BINOMIAL THEOREM

Blaise Pascal

At the age of sixteen, Pascal presented a single piece of paper to one of Mersenne's meetings in June 1639. It contained a number of projective geometry theorems, including Pascal's mystic hexagon.

Although Pascal was not the first to study the Pascal triangle, his work on the topic in *Treatise on the Arithmetical Triangle* was the most important on this topic and, through the work of Wallis, Pascal's work on the binomial coefficients was to lead Newton to his discovery of the general binomial theorem for fractional and negative powers.

In correspondence with Fermat he laid the foundation for the theory of probability. This correspondence consisted of five letters and occurred in the summer of 1654. They considered the dice problem, already studied by Cardan, and the problem of points also considered by Cardan and, around the same time, Pacioli and Tartaglia. The dice problem asks how many times one must throw a pair of dice before one expects a double six while the problem of points asks how to divide the stakes if a game of dice is incomplete. They solved the problem of points for a two player game but did not develop powerful enough mathematical methods to solve it for three or more players.

Pascal died at the age of 39 in intense pain after a malignant growth in his stomach spread to the brain.

1.1 Binomial Expression

An expression consisting of two terms with positive or negative sing between them is called a binomial expression.

e.g. $(x+y), \left(x^2 - \frac{1}{x^{1/3}}\right)\left(\frac{p^2}{x^3} + \frac{q^{3/2}}{x^7}\right)(x^2 + 3y^{2/3})$

1.2 Binomial Expansion of $(x+y)^n$ using Pascal's Triangle

Consider the following expanded powers of $(x + y)^n$, where n is a whole number.

$(x + y)^0 =$ 1

$(x + y)^1 =$ $1x + 1y$

$(x + y)^2 =$ $1x^2 + 2xy + 1y^2$

$(x + y)^3 =$ $1x^3 + 3x^2y + 3xy^2 + 1y^3$

$(x + y)^4 =$ $1x4 + 3x3y + 6x^2y^2 + 4xy^3 + 1y^4$

$(x + y)^5 =$ $1x^5 + 5x4y + 10x^3y^2 + 10x^2y^3 + 5.xy^4 + 1.y^5$

Each expansion is a polynomial. There are patterns to be noted.

(*i*) There are $(n + 1)$ terms

(*ii*) The first is x^n and last is y^n

(*iii*) The powers of x decrease by 1 and y increase by 1 from left to right.

(*iv*) The sum of the exponents in each term is n.

We can generalize the expansion of $(x + y)^n$ in which coefficients is written is symbol ${}^nC_r$ form, where ${}^nC_r = \frac{n!}{r!.(n-r)!}$.

$(x + y)^n = {}^nC_0 \,.\, x^n + {}^nC_1 \,.\, x^{n-1} \,.\, y + {}_nC^2 \,.\, x^{n-2} \,.\, y^2 + \ldots.. + {}^nC_r \,.\, x^{n-r} \,.\, y^r + \ldots. + {}^nC_n \,.\, x^0 \,.\, y^n$

 $(x+y)^n = \sum_{n-1}^{n} {}^nC_r x^{n-r} y^r$

ILLUSTRATIONS

***Illustration* 1**

Expand and simplify the expansion $(2x^3 - \sqrt{5}x^{-1})^4$.

Solution

Using formula $(X + Y)^n$ with $n = 4$, $X = 2x^3$; $Y = -\sqrt{5}x^{-1}$

Now $(2x^3 - \sqrt{5}x^{-1})^4 = \left[2x^3 + \left(-\sqrt{5}x^{-1}\right)\right]^4 = \left[2x^3 + (-\sqrt{5}x^{-1}\right]^4$

$= (2x^3)^4 + 4.(2x^3)^3(-\sqrt{5}x^{-1})$

$+6(2x^3)^2(-\sqrt{5}x^{-1})^2 + 4(2x^3)(-\sqrt{5}x^{-1})^3 + (-\sqrt{5}x^{-1})^4$

$= 16x^{12} - 32\sqrt{5}x^8 + 120x^4 - 40\sqrt{5} + 25x^{-4}.$

 General term

In the expansion of $(x + y)^n$ the $(r + 1)$th term i.e. $t_{r+1} = {}^nC_r \,.\, x_{n-r}.\, y^r$ is called the general term of the expansion.

 ***p*th Term From End**

p^{th} term from the end in the expansion of $(x + y)^n = (n - p + 2)^{th}$ term from beginning.

ILLUSTRATIONS

***Illustration* 2**

Find the 7th term in the expansion of $\left(\frac{p^2}{x^3}+\frac{q^{3/2}}{x^7}\right)^{14}$.

Solution

$$t_7 = t_{6+1} = {}^{14}C_6 \cdot \left(\frac{p^2}{x^3}\right)^{14-6} \cdot \left(\frac{q^{3/2}}{x^7}\right)^6 = {}^{14}C_6 . p^{16} . q^9 . x^{-66}.$$

***Illustration* 3**

Find the 3rd term from the end in the expansion of $\left(x^{-2/3}-\frac{3}{x^2}\right)^8$

Solution

3rd term from the end = (8 – 3 + 2)th term from beginning = 7th term from beginning.

$$= T_7 = T_{6+1} = {}^8C_6\left(x^{-2/3}\right)^{8-6} . \left(\frac{-3}{x^2}\right)^2$$

$$= 252 . x^{-16/3}.$$

1.3 Some Useful Forms of Binomial Theorem

We know that :

$$(x+y)^n = {}^nC_0 . x^n + {}^nC_1 . x^{n-1} . y + ... + {}^nC_r x^{n-r} . y^r + ... + {}^nC_n . y^n \qquad \ldots(i)$$

Put – *y* in place of *y* in equation (i) then

$$(x-y)^n = {}^nC_0 x^n + {}^nC_1 . x^{n-1}(-y) + {}^nC_2 . x^{n-2}(-y)^2 + ... + {}^nC_1 x^{n-r}(-y)^r + ... + {}^nC_n . x^0 (-y)^n .$$

$$(x-y)^n = {}^nC_0 . x^n - {}^nC_1 . x^{n-1} y + {}^nC_2 . x^{n-2} . y^2 - .... + (-1)^r . {}^nC_r . x^{n-r} y^r + ... + (-1)^n \; {}^nC_n . x^0 . y^n \qquad \ldots(ii)$$

Now replacing *x* by 1 and *y* by *x* in equation (*i*)

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_1 x + {}^nC_2 x^2 + ... + {}^nC_n . x^n \qquad \ldots(iii)$$

Now replacing *x* by 1 and *y* by (–*x*) in equation (*i*)

$$(1-x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - ...... + (-1)^n \; {}^nC_n . x^n \qquad \ldots(iv)$$

Adding (i) and (ii)

$$(x+y)^n + (x-y)^n = 2\left[{}^xC_0 \; x^n y^0 + {}^nC_2 x^{n-2} y^2 + ...\right] \qquad \ldots(v)$$

Subtract (*i*) and (*ii*)

$$(x+y)^n - (x-y)^n = 2\left[{}^nC_1 x^{n-1} y^1 + {}^nC_3 x^{n-3} a^3 + ...\right] \qquad \ldots(vi)$$

Note:

(i) If n is odd, then eq. (i) and eq. (ii) both have same number of terms equal to $\frac{n+1}{2}$.

(ii) If n is even, then eq. (i) has $\left(\frac{n}{2}+1\right)$ terms and eq. (ii) has $\frac{n}{2}$ terms.

ILLUSTRATIONS

***Illustration* 4**

If the number of terms in $\left(x+1+\frac{1}{x}\right)^n$ $(n \in I^+)$ is 401, then n is greater than

(a) 201 (b) 200 (c) 199 (d) none of these

Solution

$$\left(x+1+\frac{1}{x}\right)^n = \frac{(1+x+x^2)^n}{x^n} \qquad \ldots\text{(i)}$$

$(1 + x + x^2)^n$ is of the from $a_0 + a_1\mathrm{x} + a_2x_2 + \ldots\ldots + a_{2n}\mathrm{x}^{2n}$, which contains $(2n + 1)$ terms.

So, $\left(x+1+\frac{1}{x}\right)^n$ contains $(2n + 1)$ terms

□ $2n + 1 = 401$ □ $n = 200$

***Illustration* 5**

The coefficient of the term independent of x in the expansion of $\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}$ is

(a) 70 (b)112 (c)105 (d)210

Solution

Given expression

$$= \frac{(x^{1/3})^3+(1)^3}{x^{1/3}-x^{1/3}+1}-\frac{(x-1)}{x^{1/2}(x^{1/2}-1)}$$

$$= \frac{(x^{1/3}+1)(x^{2/3}-x^{1/3}+1)}{(x^{1/2}-x^{1/3}+1)}-\frac{(x^{1/2}+1)(x^{1/2}-1)}{x^{1/2}(x^{1/2}-1)}$$

$$= (x^{1/3}+1)-(1+x^{-1/3}) = x^{1/3}-x^{-1/2}$$

□ $$\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10} = \left(x^{1/3}-x^{-1/2}\right)^{10}$$

T_{r+1} in $\left(x^{1/3} - x^{-1/2}\right)^{10}$ is $^{10}C_r \left(x^{1/3}\right)^{10-r}.(-1)^r.\left(x^{-1/2}\right)^r$

$= (-1)^r\ ^{10}C_r x^{\left(\frac{10-r}{3}-\frac{r}{2}\right)}$ which is independant of x if $\frac{10-r}{3} = \frac{r}{2} = 0$

⇒ $20 = 5r$ ⇒ $r = 4$

Hence required coefficient $= {}^{10}C_4(-1)^4 = 210.$

⇒ (d) correct.

***Illustration* 6**

If 2nd, 3rd and 4th terms in the expansion of $(x + y)^n$ be 240, 720 and 1080, then n is

(a) 4 (b) 5 (c) 6 (d) 7

Solution

2nd term : $^nC_1 . x^{n-1}.y^1 = 240$ …(1)

3rd term : $^nC_2 . x^{n-2}.y^2 = 720$ …(2)

4th term : $^nC_3 . x^{n-3}.y^3 = 1080$ …(3)

$$\frac{(1)}{(2)} : \frac{n}{\frac{n(n-1)}{2}}.\frac{x}{y} = \frac{240}{720} \Rightarrow \frac{2}{n-1}.\frac{x}{y} = \frac{1}{3}$$

$$\frac{(2)}{(3)} : \frac{\frac{n(n-1)}{2}}{\frac{n(n-1)(n-2)}{6}}.\frac{x}{y} = \frac{720}{1080} \Rightarrow \frac{3}{n-2}.\frac{x}{y} = \frac{2}{3}$$

$$\frac{(1)/(2)}{(2)/(3)} : \frac{2}{n-1} \times \frac{n-2}{3} = \frac{1}{2} \Rightarrow 4n - 8 = 3n - 3 \Rightarrow n = 5$$

1.4 Middle Terms

Since the binomial expansion $(x + y)^n$ contains $(n + 1)$ terms. So middle term depends upon the value of n

Nature of *n*	Number of middle term	………….th term	Middle term
Even	1	$\left(\frac{n}{2}+1\right)^{th}$	$^nC_{\frac{n}{2}+1}.x^{n/2}.y^{n/2}$
Odd	2	$\left(\frac{n+1}{1}\right)^{\text{th}}$ and $\left(\frac{n+3}{2}\right)^{\text{th}}$	$^nC_{\frac{n-1}{2}}.x^{\frac{n-1}{2}}.y^{\frac{n+1}{2}}$ and $^nC_{\frac{n+1}{2}}.x^{\frac{n+1}{2}}.y^{\frac{n-1}{2}}$

ILLUSTRATIONS

***Illustration* 7**

If in the expansion of $\left(2a-\frac{a^1}{4}\right)^9$; the sum of middle terms is S, then which of the following is/are true

(a) $S=\left(\frac{63}{32}\right)a^{14}(a+8)$ (b) $S=\left(\frac{63}{32}\right)a^{14}(a-8)$ (c) $S=\left(\frac{63}{32}\right)a^{13}(a-8)$ (d) $S=\left(\frac{63}{32}\right)a^{13}(8-a)$

Solution

n is odd, so there exist two middle terms and the middle terms are 5th and 6th term.

So, $S={}^9C_4.(2a)^5\left(\frac{-a^2}{4}\right)^4+{}^9C_5(2a)^4.\left(\frac{-a^2}{4}\right)^5$

$$=\frac{9!}{4!.5!}(2a)^4\left(\frac{-a^2}{4}\right)^4\left[2a-\frac{a^2}{4}\right]=\frac{63}{32}a^{13}(8-a)$$

□ (d) is correct.

□ **Greatest Term**

If T_r and T_{r+1} be the *r*th and (*r* + 1)th terms in the expansion of $(1+x)^n$, then

$$\frac{T_{r+1}}{T_r}=\frac{{}^nC_r\ x^r}{{}^nC_{r-1}x^{r-1}}=\frac{n-r+1}{r}x$$

Let numerically, T_{r+1} be the greatest term in the above expansion. Then T_{r+1} □ Tr

or $\frac{T_{r+1}}{T_r}\geq 1$

□ $\frac{n-r+1}{r}|\ x|\ \geq 1$

or $r\leq\frac{(n+1)}{(1+|\ x|\ )}|\ x|$ …(i)

Now substituting values of *n* and *x* in (*i*), we get

$r\leq m+f$ or $r\leq m$

where *m* is a positive integer and *f* is a fraction such that $0<f<1$.

In the first cast T_{m+1} is the greatest term, while in the second case T_m and T_{m+1} are the greatest terms and both are equal.

Short Cut Method: To find the greatest term (numerically) in the expansion of $(1+x)^n$.

(a) Calculate $m=\frac{|\ x|\ (n+1)}{|\ x|\ +1}$

(b) If m is integer, then T_m and T_{m+1} are equal and both are greatest term.

(c) If m is not integer, there $T_{[m]+1}$ is the greatest term, where [.] denotes the greatest integral part.

ILLUSTRATIONS

***Illustration* 8**

Find numerically the greatest term in the expansion of (3 – 5x)11 when $x = \frac{1}{5}$.

Solution

Since $(3-5x)^{11} = 3^{11}\left(1-\frac{5x}{3}\right)^{11} = 3^{11}\left(1-\frac{1}{3}\right)^{11}$ $\left(\because \; x = \frac{1}{5}\right)$

Now, calculate $m = \frac{|x| \; (n+1)}{(|x| + 1)}$ $\left(-\frac{1}{3} < 0\right)$

$$= \frac{\left(\left|-\frac{1}{3}\right|\right)(11+1)}{\left(\left|-\frac{1}{3}\right|+1\right)} = 3$$

∴ The greatest terms in the expansion are T_3 and T_4

∴ Greatest term (when r = 2) = 3^{11} $|T_{2+1}|$

$$= 3^{11}\left|{}^{11}C_2\left(-\frac{1}{3}\right)^2\right|$$

$$= 3^{11}\left|\frac{11.10}{1.2}\times\frac{1}{9}\right| = 55\times 3^9$$

and greatest term (when r = 3) = 311 $|T_{3+1}|$

$$= 3^{11}\left|{}^{11}C_3\left(-\frac{1}{3}\right)^3\right|$$

$$= 3^{11}\left|\frac{11.10.9}{1.2.3}\times\frac{-1}{27}\right| = 55 \times 3^9$$

From above we say that the values of both greatest terms are equal.

1.5 Greatest Coefficient

(a) If n is even, then greatest coefficient = ${}^nC_{n/2}$

(b) If n is odd, then greatest coefficients are ${}^nC_{(n-1)/2}$ and ${}^nC_{(n+1)/2}$.

ILLUSTRATIONS

***Illustration* 9**

Show that if the greatest term in the expansion of $(1 + x)^{2n}$ has also the greatest coefficient then x lies between $\frac{n}{n+1}$ and $\frac{n+1}{n}$.

Solution

In the expansion of (1 + x)2n, the middle term is $\left(\frac{2n}{2}+1\right)$ th i.e. (n + 1)th term, we know that from binomial expansion, middle term has greatest coefficients. (Terms T_1, T_2, T_3, …., T_n, T_{n+1}, T_{n+2}, …)

□ $T_n < T_{n+1} > T_{n+2}$

If $T_n < T_{n+1}$

$T_{n+1} > T_n$

□ $\frac{T_{n+1}}{T_n} > 1 \quad \left(\because \frac{T_{r+1}}{T_r} = \frac{n-r=1}{r}x, \text{ here } n = 2n\right)$

□ $\frac{2n-n+1}{n}x > 1$

□ $x > \frac{n}{n+1}$ …(i)

and if $T_{n+1} > T_{n+2}$

□ $\frac{T_{n+2}}{T_{n+1}} < 1$

□ $\frac{2n-(n+1)+1}{n+1}x < 1$

□ $x < \frac{n+1}{n}$ …(ii)

From (i) and (ii), we get

$$\frac{n}{n+1} < x < \frac{n+1}{n}$$

□ **Important Result**

If $\left(\sqrt{p}+q\right)^n = I + f$, where I and n are positive integers, n being odd, and $0 \le f < 1$, If $0 < \sqrt{p} - q < 1 \quad 0 < (\sqrt{p}-q)^n < 1$

1. Now let, $\left(\sqrt{p}-q\right)^n = f'$ where $0 < f' < 1$

$I + f - f' = (\sqrt{p}+q)^n - (\sqrt{p}-q)^n$

RHS contains even powers of $\sqrt{p}$ because n is odd. Hence RHS and I are integers so $f - f'$ is also integer

□ $f - f' = 0$ since $-1 < f - f' < 1$

□ $f = f'$

$(I + f)f = (I + f)f' = \left(\sqrt{p} + q\right)^n \left(\sqrt{p} - q\right)^n$

$= (p - q^2)^n$

2. If n is even integer then $\left(\sqrt{p} - q\right)^n + \left(\sqrt{p} - q\right)^n = I + f + f'$. Here LHS and I are integers

$f + f'$ is also integer

□ $f + f' = 1$ since $0 < f + f' < 2$

□ $(I + f)\,(I - f) = (1 + f)f' = \left(\sqrt{p} + q\right)^n \left(\sqrt{p} - q\right)^n = (p - q^2)^n$

ILLUSTRATIONS

Illustration **10**

Let $R = (5\sqrt{5} + 11)^{2n+1}$ and $f = R - [R]$, where [.] denotes the greatest integer function. Prove that $Rf = 4^{2n+1}$.

Solution

Greatest integer function is defined as follow:

$[x]$ = integral part of x

∴ $f = R - [R]$ implies that f is the fraction part of R.

∴ $0 < f < 1$

Since 144 > 125 > 121, $\sqrt{125} = 5\sqrt{5}$` lies between 11 and 12.

∴ $0 < 5\sqrt{5} - 11 < 1$ and hence $(5\sqrt{5} - 11)^{2n+1}$ will also be a proper fraction

Let $g = (5\sqrt{5} - 11)^{2n+1}$

Now $[R] + f - g = R - g$

$= (5\sqrt{5} + 11)^{2n+1} - (5\sqrt{5} - 11)^{2n+1}$

$= 2\left\{ {}^{(2n+1)}C_1\, (5\sqrt{5})^{2n} .\, 11^1 + {}^{(2n+1)}C_3 (5\sqrt{5})^{2n-2} . 11^2 + \ldots \right\}$

= an even integer

Since $[R]$ is an integer, the above implies $f - g = 0$

Hence $Rf = Rg = (5\sqrt{5} + 11)^{2n+1} . (5\sqrt{5} - 11)^{2n+1}$

$= (125 - 121)^{2n+1} = 4^{2n+1}$.

1.6 SUM OF THE COEFFICIENT OF A BINOMIAL EXPANSION

If $(x + y)^n = {}^nC_0 \,.\, x^n \,.\, y^0 + {}^nC_1, x^{n-1}\, y + \ldots. + {}^nC_n \,.\, x_0 y^n$ …(1)

To find ${}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n$, we put $x = y = 1$ in equation (1) to get

□ $(1 + 1)^n = {}_nC^0 + {}_nC^1 + {}^nC_2 + \ldots\ldots.. + {}^nC_n$

□ ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots\ldots. + {}^nC_n = 2^n$

□ For finding sum of the coefficient of $(□_1 a + □_2 b + □_3 c + \ldots.. a_{26} z)^n$ put $a = b = c = \ldots.. = z = 1$.

Consider the following standard expansions

$(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \ldots\ldots\ldots. + C_n x^n$ …(E1)

$(1 + x + x^2)^n = a_0 + a_1 x + a_2 x^2 + \ldots\ldots\ldots. + a_n x^n$ …(E2)

$(3 + 4x)^n = p_0 + p_1 x + p_2 x^2 + \ldots\ldots\ldots. + p_n x^n$ …(E3)

Now, we shall discuss some techniques of manipulating these expansions to obtain identities involving the coefficients of these expansions.

1.7 MULTINOMIAL THEOREM (FOR A POSITIVE INTEGRAL INDEX)

If n is a positive integer and $a_1, a_2, a_3, \ldots., a_m \in C$ then

$$(a_1 + a_2 + a_3 + ... + a_m)^n = \sum \frac{n!}{n_1!\,n_2!\,n_3!\ldots n_m!} a_1^{n1} a_2^{n2} a_3^{n3} \ldots a_m^{nm}$$

where $n_1, n_2, n_3, \ldots.. n_m$ are all non-negative integers subject to the condition

$$n_1 + n_2 + n_3 + \ldots + n_m = n$$

Note:

The coefficient of $a_1^{n1} a_1^{n2} a_3^{n3} \ldots a_m^{nm}$ in the expansion of $(a^1 + a^2 + a_3 + \ldots + a_m)^n$ is

$$\frac{n!}{n_1!\,n_2!\,n_3!\ldots n_m!}$$

ILLUSTRATIONS

***Illustration* 11**

The sum of the coefficient of the expansion $(ax + by + 3z)^9$, is

Solution

Put $x = y = z = 1$

□ sum of coefficient

$= (a \,.\, 1 + b \,.\, 1 + 3.\, 1)^9$

$= (a + b + 3)^9$

Illustration **12**

Find the coefficient of $a^4b^3c^2d$ in the expansion of $(a - b + c - d)^{10}$.

Solution

The coefficient of $a^4b^3c^2d$ in the expansion of $(a - b + c - d)^{10}$ is

$(-1)^4 \dfrac{10!}{4!3!2!1!}$ [Q powers of b and d are 3 and 1 $(-1)^3$ (-1)}

$= 12600.$

Illustration **13**

Find the coefficient of $a^3b^4c^5$ in the expansion of $(bc + ca + ab)^6$.

Solution

In this case write $a^3b^4c^5 = (ab)^x\ (bc)^y\ (ca)^z$ say

□ $a^3b^4c^5 = a^{z+x}\ .\ b^{x+y}\ .\ c^{y+z}$

□ $z + x = 3,\ x + y = 4,\ y + z = 5$

adding all $2\,(x + y + z) = 12$

□ $x + y + z = 6$

Then $x = 1,\ y = 3,\ z = 2$ therefore

The coefficient of $a^3b^4c^5$ in the expansion of $(bc + ca + ab)^6$

or the coefficient of $(ab)^1\ (bc)^3\ (ca)^2$ in the expansion of $(bc + ca + ab)^6$ is

$$\frac{6!}{1!3!2!} = 60$$

Note:

The greatest coefficient in the expansion of $(a^1 + a^2 + a^3 + \ldots a_m)^n$ is $\dfrac{n!}{(q!)^{m-r}((q+1)!)^r}$

where q is the quotient and r is the remainder when n is divided by m.

Illustration **14**

Find the greatest coefficient in the expansion of $(a + b + c + d)^{15}$.

Solution

Here $n = 15$ and m = 4 (Q a, b, c, d four terms)

or 4) 15 (3 □ $q = 3$ and $r = 3$

12

3

Hence greatest coefficient $= \dfrac{15!}{(3!)^1(4!)^3}$

Note:

If n is a positive integer and $a_1, a_2, a_3, \ldots a_m \in C$ then coefficient of x^r in the expansion of $(a_1 + a_2 x + a_3 x^2 + \ldots + a_m x^{m-1})^n$ is

$$\sum \frac{n!}{n_1!n_2!n_3!\ldots n_m!} a_1^{n_1} a_2^{n_2} a_3^{n_3} \ldots a_m^{n_m}$$

where $n_1, n_2, n_3, \ldots, n_m$ are all non negative integers subject to the conditions

$n_1 + n_2 + n_3 + \ldots + n_m = n$ and $n_2 + 2n_3 + 3n_4 + \ldots (m-1)\, n_m = r$

***Illustration* 15**

Find the coefficient of x^7 in the expansion of $(1 + 3x - 2x^3)^{10}$.

Solution

Coefficient of x^7 in the expansion of $(1 + 3x - 2x^3)^{10}$ is $\sum \frac{10!}{n!n_2!n_3!}(1)^{n_1}(3)^{n_2}(-2)^{n_3}$

where $n_1 + n_2 + n_3 = 10$ and $n_2 + 3n_3 = 7$, the possible values of n_1, n_2 and n_3 are shown in margin.

n_1	n_2	n_3
3	7	0
5	4	1
7	1	2

 The coefficient of x^7.

$$= \frac{10!}{3!7!0!}(1)^3(3)^7(-2)^0 + \frac{10!}{5!4!1!}(1)^5(3)^4(-2)^1 + \frac{10!}{7!1!2!}(1)^7(3)^1(-2)^2$$

$= 262440 - 204120 + 4320 = 62640$.

Note: The number of distinct or dissimilar terms in the multinomial expansion $(a_1 + a_2 + a_3 + \ldots + a_m)^n$ is ${}^{n+m-1}C_{m-1}$.

***Illustration* 16**

Find the total number of terms in the expansion of $(x + y + z + w)^n$, $n \in N$.

Solution

The number of terms in the expansion of $(x + y + z + w)^n$ is ${}^{n+4-1}C_{4-1}$

$$= {}^{n+3}C_3$$

$$= \frac{(n+3)(n+2)(n+1)}{6}.$$

1.8 Use of Differentiation

This method applied only when the numerical occur as the product of the binomial coefficients.

Solution Process:

(*i*) If last term of the series leaving the plus or minus sign be *m*, then divide the *m* by *n* if *q* be the quotient and *r* be the remainder.

i.e., $m = nq + r$

then replace *x* by x^q in the given series and multiplying both sides of the expression by x^r.

(*ii*) After process (*i*), differentiate both sides w.r. to *x* and put $x = 1$ or -1 or *i* or etc.

(*iii*) If product of two numericals (or square of numericals) or three numericals (or cube of numericals) then differentiate twice or thrice.

ILLUSTRATIONS

***Illustration* 17**

If $(1 + x)^n = C_0 + C_1 x + C_2x^2 + \dots + C_nx^n$ then prove that

$$C_0 + 3C_1 + 5C_2 + \dots + (2n+1)C_n = (n+1)2^n .$$

Solution

Here last term of $C_0 + 3C_1 + 5C_2 + \dots + (2n + 1)C_n$ is $(2n + 1)\ C_n$ i.e. $(2n + 1)$ and last term with positive sign, then $2n + 1 = n.2 + 1$

or $n)\ 2n + 1(2$

$\underline{-\ 2n}$

$\underline{1}$

Here $q = 2$ and $r = 1$

and the given series is $(1 + x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$

Now replacing *x* by x^2 then $(1 + x^2)^n = C_0 + C_1x^2 + C_2x^4 + \dots + C_nx^{2n}$

multiplying both sides by x^1, we get $x\ (1 + x^2)^n = C_0x + C_1x^3 + C_2x^5 + \dots + C_nx2^{n+1}$

then differentiating both sides w.r. to *x*, we get

$$x\ .\ n(1 + x^2)^{n-1}\ .\ 2x + (1 + x^2)^n.1 = C_0 + 3C_1x^2 + 5C_2x^4 + \dots + (2n + 1)\ C_nx^{2n}$$

Putting $x = 1$, then we get

$$n.2^{n-1}\ .\ 2 + 2^n = C_0 + 3C_1 + 5C_2 + \dots + (2n + 1)\ C_n$$

or $C_0 + 3C_1 + 5C^2 + \dots + (2n + 1)\ C_n = (n + 1)\ 2^n$.

***Illustration* 18**

If $(1 + x)^n = C_0 + C_1 x + C_2x^2 + \ldots + C_nx^n$ then prove that

$1^2.C_1 + 2^2.C_2 + 3^2.C_3 + \ldots + n^2.C_n = n(n + 1) . 2^{n-2}$

Solution

Here last term of $1^2.C_1 + 2^2.C_2 + 3^2.C_3 + \ldots. + n^2.C_n$ is n^2C_n i.e., n^2 linear factor of n^2 are n and n; (start always with greater factor) and last term with positive sign, then

Here $q = 1$ and $r = 0$ and the given series is $(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots. + C_nx^n$

Differentiating both sides w.r. to x, we get

$$n(1 + x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \ldots + nC_n x^{n-1}$$

and in last term numerical is $n C_n$ i.e., n and power of x is $n - 1$.

Here $q = 1$ and $r = 1$

Now multiplying both sides by x in (i) then

$$nx(1 + x)^{n-1} = C_1 x + 2C_2x^2 + 3C_3x^3 + \ldots + nC_nx^n$$

Differentiating both sides by w.r to x, we get

$$n\{x. (n - 1)(1 + x)^{n-2} + (1 + x)^{x-1} .1\}$$

$$= C_1.1 + 2^2 C_2 x + 3^2 C_3 x^2 + \ldots + n^2 C_n x^{n-1}$$

putting $x = 1$, then

$$n\{1. (n - 1) . 2^{n-2} + 2^{n-1}\} = 1^2.C_1 + 2^2.C_2 + 3^2.C_3 + \ldots + n^2.C_n$$

or $\quad 1^2.C_1 + 2^2.C_2 + 3^2.C^3 + \ldots. + n^2.C_n = n(n + 1)2^{n-2}$.

***Illustration* 19**

If $(1 + x)^n = C_0 + C_1 x + C_2x^2 + \ldots + C_nx^n$ then prove that

$(1.2) C_2 + (2.3)C_3 + \ldots ((n - 1) . n) C_n = n(n - 1) 2^{n-2}$

Solution

Here last term of $(1.2) C_2 + (2.3) C_3 + \ldots + ((n - 1).n) C_n$ is $(n - 1) nC_n$ i.e., $(n - 1) n$ (start with greater factor here greater factor n) and last term with positive sign, then $n = n . 1 + 0$

Here $q = 1$ and $r = 0$

and the given series is $(1 + x)^n = C_0 + C_1x + C_2x^2 + C^3x_3 + \ldots + C_nx^n$

Differentiating both sides w.r. to x, we get

$$n(1+x)^{n-1} = 0 + C_1 + 2C_2x + 3C_3x^2 + \ldots + nC_nx^{n-1}$$

again differentiating both sides w.r to x, we get

$$n(n - 1)(1 + x)^{n-2} = 0 + 0 + (1.2) C_2 + (2.3) C_3x + \ldots + ((n - 1). n) C_nx^{n-2}.$$

Putting $x = 1$, then $n(n-1)(1+1)^{n-2} = (1.2)C_2 + (2.3)C_3 + \ldots + (n-1)nC_n$

or $\quad (1.2)C_2 + (2.3)C_3 + \ldots + (n-1)n \cdot C_n = n(n-1)2^{n-2}$.

***Illustration* 20**

If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$ then prove that

$$C_0 + 3C_1 + 5C_2 - \ldots + (-1)^n (2n+1) C_n = 0 .$$

Solution

The numerical value of last term of $C_0 - 3C_1 + 5C_2 - \ldots + (-1)^n (2n+1) C_n$ is $(2n+1) C_n$ *i.e.*, $(2n+1)$

Here $q = 2$ and $r = 1$

The given series is $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$ replacing x by x^2 then

$$(1+x^2)^n = C_0 x + C_1 x^3 + C_2 x^4 + \ldots + C_n x^{2n+1}$$

Multiplying both sides by x, then

$$x(1+x^2)^n = C_0 x + C_1 x^3 + C_2 x^5 + \ldots + C_n x^{2n+1}$$

Differentiating both sides w.r. to x, we get

$$x \cdot n(1+x^2)^{n-1} 2x + (1+x^2)^n \cdot 1 = C_0 + 3C_1 x^2 + 5C_2 x^4 + \ldots + (2n+1) C_n x^{2n}$$

Putting x = I in both sides, we get

$$0 + 0 = C_0 - 3C_1 + 5C_2 - \ldots + (2n+1)(-1)^n C_n$$

or $\quad C_0 - 3C_1 + 5C_2 - \ldots + (-1)n(2n+1) C_n = 0$

1.9 Use of Integration

This method is applied only when the numerical occur as the product of the binomial coefficients.

Solution Process:

If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$ then integrate both sides between the suitable limits which gives the required series.

Note :

(*i*) If the sum contains C_0, C_1, C_2, …, C_n are are positive signs, then integrate between limits 0 to 1.

(*ii*) If the sum contains alternate signs (i.e., +, –), then integrate between limits –1 to 0.

(*iii*) If the sum contains odd coefficients (i.e., C_0, C_2, C_4…..) then integrate between –1 to + 1.

(*iv*) If the sum contains even coefficients (i.e., C_1, C_3, C_5,….) then subtracting (ii) from (i) and then dividing by 2.

(*v*) If in denominator of binomial coefficient product of two numericals then integrate two times first times taken limits between 0 to x and second times take suitable limits.

ILLUSTRATIONS

***Illustration* 21**

If $C_r = {}^nC_r$, then prove that $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

Solution

Consider the expansion $(1 + x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$ …(*i*)

Integrating both sides of (*i*) within limits 0 to 1, we get

$$\int_0^1 (1+x)^n\,dx = \int_0^1 (C_0 + C_1x + C_2x^2 + ... + C_nx^n)dx$$

$$\left.\frac{(1+x)^{n+1}}{n+1}\right]_0^1 = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + ... + \left.\frac{C_nx^{n+1}}{n+1}\right]_0^1$$

□ $$\frac{2^{n+1}-1}{n+1} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1}$$

Hence $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

***Illustration* 22**

Prove that $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - ... + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$

Solution

Consider the expansion $(1 + x)^n = C_0 + C_1x + C_2x^2 + \dots\ C_nx^n$ …(i)

Integrating both sides of (*i*) within limits –1 to 0, we get

$$\int_{-1}^0 (1+x)^n\,dx = \int_{-1}^0 (C_0 + C_1x + C_2x^2 + ...C_nx^n)\,dx$$

□ $$\left.\frac{(1+x)^n}{n+1}\right]_{-1}^0 = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + ... + \left.\frac{C_nx^{n+1}}{n+1}\right]_{-1}^0$$

□ $$\frac{1}{n+1} = C_0 - \frac{C_1}{2} + \frac{C_2}{3} - ... + (-1)^n \frac{C_n}{n+1}$$

□ $$\frac{1}{n+1} = C_0 - \frac{C_1}{2} + \frac{C_2}{3} - ... + (-1)^n \frac{C_n}{n+1}$$

Hence $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - ... + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$

***Illustration* 23**

Prove that $\frac{C_0}{1} + \frac{C_2}{3} + \frac{C_4}{5} + ... = \frac{2^n}{n+1}$

Solution

Consider the expansion $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + ... + C_nx^n$ …(i)

Integrating both sides of (i) within limits –1 to 1, we get

$$\int_{-1}^{1}(1+x)^n dx = \int_{-1}^{1}(C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + ... + C_nx^n)dx$$

$$= \int_{-1}^{1}(C_0 + C_2x^2 + C_4x^4 + ...)dx + \int_{-1}^{1}(C_1x + C_3x^3 + ...)dx$$

$$= 2\int_{0}^{1}(C_0 + C_2x^2 + C_4x^4 + ...)dx + 0$$

(By Prop. of definite integral)

(since second integral contains odd function)

$$\left.\frac{(1+x)^{n+1}}{n+1}\right]_{-1}^{1} = 2\left.\left(C_0x + \frac{C_2x^3}{3} + \frac{C_4x^5}{5} + ...\right)\right]_0^1$$

$$\frac{2^{n+1}}{n+1} = 2\left(C_0 + \frac{C_2}{3} + \frac{C_4}{5} + ...\right)$$

Hence $C_0 + \frac{C_2}{3} + \frac{C_4}{5} + ... = \frac{2^n}{n+1}$

***Illustration* 24**

Prove that $\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + ... = \frac{2^n - 1}{n+1}$

Solution

We know that from examples (1) and (2).

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + \frac{C_4}{5} + \frac{C_5}{6} + ... = \frac{2^{n+1} - 1}{n+1} \quad ...(i)$$

$$C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + \frac{C_4}{5} - \frac{C_5}{6} + ... = \frac{1}{n+1} \quad ...(ii)$$

subtracting (ii) from (i), we have

$$2\left(\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + ...\right) = \frac{2^{n+1} - 2}{n+1}$$

Dividing each sides by 2, we get

$$\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + ... = \frac{2^n - 1}{n+1}.$$

***Illustration* 25**

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$, show that

$$\frac{2^2}{1.2}C_0 + \frac{2^3}{2.3}C_1\ \frac{2^4}{3.4}C_2 + \dots + \frac{2^{n+2}C_n}{(n+1)(n+2)} = \frac{3^{n+2} - 2n - 5}{(n+1)(n+2)}$$

Solution

Given $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$ …(*i*)

Integration both sides of (i) within limits 0 to x we get

$$\int_0^x (1+x)^n\,dx = \int_0^x (C_0 + C_1x + C_2x^2 + \dots + C_nx^n)dx$$

$$\left.\frac{(1+x)^{n+1}}{(n+1)}\right]_0^x = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \dots + \left.\frac{C_nx^{n+1}}{n+1}\right]_0^x$$

$$\Rightarrow \quad \frac{(1+x)^{n+1} - 1}{(n+1)} = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \dots + \frac{C_nx^{n+1}}{(n+1)} \qquad \dots(ii)$$

Integrating again both sides of (*ii*) within limits 0 to 2, we get

$$\int_0^2 \frac{(1+x)^{n+1} - 1}{(n+1)}dx = \int_0^2 \left(C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \dots + \frac{C_nx^{n+1}}{n+1}\right)dx$$

$$\Rightarrow \quad \frac{1}{(n+1)}\left.\left(\frac{(1+x)^{n+2}}{n+2} - x\right)\right]_0^2 = \frac{C_0x^2}{1.2} + \frac{C_1x^3}{2.3} + \frac{C_2x^4}{3.4} + \dots + \left.\frac{C_nx^{n+2}}{(n+1)(n+2)}\right]_0^2$$

$$\frac{1}{(n+1)}\left\{\frac{3^{n+2}}{n+2} - 2 - \frac{1}{n+2}\right\} = \frac{2^2}{1.2}C_0 + \frac{2^3}{2.3}C_1 + \frac{2^4C_2}{3.4} + \dots + \frac{2^{n+2}C_n}{(n+1)\,(n+2)}$$

Hence, $\frac{2^2}{1.2}C_0 + \frac{2^3}{2.3}C_1 + \frac{2^4C_2}{3.4} + \dots + \frac{2^{n+2}C_n}{(n+1)(n+2)} = \frac{3^{n+2} - 2n - 5}{(n+1)\,(n+2)}$

1.10 Multiplication of Binomial Expansion

When each term in summation contains the product of two binomial coefficients or square of binomial coefficients, multiplication of binomial expansions is used

In the expansion, $(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_nx^n$; ${}^nC_0, {}^nC_1, {}^nC_2, {}^nC_3, \dots$ are generally written as $C_0, C_1, C_2, C_3, \dots$, where 0, 1, 2, 3, …. Are

$(1 + x)n = C_0 + C_1x + C_2x + \dots + C_nx^n$ …(1)

$(1 + x)n = C_0x^n + C_1x^{n-1} + \dots + C_n$ …(2)

$$\left(1 + \frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \dots + \frac{C_n}{x^n} \qquad \dots(3)$$

$$(1-x)^n = C_0 - C_1x + C_2x^2 - \dots (-1)^n C_nx^n \qquad \dots(4)$$

$$\left(1-\frac{1}{x}\right)^n = C_0 - \frac{C_1}{x} + \frac{C_2}{x^2} - .... + (-1)^n \frac{C_n}{x^n} \qquad \ldots(5)$$

❑ **Types of Identities :**

1. If the difference of lower suffix of binomial coefficients in each term is same.

Case I: If each term is positive.

Working Rule: Multiply (1) and (2) or (1) and (3) and equate suitable power of x on both sides.

Case II: If terms are alternately positive and negative.

Working Rule: Multiply (4) and (2) or (5) and (2) and evaluate suitable power of x on both sides.

2. If the sum of lower suffix in each term is same :

Case I: If each term is positive.

Working Rule: Multiply (1) and (2) and equate suitable power of x on both sides.

Case II: If terms are alternately positive and negative.

Working Rule: Multiply (1) and (4) and evaluate suitable power of x on both sides.

3. If each term is the product of two binomial coefficients and…..

Case I: multiplied by an integer.

Working Rule: Multiply equation (1) with suitable power of x and differentiate w.r.t x and then multiply this equation to equation (2) or (4) and equate suitable power of x on both sides.

Case II : divided by an integer.

Working Rule : Integrate equation first limit 0 to x and then multiply this equation to equation (2) or (4) and equate suitable power of x on both sides.

ILLUSTRATIONS

***Illustration* 26**

Evaluate $C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots. + C_{n-r}C_n$.

Solution

Difference of lower suffix = r (a constant) and each term is positive

Consider (1) and (2)

$(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_{n-r}x^{n-r} + \ldots + C_n$

$(x + 1)^n = C_0x^n + \ldots. + C_rx_{n-r} + C_{r+1}x^{n-r-1} + C_{r+2}x^{n-r-2} + \ldots. + C_n$

Multiplying and comparing the coefficients of x^{n-r} on both sides

$${}^{2n}C_{n-r} = C_0C_r + C_rC_{r+1} + C_2C_{r+2} + ... + C_{n-r}C_n$$

 Required sum $= {}^{2n}C_{n-r} = (2n)!/\ (n-r)!(n+r)!$

***Illustration* 27**

Evaluate ${}^mC_r + {}^mC_{r-1}.\,{}^nC_1 + {}^mC_{r-2}.\,{}^nC_2 + ... + {}^nC_r$.

Solution

Sum of lower suffix = r

Consider $(1 + x)^m = {}^mC_0 + {}^mC_1x + \ldots + {}^mC_rx^r + \ldots. + {}^mC_m.x^m$ …(1)

$(1 + x)^n = {}^nC_0 + {}^nC_1x + \ldots\ {}^nC_rx^r + \ldots. + {}^nC_n.x^n$ …(2)

Multiply (1) and (2)

$$(1+x)^m(1+x)^n = ({}^mC_0 + ... + {}^mC_rx^r + ... + {}^mC_mx^m)$$

$$({}^nC_0 + ... + {}^nC_r.x^r + ... + {}^nC_n.x^n) \quad \ldots(a)$$

Now equate coefficients of x^r on sides in equation (A)

 $${}^{m+n}C_r = {}^mC_r.{}^nC_0 + {}^mC_{r-1}{}^nC_1 + ... + {}^mC_0.{}^nC_r$$

 required sum $= {}^{m+n}C_r$.

***Illustration* 28**

Evaluate $C_1^2 + 2C_2^2 + 3C_3^2 + ... + nC_n^2.$

Solution

Given $(1 + x)n = C_0 + C_1x + \ldots.. + C_nx^n$ …(a)

Q general term $= nC_n^*; \quad \therefore \quad n = n.1 + 0$

So, multiply both sides by x_0 and differentiate with respect to x

$$n(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + ... + nC_n.x^{n-1} \quad \ldots(1)$$

and $$(x+1)^n = C_0x^n + C_1x^{n-1} + ... + C_n \quad \ldots(2)$$

Multiply (1) and (2)

$$n(1+x)^{n-1}.(x+1)^n = (C_1 + 2C_2x + 3C_3x^2 + ... + {}^nC_n.x^{n-1}$$

And equate coefficient of x^{n-1} on both sides, we get

 $$n\ .\ {}^{2n-1}C_{n-1} = C_1^2 = 2C_2^2 + 3C_3^2 + ... + nC_n^2.$$

***Illustration* 29**

Evaluate $C_0^2 + \frac{C_1^2}{2} + \frac{C_2^2}{3} + ... + \frac{C_n^2}{n+1}$

Solution

Given $(1 + x)^n = C_0 + C_1x + \ldots + C_nx^n$

Integrating both sides w.r.t x from 0 to x.

So, $\frac{(1+x)^{n+1}-1}{(1+n)} = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + ... + \frac{C_nx^{n+1}}{n+1}$ …(1)

and $(x+1)^n = C_0x^n + C_1x^{n-1} + ... + C_n$ …(2)

Multiplying (1) and (2), we get $\frac{1}{(n+1)}\left\{(1+x)^{2n+1} - (1+x)^n\right\}$

$$= \left(C_0x + \frac{C_1x^2}{2} + ... + \frac{C_nx^{n+1}}{n+1}\right)(C_0x^n + C_1x^{n-1} + ... + C_n)$$

Now equate coefficient of x^{n+1} on both sides

∴ required sum $= \frac{1}{n+1}\left\{{}^{2n+1}C_{n+1} - 0\right\} = {}^{2n+1}C_{n+1} \cdot \frac{1}{n+2}$

1.11 Divisibility problems

To show that an expression is divisible by an integer.

Solution Process :

(i) If a, p, n, r are positive integers, then first of all write

$$a^{pn+r} = a^{pn}.a^r = (a^p)^n.a^r$$

(ii) If we will show that the given expression is divisible by c, then express

$a^p = \{1 + (a^p - 1)\}$, if some power of $(a^p - 1)$ has c is a factor

or $a^p = \{2 + (a^p - 2)\}$, if some power of $(a^p - 2)$ has c is a factor

$a^p = \{3 + (a^p - 3)\}$, if some power of $(a^p - 3)$ has c is a factor

…………………………..

or $a^p = \{k + (a^p - k)\}$, if some power of $(a^p - k)$ has c is a factor.

ILLUSTRATIONS

***Illustration* 30**

If n is any positive integer, show that $2^{3n+3} - 7n - 8$ is divisible by 49?

Solution

The given expression

$= 2^{3n+3} - 7n - 8 = 2^{3n} . 2^3 - 7n - 8$

$= 8^n . 8 - 7n - 8 = 8(1+7)^n - 7n - 8$

$= 8\{1 + {}^nC_1 7 + {}^nC_2 7^2 + … + {}^nC_n 7^n\} - 7n - 8$

$= 8 + 56n + 8({}^nC_2 \cdot 7^2 + … + {}^nC_n 7^n\} - 7n - 8$

$= 49n + 8({}^nC_2 . 7^2 + … + {}^nC_n 7^n)$

$= 49\ \{n + 8\ ({}^nC_2 + \ldots + {}^nC_n 7n^{-2})\}$

Hence $2^{3n+3} - 7n - 8$ is divisible by 49.

***Illustration* 31**

Find the greater number in 100^{300} and 300 !

Solution

Using important result $\left(\frac{n}{3}\right)^n < n!$

Putting $n = 300$,

□ $(100)^{300} < 300\,!$

Hence the greater number is 300 !

***Illustration* 32**

Find the greater number in 300 ! and $\sqrt{300^{300}}$.

Solution

From example (1) $300\,! > (100)300$

Clearly $(100)^{150} > 3^{150}$

□ $(100)^{150}\,(100)^{150} > 3^{150}\,.\,(100)^{150}$

□ $(100)^{300} > (300)^{150}$

□ $(100)^{300} > \sqrt{300^{300}}$

From (*i*) and (*ii*), we get

$$300! > (100)^{300} > \sqrt{300^{300}}$$

Hence $300! > \sqrt{300^{300}}$

Hence the greater number is 300 !.

***Illustration* 33**

(*i*) If $a_n = \left(1 + \frac{1}{n}\right)^n$ then prove that $2 < a_n < 3\ n\ \forall\ N$.

(*ii*) Using binomial theorem, prove the inequality $n^{n+1} > (n+1)^n;\ n \geq 3,\ n \in N$

(*iii*) Using binomial theorem, prove the inequality $n^n > (n+1)^{n-1};\ n \geq 3,\ n \in N$.

Solution

(*i*) We have $a_n = \left(1 + \frac{1}{n}\right)^n$

$$= 1 + n.\frac{1}{n} + \frac{n(n-1)}{1.2}.\frac{1}{n^2} + \frac{n(n-1)\,(n-2)}{1.2.3}.\frac{1}{n^3} + \ldots$$

$$= 1 + 1 + \frac{\left(1 - \frac{1}{n}\right)}{2!} + \frac{\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)}{3!} + \ldots$$

= 2 + positive quantity (Q $n \in N$)

□ $a_n > 2$ …(1)

Also $a_n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots. + \frac{1}{n!}$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}$$

$$< 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{2^2} + \ldots\infty$$

$$< 1 + \frac{1}{1 - \frac{1}{2}}$$

< 3 …(2)

from (1) and (2), we get

$2 < a_n < 3$

(*ii*) from part (*i*), $\left(1 + \frac{1}{n}\right)^n < 3$

□ $\left(1 + \frac{1}{n}\right)^n < n$ (Q $n \geq 3$)

□ $\frac{(n+1)^n}{n^n} < n$

□ $(n+1)^n < n^{n+1}$

$\frac{(n+1)^n}{n^n} < n$

□ $n^n > \frac{(n+1)^n}{n} > \frac{(n+1)^n}{n+1}$ $\left(\text{since } \frac{1}{n} > \frac{1}{n+1}\right)$

□ $n^2 > \frac{(n+1)^n}{(n+1)}$

□ $n^2 > (n+1)^{n-1}$

***Illustration* 34**

Find the

(*i*) last digit (*ii*) last two digits and (*iii*) last three digits of 17^{256}.

Solution

Since $17^{256} = (17^2)^{128} = (289)^{128} = (290 - 1)^{128}$

□ $17^{256} = {}^{128}C_0\,(290)^{128} - {}^{128}C_1\,(290)^{127} + {}^{128}C_2(290)^{126} - \ldots$

$$- {}^{128}C_{125}(290)^3 + {}^{128}C_{126}(290)^2 - {}^{128}C_{127}(290) + 1$$

$$= [{}^{128}C_0(290)^{128} - {}^{128}C_1(290)^{127} + {}^{128}C_2(290)^{126} - \ldots - {}^{128}C_{125}(290)^3]$$

$$+ {}^{128}C_{126}(290)^2 - {}^{128}C_{127}(290) + 1$$

$$= 1000\,m + {}^{128}m_2(290)^2 - {}^{128}C_1(290) + 1$$

$$= 1000\,m + \frac{(128)(127)}{2}(290)^2 - 128 \times 290 + 1$$

$$= 1000\,m + {}^{128}C_2\,(290)^2 - {}^{128}C_1\,(290) + 1 \qquad (m \in I_+)$$

$$= 1000\,m + \frac{(128)(127)}{2}(290)^2 - 128 \times 290 + 1$$

$$= 1000\,\text{m} + (128)\,(127)\,(290)\,(145) - 128 \times 290 + 1$$

$$= 1000\,\text{m} + (128)\,(290)\,(127 \times 145 - 1) + 1$$

$$= 1000\,\text{m} + 683527680 + 1$$

$$= 1000\,\text{m} + 683527000 + 680 + 1$$

$$= 1000\,(\text{m} + 683527) + 681$$

Hence last three digits of 17^{256} must be 681, as a result last two digits of 17256 or 81 and last digit of 17^{256} is 1.

***Illustration* 35**

If 7 divides $32^{32^{32}}$ then find the remainder.

Solution

We have $32 = 2^5$

□ $(32)^{32} = (2^5)^{32} = 2^{160}$

$(32)^{32} = (3 - 1)^{160}$

$$= {}^{160}C_0\,3^{160} - {}^{100}C_1\,3^{159} + \ldots - {}^{160}C_{159}\,3 + {}^{160}C_{160}\,1.$$

$$= 3\,(3^{159} - {}^{160}C_1\,3^{158} + \ldots - {}^{160}C_{159}) + 1$$

$$= 3m + 1,\ \ m \in \text{I}_+$$

Now, $32^{32^{32}} = 32^{3m+1} = 2^{5(3m+1)} = 2^{15m+5}$

□ $32^{32^{32}} = 2^{3(5m+1)}.2^2$

$$= 4.(8)^{5m+1}$$

$$= 4.(7+1)^{5m+1}$$

$$= 4.({}^{5m+7}C_0(7)^{5m+1} + {}^{5m+1}C_1(7)^{5m} + {}^{5m+1}C_2(7)^{5m-1} + ... + {}^{5m+1}C_{5m}7 + {}^{5m+1}C_{5m+1})$$

$$= 4[7\{{}^{5m+1}C_0 7^{5m} + {}^{5m+1}C_1 7^{5m-1} + {}^{5m+1}C_2 7^{5m-2} + ... + {}^{5m+1}C_{5m}\} + 1]$$

$$= 4\,[7n+1], \quad n \in I_+$$

$$= 28\,n + 4$$

This show that where $32^{32^{32}}$ is divided by 7, then remainder is 4.

***Illustration* 36**

Prove that $2222^{5555} + 5555^{2222}$ is divisible by 7.

Solution

We have $2222^{5555} + 5555^{2222} =$

$$= (2222^{5555} + 4^{5555}) + (5555^{2222} - 4^{2222}) - (4^{5555} - 4^{2222}) \quad \ldots(1)$$

The number $2222^{5555} + 4^{5555}$ is divisible by 2222 + 4 = 2226 = 7.318 which is divisible by 7. (because a sum of two odd powers is always divisible by the sum of the b bases of the powers), the difference $5555^{2222} - 4^{2222}$ is also divisible by 7 since it is divisible by 5555 – 4 = 5551 = 7.793 (because the difference of any integral powers with equal exponent is divisible by the difference of the bases) as to the difference $4^{5555} - 4^{2222}$, it can be rewritten as

$$4^{2222}\,(4^{3333} - 1) = 4^{2222}\,(2^{6666} - 1) = 4^{2222}\,(64^{1111} - 1),$$

this expression is divisible by 64 – 1 = 63 = 7.9. Hence $4^{5555} - 4^{2222}$ is also divisible by 7. Hence each brackets of (1) are divisible by 7. Hence $2222^{5555} + 5555^{2222}$ is divisible by 7.

*****

MISCELLANEOUS PROBLEMS

OBJECTIVE TYPE

Example 1

The sum to (n+1) terms of the series $\frac{C_0}{2}-\frac{C_1}{3}+\frac{C_2}{4}-\frac{C_3}{5}+...$ is

(a) $\frac{1}{n+1}$ (b) $\frac{1}{n+2}$ (c) $\frac{1}{n(n+1)}$ (d) none of these

Solution

We have

$$(1-x)^n = C_0 - C_1x + C_2x^2 - C_3x^3 + ..$$

$$\Rightarrow \quad x(1-x)^n = C_0x - C_1x^2 + C_2x^3 - C_3x^4 ...$$

$$\int_0^1 x(1-x)^n dx = \int_0^1 x(1-x)^n dx = \int_1^0 (1-t)t^n(-1)dt \quad \text{[Put 1-x=t]}$$

$$= \int_0^1 \left(t^n - t^{n+1}\right)dt = \left|\frac{t^{n+1}}{n+1} - \frac{t^{n+2}}{n+2}\right|_0^1$$

$$= \frac{1}{n+1} - \frac{1}{n+2} = \frac{1}{(n+1)(n+2)}$$

Integrating R.H.S of (a) we get $\left(\frac{C_0x^2}{2} - \frac{C_1x^3}{3} + \frac{C_2x^4}{4} - ..\right)_0^1$

$$= \frac{C_0}{2} - \frac{C_1}{3} + \frac{C_2}{4} - ...$$

Thus $\frac{C_0}{2} - \frac{C_1}{3} + \frac{C_2}{4} - ... = \frac{1}{(n+1)(n+2)}$

$\therefore$ **Ans. (c)**

Example 2

The Value of ${}^{50}C_4 + \sum_{r=1}^{6} {}^{56-r}C_3$ is

(a) ${}^{55}C_3$ (b) ${}^{55}C_4$ (c) ${}^{56}C_4$ (d) ${}^{56}C_3$

Solution

$${}^{50}C_4 + \left({}^{50}C_3 + {}^{51}C_3 + {}^{52}C_3 + ... + {}^{55}C_3\right)$$

Taking first two terms together and adding them and following the same pattern, we get ${}^{56}C_4$

$[As\ ^nC_r + ^nC_{r-1} = ^{n+1}C_r]$

∴ **Ans. (c)**

Example 3

The digit as unit place in the number $17^{1995} + 11^{1995} - 7^{1995}$

(a) 0 (b) 1 (c) 2 (d) 3

Solution

We have

$17^{1995} + 11^{1995} - 7^{1995} = (7 + 10)^{1995} + (1 + 10)^{1995} - 7^{1995}$

$[7^{1995} + {}^{1995}C_1 7^{1994} . 10^1 + {}^{1995}C_2 7^{1993} . 10^2 + ... {}^{1995}C_{1995} . 10^{1995}] +$

$[{}^{1995}C_0 + {}^{1995}C_1 10^1 . {}^{1995}C_2 10^2 + ...+ {}^{1995}C_{1995}\ 10^{1995}] - 7^{1995}$

$= [{}^{1995}C_1 7^{1994} . 10^1 + ...+ 10^{1995}] + [{}^{1995}C_1 10^1 + ...+ {}^{1995}C_{1995} . 10^{1995}] + {}^{1995}C_0$

= (a multiple of 10) + 1

Thus, the units place digits is 1.

∴ **Ans. (b)**

Example 4

The number of terms in the expansion of $(a + b + c)^n$, where n□N is

(a) $\frac{(n+1)(n+2)}{2}$ (b) $n + 1$ (c) $n + 2$ (d) $(n + 1)\ n$

Solution

$(a + (b + c))^n = a^n + {}^nC_1\ a^{n-1}\ (b + c)^1 + {}^nC_2 a^{n-2}(b + c)^2 +...+ {}^nC_n(b + c)^n$

Further expanding each term of R.H.S.,

First term on expansion gives one term

Second term on expansion gives two terms and so on.

□ total number of terms = 1 + 2 + 3 + ... + $(n + 1)$ = $\frac{(n+1)(n+2)}{2}$

∴ **Ans. (a)**

Example 5

The value of $\frac{(18^3 + 7^3 + 3.18.7.25)}{3^6 + 6.243.2 + 15.81.4 + 20.27.8 + 15.9.16 + 6.3.32 + 64}$

(a) 1 (b) 2 (c) 3 (d) none of these

Solution

The numerator is of the form

$$a^3+b^3+3ab(a+b)=(a+b)^3$$

Where a = 18, and b = 7

$$\therefore N^r=(18+7)^3=(25)^3$$

Denominator can be written as

$$3^6+{}^6C_1.3^5.2^1+{}^6C_3.3^4.2^2+{}^6C_3.3^3.2^3+{}^6C_4.3^2.2^4+{}^6C_5.3.2^5+{}^6C_6 2^6$$

$$=(3+2)^6=5^6=(25)^3$$

$$\therefore \frac{Nr}{Dr}=\frac{(25)^3}{(25)^3}=1$$

$\therefore$ **Ans. (a)**

Example 6

The coefficient of x^5 in the expansion of $(1+x)^{21}+(1+x)^{22}+\ldots..+(1+x)^{30}$ is

(a) ${}^{51}C_5$ (b) ${}^{9}C_5$ (c) ${}^{31}C_6-{}^{21}C_6$ (d) ${}^{30}C_5.{}^{20}C_5$

Solution

$$(1+x)^{21}+(1+x)^{22}+\ldots..+(1+x)^{30}$$

$$=(1+x)^{21}\left(\frac{(1+x)^{10}-1}{(1+x)-1}\right)=\frac{1}{x}\left[(1+x)^{31}-(1+x)^{21}\right]$$

$\therefore$ coefficient of x^5 in the given expression

= coefficient of x^5 in $\left[\frac{1}{x}\left\{(1+x)^{31}-(1+x)^{21}\right\}\right]$

= coefficient of x^6 in $\left\{(1+x)^{31}-(1+x)^{21}\right\}={}^{31}C_6-{}^{21}C_6$

$\therefore$ **Ans. (c)**

Example 7

If $f(x)=x^n$, then the value of $f(1)+\frac{f^1(1)}{1}+\frac{f^2(1)}{2!}+\ldots..+\frac{f^n(1)}{n!}$, where $f^r(x)$ denotes the rth order derivative of $f(x)$ with respect of x is

(a) n (b) 2^n (c) 2^{n-1} (d) none of these

Solution

We have $f(x)=x^n$ So,

$$f^r(x)=\frac{n!}{(n-r)!}x^{n-r}\Rightarrow f^r(1)=\frac{n!}{(n-r)!}$$

Now,

$$f(1)+\frac{f^1(1)}{1}+\frac{f^2(1)}{2!}+\frac{f^3(1)}{3!}+.....+\frac{f^n(1)}{n!}$$

$$\sum_{r=0}^{n}\frac{f^r(1)}{r!}=\sum_{r=0}^{n}\frac{n!}{(n-r)!r!}=\sum_{r=0}^{n}{}^nC_r=2^n$$

$\therefore$ **Ans. (b)**

Example 8

If the sum of the coefficients in the expansion of $(1+2x)^n$ is 6561, the greatest term in the expansion for $x=\frac{1}{2}$ is

(a) 4^{th} (b) 5^{th} (c) 6^{th} (d) none of these

Solution

Sum of the coefficient in the expansion of $(1+2x)^n=6561$

$\Rightarrow$ $(1+2x)^n=6561$, when $x = 1$

$\Rightarrow$ $3^n=6561$

$\Rightarrow$ $3^n=2^8\Rightarrow n=8$

Now, $\frac{T_{r+1}}{T_r}=\frac{{}^8C_r(2x)^r}{{}^8C_{r-1}(2x)^{r-1}}=\frac{9-r}{r}\times 2x$

$\Rightarrow$ $\frac{T_{r+1}}{T_r}=\frac{9-r}{r}$ [$\because$ x = ½]

$\therefore$ $\frac{T_{r+1}}{T_r}>1\Rightarrow\frac{9-r}{r}>1\Rightarrow 9-r>r\Rightarrow 2r<9\Rightarrow r<4\frac{1}{2}$

Hence, 5^{th} term is the greatest term.

$\therefore$ **Ans. (b)**

Example 9

The positive integer which is just greater than $(1 + 0.0001)^{1000}$

(a) 3 (b) 4 (c) 5 (d) 2

Solution

Expression on expansion gives

$$1+1000\times10^{-4}+\frac{1000\times999}{2}10^{-8}+{}^{1000}C_3 10^{-12}$$

$$<1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+.....=\frac{1}{1-\frac{1}{10}}=\frac{10}{9}$$

So integer just greater than the given expression must be 2.

$\therefore$ **Ans. (d)**

SUBJECTIVE TYPE

Example 1

Find the number of terms which are free from radical signs in the expansion of $(y^{1/5}+x^{1/10})^{55}$

Solution

The general term in the expansion of $(y^{1/5}+x^{1/10})^{55}$ is

$$T_{r+1} = {}^{55}C_r (y^{1/5})^{55-r}(x^{1/10})^r$$

$$^{55}C_r \; y^{11-\frac{r}{5}} x^{\frac{r}{10}}$$

Clearly, T_{r+1} will be independent of radical signs if r/5 and r/10 are integers, where

$r \le r \le 55$ r = 0, 10, 20, 30, 40, 50

hence total terms are 6.

Example 2

Find the greatest integer less than or equal to $(\sqrt{2}+1)^6$

Solution

Let $(\sqrt{2}+1)^6 = I + f$, where $I \in N$ and $0 < f < 1$ clearly, F is the greatest integer less than or equal to $(\sqrt{2}+1)^6$, Let $G = (\sqrt{2}-1)^6$ then

$$I + F + G = \left\{2^6 + {}^6C_2\left(\sqrt{2}\right)^{6-2} + {}^6C_4\left(\sqrt{2}\right)^{6-4} + {}^6C_6\left(\sqrt{2}\right)^{6-6}\right\}\{F + G = 1\}$$

$I + 1 = 198$

$I = 197$

Example 3

If n is an odd natural number, then $\sum_{r=0}^{n} \frac{(-1)^r}{{}^nC_r}$ will be

Solution

$$\sum_{r=0}^{n} \frac{(-1)^r}{{}^nC_r}$$

$$= \sum_{r=0}^{\frac{n+1}{2}} \left\{\frac{(-1)^r}{{}^nC_r} + \frac{(-1)^{n-r}}{{}^nC_{n-r}}\right\}$$

$$= \sum_{r=0}^{\frac{n+1}{2}} (-1)^r \left\{\frac{1}{{}^nC_r} + \frac{(-1)^n}{{}^nC_r}\right\}$$

$= \quad 0$

Example 4

Find sum of coefficients of the terms of degree *m* in the expansion of $(1+x)^n(1+y)^n(1+z)^n$

Solution

$$(1+x)^n(1+y)^n(1+z)^n$$

$$= \sum_{r=0}^{n} {}^nC_r x^r . \sum_{s=0}^{n} {}^nC_s y^s . \sum_{t=0}^{n} {}^nC_t z^t$$

$$= \sum_{0 \le r, s, t, \le n} {}^nC_r \ {}^nC_s \ {}^nC_t x^r y^s z^t$$

For sum of the coefficients of degree *m*, we must have $r + s + t = m$

where *r, s, t* are integers with $r, s, t \ge 0$

sum of such coefficients

$$= \sum_{\substack{r, s, t \ge 0 \\ r+s+t=m}} \left({}^nC_r\right)\left({}^nC_s\right)\left({}^nC_t\right)$$

= the number of ways of choosing a total number of *m* balls out of *n* white *n* green and *n* black.

$= \quad {}^{3n}C_m$

Example 5

Show that: $2 . C_0 + 2^2 . \frac{C_1}{2} + 2^3 . \frac{C_2}{3} + ... + 2^{n+1} \frac{{}^{(n+1)}C_n}{n+1} = \frac{3^{(n+1)} - 1}{n+1}$

Solution

Integrating the expansion of $(1 + x)^n$ between the limits 0 to 2.

$$\int_0^2 (1+x)^n \, dx = \int_0^2 (C_0 + (C_1 x + ... + C_n x^n) \, dx$$

$$\Rightarrow \quad \left.\frac{(1+x)^{n+1}}{n+1}\right|_0^2 = \left. C_0 x + C_1 \frac{x^2}{2} + ... + C_n \frac{x^{n+1}}{n+1}\right|_0^2$$

$$\Rightarrow \quad 2 . C_0 + 2^2 . \frac{C_1}{2} + 2^3 . \frac{C_2}{3} + ... + 2^{n+1} \frac{{}^{(n+1)}C_n}{n+1} = \frac{3^{(n+1)} - 1}{n+1}$$

Alternative method:

LHS. $= \frac{1}{n+1}\left\{{}^{(n+1)}C_1 . 2 + {}^{(n+1)}C_2 . 2^2 + {}^{(n+1)}C_3 . 2^3 + ... + {}^{(n+1)}C_{n+1} . 2^{(n+1)}\right\}$

$$= \quad \frac{1}{n+1}\left\{1+{}^{(n+1)}C_1\,.\,2+{}^{(n+1)}C_2\,.\,2^2+\ldots+{}^{(n+1)}C_{n+1}\,.\,2^{(n+1)}-1\right\}$$

$$= \quad \frac{1}{(n+1)}\left\{(1+2)^{n+1}-1\right\}$$

Example 6

Show that:

$$C_0C_r+C_1C_{r+1}+C_2C_{r+2}+\ldots+C_{n-r}C_n=\frac{(2n)!}{(n-r)!\,(n+r)!}$$

Solution

Consider $(1+x)^n=C_0+C_1x+C_2x^2+\ldots+C_rx^r+C_{r+1}x^{r+1}+\ldots+C_nx^n$

and $\left(1+\frac{1}{x}\right)^n=C_0+\frac{C_1}{x}+\frac{C_2}{x^2}+\ldots+\frac{C_r}{x^r}+\frac{C_{r+1}}{x^{r+1}}+\ldots+\frac{C_n}{x^n}$

In the product of these two expansions, collecting the coefficient of x^r

$C_0C_r+C_1C_{r+1}+C_2C_{r+2}+\ldots+C_{n-r}C_n=$ coefficient of x^r in $\frac{(1+x)^{2n}}{x^n}$

= coefficient of x^{n+r} in $(1+x)^{2n}$

$= \quad {}^{2n}C_{n+r}=\frac{(2n)!}{(n+r)!(n-r)!}$

Example 7

Find the sum of the series

$$\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left(\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\ldots\ldots\text{ upto } m \text{ terms}\right)$$

Solution

$$\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left(\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\ldots\ldots\text{ upto } m \text{ terms}\right)$$

$$\sum_{r=0}^{n}(-1)^r\,{}^nC_r\frac{1}{2^r}+\sum_{r=0}^{n}(-1)^r\,{}^nC_r\frac{3^r}{2^{2r}}+\sum_{r=0}^{n}(-1)^r\,{}^nC_r\frac{7^r}{2^{3r}}+\ldots.$$

$$\left(1-\frac{1}{2}\right)^n+\left(1-\frac{3}{4}\right)^n+\left(1-\frac{7}{8}\right)^n+\ldots\ldots\text{upto } m \text{ terms}$$

$$\frac{1}{2^n}+\frac{1}{2^{2n}}+\frac{1}{2^{3n}}+\ldots\ldots\text{ upto } m \text{ terms}$$

$$\frac{2^{mn}-1}{2^{mn}(2^n-1)}$$

Example 8

When $32^{(32^{32})}$ is divided by 7, the remainder is _____.

Solution

We have $32^{32} = (2^5)^{32} = 2^{60}$

$= (3-1)^{160} = {}^{160}C_0 3^{160} - {}^{160}C_1 3^{159} + ... - {}^{160}C_{159.} 3 + 1$

= 3m+1, where m is some positive integer

Now, $32^{(32^{32})} = (2^5)^{3m+1} = (2^3)^{5m+1}.2^2 = (7+1)^{5m+1}.4$

But $(7+1)^{5m+1} = 7n+1$ for some positive integer n. Thus, N = $32^{(32^{32})} = 28n+4$

Thus shows that when N is divided by 7, the remainder is 4.

Example 9

If $(1+x)^n = \sum_{r=0}^{n} C_r \ x^r$, find the value of

$(i)\ \sum_{0 \le r < s \le n} (C_r \cdot C_s)$ $\quad (ii)\ \sum\sum_{o \le r < s \le n} (C_r - C_s)^2$ $\quad (iii)\ \sum\sum_{0 \le r < s \le n} (C_r + C_s)^2$

Solution

(i) We have

$(C_0 + C_1 + C_2 + ...C_n)^2$

$= \sum_{r=0}^{n} C_r^2 + 2 \sum\sum_{o \le r < s \le n} C_r C_s$

$\Rightarrow \quad 2\sum\sum_{0 \le r < s \le n} C_r C_s = (2^n)^2 - {}^{2n}C_n$

$\Rightarrow \quad \sum\sum_{o \le r < s \le n} C_r C_s = 2^{2n-1} - \frac{1}{2}\ {}^{2n}C_n$

(ii) $\sum\sum_{o \le r < s \le n} (C_r - C_s)^2$

$= (C_0 - C_1)^2 + (C_0 - C_2)^2 + ...(C_0 - C_n)^2$

$(C_1 - C_2)^2 + (C_1 - C_3)^2 + ...(C_1 - C_n)^2 + ...$

$= n\left(C_0^2 + C_1^2 + ...C_n{}^2\right) - 2\sum\sum_{0 \le r < s \le n} C_r C_s$

$= \quad n.\ {}^{2n}C_n - 2^{2n} + {}^{2n}C_n$ (by (a))

$= \quad (n+1)\ {}^{2n}C_n - 2^{2n}$

(iii) $\quad \sum\sum_{0\le r<s\le n} (C_r + C_s)^2 = n\left(\sum_{i=o}^{n} C_i^2\right) + 2\sum\sum_{0\le r<s\le n} C_r C_s$

$= (n-1)\ {}^{2n}C_n + 2^{2n}$

*****

Exercise - I

OBJECTIVE TYPE QUESTIONS

Multiple choice questions with ONE option correct

1. Greatest term in the expansion of $(2+3x)^{12}$, when x=5/6 is

 (a) ${}^{12}C_7.5^5.4$ (b) ${}^{12}C_8\,\frac{5^8}{2^4}$ (c) ${}^{12}C_7\left(\frac{5}{2}\right)^7.2^5$ (d)none of these

2. If the coefficients of x^6 and x^5 is the expansion of $\left(3+\frac{x}{4}\right)^n$ are equal, then n =

 (a) 17 (b) 47 (c) 77 (d) 67

3. If $(1+x-2x^2)^6 = 1 + a_1x + a_2x^2 + ... + a_{12}x^{12}$, then the value of $a_1 + a_3 + a_5 + ...a_{11}$ is equal to

 (a) 32 (b) 64 (c) -32 (d) -63

4. The greatest coefficient in the expansion of $(1+x)^{10}$ is

 (a) 10!/5!.6! (b) $10!/(5!)^2$ (c) 10!/5!7! (d)none of these

5. If $\sum_{r=1}^{n} r^3\left(\frac{{}^nC_r}{{}^nC_{r-1}}\right)^2 = 14^2$, then n=

 (a) 14 (b) 7 (c) 6 (d)none of these

6. If $a_n = \sum_{r=0}^{n}\frac{1}{{}^nC_r}$, $a_n = \sum_{r=0}^{n}\frac{r}{{}^nC_r}$ equals

 (a) $(n-1)a_n$ (b) na_n (c) $\frac{1}{2}na_n$ (d)none of these

7. Let $f(n) = 10^n + 3.4^{n+2} + 5, n \in N$. The greatest integer which divides $f(n)$ for all n is

 (a) 27 (b) 9 (c) 3 (d)none of these

8. If the 4th term in the expansion of $\left(px+x^{-1}\right)^m$ is 2.5 for all $x \in R$, then

(a) $p=\frac{5}{2}, m=3$ (b) $p=\frac{1}{2}, m=6$ (c) $p=-\frac{1}{2}, m=6$ (d)none of these

9. $\sum_{i=0}^{10}\sum_{j=1}^{10} {}^{10}C_j \, {}^{j}C_i$ equals

(a) 13^{10} (b) $3^{10}-1$ (c) $3^{10}-2$ (d) $3^{10}-3$

10. If the coefficient of 4th term in the expression of $\left(x+\frac{\alpha}{2x}\right)^n$ is 20, then the respective values of □ and n are

(a) 2, 7 (b) 5, 8 (c) 3, 6 (d) 2, 6

11. The coefficient of x^n in the expansion of $(1-2x+3x^2-4x^3+.....)^{-n}$ is

(a) $\frac{(2n)!}{n!}$ (b) $\frac{(2n)!}{(n!)^2}$ (c) $\frac{1(2n)!}{2(n!)^2}$ (d)none of these

12. The coefficient of x^n in the expansion of $\frac{1}{(1-x)(3-x)}$ is

(a) $\frac{2^{n+1}-1}{2.3^{n+1}}$ (b) $\frac{3^{n+1}-1}{3^{n+1}}$ (c) $2\left(\frac{3^{n+1}-1}{3^{n+1}}\right)$ (d)none of these

13. The two consecutive terms in the expansion of $(3+2x)^{74}$ whose coefficients are equal is

(a) 30th and 31st term terms (b) 29th and 30th terms

(c) 31st and 32nd terms (d) 28th and 29th terms

14. If the sum of the coefficients in the expansion of $(a+b)^n$ is 4096, then the greatest coefficient in the expansion is

(a) 924 (b) 729 (c) 1594 (d)none of these

15. If $|x| < 1$, then the coefficient of x^n in the expansion of $(1+x+x^2+x^3+...)^2$ is

(a) n (b) $n-1$ (c) $n+2$ (d) $n+1$

Multiple choice questions with ONE or MORE THAN ONE option correct

1. The value of x, for which the 6th term in the expansion $\left\{2^{\log_2\sqrt{9^{x-1}+7}}+\frac{1}{2^{\left[\frac{1}{5}\log_2(3^{x-1}+1)\right]}}\right\}^7$ is 84, is equal to

(a) 4 (b) 3 (c) 2 (d) 1

2. For a positive integer *n*, if the expansion of $\left(\frac{5}{x^2}+x^4\right)^n$ has a term independent of *x*, then *n* can be

(a) 18 (b) 21 (c) 27 (d) 95

3. The value of the expression $C_0^2 - C_1^2 + C_2^2 - C_3^2 ......(-1)^n C_n^2$ is

(a) 0 if *n* is odd (b) $(-1)^n$ if *n* is odd

(c) $(-1)^{n/2}\ {}^nC_{\frac{n}{2}}$ (d) $(-1)^{n-1}\ {}^nC_{n-1}$ if *n* is even

4. If the second, third and fourth terms in expansion of $(a + b)^n$ are 135, 30 and $\frac{10}{3}$ respectively, then

(a) $a = 3$ (b) $b = -\frac{1}{3}$ (c) $n = 5$ (d) $n = 7$

5. In the expansion of $\left(\sqrt[8]{5}+\sqrt[6]{2}\right)^{100}$, there are

(a) 4 rational terms (b) 96 irrational terms

(c) 97 irrational terms (d) 5 rational terms

6. If $T_9 = 495$ in the binomial expansion of $\left(\frac{1}{x^2}+\frac{x}{2}\log_2 x\right)^{12}$, then x is equal to

(a) 1 (b) an integer > 1 (c) fraction (d) none of these

7. The middle term in the expansion of $\left(\frac{x}{2}+2\right)^8$ is 1120; then x is equal to

(a) 2 (b) 3 (c) -2 (d) -3

8. The middle term in the expansion of $(1+x)^{2n}$ is

(b) ${}^{2n}C_n x^n$ (b) ${}^{2n}C_{n+1} x^{n+1}$

(c) ${}^{2n}C_{n-1} x^{n-1}$ (d) $\frac{1.3.5...(2n-1)}{n!}2^n x^n$

9. If the coefficients of *rth*, *(r+1)*th and *(r+2)*th terms in the expansion of $(1+x)^{14}$ are in A.P.. then r =

(a) 5 (b) 7 (c) 9 (d) 11

10. Co-efficient of x^5 in the expansion of $(1+x^2)^5(1+x)^4$ is

(a) 40 (b) 50 (c) 30 (d) 60

*****

Exercise - II

ASSERTION & REASON , COMPREHENSIVE & MATCHING TYPE

Assertion & Reason Type

In the following question, a statement of Assertion (A) is given which is followed by a corresponding statement of reason (R). Mark the correct answer out of the following options/codes.

(a) If both (A) and (R) are true and (R) is the correct explanation of (A).

(b) If both (A) and (R) are true but (R) is not correct explanation of (A).

(c) If (A) is true but (R) is false.

(d) If both (A) and (R) are false.

(e) If (A) is false but (R) is true.

1. A: $^{n}C_0 + {}^{n+1}C_1 + {}^{n+2}C_2 + ... + {}^{n+r}C_r = {}^{n+r+1}C_r \;\; n \in N$

   R: The coefficient of x^n in $\frac{1}{x}\left[(1+x)^{n+r+1} - (1+x)^n\right]$ is $^{n+r+1}C_r$

2. A: The sum of $(n + 1)$ term of the series $\frac{C_0}{2} - \frac{C_1}{3} + \frac{C_2}{4} - \frac{C_3}{5} .... = \frac{1}{(n+1)(n+2)}$

   R: $\int_0^1 x(1-x)^n\, dx = \frac{1}{(n+1)(n+2)}$

3. A: The number of distinct term in the expansion of $(x_1 + x_2 + ... + x_r)^n$ is $^{n+r-1}C_{r-1}$

   R: The number of ways of distributing n identical objects among r persons so that each may receive 0, 1, 2, 3 or all the n objects is $^{n+r-1}C_{r-1}$

4. A: If $C_r = {}^nC_r$ then $\sum_{k=1}^{n} k^3 \left(\frac{C_k}{C_{k-1}}\right)^2 = \frac{1}{12} n(n+1)^2(n+2)$

   R: $\frac{C_k}{C_{k-1}} = \frac{n+k-1}{k}$ for k = 1, 2, ….. n

5. A: $1^2 C_1^2 + 2^2 C_2^2 + 3^3 C_3^2 + ... + n^2 C_n^2 = n^2 \;\; {}^{2n-2}C_{n-1}$

   R: The LHS in (A) term independent of x in $n^2(1+x)^{n-1}\left(1+\frac{1}{x}\right)^{n-1}$

Passage Based Questions

Passage – I

If $(1+x)^n = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$ then

1. $a_0 - a_2 + a_4 - a_6 + ... =$

(a) $2^{n/2} \sin \frac{n\pi}{4}$ (b) $2^{n/2} \cos\left(\frac{n\pi}{4}\right)$ (c) $2^{n/2} \cos\left(\frac{n\pi}{3}\right)$ (d) $2^{n/2} \sin\left(\frac{n\pi}{3}\right)$

2. $a_1 - a_3 + a_5 - a_7 + ... =$

(a) $2^{n/2} \sin \frac{n\pi}{4}$ (b) $2^{n/2} \cos\left(\frac{n\pi}{4}\right)$ (c) $2^{n/2} \cos\left(\frac{n\pi}{3}\right)$ (d) $2^{n/2} \sin\left(\frac{n\pi}{3}\right)$

3. $a_0 + a_3 + a_6 + a_9 + ... =$

(a) $\frac{1}{3}\left[2^n + 2\sin\left(\frac{n\pi}{3}\right)\right]$ (b) $\frac{1}{3}\left[2^n \cos\left(\frac{n\pi}{3}\right)\right]$

(c) $\frac{1}{3}\left[2^n \sin\left(\frac{n\pi}{3}\right)\right]$ (d) $\frac{1}{3}\left[2^n + 2\cos\left(\frac{n\pi}{3}\right)\right]$

4. $a_1 + a_4 + a_7 + a_{10} + ... =$

(a) $\frac{1}{3}\left[2^n + 2\sin\left(\frac{(n-2)\pi}{3}\right)\right]$ (b) $\frac{1}{3}\left[2^n \cos\left(\frac{(n+2)\pi}{3}\right)\right]$

(c) $\frac{1}{3}\left[2^n \sin\left(\frac{n\pi}{3}\right)\right]$ (d) $\frac{1}{3}\left[2^n + 2\cos\left(\frac{n\pi}{3}\right)\right]$

5. $a_3 + a_7 + a_{11} + a_{15} + ... =$

(a) $\frac{1}{2}\left[2^{n-1} - 2^{n/2} \sin\left(\frac{n\pi}{4}\right)\right]$ (b) $\frac{1}{2}\left[2^{n-1} + 2^{n/2} \cos\left(\frac{n\pi}{4}\right)\right]$

(c) $2^{\frac{n}{2}-1} \sin\left(\frac{n\pi}{4}\right)$ (d) $2^{\frac{n}{2}-1} \cos\left(\frac{n\pi}{4}\right)$

Passage – II

If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + ...C_n x^n$

1. The sum of the products of the binomial coefficient $C_0, C_1, ..., C_n$, taken two at a time is

(a) $2^{2n} - {}^{2n}C_n$ (b) $\frac{1}{2}\left(2^{2n} - {}^{2n}C_n\right)$ (c) $\frac{1}{2}\left(2^{2n} - 2n\right)$ (d) $2^{2n-1} - {}^{2n}C_n$

2. $\sum_{0 \le i < j \le n} \left(C_i + C_j\right)^2$

(a) $(n-1)^{2n}C_n + 2^{2n}$ (b) $(n-1)^{2n}C_n - 2^{2n}$ (c) $n^{2n}C_n - 2^n$ (d) $n^{2n}C_n - 2^n$

3. $\sum_{0 \le i < j \le n} (i+j) C_i C_j =$

(a) $n(2^{2n-1} - {}^{2n-1}C_{n-1})$ (b) $n(2^{2n-1} + {}^{2n-1}C_{n-1})$

(c) $n\,2^{2n} - {}^{2n}C_n$ (d) None of these

4. $\sum_{i=0}^{n} C_i^2$

(a) ${}^{2n}C_n - {}^{n}C_n$ (b) $\frac{1}{2}\,{}^{2n}C_n$ (c) $\left({}^{2n}C_n\right)^2$ (d) ${}^{2n}C_n$

Matching Type Questions

1. **Column I** **Column II**

(A) The coefficient of x^6 in the expansion of $(1+x)^{21} + (1+x)^{22} + ...(1+x)^{30}$ is (p) ${}^{18}C_8$

(B) The coefficient of x^8y^{10} in $(x + y)^{18}$ (Q) ${}^{31}C_7 - {}^{21}C_7$

(C) The sum of the series $\sum_{r=10}^{10} {}^{20}C_r$ is (R) $2^{19} + \frac{1}{2}\,{}^{20}C_{10}$

(D) In the expansion of $(1 + x)^{50}$ the sum of the coefficient of odd power of x is (S) 2^{49}

(e) The value of ${}^{14}C_1 + {}^{14}C_3 + {}^{14}C_5 + ... + {}^{14}C_{11}$ (T) $2^{13} - 14$

(a) A–P, B–Q, C–R, D–S, E–T (b) A–Q, B–P, C–R, D–S, E–T

(c) A–Q, B–R, C–P, D–S, E–T (c) A–Q, B–P, C–R, D–T, E–S

2. **Column I** **Column II**

(A) For $1 \le r \le n,$ the value of $\sum_{r=0}^{n} \left(\frac{r+2}{r+1}\right) {}^{n}C_r$ is (P) $\frac{2^n(n+3)-1}{(n+1)}$

(B) $\sum_{r=1}^{n} r\ {}^{n}C_r$ is (Q) $n\,2^{2n-1}$

(C) $\sum_{r=0}^{n} \frac{{}^{n}C_r}{{}^{n}C_r + {}^{n}C_{r+1}}$ (R) $\frac{n}{2}$

(D) $\frac{C_0}{1} + \frac{C_2}{3} + \frac{C_4}{5} + \frac{C_6}{7}$ (S) $\frac{2^n}{n+1}$

(a) A–P, B–S, C–R, D–Q (b) A–S, B–Q, C–R, D–P

(c) A–P, B–Q, C–R, D–S (d) A–Q, B–P, C–R, D–S

*****

Exercise - III

SUBJECTIVE TYPE

1. Find the coefficient of x^n in the expansion of $\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+....+\frac{x^n}{n!}\right)^2$

2. Find the coefficient of x^{50} in the expression
$(1+x)^{1000}+2x(1+x)^{999}+3x^2(1+x)^{998}+....1001\ x^{1000}$

3. If a_1, a_2, a_3, a_4 are the coefficients of any four consecutive terms in the expansion of $(1 + x)^n$, prove that $\frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}=\frac{2a_2}{a_2+a_3}$

4. If $(1 + x + x^2)^n = a_0+a_1x+a_2x^2+...a_nx^{2n}$ then prove that
   (i) $a_0^2-a_1^2+a_2^2-a_3^2+....+a_{2n}^2=a_n$
   (ii) $a_0a_2-a_1a_3+a_2a_4-....+a_{2n-2}a_{2n}=a_{n+1}$

5. Given $p + q = 1$ evaluate $\sum_{r=0}^{n} r^3\ {}^nC_r p^r q^{n-r}$

6. If $(2+\sqrt{3})^n=I+f,$ where I and n are positive integer and $(1-f)\,(I+f)=1.$

7. Prove that ${}^nC_o-{}^nC_1+{}^nC_2......+(-1)^{m-1}\ {}^nC_{m-1}=(-1)^{m-1}\ {}^{n-1}C_{m-1}$

8. Prove that the coefficient of x^r in the expansion of
$(x+3)^{n-1}+(x+3)^{n-2}(x+2)+(x+3)^{n-3}(x+2)^2+.....\ (x+2)^{n-1}$ is $(3^{n-r}-2^{n-r})\ {}^nC_r$

9. Prove that $(n!)!$ is divisible by $(n!)^{(n-1)!}$

10. Prove that ${}^{n-1}C_r+{}^{n-2}C_r+{}^{n-3}C_r+.......{}^rC_r={}^nC_{r-1}$

11. If x is so small that its square and higher powers may be neglected, show that
$\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}}=1-\left(\frac{41}{24}\right)x$

12. Prove the following it $C_0, C_1, C_2.....C_n$ are the combinational coefficients in the expansion of $(1+x)^n,\ n\in N,$ then prove the following :
   (i) $C_0^2+C_1^2+C_2^2+.......C_n^2=\frac{(2n)!}{n!\ n!}$
   (ii) $C_0C_1+C_1C_2+......C_{n-1}C_n=\frac{(2n)!}{(n+1)!(n-1)!}$
   (iii) $C_o+2C_1+3C_2+........(n+1)C_n=(n+2)\ 2^{n-1}$
   (iv) $(C_0+C_1)(C_1+C_2)(C_1+C_3).........(C_{n-1}+C_n)$

$$= \frac{C_0.C_1.C_2......C_{n-1}(n+1)^n}{n!}$$

(*v*) $2.C_0 + \frac{2^2 C_1}{2} + \frac{2^3 C_2}{3} + \frac{2^4 C_3}{4} + ... \frac{2^{n+1} C_n}{n+1} = \frac{3^{n+1}-1}{n+1}$

(*vi*) $1.C_0^2 + 3.C_1^{\ 2} + 5.C_2^{\ 2} + ...(2n+1)C_n^{\ 2} = \frac{(n+1)(2n)!}{n!n!}$

13. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n$, then prove that

$$\frac{C_1}{C_0} + 2\frac{C_2}{C_1} + 3\frac{C_3}{C_2} + ... + n\frac{C_n}{C_{n-1}} = \frac{n(n+1)}{2}.$$

14. If $S_n = {}^nC_0\ {}^nC_1 + {}^nC_2 + ... + {}^nC_{n-1}\ {}^nC_n$, and if $\frac{S_{n+1}}{S_n} = \frac{15}{4}$, find *n*.

15. Show that: $C_0\ {}^{2n}C_n - C_1\ {}^{2n-2}C_n + C_2\ {}^{2n-4}C_n - .... = 2^n.$

16. Prove that $2 < f(n) < 3$ for all $n \in N, n \geq 2$ if $f(n) = \left(1 + \frac{1}{n}\right)^n$

17. If x^p occurs in the expansion of $(x^2 + 1/\ x)^{2n}$, prove that its coefficients is

$$\frac{(2n)!}{\left[\frac{1}{3}(4n-p)\right]!\left[\frac{1}{3}(2n+p)\right]!}$$

18. Prove that $C_1 - \frac{1}{2}C_2 + \frac{1}{3}C_3 - ... + \frac{(-1)^{n-1}.C_n}{n} = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}.$

19. Let $f(n) = \frac{C_0}{3} - \frac{C_1}{4} + \frac{C_2}{5} - ....(n+1)$ terms.

Evaluate $\int_0^1 f(x)dx$, where $f(x)$ is extension of $f(n)$ over [0,1].

*****

Exercise - IV

IIT – JEE PROBLEMS

A. Fill in the blanks

1. The sum of the coefficients of the polynomial $(1 + x - 3x^2)^{2163}$ is …..

2. The coefficient of x^{99} in the polynomial $(x - 1)(x - 2) \ldots (x - 100)$ is …..

3. If $(1 + ax)^n = 1 + 8x + 24x^2 + \ldots$, then a = ….. and n = …..

4. Let n be a positive integer. If the coefficients of 2nd, 3rd, and 4th terms in the expansion of $(1 + x)^n$ are in A.P., then the value of n is

B. Multiple Choice Questions with ONE correct answer

5. Given positive integers $r > 1$, $n > 2$ and the coefficient of $(3r)$ th and $(r + 2)$ th terms in the binomial expansion of $(1 + x)^{2n}$ are equal. Then

   (a) $n = 2r$ (b) $n = 2r + 1$ (c) $n = 3r$ (d)none of these

6. The coefficient of x^4 in $\left(\frac{x}{2} - \frac{3}{x^2}\right)^{10}$ is

   (a) $\frac{405}{256}$ (b) $\frac{504}{259}$ (c) $\frac{405}{263}$ (d)none of these

7. If C_r stands for ${}^nC_r$, then the sum of the series

   $\frac{2\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)}{n!}[C_0^2 - 2C_1^2 + 3C_2^2 - \ldots + (-1)^n (n+1) C_n^2]$ where n is an even positive integer, is equal to

   (a) $(-1)^{n/2}(n + 2)$ (b) $(-1)^n(n + 1)$ (c) $(-1)^{n/2}(n + 1)$ (d)none of these

8. If $a_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$, then $\sum_{r=0}^{n} \frac{r}{{}^nC_r}$ equals

   (a) $(n - 1)\, a_n$ (b) $n\, a_n$ (c) $\frac{1}{2} n a_n$ (d)none of these

9. If in the expansion of $(1 + x)^m.(1 - x)^n$, the coefficients of x and x^2 are 3 and –6 respectively, then m is

   (a) 6 (b) 9 (c) 12 (d) 24

10. For $2 \le r \le n, \binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2}$ is equal to

    (a) $\binom{n+1}{r-1}$ (b) $2\binom{n+1}{r+1}$ (c) $2\binom{n+2}{r}$ (d) $\binom{n+2}{r}$

11. In a binomial expansion of $(a - b)^n$, $n \geq 5$ the sum of the 5^{th} and 6^{th} terms is zero. Then a/b equals

(a) $\frac{n-5}{6}$ (b) $\frac{n-4}{5}$ (c) $\frac{5}{n-4}$ (d) $\frac{6}{n-5}$

12. Let T_n denote the number of triangles which can be formed using the vertices of a regular polygon of n sides. If $T_{n+1} - T_n = 21$, then n equals

(a) 5 (b) 7 (c) 6 (d) 4

13. Coefficient of t^{24} in $(1 + t^2)^{12} (1 + t^{12}) (1 + t^{24})$ is

(a) ${}^{12}C_6 + 3$ (b) ${}^{12}C_6 + 1$ (c) ${}^{12}C_6$ (d) ${}^{12}C_6 + 2$

14. $\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + ... + \binom{30}{20}\binom{30}{30} =$

(a) ${}^{30}C_{11}$ (b) ${}^{60}C_{10}$ (c) ${}^{30}C_{10}$ (d) ${}^{65}C_{55}$

C. Subjective Questions

15. If $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$, then show that the sum of the products of the Ci's taken two at a time represented by $\Sigma\, \Sigma\, C_i\, C_j$ is equal to $0 \leq i < j \leq n$ $2^{2n-1} - \frac{(2n!)}{2(n!)^2}$

16. Given $S_n = 1 + q + q^2 + ... + q^n$ $S_n = 1 + \frac{q+1}{2} + \left(\frac{q+1}{2}\right)^2 + ... + \left(\frac{q+1}{2}\right)^n, q \neq 1$ Prove that

${}^{n+1}C_1 + {}^{n+1}C_2 s_1 + {}^{n+1}C_3 s_2 + ... + {}^{n+1}C_n s_n = 2^n S_n$

17. Let n be a positive integer and $(1 + x + x^2)^n = a_0 + a_1x + ... + a_{2n}x^{2n}$. Show that $a_0^2 - a_1^2 + ... + a_{2n}^2 = a_n$

18. Let n be any positive integer. Prove that $\sum_{k=0}^{m} \frac{\binom{2n-k}{k}}{\binom{2n-k}{n}} \cdot \frac{(2n-4k+1)}{(2n-2k+1)} 2^{n-2k} = \frac{\binom{n}{m}}{\binom{2n-2m}{n-m}} 2^{n-2m}$ for each non negative integer $m \leq n \left(\text{Here} \binom{p}{q} = {}^pC_q\right)$.

19. For any positive integers m, n (with $n \geq m$) let $\binom{n}{m} = {}^nC_m$. Prove that

$$\binom{n}{m} + \binom{n-1}{2} + \binom{n-2}{m} + ... + \binom{m}{m} = \binom{n+1}{m+1}$$

Hence, or otherwise, prove that

$$\binom{n}{m} + 2\binom{n-1}{m} + 3\binom{n-2}{m} + ... + (n-m+1)\binom{m}{m} = \binom{n+2}{m+2}$$

20. Prove that $2^k\binom{n}{0}\binom{n}{k}-2^{k-1}\binom{n}{1}\binom{n-1}{k-1}+2^{k-2}\binom{n}{2}\binom{n-2}{k-2}-.....(-1)^k\binom{n}{k}\binom{n-k}{0}=\binom{n}{k}$

21. Prove that $\frac{3!}{2(n+3)}=\sum_{r=0}^{n}(-1)^r\left(\frac{{}^nC_r}{{}^{r+3}C_r}\right)$

22. Prove that $\sum_{r=1}^{k}(-3)^{r-1}\ {}^{3n}C_{2r-1}=0$ where $k=\frac{3n}{2}$ and n is an even positive integer

23. If $\sum_{r=0}^{2n} a_r(x-2)^r=\sum_{r=0}^{2n} b_r(x-3)^r$ and $a_k=1$ for all $k\geq n$, then show that $b_n={}^{2n+1}C_{n+1}$

*****

Answers

Exercise - I

Only One Option is correct

1.	(c)	2.	(c)	3.	(c)	4.	(d)	5.	(c)
6.	(c)	7.	(b)	8.	(b)	9.	(b)	10.	(d)
11.	(b)	12.	(a)	13.	(a)	14.	(a)	15.	(d)

More Than One Choice Correct

1.	(c, d)	2.	(a, b, c, d)	3.	(a, c)	4.	(a, b, c)	5.	(a, c)
6.	(b, c)	7.	(a, c)	8.	(a, d)	9.	(a, c)	10.	(c, d)

Exercise - II

Assertion and Reason

1.	(a)	2.	(b)	3.	(a)	4.	(c)	5.	(a)

Passage – I

1.	(b)	2.	(a)	3.	(d)	4.	(a)	5.	(a)

Passage – II

1.	(b)	2.	(a)	3.	(a)	4.	(d)

Matching Type Questions

1.	(b)	2.	(c)

Exercise - III

Subjective Type

1. $\dfrac{2^n}{n!}$ 2. $\dfrac{1002!}{50!\,952!}$ 5. $np\,[1+3(n-1)p+(n-1)\,(n-2)\,p^2]$

12. $\dfrac{n}{2}(2^{2n}-{}^{2n}C_n)$ 19. $\ln\left(\dfrac{32}{57}\right)$

Exercise - IV

IIT-JEE Level Problem

Section - A

1.	1	2.	5050	3.	$a=2$, $n=4$	4.	$n=7$

Section - B

1.	(a)	2.	(a)	3.	(a)	4.	(c)	5.	(c)
6.	(d)	7.	(c)	8.	(b)	9.	(d)	10.	(c)

*****