G=\left(\frac{7+2\sqrt{6}}{3}+\frac{8i}{3}\right)$$



$$OG=\sqrt{\left(\frac{7+2\sqrt{6}}{3}\right)^2+\frac{64}{9}}=\frac{1}{3}\sqrt{137+28\sqrt{6}}$$

□ $OP = 3OG = \sqrt{137+28\sqrt{6}}$

∴ **Ans. (b)**

Example 2

The value of the expression

$$2\left(1+\frac{1}{\omega}\right)\left(1+\frac{1}{\omega^2}\right)+3\left(2+\frac{1}{\omega}\right)\left(2+\frac{1}{\omega^2}\right)+4\left(3+\frac{1}{\omega}\right)\left(3+\frac{1}{\omega^2}\right)+\ldots\ldots+(n+1)\left(n+\frac{1}{\omega}\right)\left(n+\frac{1}{\omega^2}\right)$$

where ω is an imaginary cube root of unity, is

(a) $\frac{n(n^2-2)}{3}$ (b) $\frac{n(n^2+2)}{3}$ (c) $\frac{n^2(n+1)^2+4n}{4}$ (d) None of these

Solution

nth term of the expression

$$t_n=(n+1)\left(n+\frac{1}{\omega}\right)\left(n+\frac{1}{\omega^2}\right)$$

$$= n^3+n^2\left(\frac{1}{\omega^2}+\frac{1}{\omega}+1\right)+n\left(1+\frac{1}{\omega^2}+\frac{1}{\omega}\right)+1$$

$$= n^3 + n^2(\omega + \omega^2 + 1) + n(\omega + \omega^2 + 1) + 1$$

$$= n^3 + 1$$

□ $S_n=\sum_{r=1}^{n} t_r=\sum_{r=1}^{n}(r^3+1)=\frac{n^2(n+1)^2}{4}+n$ ∴ **Ans. (c)**

Example 3

If $|z| < 4$, then $|iz + 3 - 4i|$ is less than

(a) 4 (b) 5 (c) 6 (d) 9

Solution

$|iz + (3 - 4i)|$ □ $|iz| + |3 - 4i| = |z| + 5 < 4 + 5 = 9$

∴ **Ans. (d)**

Example 4

If z_1 and z_2 are two complex numbers satisfying the equation $\left|\frac{z_1 + iz_2}{z_1 - iz_2}\right| = 1$, then $\frac{z_1}{z_2}$ is a

(a) purely real (b) of unit modulus (c) purely imaginary (d) none of these

Solution

$$(z_1 + iz_2)(\bar{z}_1 - i\bar{z}_2) = (z_1 - iz_2)(\bar{z}_1 + i\bar{z}_2)$$

□ $\bar{z}_1 z_2 = z_1 \bar{z}_2$ □ $\frac{z_1}{z_2} = \frac{\bar{z}_1}{\bar{z}_2}$

□ $\frac{z_1}{z_2}$ is purely real

∴ **Ans. (a)**

Example 5

The points z_1, z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram taken in order if and only if

(a) $z_1 + z_4 = z_2 + z_3$ (b) $z_1 + z_3 = z_2 + z_4$ (c) $z_1 + z_2 = z_3 + z_4$ (d) None of these

Solution

Let A, B, C, D are vertices of parallelogram represented by z_1, z_2, z_3 & z_4 respectively

Since, In parallelogram, diagonals bisects each other, thus mid-point of AC and BD should be same.

□ $\frac{z_1 + z_3}{2} = \frac{z_2 + z_4}{2}$ □ $z_1 + z_3 = z_2 + z_4$

∴ **Ans. (b)**

Example 6

If the equation $|z - z_1|^2 + |z - z_2|^2 = k$, represents the equation of a circle, where $z_1 = 2 + 3i$, $z_2 = 4 + 3i$ are the extremities of a diameter, then the value of k is

(a) 1/4 (b) 4 (c) 2 (d) None of these

Solution

As z_1 and z_2 are the extremities of diameter, hence equation of circle will be

□ $|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$

□ $k = |z_1 - z_2|^2 = |2 + 3i - 4 - 3i|^2 = |-2|^2 = 4$

∴ **Ans. (b)**

Example 7

For positive integer n_1, n_2 the value of the expression

$(1+i)^{n_1} + \left(1+i^3\right)^{n_1} + \left(1+i^5\right)^{n_2} + \left(1+i^7\right)^{n_2}$, where $i=\sqrt{-1}$ is a real number if and only if

(a) $n_1 = n_2 + 1$ (b) $n_1 = n_2$ (c) $n_1 = n_2 - 1$ (d) $n_1 > 0,\ n_1 > 0$

Solution

$$(1+i)^{n_1} + \left(1+i^3\right)^{n_1} + \left(1+i^5\right)^{n_2} + \left(1+i^7\right)^{n_2}$$

$$= (1+i)^{n_1} + (1-i)^{n_1} + (1+i)^{n_2} + (1-i)^{n_2}$$

$$= \left(\sqrt{2}\right)^{n_1}\left(e^{\frac{in_1\pi}{4}} + e^{-\frac{in_1\pi}{4}}\right) + \left(\sqrt{2}\right)^{n_2}\left(e^{\frac{in_2\pi}{4}} + e^{-\frac{in_2\pi}{4}}\right)$$

$$= 2^{\frac{n_1}{2}}.2\cos\frac{n_1\pi}{4} + 2^{\frac{n_2}{2}}.2\cos\frac{n_2\pi}{4}$$

= Real number. Thus given expression would assume real values for every positive integral values of n_1 and n_2.

∴ **Ans. (d)**

Example 8

If $z^2 + z + 1 = 0$ then the value of $\left(z+\frac{1}{z}\right)^2 + \left(z^2+\frac{1}{z^2}\right)^2 + \left(z^3+\frac{1}{z^3}\right)^2 \ldots\ldots + \left(z^{21}+\frac{1}{z^{21}}\right)^2$ is equal to

(a) 21 (b) 42 (c) 0 (d) None of these

Solution

$z^2 + z + 1 = 0$ ⇒ $z = \omega, \omega^2$

If $z = \omega$, then $\frac{1}{z} = \omega^2$ and if $z = \omega^2$, then $\frac{1}{z} = \omega$. So, we may take $z = \omega$

When n is a multiple of 3, then $\left(z^n + \frac{1}{z^n}\right)^2 = \left(\omega^n + \frac{1}{\omega^n}\right)^2 = (1 + 1)^2 = 4$

and when n is not a multiple of 3, then

$$\left(z^n + \frac{1}{z^n}\right)^2 = \left(\omega^n + \frac{1}{\omega^n}\right)^2 = \left(\omega^n + \frac{\omega^{3n}}{\omega^n}\right)^2$$

$= (\omega^n + \omega^{2n})^2 = (-1)^2 = 1$ (∵ when n is not a multiple of 3, then $\omega^n + \omega^{2n} = -1$)

This means that in the given series 7 brackets have value 4 each and the remaining 14 brackets have value 1 each. So, the sum of the series = 7 × 4 + 14 × 1 = 42.

∴ **Ans. (b)**

Example 9

If ω is a non-real cube root of unity then $(a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3$ is equal to

(a) $(a + b - c)(b + c - a)(c + a - b)$ (b) $(a - b - c)(b - c - a)(c - a - b)$

(c) $(2a - b - c)(2b - c - a)(2c - a - b)$ (d) None of these

Solution

We know that $A^3 + B^3 = (A + B)(A□ + B□^2)(A□^2 + B□)$

Substituting $A = a + b□ + c□^2$ and $B = a + b□^2 + c□$, we obtain

$(a + b□ + c□)^3 + (a + b□^2 + c□)^3$

$= (a + b□ + c□^2 + a + b□^2 + c□) \times (a□ + b□^2 + c□^3 + a□^2 + b□^4 + c□^3) \times (a□^2 + b□^3 + c□^4 + a□^4 + b□^3 + c□^2)$

$= (2a - b - c)(2c - a - b)(2b - a - c)$ (Q $□^4 = □^3□ = □$ and $□^2 + □ = -1$)

∴ **Ans. (c)**

Example 10

If 1, $□_1$, $□_2$, $□_3$, ……, $□_{n-1}$ are the *n*, *n*th roots of unity then

$(1 + □_1)(1 + □_2)(1 + □_3) \ldots\ldots (1 + □_{n-1})$ is equal to

(a) 1 (b) $\frac{1+(-1)^n}{2}$ (c) $\frac{1-(-1)^n}{2}$ (d) – 1

Solution

As 1, $□_1$, $□_2$, $□_3$, ……, $□_{n-1}$ are the *n*, *n*th roots of unity, therefore,

$x^n - 1 = (x - 1)(x - □_1)(x - □_2) \ldots\ldots (x - □_{n-1})$

□ $(x - □_1)(x - □_2) \ldots\ldots (x - □_{n-1}) = \frac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \ldots\ldots + x^2 + x + 1$

Put $x = -1$, to obtain

$(-1 - □_1)(-1 - □_2)(-1 - □_3) \ldots\ldots (-1 - □_{n-1})$

$= (-1)^{n-1}(1 + □_1)(1 + □_2) \ldots\ldots (1 + □_{n-1}) = \frac{(-1)^{n-1}\left(1-(-1)^n\right)}{1-(-1)}$

□ $(1 + □_1)(1 + □_2) \ldots\ldots (1 + □_{n-1}) = \frac{1-(-1)^n}{2}$

∴ **Ans. (c)**

Example 11

If a = cis □, b = cis □, c = cis □ and $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=1$, then cos (□ – □) + cos (□ – □) + cos (□ – □) =

(a) 3/2 (b) – 3/2 (c) 0 (d) 1

Solution

$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=1$

□ $\frac{\text{cis }\alpha}{\text{cis }\beta}+\frac{\text{cis }\beta}{\text{cis }\gamma}+\frac{\text{cis }\gamma}{\text{cis }\alpha}=1$

□ cis (□ – □) + cis (□ – □) + cis (□ – □) = 1

Equating real parts of both sides

□ cos (□ – □) + cos (□ – □) + cos (□ – □) = 1

∴ **Ans. (d)**

Example 12

If $| z - 2 - 2i | = 1$ then the minimum value of $| z |$ is

(a) $2\sqrt{2} - 1$ (b) $2\sqrt{2}$ (c) $2\sqrt{2} + 1$ (d) $2\sqrt{2} - 2$

Solution

Given $| z - 2 - 2i | = 1$

Now $| 2 + 2i | = | z - (z - 2 - 2i)| \le | z | + | z - 2 - 2i |$

$\Rightarrow \sqrt{2^2 + 2^2} \le | z| + 1$

$\Rightarrow | z| \ge 2\sqrt{2} - 1$

Hence minimum value of $| z |$ is $2\sqrt{2} - 1$

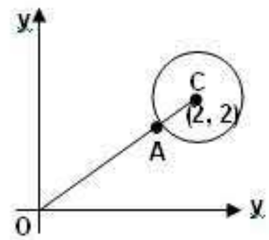
Alternatively,

$| z - (2 + 2i)| = 1$ is a circle with centre at C(2, 2) and radius 1. Let OC meet the circle at A

then minimum $| z | = | \text{OA} |$

$= | \text{OC} | - | \text{AC} |$

$= \sqrt{2^2 + 2^2} - 1 = 2\sqrt{2} - 1$



$\therefore$ **Ans. (a)**

Example 13

The point represented by the complex number $2 - i$ is rotated about origin through an angle of $\pi/2$ in clockwise direction. The new position of the point is

(a) $1 + 2i$ (b) $-1 - 2i$ (c) $2 + i$ (d) $-1 + 2i$

Solution

Let Amp $(2 - i) = \theta$, then $2 - i = \sqrt{2^2 + 1^2} \text{ cis } \theta = \sqrt{5} \text{ cis } \theta$

$\Rightarrow \sqrt{5} \cos \theta = 2$ and $\sqrt{5} \sin \theta = -1$

New number $= \sqrt{5} \text{ cis}\left(\theta - \frac{\pi}{2}\right)$ ($\because$ rotation is clockwise, amplitude is reduced by $\pi/2$)

$= \sqrt{5}\left(\cos\left(\theta - \frac{\pi}{2}\right) + i \sin\left(\theta - \frac{\pi}{2}\right)\right)$

$= \sqrt{5} (\sin \theta - i \cos \theta)$

$= \sqrt{5} \sin \theta - i\sqrt{5} \cos \theta = -1 - 2i$

Alternatively, the number is divided by i; so the new number $= \frac{2-i}{i} = -1 + \frac{2}{i} = -1 - 2i$

$\therefore$ **Ans. (b)**

Example 14

For all complex numbers z_1, z_2 satisfying $| z_1 | = 12$ and $| z_2 - 3 - 4i | = 5$, the minimum value of $| z_1 - z_2 |$ is

(1) 0 (2) 2 (3) 7 (4) 17

Solution

First, we note that

$$| z_2 | = |(z_2 - 3 - 4i) + (3 + 4i) | \le | z_2 - 3 - 4i | + | 3 + 4i | = 5 + 5 = 10$$

Hence $| z_1 - z_2 | \ge || z_1 | - | z_2 ||$

$$= | z_1 - z_2| \ge | 12 - 10| = 2 \qquad (\because | z_2 | \le 10 \Rightarrow - | z_2 | \ge - 10)$$

$\therefore$ **Ans. (b)**

Example 15

The locus of the centre of a circle which touches the circles $| z - z_1 | = a$ and $| z - z_2 | = b$ externally (z, z_1 and z_2 are complex numbers) will be

(a) an ellipse (b) a hyperbola (c) a circle (d) none of these

Solution

Let A (z_1), B (z_2) be the centres of given circle and P be the centre of the variable circle which touches given circles externally, then

$| AP | = a + r$ and $| BP | = b + r$, where r is the radius of the variable circle. On subtraction, we get

$$| AP | - | BP | = a - b$$

$\Rightarrow$ $|| AP || - | BP || = | a - b |$, a constant. Hence locus of P is

(i) right bisector of [AB] if $a = b$

(ii) a hyperbola if $| a - b | < | AB | = | z_2 - z_2 |$

(iii) an empty set if $| a - b | > | AB | = | z_2 - z_1 |$

(iv) set of all points on line AB except those which lie between A and B if $| a - b | = | AB | \ne 0$.

$\therefore$ **Ans. (d)**

SUBJECTIVE TYPE

Example 1

The value of

$$1+\sum_{k=0}^{14}\left\{\cos\frac{(2k+1)\pi}{15}+i\sin\frac{(2k+1)\pi}{15}\right\}$$ **is**

Solution

$$1+\sum_{k=0}^{14}\left\{\cos\frac{(2k+1)\pi}{15}+i\sin\frac{(2k+1)\pi}{15}\right\}$$

$$= 1+\sum_{k=0}^{14}e^{i\frac{(2k+1)\pi}{15}}$$

$$= 1+\sum_{k=0}^{14}\alpha^{2k+1} \qquad (\text{where } \alpha = e^{i\pi/15})$$

$$= 1 + (\alpha + \alpha^3 + \alpha^5 + \ldots\ldots\ \alpha^{29})$$

$$= 1+\frac{\alpha\left(1-(\alpha^2)^{15}\right)}{1-\alpha^2}$$

$$= 1+\frac{\alpha\left(1-\alpha^{30}\right)}{1-\alpha^2}$$

$$= 1+\frac{\alpha(1-1)}{(1-\alpha^2)} \qquad (\text{since } \alpha^{30} = e^{i2\pi} = 1)$$

$$= 1$$

Example 2

Find the maximum value of $|z|$ when z satisfies the condition $\left|z+\frac{2}{z}\right|=2$.

Solution

We have

$$|z| = \left|z+\frac{2}{z}-\frac{2}{z}\right|$$

$$|z| \le \left|z+\frac{2}{z}\right|+\left|\frac{-2}{z}\right|$$

$$|z| \le 2+\frac{2}{|z|}$$

$$|z|^2 - 2|z| - 2 \le 0$$

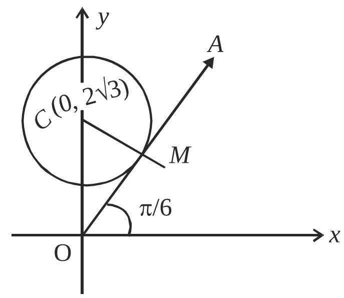
$$\Rightarrow \quad 1-\sqrt{3} \le |z| \le 1+\sqrt{3}$$

Hence max of | z | = 1 + $\sqrt{3}$

Example 3

Find the least value of p for which the two curves arg (z) = $\frac{\pi}{6}$ and $|z - 2\sqrt{3}\,i| = p$ intersects.

Solution



$|z - 2\sqrt{3}\,i| = p$ represents a circle of radius p having centre at $\left(0, 2\sqrt{3}\right)$ and arg (z) = $\frac{\pi}{6}$ is a line making an angle of 30° with OX and lying in first quadrant.

Let CM be perpendicular from C on OA. Then CM = OC sin □/3 = $2\sqrt{3} \times \frac{\sqrt{3}}{2} = 3$ Now, two curves will intersect if

CM □ p □ p □ 3

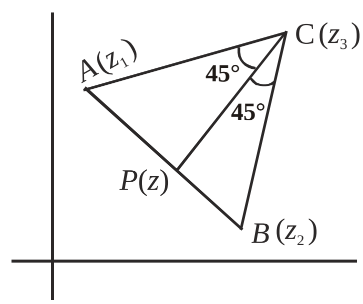
Hence least value of p is 3.

Example 4

A point z is equidistant from three distinct points z_1, z_2 and z_3 in the argand plane. If z, z_1 and z_2 are collinear then find arg $\left(\frac{z_3 - z_1}{z_3 - z_2}\right)$. ($z_1$, z_2, z_3 are in anticlockwise sense)

Solution

Let $P(z)$, $A(z_1)$, $B(z_2)$ & $C(z_3)$ be the given points. It is given that P, A and B are collinear such that $PA = PB$, therefore, P lies on the perpendicular bisector of AB.



$PA = PB = PC$

P is the mid point of AB (P is collinear to A and B)

□ACP = □BCP = 45°

□ACB = 90°

arg $\left(\frac{z_2 - z_2}{z_1 - z_3}\right) = \frac{\pi}{2}$

□ arg $\left(\frac{z_3 - z_1}{z_3 - z_2}\right) = -\frac{\pi}{2}$

Example 5

Find all non-zero complex numbers satisfying $\bar{Z} = iZ^2$.

Solution

Let $Z = x + iy, \bar{Z} = x - iy; Z^2 = x^2 - y^2 + 2ixy$

 The equation is $x - iy = i\,(x^2 - y^2 + 2ixy)$

 Equating real and imaginary parts

$x = -2xy$ ... (i)

$-y = x^2 - y^2$ ... (ii)

(i) gives either $x = 0$, in that case $y = 0$; $y = 1$

or $y = -\frac{1}{2}$, in that case $\frac{1}{4} + \frac{1}{2} = x^2$

 $x = \pm\frac{\sqrt{3}}{2}$

 The non-zero Z, satisfying the equation are

$$Z_1 = i; Z_2 = \frac{\sqrt{3}}{2} - \frac{1}{2}i; Z_3 = \frac{-\sqrt{3}}{2} - \frac{1}{2}i$$

Example 6

Find the complex number with least value of $|z|$ given that $|z - 2 + 2i| = 1$.

Solution

$|z - 2 + 2i| = 1$, hence $z - 2 + 2i = \cos\theta + i\sin\theta$

 $z = (2 + \cos\theta) + i(\sin\theta - 2)$

$$|z| = \sqrt{4 + 4\cos\theta + \cos^2\theta + 4 - 4\sin\theta + \sin^2\theta}$$

$$= \sqrt{9 + 4(\cos\theta - \sin\theta)}$$

$$= \sqrt{9 + 4\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)}$$

 Least $|z| = \sqrt{9 - 4\sqrt{2}} = \sqrt{8 + 1 - 2\sqrt{8}} = \sqrt{\left(\sqrt{8} - 1\right)^2} = \sqrt{8} - 1 = 2\sqrt{2} - 1$

and then $z = 2 - 2i - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i = \left(2 - \frac{1}{\sqrt{2}}\right) + i\left(\frac{1}{\sqrt{2}} - 2\right)$

Example 7

Find the complex numbers z which simultaneously satisfy the equations

$$\left|\frac{z - 12}{z - 8i}\right| = \frac{5}{3}, \left|\frac{z - 4}{z - 8}\right| = 1$$

Solution

Now $\left|\frac{z - 4}{z - 8}\right| = 1$ $\left|\frac{x - 4 + iy}{x - 8 + iy}\right| = 1$

□ $(x-4)^2 + y^2 = (x-8)^2 + y^2$ □ $x = 6$

With $x = 6$, $\left|\dfrac{z-12}{z-8i}\right| = \dfrac{5}{3}$ □ $\left|\dfrac{-6+iy}{6+i(y-8)}\right| = \dfrac{5}{3}$

□ $9(36 + y^2) = 25[36 + (y-8)^2]$ □ $y^2 - 25y + 136 = 0$

□ $y = 17, 8$

Hence the required numbers are $z = 6 + 17i$, $6 + 8i$.

*****

Exercise - I

OBJECTIVE TYPE QUESTIONS

Multiple Choice Questions With ONE Option Correct

1. $\left(\dfrac{i+\sqrt{3}}{2}\right)^{200} + \left(\dfrac{i-\sqrt{3}}{2}\right)^{200}$ is equal to

   (a) 1 (b) – 1 (c) 0 (d) None of these

2. The value of $\sum_{k=1}^{6}\left(\sin\dfrac{2k\pi}{7} - i\cos\dfrac{2k\pi}{7}\right)$ is

   (a) – 1 (b) 1 (c) 0 (d) i

3. If $|Z_1 + Z_2| = |Z_1| + |Z_2|$, then $\arg Z_1 - \arg Z_2$ is equal to

   (a) □ (b) $-\dfrac{\pi}{2}$ (c) $\dfrac{\pi}{2}$ (d) 0

4. If 1, □, $\square^2$ are the cube roots of unity, then the roots of $(x-1)^3 + 8 = 0$ are

   (a) $-1, 1 + 2\square, 1 + 2\square^2$ (b) $-1, 1 - 2\square, 1 - 2\square^2$ (c) $-1, -1, -1$ (d) None of these

5. The minimum value of $|Z| + |Z - 3|$ is

   (a) 0 (b) 3 (c) 1 (d) 2

6. $|Z + 4| > |Z + 2|$ represents the region given by

   (a) Re Z > 0 (b) Re Z > 3 (c) Re Z > – 3 (d) Re Z > 1

7. If $\dfrac{Z-1}{Z+1}$ is purely imaginary, then $|Z|$ is

   (a) equal to 1 (b) > 1 (c) < 1 (d) > 2

8. $\arg\left(-\frac{3}{2}\right)$ equals

(a) $\frac{\pi}{2}$ (b) $-\frac{\pi}{2}$ (c) 0 (d) None of these

9. If $x+\frac{1}{x}=1$, then $x^{2000}+\frac{1}{x^{2000}}$ is equal to

(a) 1 (b) – 1 (c) 0 (d) None of these

10. The equation $z\bar{z}+a\bar{z}+\bar{a}z+b=0$, b □ R, represents a circle, if

(a) $|a| > b$ (b) $|a|^2 > b$ (c) $|a|^2 < b$ (d) None of these

11. If the complex numbers z_1, z_2, z_3 are in A.P., then they lie on a

(a) circle (b) parabola (c) line (d) ellipse

12. If z_1, z_2, z_3 are complex numbers such that $|z_1| = |z_2| = 1$, then $|z_1 - z_2|^2 + |z_2 - z_3|^2 + |z_3 - z_1|^2$ can not exceed

(a) 6 (b) 9 (c) 12 (d) None of these

13. Dividing $f(z)$ by z – i, we obtain the remainder i and dividing it by $z + i$, we get the remainder $1 + i$. The remainder upon the division of $f(z)$ by $z^2 + 1$ is

(a) $\frac{1}{2}(z+1)+i$ (b) $\frac{1}{2}(iz+1)+i$ (c) $\frac{1}{2}(iz-1)+i$ (d) $\frac{1}{2}(z+i)+i$

14. Let z be a complex number satisfying $|z-3|$ □ $|z-5|$, $|z-i|$ □ $|z+i|$ and $|z-i|$ □ $|z-5i|$. The area of the region in which z can lie is

(a) 4 (b) 6 (c) 8 (d) None of these

15. z_1 and z_2 are any two distinct complex numbers in an argand plane. If □□ $|z_1|$ = □□ $|z_2|$, then the complex number $\frac{\alpha\beta z_1}{\gamma\delta z_2}+\frac{\gamma\delta z_2}{\alpha\beta z_1}$ lies on the

(a) Line segment [– 2, 2] on the real axis (b) Line segment [– 2, 2] on the imaginary axis

(c) Unit circle $|z| = 1$ (d) None of these

Multiple Choice Questions With MORE THAN ONE Option Correct

1. Roots of the equation $x^n - 1 = 0$, n □ i,

(a) are collinear (b)lie on a circle

(c) form a regular polygon of unit circum-radius (d)are non-collinear

2. The value of $169e^{i\left(\pi+\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{5}{13}\right)}$ is

(a) $119 - 120i$ (b) $-i(120 + 119i)$ (c) $119 + 120i$ (d) None of these

3. If $\left|\dfrac{z_1 z - z_2}{z_1 z + z_2}\right| = k$, ($z_1$, z_2 □ 0) then

(a) for $k = 1$, locus of z is a straight line

(b) for k □ {1, 0}, z lies on a circle

(c) for $k = 0$, z represents a point

(d) for $k = 1$, z lies on the perpendicular bisector of the line segment joining $\dfrac{z_2}{z_1}$ and $-\dfrac{z_2}{z_1}$

4. If from a point P representing the complex number z_1 on the curve $|z| = 2$, pair of tangents are drawn to the curve $|z| = 1$, meeting at point Q(z_2) and R(z_3), then

(a) Complex number $\dfrac{z_1 + z_2 + z_3}{3}$ will lie on the curve $|z| = 1$

(b) $\left(\dfrac{4}{\bar{z}_1} + \dfrac{1}{\bar{z}_2} + \dfrac{1}{\bar{z}_3}\right)\left(\dfrac{4}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3}\right) = 9$

(c) $\arg\left(\dfrac{z_2}{z_3}\right) = \dfrac{2\pi}{3}$

(d) Orthocentre and circumcentre of □PQR will coincide

5. If $f(x)$ and $g(x)$ are two polynomials such that $h(x) = xf(x^3) + x^2 g(x^6)$ is divisible by $x^2 + x + 1$, then

(a) $f(1) = g(1)$ (b) $f(1) = -g(1)$ (c) $f(1) = g(1)$ □ 0 (d) $f(1) = -g(1)$ □ 0

6. The complex numbers satisfying $|z + 2| + |z - 2| = 8$ and $|z - 6| + |z + 6| = 12$

(a) $4i$ (b) $-4i$ (c) 4 (d) -4

7. If the points z_1, z_2, z_3 are the affixes of vertices of an equilateral triangle, then

(a) $\dfrac{1}{z_1 - z_2} + \dfrac{1}{z_2 - z_3} + \dfrac{1}{z_3 - z_1} = 0$ (b) $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$

(c) $(z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_1)^2 = 0$ (d) $z_1^3 + z_2^3 + z_3^3 + 3z_1 z_2 z_3 = 0$

8. The inequality $\sin|z| > 0$ may represent

(a) A circle whose centre is origin and whose radius is □

(b) A parabola whose vertex is (0, 0)

(c) An annular region between two concentric circle centred at (0, 0) and having radii 2□ and 3□

(d) An ellipse of semi-axes □ and 2□

9. If z_1, z_2, z_3 are the affixes of vertices of an equilateral triangle and z_0 is the affix of the circumcentre, then

(a) $z_0 = z_1 + z_2 + z_3$

(b) $| z_0 - z_1 | = | z_0 - z_2 | = | z_0 - z_3 |$

(c) $z_0^2 = z_1z_2 + z_2z_3 + z_3z_1$

(d) $z_0 = \frac{z_1 + z_2 + z_3}{3}$

10. If $| z - 1 | + | z + 1 | = 4$, then

(a) Locus of z is ellipse and eccentricity of ellipse is 1/2

(b) Area bounded by figure is $2\pi\sqrt{3}$

(c) Locus of z represents a circle

(d) None of these

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Exercise - II

ASSERTION & REASON , COMREHENSION & MATCHING TYPE

Assertion and Reason

(a) Both A and R are true and R is the correct explanation of A.

(b) Both A and R are true but R is not the correct explanation of A

(c) A is true but R is false.

(d) A is false but R is true.

1. A : The locus of the centre of a circle which touches the circles $|z - z_1| = a$ and $|z - z_2| = b$ externally (z, z_1 and z_2 are complex numbers) will be hyperbola.

   R : $|z - z_1| - |z - z_2| < |z_2 - z_1|$ □ z lies on hyperbola.

2. A : If z_1 and z_2 are two complex numbers such that $|z_1| = |z_2| + |z_1 - z_2|$, then $\text{Im}\left(\frac{z_1}{z_2}\right) = 0$.

   R : arg $(z) = 0$ □ z is purely real.

3. A : consider an ellipse having its foci at A(z_1) and B(z_2) in the Argand plane. If the eccentricity of the ellipse be 'e' and it is known that origin is an interior point of the ellipse, then $e \in \left(-\frac{|z_1 + z_2|}{|z_1| + |z_2|}, \frac{|z_1 + z_2|}{|z_1| + |z_2|}\right)$.

   R : If z_0 is the point interior to curve $|z - z_1| + |z - z_2| =$ □

   □ $|z_0 - z_1| + |z_0 - z_2| <$ □

4. A : The equation $|z - i| + |z + i| = k$, $k > 0$, can represent an ellipse, if $k > 2i$.

   R : $|z - z_1| + |z - z_2| = k$, represents ellipse, if $|k| > |z_1 - z_2|$

5. A : The equation $|z - z_1| + |z - z_2| = k$, where $k > 0$ can represent a hyperbola, if $k < 2i$.

   R : $|z - z_1| - |z - z_2| = k$, represents a hyperbola, if $|k| < |z_1 - z_2|$

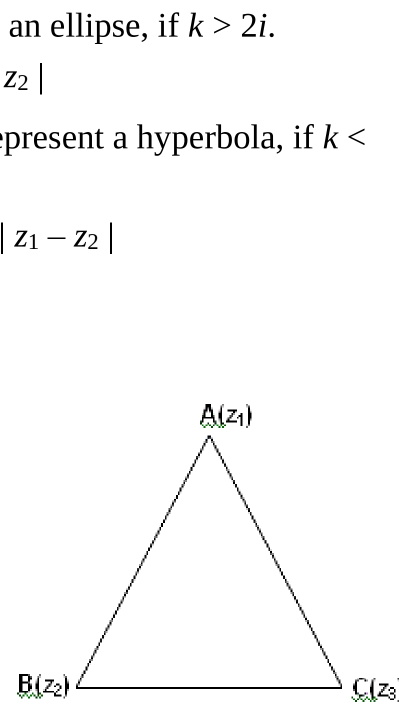
Passage Based Questions

Passage – I

Suppose z_1, z_2 and z_3 represent the vertices A, B and C of an equilateral triangle ABC on the Argand plane.

Then AB = BC = CA □ $|z_2 - z_1| = |z_3 - z_2| = |z_1 - z_3|$

Also, $\angle CAB = \frac{\pi}{3} \Rightarrow \arg\frac{z_3 - z_1}{z_2 - z_1} = \pm\frac{\pi}{3}$



□ $$\frac{z_3 - z_1}{z_2 - z_1} = \left|\frac{z_3 - z_1}{z_2 - z_1}\right| \left\{\cos\left(\pm\frac{\pi}{3}\right) + i\sin\left(\pm\frac{\pi}{3}\right)\right\} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

□ $$\frac{z_3 - z_1}{z_2 - z_1} - \frac{1}{2} = \pm\frac{\sqrt{3}}{2} i \Rightarrow \frac{2z_3 - z_1 - z_2}{2(z_2 - z_1)} = \pm\frac{\sqrt{3}}{2} i$$

On squaring, we get $(2z_3 - z_1 - z_2)^2 = -3(z_2 - z_1)^2$

1. If the complex numbers z_1, z_2, z_3 represent the vertices of an equilateral triangle such that $|z_1| = |z_2| = |z_3|$, then $z_1 + z_2 + z_3$ is equal to

   (a) 0 (b) 3 (c) □ (d) □^{2}

2. If a and b are two real numbers lying between 0 and 1 such that $z_1 = a + i$, $z_2 = 1 + bi$ and $z_3 = 0$ form an equilateral triangle, then

   (a) $a = 2 + \sqrt{3}$ (b) $b = 4 - \sqrt{3}$ (c) $a = b = 2 - \sqrt{3}$ (d) $a = 2$, $b = \sqrt{3}$

3. Let the complex numbers z_1, z_2 and z_3 be the vertices of an equilateral triangle. Let z_0 be the circumcentre of the triangle, then $z_1^2 + z_2^2 + z_3^2$ is equal to

   (a) z_0^2 (b) $3z_0^2$ (c) $9z_0^2$ (d) 0

4. If the complex numbers z_1, z_2 and z_3 represent the vertices of an equilateral triangle inscribed in the circle $|z| = 2$. If $z_1 = 1 + i\sqrt{3}$, then

   (a) $z_1 = 1$, $z_3 = 1 - i\sqrt{3}$ (b) $z_2 = 1 - i\sqrt{3}$, $z_3 = -1 - i\sqrt{3}$

   (c) $z_2 = 1 - i\sqrt{3}$, $z_3 = -1 + i\sqrt{3}$ (d) $z_2 = -2$, $z_3 = 1 - i\sqrt{3}$

Passage – II

Let z_0 be the centre and r □ R be the radius, then for any point $P(z)$ on the circle we have

$$|z - z_0| = r$$

Squaring both sides $|z - z_0|^2 = r^2$

i.e., $(z - z_0)(\bar{z} - \bar{z}_0) = r^2$

So, $z\bar{z} - z\bar{z}_0 - z_0\bar{z} + z_0\bar{z}_0 = r^2$

□ $z\bar{z} - z\bar{z}_0 - z_0\bar{z} + (|z_0|^2 - r^2) = 0$ … (i)

Observing the equation (i) we can see that the equation of a circle in the Argand plane is in the form

$z\bar{z} + \bar{a}z + a\bar{z} + b = 0$ where b is purely real

The circle represented by equation (i) has centre at $-a$ and radius $= \sqrt{|a|^2 - b}$.

1. If z is non-real complex number lying on the circle $|z| = 1$ then $\tan\left(\frac{\arg(z)}{2}\right)$ is equal to

(a) $\frac{i(1+z)}{1-z}$ (b) $\frac{i(1-z)}{1+z}$ (c) $\frac{i^2(1-\bar{z})}{1-z}$ (d) None of these

2. The curve represented by Re $(1/z) = c$ (where c □ 0) is a/an

(a) Straight line (b) Ellipse (c) Parabola (d) Circle

3. If $z = (□ + 2) + i\left(4+\sqrt{25-\lambda^2}\right)$, then the locus of z is

(a) A circle (b) Point circle (c) Part of a circle (d) None of these

4. If $|z - 3 - 4i| - |z + 3 + 4i| = 0$, then z lies on

(a) An ellipse (b) A hyperbola

(c) Portion of a circle (d) Portion of a straight line

Matching Type Questions

1. If the vertices of a triangle are three complex numbers z_1, z_2 and z_3 lying on the unit circle $|z| = 1$, then

List I	List II
(A) If the triangle is equilateral	(P) $\lvert z_1 \rvert = \lvert z_2 \rvert = \lvert z_3 \rvert$
(B) If the triangle is isosceles with AB = AC	(Q) $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$
(C) If the triangle is scalene	(R) $z_1 + z_2 + z_3 = 0$
	(S). $z_1^2 = z_2 z_3$

(a) A-P, Q, R, S, B-P, R, S, C-P (b) A-P, Q, R, S, B-P, R, S, C-P, Q

(c) A-P, R, S, B-P, R, S, C-P (d) A-P, Q, R, S, B-P, R, Q, S, C-P

2. If x_i's are the nth roots of unity (x_i □ 1), then

List – I	List – II
(A) $x_1 + x_2 + \ldots\ldots + x_{n-1}$	(P) 0
(B) $x_1 x_2 \ldots\ldots\ldots x_{n-1}$ (n even)	(Q) n
(C) $(1 - x_1)(1 - x_2) \ldots\ldots\ldots (1 - x_{n-1})$	(R) 1

(a) A-P, B-R, P, C-Q (b) A-P, B-R, C-Q

(c) A-P, R, B-R, C-Q (d) A-P, Q, B-R, C-Q

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Exercise - III

SUBJECTIVE TYPE

1. Find the value of the expression $1(2 - □)(2 - □^2) + 2(3 - □)(3 - □^2) + \ldots\ldots + (n-1)(n - □)(n - □^2)$ where □ is an imaginary cube root of unity.
2. The points A, B, C represent the numbers z_1, z_2 and z_3 respectively and the angles of the triangle ABC at B and C are both $\frac{1}{2}$ (□ – □) then prove that
$$(z_3 - z_2)^2 = 4(z_3 - z_1)(z_1 - z_2)\sin^2\frac{\alpha}{2}$$
3. Let s denote the set of all complex numbers z satisfying the inequality $|z - 5i|$ □ 3. Find the complex numbers z in s having
   (i) least positive argument
   (ii) maximum positive argument
   (iii) least modulus
   (iv) maximum modulus
4. ABCD is a rhombus described in the clockwise direction in the Argand plane. If the vertices A, B, C and D are given by z_1, z_2, z_3, z_4 respectively and □CBA = 2□/3 then show that
$$2\sqrt{3}\, z_2 = \left(\sqrt{3} - i\right)z_1 + \left(\sqrt{3} + i\right)z_3$$
$$2\sqrt{3}\, z_4 = \left(\sqrt{3} + i\right)z_1 + \left(\sqrt{3} - i\right)z_3$$
5. Find the equation of the circle in complex form which touches the line $iz + \bar{z} + 1 + i = 0$ and for which the lines $(1 - i)z = (1 + i)\bar{z}$ and $(1 + i)z + (i - 1)\bar{z} - 4i = 0$ are the normals.
6. (i) Locate the complex number $z = x + iy$ for which
   (a) $\log_{1/2}|z - 2| > \log_{1/2}|z|$
   (b) $\log_{1/\sqrt{3}}\dfrac{|z|^2 - |z| + 1}{2 + |z|} > -2$

   (ii) If the equation $ax^2 + bx + c = 0$, $(0 < a < b < c)$ has complex roots z_1 & z_2 then show that $|z_1| > 1$, $|z_2| > 1$.
7. If z_1 and z_2 are two complex numbers such that $\left|\dfrac{z_1 - z_2}{z_1 + z_2}\right| = 1$, prove that $\dfrac{iz_1}{z_2} = k$, where k is a real number.
8. If Z is represented on the complex plane by a point on the circle $|Z - 1| = 1$, then prove that $\dfrac{Z - 2}{Z} = i\tan(\arg Z)$.
9. Let Z_1 and Z_2 be the roots of $Z^2 + pZ + q = 0$ where the coefficients p and q may be complex numbers. Let A and B represent Z_1 and Z_2 in the complex plane. If $\underline{|AOB} = \alpha \neq 0$ and OA = OB, where O is the origin, prove that $p^2 = 4q\cos^2\left(\dfrac{\alpha}{2}\right)$.
10. Assume that A_i $(i = 1, 2, 3, \ldots\ldots, n)$ are the vertices of a regular polygon of n sides inscribed in a circle of radius unity. Show that
   (i) $|A_1A_2|^2 + |A_1A_3|^2 + \ldots\ldots + |A_1A_n|^2 = 2n$ (ii) $|A_1A_2| \,.\, |A_1A_3| \ldots\ldots |A_1A_n| = n$.

Exercise - IV

IIT – JEE PROBLEMS

A. Fill in the Blanks

1. If the expression $\dfrac{\left[\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)-i\tan(x)\right]}{\left[1+2i\sin\left(\frac{x}{2}\right)\right]}$ is real, then the set of all possible values of x is ……………… .

2. For any two complex numbers z_1, z_2 and any real numbers a and b. $|\, az_1 - bz_2 \,|^2 + |\, bz_1 + az_2 \,|^2 =$ ………… .

3. If a and b are real numbers between 0 and 1 such that the points $z_1 = a + i$, $z_2 = 1 + bi$ and $z_3 = 0$ form an equilateral triangle, then $a =$ ………… and $b =$ …………… .

4. ABCD is a rhombus. Its diagonals AC and BD intersect at the point M and satisfy BD = 2AC. If the points D and M represent the complex numbers $1 + i$ and $2 - i$ respectively, then A represents the complex number ……………… or …………… .

5. Suppose z_1, z_2, z_3 are the vertices of an equilateral triangle inscribed in the circle $|\, z \,| = 2$. If $z_1 = 1 + i\sqrt{3}$, then $z_2 =$ ……………, $z_3 =$ …………… .

6. The value of the expression
$1(2 - □)\,(2 - □^2) + 2(3 - □)\,(3 - □^2) + \ldots\ldots\ldots + (n - 1)\,.\,(n - □)\,(n - □^2)$, where □ is an imaginary cube root of unity, is ……………… .

B. True / False

7. For complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we write z_1 □ z_2, if x_1 □ x_2 and y_1 □ y_2. Then for all complex numbers z with 1 □ z, we have $\dfrac{1-z}{1+z} \cap 0$.

8. If the complex numbers, z_1, z_2 and z_3 represent the vertices of an equilateral triangle such that $|\, z_1 \,| = |\, z_2 \,| = |\, z_3 \,|$, then $z_1 + z_2 + z_3 = 0$.

9. The cube roots of unity when represented on argand diagram form the vertices of an equilateral triangle.

C. Multiple Choice Questions with ONE Correct Answer

10. The smallest positive integer n for which $\left(\dfrac{1+i}{1-i}\right)^n = 1$, is

    (a) 8 (b) 16 (c) 12 (d) None of these

11. The complex numbers $z = x + iy$ which satisfy the equation $\left|\dfrac{z-5i}{z+5i}\right| = 1$, lie on

    (a) The x-axis (b) The straight line $y = 5$
    (c) A circle passing through the origin (d) None of these

12. If $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$, then

(a) Re (z) = 0 (c) Im (z) = 0

(c) Re (z) > 0, Im (z) > 0 (d) Re (z) > 0, Im (z) < 0

13. The inequality | z – 4 | < | z – 2 | represents the region given by

(a) Re (z) □ 0 (b) Re (z) < 0 (c) Re (z) > 0 (d)None of these

14. If $z = x + iy$ and $w = \frac{(1 - iz)}{(z - i)}$, then | w | = 1 implies that, in the complex plane

(a) z lies on the imaginary axis (b) z lies on the real axis

(c) z lies on the unit circle (d) None of these

15. The points z_1, z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram taken in order, if and only if

(a) $z_1 + z_4 = z_2 + z_3$ (b)$z_1 + z_3 = z_2 + z_4$

(c) $z_1 + z_2 = z_3 + z_4$ (d)None of these

16. If a, b, c and u, v, w are complex numbers representing the vertices of two triangle such that $c = (1 – r)\, a + rb$ and $w = (1 – r)\, u + rv$, where r is a complex number, then the two triangles

(a) have the same area (b) are similar

(c) are congruent (d) None of these

17. The value of $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7}\right)$ is

(a) – 1 (b) 0 (c) – i (d) i

18. If z_1 and z_2 are two non zero complex numbers such that $| z_1 + z_2 | = | z_1 | + | z_2 |$, then $\arg z_1 – \arg z_2$ is equal to

(a) – □ (b) $-\frac{\pi}{2}$ (c) 0 (d) $\frac{\pi}{2}$

19. The complex numbers $\sin x + i \cos 2x$ and $\cos x – i \sin 2x$ are conjugate to each other, for

(a) $x = n$□ (b) $x = 0$ (c)$x = (n + 1/2)$ □ (d)No value of x

20. If □ (□ 1) is a cube root of unity and (1 + □)^7 = A + B□, then A and B are respectively

(a) 0, 1 (b) 1, 1 (c) 1, 0 (d) – 1, 1

21. Let z and w be two non zero complex numbers such that | z | = | w | and arg z + arg w = □, then z equals

(a) w (b) – w (c) $\bar{w}$ (d) $-\bar{w}$

22. Let z and w be two complex numbers such that | z | □ 1, | w | □ 1 and $| z + iw | = | z – i\bar{w} | = 2$, then z equals

(a)1 or i (b)i or – i (c)1 or – 1 (d)i or – 1

23. For positive integers n_1, n_2 the value of expression

$(1+i)^{n_1}+\left(1+i^3\right)^{n_1}+\left(1+i^5\right)^{n_2}+\left(1+i^7\right)^{n_2}$, here $i=\sqrt{-1}$, is a real number, if and only if

(a) $n_1 = n_2 + 1$ (b) $n_1 = n_2 - 1$ (c) $n_1 = n_2$ (d) $n_1 > 0$, $n_2 > 0$

24. If □ is an imaginary cube root of unity, then (1 + □ – □^{2})^7 is equal to

(a) 128 □ (b) – 128 □ (c) 128 □^{2} (d) – 128 □^{2}

25. The value of sum $\sum_{n=1}^{13}\left(i^n+i^{n+1}\right)$, where $i=\sqrt{-1}$ equals

(a) i (b) $i - 1$ (c) $-I$ (d) 0

26. If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x+iy$, then

(a) $x = 3, y = 1$ (b) $x = 1, y = 1$ (c) $x = 0, y = 3$ (d) $x = 0, y = 0$

27. If $i=\sqrt{-1}$, then $4+5\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{334}+3\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{365}$ is equal to

(a) $1-i\sqrt{3}$ (b) $-1+i\sqrt{3}$ (c) $i\sqrt{3}$ (d) $-i\sqrt{3}$

28. If arg (z) < 0, then arg (– z) – arg (z) =

(a) □ (b) – □ (c) – □/2 (d) □/2

29. If z_1, z_2 and z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right| = 1$,

then $|z_1 + z_2 + z_3|$ is

(a) equal to 1 (b) less than 1 (c) greater than 3 (d) equal to 3

30. Let z_1 and z_2 be n^{th} roots of unity which subtend a right angle at the origin, then n must be of the form

(a) $4k + 1$ (b) $4k + 2$ (c) $4k + 3$ (d) $4k$

31. The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1-z_3}{z_2-z_3}=\frac{1-i\sqrt{3}}{2}$ are the vertices of a triangle which is

(a) of area zero (b) right-angled isosceles
(c) equilateral (d) obtuse-angle isosceles

32. Let $\omega=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$, then value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$ is

(a) 3 □ (b) 3 □ (□ – 1) (c) 3 □^{2} (d) 3 □ (1 – □)

33. For all complex numbers z_1, z_2 satisfying $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$, the minimum value of $|z_1 - z_2|$ is

(a) 0 (b) 2 (c) 7 (d) 17

34. If $|z| = 1$ and $w = \frac{z-1}{z+1}$ (where z □ – 1), then Re (w) is

(a) 0 (b) $\frac{-1}{|z+1|^2}$ (c) $\left|\frac{z}{z+1}\right| . \frac{1}{|z+1|^2}$ (d) $\frac{\sqrt{2}}{|z+1|^2}$

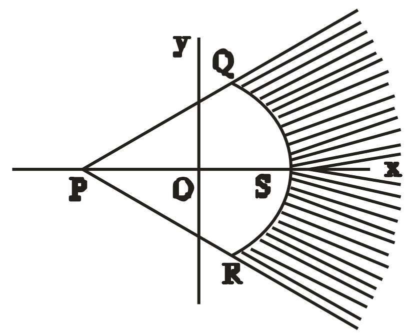
35. If □ (□ 1) be a cube root of unity and $(1 + □^2)^n = (1 + □^4)^n$, then the least positive value of n is

(a) 2 (b) 3 (c) 5 (d) 6

36. The minimum value of $|a + b□ + c□^2|$, where a, b and c are all not equal integers and □ (□ 1) is a cube root of unity, is

(a) $\sqrt{3}$ (b) $\frac{1}{2}$ (c) 1 (d) 0

37. The shaded region, where P = (– 1, 0), Q = $\left(-1+\sqrt{2}, \sqrt{2}\right)$, R = $\left(-1+\sqrt{2}, -\sqrt{2}\right)$, S = (1, 0) is represented by



(a) $|z + 1| > 2$, $|\arg (z + 1)| < \frac{\pi}{4}$

(b) $|z + 1| < 2$, $|\arg (z + 1)| < \frac{\pi}{2}$

(c) $|z – 1| > 2$, $|\arg (z – 1)| > \frac{\pi}{4}$

(d) $|z – 1| < 2$, $|\arg (z – 1)| > \frac{\pi}{2}$

38. If w = □ + i□, where □ □ 0 and z □ 1, satisfies the condition that $\left(\frac{w - \bar{w}z}{1-z}\right)$ is purely real, then the set of values of z is

(a) $|z| = 1$, z □ 2 (b) $|z| = 1$ and z □ 1 (c) $z = \bar{z}$ (d) None of these

39. A man walks a distance of 3 units from the origin towards the north-east (N 45° E) direction. From there, he walks a distance of 4 units towards the north-west (N 45° W) direction to reach a point P. Then the position of P in the Argand plane is

(a) $3e^{i□/4} + 4i$ (b) $(3 – 4i)\, e^{i□/4}$

(c) $(4 + 3i)\, e^{i□/4}$ (d) $(3 + 4i)\, e^{i□/4}$

40. If $|z| = 1$ and z □ ± 1, then all the values of $\frac{z}{1-z^2}$ lie on

(a) A line not passing through the origin (b) $|z| = \sqrt{2}$

(c) The x-axis (d) The y-axis

D. Multiple Choice Questions with ONE or MORE THAN ONE correct answer

41. If $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers such that $|z_1| = |z_2| = 1$ and Re $(z_1\bar{z}_2) = 0$, then the pair of complex numbers $w_1 = a + ic$ and $w_2 = b + id$ satisfies

(a) $|w_1| = 1$ (b) $|w_2| = 1$ (c) Re $(w_1\bar{w}_2) = 0$ (d) None of these

42. Let z_1 and z_2 be complex numbers such that z_1 □ z_2 and $|z_1| = |z_2|$. If z_1 has

positive real part and z_2 has negative imaginary part, then $\frac{z_1+z_2}{z_1-z_2}$ may be

(a) Zero (b) Real and positive

(c) Real and negative (d) Purely imaginary

E. Subjective Type Question

43. It is given that n is an odd integer greater than 3, but n is not a multiple of 3. Prove that $x^3 + x^2 + x$ is factor of $(x + 1)^n - x^n - 1\cdot$

44. Find the real values of x and y for which the following equation is satisfied:
$$\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i$$

45. Let the complex numbers z_1, z_2 and z_3 be the vertices of an equilateral triangle. Let z_0 be the circumcentre of the triangle. Then prove that $z_1^2+z_2^2+z_3^2=3z_0^2$

46. A relation R on the set of complex numbers is defined by z_1 R z_2, if and only if $\frac{z_1-z_2}{z_1+z_2}$ is real.
Show that R is an equivalence relation.

47. Prove that the complex numbers z_1, z_2 and the origin form an equilateral triangle only if $z_1^2+z_2^2-z_1z_2=0$.

48. If 1, a_1, a_2, ……, a_{n-1} are the n roots of unity, then show that
$$(1 - a_1)(1 - a_2)(1 - a_3) \ldots\ldots (1 - a_{n-1}) = n.$$

49. Show that the area of the triangle on the argand diagram formed by the complex number z, iz and $z + iz$ is $\frac{1}{2}|z|^2$.

50. Complex numbers z_1, z_2, z_3 are the vertices A, B, C respectively of an isosceles right angled triangle with right angle at C. Show that
$$(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$$

51. Let $z_1 = 10 + 6i$ and $z_2 = 4 + 6i$. If z is any complex number such that the argument of $\frac{(z-z_1)}{(z-z_2)}$ is $\frac{\pi}{4}$, then prove that $|z - 7 - 9i| = 3\sqrt{2}$.

52. If $iz^3 + z^2 - z + i = 0$, then show that $|z| = 1$.

53. $|z|$ □ 1, $|w|$ □ 1, show that $|z - w|^2$ □ $(|z| - |w|)^2 + (\arg z - \arg w)^2$.

54. Find all non-zero complex numbers z satisfying $\bar{z}=iz^2$.

55. Let z_1 and z_2 be roots of the equation $z^2 + pz + q = 0$, where the coefficients p and q may be complex numbers. Let A and B represent z_1 and z_2 in the complex plane. If □AOB = □ □ 0 and OA = OB, where O is the origin prove that $p^2 = 4q\cos^2\left(\frac{\alpha}{2}\right)$.

56. Let $\bar{b}z+b\bar{z}=c$, b □ 0, be a line in the complex plane, where $\bar{b}$ is the complex conjugate of b. If a point z_1 is the reflexion of the point z_2 through the line, then show that $c=\bar{z}_1b+z_2\bar{b}$.

57. For complex numbers z and w, prove that $|z|^2 w - |w|^2 z = z - w$, if and only if $z = w$ or $z\bar{w} = 1$.

58. Let a complex number □, □ □ 1, be a root of the equation $z^{p+q} - z^p - z^q + 1 = 0$. Where p, q are distinct primes. Show that either

1 + □ + □^{2} + …… + □$^{p-1}$ = 0 or 1 + □ + □^{2} + …… + □$^{q-1}$ = 0

but not both together.

59. If z_1 and z_2 are two complex numbers such that $|z_1| < 1 < |z_2|$, then prove that $\left|\frac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1$.

60. Prove that there exists no complex number z such that $|z| < \frac{1}{3}$ and $\sum_{r=1}^{n} a_r z^r = 1$, where $|a_r| < 2$.

61. Find the centre and radius of the circle formed by all the points represented by $z = x + iy$ satisfying the relation $\left|\frac{z-\alpha}{z-\beta}\right| = k$ (k □ 1), where □ and □ are constant complex numbers given by □ = □$_1$ + i□$_2$, □ = □$_1$ + i□$_2$.

62. If one of the vertices of the square circumscribing the circle $|z - 1| = \sqrt{2}$ is $2 + \sqrt{3}\,i$. Find the other vertices of square.

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Answers

Exercise - I

Only One Option is correct

Level - I

1.	(b)	2.	(d)	3.	(d)	4.	(b)	5.	(b)
6.	(c)	7.	(a)	8.	(d)	9.	(b)	10.	(b)
11.	(c)	12.	(b)	13.	(b)	14.	(b)	14.	(a)

More Than One Choice Correct

1.	(b, c, d)	2.	(a, b)	3.	(a, b, c, d)	4.	(a, b, c, d)	5.	(a, b)
6.	(a, b)	7.	(a, b, c)	8.	(a, c)	9.	(b, d)	10.	(a, b)

Exercise - II

Assertion and Reason

1.	(d)	2.	(a)	3.	(d)	4.	(d)	5.	(d)

Passage – I

1.	(a)	2.	(c)	3.	(b)	4.	(d)

Passage – II

1.	(b)	2.	(d)	3.	(c)	4.	(d)

Matching Type Questions

1.	(a)	2.	(b)

Exercise - III

Subjective Type

1. $(\frac{n}{4}\ (n-1)(n^2+3n+4))$

3. (i) $\frac{12}{5}+\frac{16}{5}i$ (ii) $-\frac{12}{5}+\frac{16}{5}i$ (iii) $2i$ (iv) $8i$

5. $(|z-(1+i)| = \frac{1}{\sqrt{2}})$

6. (i) All points on the right of $x > 1$ except the point (2, 0)

   (ii) All points inside the circle $|z| = 5$

Exercise - IV

IIT-JEE Level Problem

Section A

1. $x = 2n\square + 2\square$, $\square = \tan^{-1} k$, where $k \square (1, 2)$ or $x = 2n\square$

2. $(a^2 + b^2)(|z_1|^2 + |z_2|^2)$

3. $a = b = 2 \pm \sqrt{3}$

4. $3 - \frac{i}{2}$ or $1 - \frac{3i}{2}$

5. $z_2 = -2$, $z_3 = 1 - i\sqrt{3}$

6. $\frac{1}{4} n(n-1)(n^2 + 3n + 4)$

Section B

7.	True	8.	True	9.	True

Section C

10.	(d)	11.	(a)	12.	(b)	13.	(d)	14.	(b)
15.	(b)	16.	(b)	17.	(d)	18.	(c)	19.	(d)
20.	(b)	21.	(d)	22.	(c)	23.	(d)	24.	(d)
25.	(b)	26.	(d)	27.	(c)	28.	(a)	29.	(a)
30.	(d)	31.	(c)	32.	(b)	33.	(b)	34.	(a)
35.	(b)	36.	(c)	37.	(a)	38.	(b)	39.	(d)
40.	(d)								

Section D

41. (a, b, c,)   42. (a, d)

Section E

44. $x = 3$ and $y = -1$

54. $z = i, \pm \frac{\sqrt{3}}{2} - \frac{i}{2}$

61. Centre = $\frac{\alpha - k^2\beta}{1 - k^2}$, Radius = $\left|\frac{k(\alpha - \beta)}{1 - k^2}\right|$

62. $z_2 = -\sqrt{3}\, i$, $z_3 = \left(1 - \sqrt{3}\right) + i$ and $z_4 = \left(1 + \sqrt{3}\right) - i$

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