APPLICATION OF DERIVATIVE

LAGRANGE, JOSEPH LOUIS DE

The French physicist Joseph Louis, comte de Lagrange, (1736-1813) was one of the most important mathematical and physical scientists of the late 18th century. He invented and brought to maturity the calculus of variations and later applied the new discipline to CELESTIAL MECHANICS, especially to finding improved solutions to the THREE-BODY PROBLEM. He also contributed significantly to the numerical and algebraic solution of equations and to number theory. In his classic Mecanique analytique (Analytical Mechanics, 1788), he transformed mechanics into a branch of mathematical analysis. The treatise summarized the chief results in mechanics known in the 18th century and is notable for its use of the theory of differential equations. Another central concern of Lagrange was the foundations of calculus. In a book published in 1797 he stressed the importance of Taylor series and the concept of function.

IIT-JEE Syllabus

Geometrical interpretation of the derivative, tangents and normals, increasing and decreasing functions, maximum and minimum values of a function, Rolle's Theorem and Lagrange's Mean Value Theorem.

INTRODUCTION

Let $y = f(x)$ be any curve then any line touching the curve at unique point is called tangent to the curve whereas any line perpendicular to the given tangent is called normal to the curve at the given point.

3.1 TANGENTS TO A CURVE

- **Slope of Tangent to The Curve $y = f(x)$**

Let $P(x, y)$ be any point on the curve $y = f(x)$ and tangent at point P makes angle θ with +ve direction of x-axis in anti-clock-wise direction then gradient or slope of tangent to the curve at $P(x, y)$ is equal to tan □. *i.e.* slope of tangent at $(a, f(a))$ is tan □.

The derivative of $y = f(x)$ at the point x = a is known as slope of the tangent to the curve $y = f(x)$ at the point $(a, f(a))$. Thus $\tan\theta = \left(\frac{dy}{dx}\right)_p$

- **Equation of Tangent**

Let $P \equiv (x_1, y_1)$ be any point the curve $y = f(x)$ then equation of tangent at P is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$

3.2 Normals To A Curve

❑ Equation of Normal

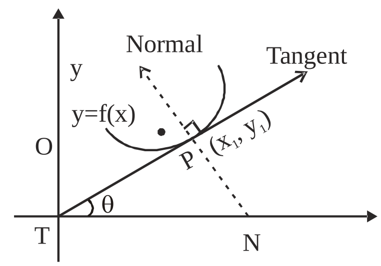
Normal to the curve is a line which is perpendicular to tangent at the point of contact.

Slope of normal $= \dfrac{-1}{\text{slope of tangents}}$

Equation of normal at P is

$$y - y_1 = \left(-\frac{dx}{dy}\right)_{(x_1, y_1)} (x - x_1)$$

PT is the tangent at the point *P* to the curve and *PN* is the normal at the point *P* to the curve.



(i) If $\frac{dy}{dx} > 0$ the tangents makes acute angle with +ve *x-axis*.

(ii) If $\frac{dy}{dx} < 0$ the tangent makes obtuse angle with +ve x-axis.

(iii) $\frac{dy}{dx} = 0$ tangent is parallel to x-axis.

(iv) $\frac{dy}{dx} = \infty$ tangent is perpendicular to x-axis.

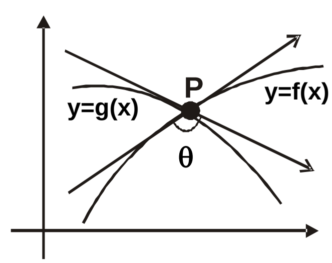
(v) $\frac{dy}{dx} = \pm 1$ tangent is equally inclined with axis i.e it makes an angle of either 45° or 135° with the positive direction of *x*–axis in anticlockwise direction.

3.3 Application of Tangents And Normals

❑ Angle of Intersection of Two Curve

Let $y = f(x)$ and $y = g(x)$ be two curve intersecting at a point *P* and is the angle between the tangent at point *P* then $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

Where m_1 and m_2 are the slope of tangents to the curves $y = f(x)$ and $y = g(x)$ respectively at the point *P*.



NOTE:

(i). Two curve are said to be orthogonal (□ = 90°) at a point P if $m_1.m_2 = -1$

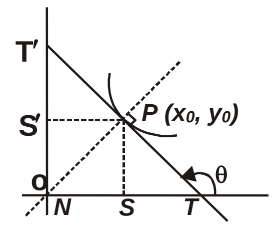
(ii) Two curves touch each other (□ = 0°) at P, if $m_1 = m_2$.

- **Length of Tangent, Normal, Sub-tangent & Sub-normal**

Let $P(x_0, y_0)$ be any point on the curve $y = f(x)$.

Then slope of tangent at the point P is given by

$$m = \tan\theta = \left(\frac{dy}{dx}\right)_P$$



In the fig. the length PT denotes length of tangent and ST denotes length of sub-tangent,

Form Δ PST, PS = y_0 and $\angle PTS = \pi - \theta$

Thus $\frac{PS}{PT}$ = sin (□ – □) □□ PT = |PS cosec □|

□ $$PT = \left|y_0\sqrt{1+\cot^2\theta}\right| = \left|\frac{y_0\sqrt{1+(dy/dx)_p^2}}{(dy/dx)_p}\right|$$

also $\frac{PS}{ST}$ = tan (□ – □) □ ST = |PS cot □|

⇒ $$ST = \left|\frac{y_0}{\tan\theta}\right| = \left|\frac{y_0}{\left(\frac{dy}{dx}\right)_p}\right| = \left|y_0\left(\frac{dx}{dy}\right)_p\right|$$

In the fig. the length PN denotes length of normal & SN denotes length of sub-normal from □PSN, $PS = y_0$ and $\angle SPN = \pi - \theta$

Thus $\frac{PS}{PN}$ = cos (□ – □) □ PN = |PS sec □|

□ $$PN = \left|y_0\sqrt{1+\tan^2\theta}\right| = \left|y_0\sqrt{1+\left(\frac{dy}{dx}\right)_p^2}\right|$$

also $\frac{PS}{SN}$ = cot □ □ $SN = |PS\tan\theta|$

$$SN = \left|y_0\left(\frac{dy}{dx}\right)_p\right|$$

- **Intercepts Made by Tangent on the axes**

The intercept made by tangents on x-axis and y-axis is the length OT and OT □ respectively in the figure. Thus

$$OT = \left| x_0 - \frac{y_0}{\left(\frac{dy}{dx}\right)_p} \right| \text{ and } OT' = \left| y_0 - x_0\left(\frac{dy}{dx}\right)_p \right| .$$

ILLUSTRATIONS

***Illustration* 1**

The normal to the curve at (0, 3) be given by the equation 3 x – y + 3 = 0, then find the value of $\lim\limits_{x\to 0} x^2\left\{f\left(x^2\right)-5f\left(4x^2\right)+4f\left(7x^2\right)\right\}^{-1}$

Solution

The slope of normal is 3 therefore $f\,'(0) = -\frac{1}{3}$

Now, limit $L = \lim\limits_{x\to 0}\frac{x^2}{f\left(x^2\right)-5f\left(4x^2\right)+4.f\left(7x^2\right)}$

$$= \lim_{x\to 0}\frac{2x}{f\,'\left(x^2\right)\times 2x - 5f\,'\left(4x^2\right)\times 8x + 4.f\,'\left(7x^2\right)\times 14x}$$

$$= \lim_{x\to 0}\frac{2}{2.f\,'\left(x^2\right)-40f\,'\left(4x^2\right)+56f\,'\left(7x^2\right)} = -\frac{1}{3}$$

***Illustration* 2**

Find the acute angle between the curve $y=|x^2-1|$ and $y=|x^2-3|$ at their point of intersection.

Solution

Q $y=|x^2-1|$ and $y=|x^2-3|$ intersect each other

$\Rightarrow |x^2-1|=|x^2-3|$

$\Rightarrow \left(x^2-1\right)=\pm\left(x^2-3\right)$

$\Rightarrow$ point of intersection is $\left(\pm\sqrt{2},1\right)$

Q at $\left(\sqrt{2},1\right)$ slope of tangent to the curve $y=|x^2-1|=2x=2\sqrt{2}$

slope of tangent to the curve $y=|x^2-3|=-2x=-2\sqrt{2}$

$\Rightarrow \tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\left|\frac{2\sqrt{2}+2\sqrt{2}}{1-8}\right|=\frac{4\sqrt{2}}{7}$

$\Rightarrow \theta=\tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$

PRACTICE EXERCISE

1. The equation of tangent to the curve $(1 + x^2)y = 1$ at the points of its intersection with the curve $(x + 1)y = 1$ are given by ____

2. Equation of normal to the curve $x + y = x^y$, where it cuts x-axis is

3. The angle of intersection of curves $y = x^2$ and $6y = 7 - x^3$ at (1, 1) is ____.

4. The sum of intercepts made on the co-ordinate axes by any tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is equal to ____

5. The length of normal to the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ at 'θ' $= \pi/2$ is

Answers

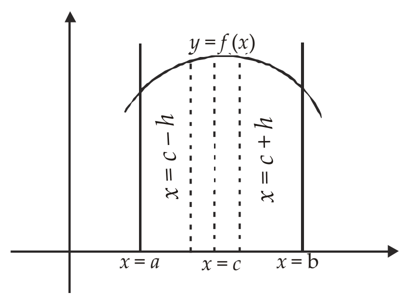
1. $y = 1, x + 2y = 2$ **2.** $x - y - 1 = 0$ **3.** $\frac{\pi}{2}$ **4.** a **5.** $a\sqrt{2}$

MAXIMA & MINIMA

❑ Local Maxima & Local Minima

Definition

A function is said to have local maxima at $x = c$, if there is an open interval (a, b) containing c on which $f(x)$ is the largest value i.e. $f(c) \geq f(x)$ for all x in the interval (a, b). Similarly f is said to have local minima at $x = c_1$ in (a, b) if $f(c_1) \leq f(x)$ for all x in (a, b)



Let function $y = f(x)$ which is continuous in $[a, b]$ is said to have local maxima at $x = c$. *iff*

$f(c) > f(c + h)$ & $f(c) > f(c - h)$ where $h > 0$

and local min at $x = c$. *iff*

$f(c) < f(c + h)$ & $f(c) < f(c - h)$ where $h > 0$

NOTE:

(*i*) The points at which a function attains either local max or local min are known as extreme point or simply extrema

(*ii*) The local maximum and local minimum values are known as relative maximum and relative minimum value.

(iii) Suppose that f is a function defined on an open interval containing the number x_0. If f has a relative extremum at $x = x_0$, then either $f'(x_0) = 0$ or f is not differentiable at x_0.

- **Critical Point**

  Those value of domain of a function $y = f(x)$ at which either $f'(x) = 0$ or f is not differentiable are known as critical point where concavity of curve is changed.

3.4 Local Maxima & Local Minima

- **Local Maxima And Local Minima By First Derivative**

  Let $f(x)$ be continuous at critical point $x = c$ and h be a small positive real value, then

$f'(c-h)$	$f'(c+h)$	Local Maxima/Local Minima
+	–	Local Maxima
–	+	Local Minima
+	+	Neither Minima Nor Maxima
–	–	Neither Minima Nor Maxima

- **Local Maxima And Local Minima By Second Derivative**

Step I:

Let $y = f(x)$, find $f'(x)$ and $f''(x)$

Step II:

Solve $f'(x) = 0$ and find all real solutions.

let $x = a$, b and c these points known as stationary point or critical points.

$f''(x)$ must be exists at all x where $f'(x) = 0$

Step III:

(i) $f''(a) > 0$ then at $x = a$ is point of local minima.

(ii) $f''(b) < 0$ then at $x = b$ is point of local maxima.

(iii) If $f''(c) = 0$ and $f'(c) = 0$. In such a situation we shall have to depends on following test.

nth derivative test

Let f be a function such that

(i) $f'(c) = f''(c) = ...f^{n-1}(c) = 0$

(ii) $f^{n}(c) = 0$

(a) $f(x)$ has local maximum at $x = c$ if n is even and $f^{n}(c) < 0$.

(b) $f(x)$ has local minimum at $x = c$ if n is even and $f^{n}(c) > 0$.

(c) $f(x)$ has no local extremum at $x = c$ if n is odd.

- **Finding Greatest & Least Value of a Continuous function in [*a, b*]**

  Step - I Find the critical point of $f(x)$ in (a, b).

  Step – II Evaluate $f(x)$ at all critical point and at the end points a and b.

Step -III The largest of the value in step - II is the absolute maximum value and smallest value is absolute minimum value.

PRACTICE EXERCISE

6. The co-ordinates of the point on the curve $x^3 = y(x - a)^2$, where the ordinate is minimum is ____

7. The minimum value of a secθ – $b$tan□ is ____

8. The least value of $2\log_{10} x - \log_x$ (0.1) for $x > 1$ is ____

9. The minimum value of $\log_{10}(4x^3 - 12x^2 + 11x - 3)$ is ______

10. The maximum value of tan A tan B if $A > 0$, $B > 0$ and $A + B = \pi/3$ is _____

Answers

6. $\left(3a, \dfrac{27a}{4}\right)$ **7.** $\sqrt{a^2 - b^2}$ **8.** 4 **9.** **5**

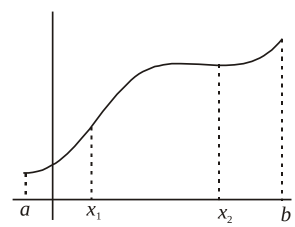
10. 1/2

Increasing & Decreasing Functions

❑ **Increasing Function:**

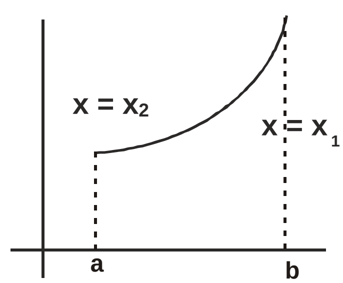
A function $y = f(x)$, is said to be an increasing function in $[a, b]$ if $x_1 > x_2$ □ $f(x_1)$ □ $f(x_2)$.

Where x_1, x_2 □ $[a, b]$



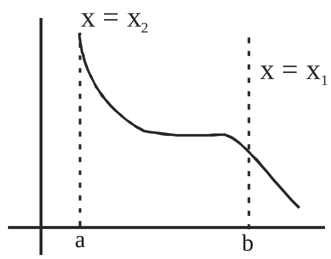
❑ **Strictly Increasing Function**

A function, $y = f(x)$ is said to strictly increasing in $[a, b]$ if $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$. Where $x_1, x_2 \in [a, b]$



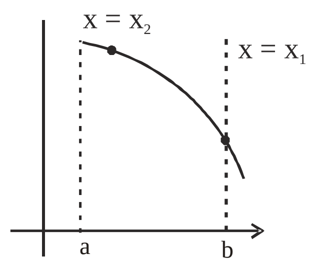
❑ Decreasing Function

A function $y = f(x)$ is said to be decreasing in $[a, b]$ if $x_1 > x_2$ *then* $f(x_1) \le f(x_2)$ where $x_1, x_2 \in [a,b]$



❑ Strictly Decreasing Function

A function $y = f(x)$ is said to be strictly decreasing function in $[a, b]$ if $x_1 > x_2 \Rightarrow f(x_1) < f(x_2), x_1, x_2 \in [a,b]$



3.5 Monotonic Function

A function $y = f(x)$ is said to be monotonic in a given interval if it is either strictly increasing or strictly decreasing

❑ Test of Monotonic Function

(*i*) $f(x)$ is said to be increasing in $[a,b]$ if $f'(x) \ge 0$ in $[a,b]$

(*ii*) $f(x)$ is said to be strictly increasing in $[a,b]$ if $f'(x) > 0$ in (a,b)

(*iii*) $f(x)$ is decreasing in $[a,b]$ if $f'(x) \le 0$ in $[a, b]$

(*iv*) $f(x)$ is said to be strictly decreasing in $[a,b]$ of $f'(x) < 0$ in (a,b)

Note:

1. If a function $f(x)$ is said to increasing in $[a,b]$ then f^{-1} exists and is also increasing in $[a, b]$.
2. If $f(x)$ and $g(x)$ both are increasing function then $(gof)(x)$ is an increasing function .
3. If the function $f(x)$ is increasing and $g(x)$ is decreasing then composite $(gof)(x)$ is decreasing.

4. A function may be increasing in some interval I_1 and decreasing in some other interval I_2.

3.6 Rolle's And Lagrange's Mean Value Theorem

❑ Rolle's Theorem

If a function $y = f(x)$ is defined in $[a,b]$ and

(i) $f(x)$ is continuous in $[a,b]$

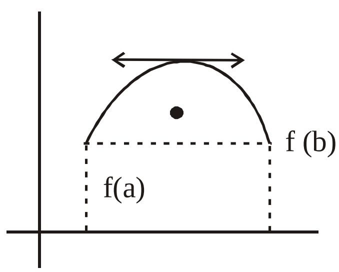
(ii) $f(x)$ is differentiable in (a,b) and

(iii) $f(a) = f(b)$

Then there will be atleast one value of $c \in (a, b)$ such that $f'(c) = 0$

❑ Geometrical Meaning of Rolle's Theorem

Let a curve be continuous in [a, b] and differentiable in (a, b) then there will be atleast one tangents parallel to x–axis.



❑ Lagrange's Mean Value Theorem

If a function $f(x)$ is said to be defined on [a,b] and

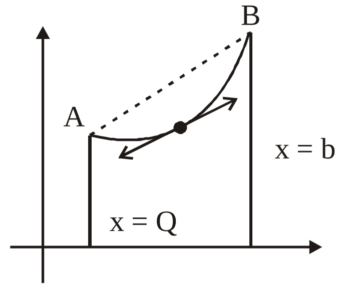
(i) Continuous in $[a, b]$

(ii) differentiable in (a, b)

then there will be atleast one value $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

❑ Geometric Meaning

If $f(x)$ is continuous and differentiable in $[a, b]$ then there will be atleast one tangent to the curve which will be parallel to the chord joining the point A and B.



PRACTICE EXERCISE

11. If the function $f(x) = x \sec^2 x - \tan x$ in x □ [0, 2□] is an increasing function, then x□□□□□□□□□□□□

12. The function $f(x) = \log x$ is strictly increasing for x □□□□□□□□□□□

13. The interval in which $f(x) = x^3 \log_3 x$ is increasing is □□□□□□□□

14. Verify Rolle's theorem for $f(x) = \tan x$ in [0, □].

15. Lagrange's mean value theorem for $f(x) = x(x-1)(x-2)$ in [0, 1/2] is applicable at x =

______ .

Answers

11. $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ 12**.** (□, □) **13.** ($e^{-1/3}$, □) **14.** Not applicable

15. 0.24.

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MISCELLANEOUS PROBLEMS

OBJECTIVE TYPE

Example 1

The points on the curve $y = x^3 - 2x^2 - x$ at which the tangent lines are parallel to the line $y = 3x - 2$.

(a) (2, 2) (b)(2, –2) (c) $\left(\frac{-2}{3}, \frac{-14}{27}\right)$ (d) $\left(\frac{2}{3}, \frac{-14}{27}\right)$

Solution

Let $P(x_1, y_1)$ be the required point. The given cure is

$y = x^3 - 2x^2 - x$

$\Rightarrow \frac{dy}{dx} = 3x^2 - 4x - 1 \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 3x_1^2 - 4x_1 - 1$

Since the tangent at (x_1, y_1) is parallel to the line $y = 3x - 2$.

$\therefore$ Slope of the tangent at (x_1, y_1) = slope of the line $y = 3x - 2$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 3 \Rightarrow 3x_1^2 - 4x_1 - 1 = 3.$

$\Rightarrow 3x_1^2 - 4x_1 - 4 = 0 \Rightarrow (x_1 - 2)(3x_1 + 2) = 0 \Rightarrow x_1 = 2, \frac{-2}{3}.$

Since (x_1, y_1) lies on (i), therefore $y_1 = x_1^3 - 2x_1^2 - x_1$

When $x_1 = 2; y_1 = 2^3 - 2(2)^2 - 2 = -2.$

When $x_1 = \frac{-2}{3}; y_1 = \left(\frac{-2}{3}\right)^3 - 2\left(-\frac{2}{3}\right)^2 + \frac{2}{3} \frac{-14}{27}.$

Thus, The required points are (2, –2) and $\left(\frac{-2}{3}, \frac{-14}{27}\right)$,

 Ans. (b, c)

Example 2

The slope(s) of the tangent and the normal to the curve $x^2 + 3y + y^2 = 5$ at (1, 1) is/are

(a) $\frac{-2}{5}$ (b) $\frac{-5}{2}$ (c) $\frac{5}{2}$ (d) $\frac{2}{5}$

Solution

The equation of the curve is $x^2 + 3y + y^2 = 5$.

Differentiating w.r.t. x, we get

$2x + 3\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-2x}{2y+3} \Rightarrow \left(\frac{dy}{dx}\right)_{(1,\,1)} = -\frac{2}{5}$

$\therefore$ Slope of the tangent at (1, 1) = $\left(\frac{dy}{dx}\right)_{(1,1)} = \frac{-2}{5}$.

And, Slope of the normal at (1, 1) = $\frac{-1}{\left(\frac{dy}{dx}\right)_{(1,1)}} = \frac{-1}{\frac{-2}{5}} = \frac{5}{2}$.

$\therefore$ **Ans. (c)**

Example 3

The point on the curve $y = x^3 - 11x + 5$ at which the tangent has the equation $y = x - 11$ is

(a) (2, 9) (b)(2,–9) (c) (–2, 9) (d) (–2, –9)

Solution

Let the required point be $P(x_1, y_1)$. Then

$y_1 = x_1^3 - 11x_1 + 5$ …(i) [$\because$ (x_1, y_1) lies on $y = x^3 - 11x + 5$]

Now, $y = x^3 - 11x + 5$

$\Rightarrow \frac{dy}{dx} = 3x^2 - 11 \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = 3x_1^2 - 11$

Since the line $y = x - 11$ is tangent at the point (x_1, y_1), therefore,

Slope of the tangent at (x_1, y_1) = slope of the line $y = x - 11$.

$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)}$ = slope of the line $x - y - 11 = 0$

$\Rightarrow 3x_1^2 - 11 = \frac{-1}{-1} \Rightarrow 3x_1^2 = 12 \Rightarrow x_1 = \pm 2$

When $x_1 = 2, y_1 = 2^3 - 22 + 5 = -9$ [Using (i)]

When $x_1 = -2, y = (-2)^3 - 11(-2) + 5 = 19$ [Using (i)]

So, two points are (2, –9) and (–2, 19). Of these two point (–2, 19) does not lie on $y = x - 11$. Therefore the required point is (2, –9).

$\therefore$ **Ans. (b)**

Example 4

The equations of the tangent to the parabola $y^2 = 4a\,x$ at the point $(at^2, 2at)$ is

(a) $x + y = at^2$ (b) $tx = y + at^2$ (c) $ty = x + at^2$ (d) $x + ty = 0$

Solution

The equation of the given curve is

$y^2 = 4ax$ ...(i)

Differentiating (i) w.r.t. x, we get

$$2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y} \Rightarrow \left(\frac{dy}{dx}\right)_{(at^2,\ 2at)} = \frac{2a}{2at} = \frac{1}{t}$$

So, the equation of the tangent at $(at^2, 2at)$ is

$$y - 2at = \frac{1}{t}\left(\frac{dy}{dx}\right)_{(at^2,\ 2at)} (x - at^2)$$

$$\Rightarrow \ y - 2at = \frac{1}{t}(x - at^2) \Rightarrow \ ty = x + at^2$$

□ **Ans. (c)**

Example 5

The function $f(x) = \frac{x}{\log x}$ increases on the interval

(a) $(0, \infty)$ (b) $(0, e)$ (c) (e, ∞) (d) none of these

Solution

Clearly, $f(x)$ is defined for $x > 0$.

Now, $f(x) = \frac{x}{\log x} \Rightarrow f'(x) = \frac{\log x - 1}{(\log x)^2}$

$\therefore \ f'(x) > 0 \Rightarrow \log x - 1 \Rightarrow \log x > 1 \Rightarrow x > e \Rightarrow x \in (e, \infty)$.

□ **Ans. (c)**

Example 6

The function $f(x) = 2 \log (x - 2) - x^2 + 4x + 1$ increase on the interval

(a) $(1, 2)$ (b)$(2, 3)$ (c) $(5/2, 3)$ (d) $(2, 4)$

Solution

$f(x) = 2 \log (x - 2) - x^2 + 4x + 1$

$$\Rightarrow \ f'(x) = \frac{2}{x-2} - 2x + 4$$

$$\Rightarrow \quad f'(x) = 2\left[\frac{1-(x-2)^2}{x-2}\right] = -2\frac{(x-1)\,(x-3)}{x-2}$$

$$\Rightarrow \quad f'(x) = \frac{-2\,(x-1)\,(x-3)\,(x-2)}{(x-2)^2}$$

$\therefore \quad f'(x) > 0 \Rightarrow -2\,(x-1)\,(x-3)\,(x-2) > 0$

$\Rightarrow \quad (x-1)\,(x-2)\,(x-3) < 0$

$\Rightarrow \quad x \in (-\infty, 1) \cup (2, 3).$

Thus, $f(x)$ is increasing

on $(-\infty, 1) \cup (2, 3)$.

□ **Ans. (b, c)**

Example 7

For $x > 1$, $y = \log x$ satisfies the inequality

(a) $x - 1 > y$ (b) $x^2 - 1 > y$ (c) $y > x - 1$ (d) $\frac{x-1}{x} < y$

Solution

Let $f(x) = \log x - (x - 1)$

Then, $f'(x) = \frac{1}{x} - 1 = \frac{1-x}{x}$

Clearly, $f'(x) < 0$ for $x > 1$.

$\Rightarrow \quad f(x)$ is decreasing function for $x > 1$

$\Rightarrow \quad f(x) < f(1)$ for $x > 1$

$\Rightarrow \quad \log x - (x - 1) < 0$ for $x > 1$

$\Rightarrow \quad \log x < x - 1$ for $x > 1$

But $x^2 - 1 > x - 1$ for $x > 1$.

$\therefore \quad x^2 - 1 > x - 1$ and $\log x < x - 1 \Rightarrow \log x < x^2 - 1$.

Similarly, it can be proved that $\frac{x-1}{x} < \log x$.

□ **Ans. (a, b, d)**

Example 8

Let $f(x) = x^3 + ax^2 + bx + 5 \sin^2 x$ be an increasing function on the set R. Then a and b satisfy

(a) $a^2 - 3b - 15 > 0$ (b) $a^2 - 3b + 15 > 0$ (c) $a^2 - 3b + 15 < 0$ (d) $a > 0$ and $b > 0$

Solution

$f(x) = x^3 + ax^{2+} + bx + 5 \sin^2 x$ is increasing on R.

$\Rightarrow$ $f'(x) > 0$ for all $x \in R$

$\Rightarrow$ $3x^2 + 2ax + b + 5 \sin 2x > 0$ for al l $x \in R$

$\Rightarrow$ $3x^2 + 2ax + (b - 5) > 0$ for all $x \in R$

$\Rightarrow$ $(2a)^2 - 4\ \ 3\ \ (b - 5) < 0$

$\Rightarrow$ $a^2 - 3b + 15 < 0$

 Ans. (c)

Example 9

The value of 'a' for which the function $f(x) = a \sin x + (1/3) \sin 3x$ has an extremum at $x = \frac{\pi}{3}$ is

(a) 1 (b) –1 (c) *0* (d) -1

Solution

If $f(x)$ has an extremum at $x = \pi/3$, then

$f'(x) = 0$ at $x = \pi/3$.

Now, $f(x) = a \sin x + \frac{1}{3} \sin 3x$

$\Rightarrow$ $f'(x) = a \cos x + \cos 3x$

$\therefore$ $f'(\pi/3) = 0 \Rightarrow a \cos(\pi/3) + \cos \pi = 0 \Rightarrow a = 2$.

 Ans. (d)

Example 10

If $f(x) = a \log |x| + bx^2 + x$ has its extremum values at $x = -1$ and $x = 2$, then

(a) $a = 2, b = -1$ (b) $a = 2, b = -1/2$ (c) $a = -2, b = 1/2$ (d) none of these

Solution

We have, $f(x) = a \log |x| + b x^2 + x$

$\therefore$ $f'(-1) = 0$ and $f'(2) = 0$

$\Rightarrow$ $-a - 2b + 1 = 0$ and $\frac{a}{2} + 4b + 1 = 0 \Rightarrow a = 2$ and $b = -1/2$.

 Ans. (b)

Example 11

The difference between the greatest and least values of the function $f(x) = \sin 2x - x$ on $[-\pi/2, \pi/2]$ is

(a) $\frac{\sqrt{3}+\sqrt{2}}{2}$ (b) $\frac{\sqrt{3}+\sqrt{2}}{2}+\pi/6$ (c) $\frac{\sqrt{3}}{2}+\frac{\pi}{3}$ (d) π

Solution

We have, $f'(x) = 2\cos 2x - 1$

$$\therefore \quad f'(x) = 0 \Rightarrow 2\cos 2x - 1 = 0 \Rightarrow \cos 2x = \frac{1}{2}$$

$$\Rightarrow 2x = -\pi/3, \pi/3 \Rightarrow x = -\pi/6, \pi/6$$

Now, $f(-\pi/2) = \pi/2$, $f(\pi/2) = -\pi/2$,

$$f(-\pi/6) = -\frac{\sqrt{3}}{2}+\frac{\pi}{6} \text{ and } f(\pi/6) = \frac{\sqrt{3}}{2}-\frac{\pi}{6}.$$

Clearly, $\frac{\pi}{2}$ is the greatest value of $f(x)$ and its least value is $-\pi/2$. Hence, the required difference is

$$\pi/2 - (-\pi/2) = \pi.$$

□ **Ans. (d)**

Example 12

The value of a so that the sum of the squares of the roots of the equation $x^2 - (a-2)x - a + 1 = 0$ assume least value, is

(a) 2 (b) 0 (c) 3 (d) 1

Solution

Let α and β be the roots of the given equation, so that

$$\alpha + \beta = a - 2 \text{ and } \alpha\beta = -(a-1).$$

Let $S = \alpha^2 + \beta^2$. Thus,

$$S = (\alpha^2 + \beta)^2 - 2\alpha\beta = (a-2)^2 + 2(a-1) = a^2 - 2a + 2.$$

$$\therefore \quad \frac{ds}{da} = 2a - 2.$$

Now, $\frac{dS}{da} = 0 \Rightarrow a = 1$. Also, $\frac{d^2S}{da^2} = 2 > 0$.

Hence, S is minimum when $a = 1$.

□ **Ans. (d)**

SUBJECTIVE TYPE

Example 1

Find the interval of monotonocity of the following function's

(i) $f\left(x\right)=\dfrac{x}{x^2-6x-16}$ *(ii)* $f\left(x\right)=\dfrac{1-x+x^2}{1+x+x^2}$

Solution

(i) ∵ $f\left(x\right)=\dfrac{x}{x^2-6x-16}$

$$\Rightarrow f'\left(x\right)=\frac{\left(x^2-6x-16\right)-x\left(2x-6\right)}{\left(x^2-6x-16\right)^2}$$

$$=\frac{x^2-6x-16-2x^2+6x}{\left(x-8\right)^2\left(x-2\right)^2}=\frac{-\left(x^2+16\right)}{\left(x-8\right)^2\left(x-2\right)^2}$$

f(x) is decreasing in $\left(-\infty,-2\right)\cup\left(-2,8\right)\cup\left(8,\infty\right)$

(ii) ∵ $f(x)=\dfrac{1-x+x^2}{1+x+x^2}$

$$f'(x)=\frac{2\left(x^2-1\right)}{\left(1+x+x^2\right)}$$

f(*x*) is increasing in $\left(-\infty,-1\right)\cup\left(1,\infty\right)$ and decreasing in (–1,1)

Example 2

If '*a*' is a positive constant. Find the interval in which $f'(x)$ is increasing.

where $f(x)=\begin{cases} x.e^{ax} & x\le 0 \\ x+ax^2-x^3 & x>0 \end{cases}$

Solution

f(x) is continuous and differentiable at $x = 0$.

$$f'(x)=\begin{cases} \left(1+ax\right)e^{ax} & x\le 0 \\ 1+2ax-3x^2 & x>0 \end{cases}$$

⇒ $f'(x)$ is also continuous and differentiable at $x = 0$

$$f''(x)=\begin{cases} \left(2+ax\right)a.e^{ax} & x\le 0 \\ 2a-6x & x>0 \end{cases}$$

For $f'(x)$ is increasing function we must have $f''(x)>0$

Case-I:

$x \le 0$ In this case $f''(x) > 0$

$\Rightarrow$ $(2+ax)a.e^{ax} > 0$

$\Rightarrow$ $2+ax > 0$ [as $e^{ax} > 0$ x □ R]

$\Rightarrow$ $x > -2/a$

$\Rightarrow$ $x \in \left(-\frac{2}{a}, 0\right)$ ...(i)

Case-II:

When

$f''(x) > 0$

$\Rightarrow$ $2a - 6x > 0$

$\Rightarrow$ $2a > 6x$

$\Rightarrow$ $x < a/3$

$\Rightarrow$ $x \in \left(0, \frac{a}{3}\right)$ ...(ii)

From (i) and (ii) we get $f'(x)$ will be increasing if $x \in \left(\frac{-2}{a}, 0\right) \cup \left(0, \frac{a}{3}\right)$

Example 3

Prove the following in equalities

(i) $x - \frac{x^3}{6} < \sin x < x$ for all $x \in \left(0, \frac{\pi}{2}\right)$

(ii) $x + \frac{x^3}{6} > \tan^{-1} x$ for all $x \in (0,1)$

Solution

(i) Let $f(x) = \sin x - x$

$\Rightarrow$ $f'(x) = \cos x - 1$

Q $x \in \left(0, \frac{\pi}{2}\right), 0 < \cos x < 1$

$\Rightarrow$ $f'(x) < 0$

Hence $f(x)$ is a decreasing functions.

Now, let $x > 0$

$\Rightarrow$ $f(x) < f(0)$

$\Rightarrow \quad \sin x - x < 0$

$\Rightarrow \quad \sin x < x$ for all $x \in \left(0, \frac{\pi}{2}\right)$ ....(i)

Let $f(x) = x - \frac{x^3}{6} - \sin x$

$\Rightarrow f'(x) = 1 - \frac{3x^2}{6} - \cos x = \left(1 - \frac{x^2}{2} - \cos x\right)$

Let $g(x) = 1 - \frac{x^2}{2} - \cos x$

$\Rightarrow g'(x) = -x + \sin x$

$\because g'(x) < 0$ in $\left(0, \frac{\pi}{2}\right)$ hence $g(x)$ is a decreasing function.

Let $x > 0$

$\Rightarrow \quad g(x) < g(0)$

$\Rightarrow \quad x - \frac{x^3}{6} < \sin x$ ...(ii)

From (1) and (2), we get

$x - \frac{x^3}{6} < \sin x < x$ for all $x \in \left(0, \frac{\pi}{2}\right)$

(ii) Let $f(x) = x + \frac{x^3}{6} - \tan^{-1} x$

$f'(x) = 1 + \frac{x^2}{2} - \frac{1}{1+x^2}$

$= \frac{2+x^2}{2} - \frac{1}{1+x^2}$

$= \frac{(2+x^2)(1+x^2) - 2}{2(1+x^2)} > 0$ for all x

Hence $f(x)$ is an increasing function.

Let $x > 0$

$\Rightarrow \quad f(x) > f(0)$

$\Rightarrow \quad x + \frac{x^3}{6} - \tan^{-1} x > 0$

$\Rightarrow \quad x + \frac{x^3}{6} > \tan^{-1} x$ for all $x \in (0,1)$

Example 4

If P (1) = 0 and $\frac{d}{dx}P(x) >$ $P(x)$ for all $x \geq 1$ then prove that $P(x) > 0 \ \forall \ x \geq 1$

Solution

$$e^{-x}.\frac{dP(x)}{dx} > e^{-x}.P(x)$$

$$\Rightarrow \quad e^{-x}.\frac{dP(x)}{dx} - e^{-x}.P(x) > 0$$

$$\Rightarrow \quad \frac{d}{dx}\left\{P(x).e^{-x}\right\} > 0 \quad \text{for all } x \geq 1$$

$\Rightarrow$ $P(x)$. e^{-x} is an increasing function for all $x \geq 1$.

$\Rightarrow$ $P(x) > 0$ for all $x \geq 1$ proved.

Example 5

Find the interval in which the function $f(x) = \sin(\log x) + \cos(\log x)$ is decreasing.

Solution

$$f(x) = \sin(\log x) + \cos(\log x)$$

$$\Rightarrow f'(x) = \frac{\cos(\log x) - \sin(\log x)}{x} < 0$$

Q $f(x)$ is decreasing function

as $x > 0$, to be a decreasing function

$$\Rightarrow \cos(\log x) - \sin(\log x) < 0$$

$$\Rightarrow \cos(\log x) < \sin(\log x)$$

$$2\pi + \frac{\pi}{4} < \log x < 2\pi k + \frac{5\pi}{4}$$

$$\Rightarrow e^{2\pi k + \frac{\pi}{4}} < x < e^{2\pi k + \frac{5\pi}{4}} \qquad \text{where } k \pm 0 \pm 1 \pm 2........$$

Example 6

Find the maximum value of $f(x) = \frac{40}{3x^4 + 8x^3 - 18x^2 + 60}$

Solution

Given $f(x) = \frac{40}{3x^4 + 8x^3 - 18x^2 + 60}$ is maximum or minimum according to whether $3x^4+8x^3-18x^2 + 6$ is minimum or maximum respectively.

Let $g(x) = 3x^4 + 8x^3 - 18x^2 + 60$

$g\square(x) = 12x^3 + 12x^2 - 36x = 12x(x+3)(x-1)$

$g(-3) = -75 \quad g(1) = 53$

$g(0) = 60$

$g(x)$ is maximum at $x = 0$

$g(x)$ is minimum at $x = -3, 1$

$\Rightarrow$ Maximum value of $g(x)$ = minimum value of $f(x)$ = 2/3. Maximum value of $f(x)$.

Maximum value of $f(x) = \frac{40}{53}, \frac{-8}{15}$

Example 7

Find the maxima and minima for the function given by $f(x) = \sin x + \cos 2x$ in $[0, 2\pi]$

Solution

Give $f(x) = \sin x + \cos 2x$

$\Rightarrow f'(x) = \cos x - 2\sin 2x$

$= \cos x - 4\sin x.\cos x$

$= \cos(1 - 4\sin x)$

$\Rightarrow f''(x) = -\sin x - 4\cos 2x$

When $f'(x) = 0$ we get $\cos x = 0, \sin x = \frac{1}{4}$

$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}, \sin^{-1}\left(\frac{1}{4}\right), \pi - \sin^{-1}\frac{1}{4}$

When $\Rightarrow f''\left(\frac{\pi}{2}\right) = 3 > 0$

$\Rightarrow f''\left(\frac{3\pi}{2}\right) = 5 > 0$

$x = \sin^{-1}\left(\frac{1}{4}\right) \quad \Rightarrow f''(x) = \frac{-15}{4} > 0$

$x = \pi - \sin^{-1}\left(\frac{1}{4}\right) \quad \Rightarrow f''(x) = \frac{-15}{4} > 0$

$f(x)$ will be max at $x = \sin^{-1}\frac{1}{4}, \pi - \sin^{-1}\frac{1}{4}$

$f(x)$ will be min at $x = \frac{\pi}{2}, \frac{3\pi}{2}$

maximum value of $f(x) = \frac{9}{8}$ and minimum value is 0, 2.

minimum value of $f(x) = 0, -2$

Example 8

A box of maximum volume with top open is to be made by cutting out four equal square's from four corners of a square tin sheet of side length a ft. and folding up the flaps. Find the sides of square.

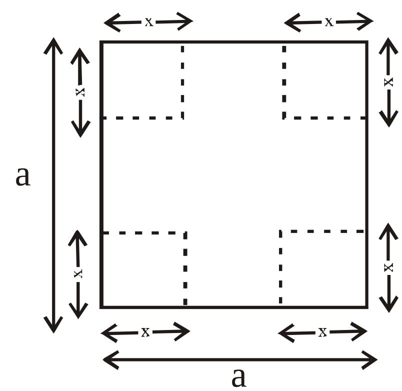
Solution

Let sides of square a is x.

$$\Rightarrow V = (a-2x)^2 .x$$

$$\Rightarrow \frac{dV}{dx} = (a-2x)^2 - 4x(a-2x)$$

$$\Rightarrow \frac{dv}{dx} = 0$$ for maxima and minima.



$$\Rightarrow x = \frac{a}{2}, \frac{a}{6}$$

$$\frac{d^2v}{dx^2} = -2(a-6x) - 6(a-2x)$$

at $x = \frac{a}{2}$ $\frac{d^2v}{dx^2} < 0$ but this is not possible

at $x = \frac{a}{6}$ $\frac{d^2v}{dx^2} < 0$ at $x = a/6$

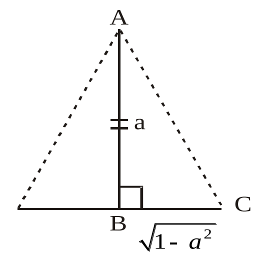
Hence volume is max at $x = a/6$

Example 9

Find the greatest volume of a right circular cone that can be described by revolution about a side of a right angled triangle of hypotenuse 1 ft.

Solution

Let *ABC* is a right angled triangle and length of side *AB* is *a*



□ $BC = \sqrt{1-a^2}$

and cone be revolved about *AB*

□ $V = \frac{1}{3}\pi\left(1-a^2\right)a$

$= \frac{\pi}{3}\left(a-a^3\right)$

$\frac{dv}{da} = \frac{\pi}{3}\left(1-3a^2\right)$

$\frac{d^2v}{da^2} = -2\pi a < 0$

when $\frac{dv}{da} = 0$ we get $a = \frac{1}{\sqrt{3}}$

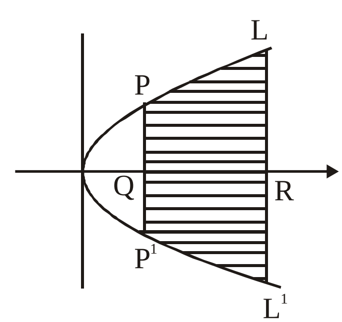
putting $a = \frac{1}{\sqrt{3}}$ we get $\frac{d^2v}{da^2} < 0$.

Hence maximum volume is $\frac{2\sqrt{3}\pi}{27}$.

Example 10

Let *LL*' is the latus rectum of the parabola $y^2 = 4ax$ and *PP*' is a double ordinate between the vertex and latus return. Show that area of trapezium *PP*'*LL*' is maximum when distance of *PP*' from the vertex is *a*/9.

Solution



Let double ordinate *PP*' drawn at $x = at^2$ from the vertex

Thus P □ $(at^2, 2at)$

Now Area = $\frac{1}{2}\left(PP' + LL'\right)QR$

□ $A = 2a^2 (t + 1)(1 - t^2)$

$$\frac{dA}{dt} = 2a^2\left\{\left(1-t^2\right)-2t^2-2t\right\} \qquad = 0, \text{ we get } t = 1, 1/3$$

$$\frac{d^2A}{dt^2} = 2a^2\{-6t-2\} < 0 \text{ at } t = 1/3.$$

∴ Area is max at $t = 1/3$

Hence x = at^2 is a point at which area is maximum = $\frac{a}{9}$

Example 11

$A\,(x_1,\, y_1)$ & $B\,(x_2,\, y_2)$ are the two points on the curve $f(x) = ax^2 + bx + c$. Find a point $P\,(x_3, y_3)$ on the curve such that slope of tangent at point P becomes equal to slope of chord AB i.e. $m\,(T_p) = m\,(AB)$.

Solution

Let's apply Lagbrange's mean value theorem for the function . $f(x) = ax^2 + bx + c$ in $[x_1,\, x_2]$ The function $f(x) = ax^2 + bx + c$ is a polynomial function of x and is therefore continuous in $[x_1, x_2]$.

$f'(x) = 2ax + b,$ therefore $f'(x)$ exists in the open interval (x_1, x_2).

Thus both the conditions of Lagrange's mean value theorem are satisfied, so their must exists a point $x_3 \in (x_1,\, x_2)$ such that

$$f'(x_3) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{a(x_2^2 - x_1^2) + b(x_2 - x_1)}{(x_2 - x_1)} = a(x_2 + x_1) + b$$

$$\Rightarrow\ 2ax_3 + b = ax_2 + b \qquad \Rightarrow\ x_3 = \frac{(x_2 + x_1)}{2}$$

$$\&\ y_3 = a\left(\frac{x_2 + x_1}{2}\right)^2 + b\left(\frac{x_2 + x_1}{2}\right) + c$$

*****

Exercise - I

OBJECTIVE TYPE QUESTIONS

Single Choice Questions

1. The tangent to the curve $y = x^3 - 6x^2 + 9x + 4$, $0 \le x \le 5$ has maximum slope at x which is equal to

   (a) 4 (b) 2 (c) 3 (d) None of these

2. The number of tangents to the curve $y^2 - 2x^3 - 4y + 8 = 0$ that passes through (1, 2) is

   (a) 1 (b) 3 (c) 6 (d) 2

3. The equation of the tangent to the curve $y = e^{-|x|}$ at the point where the curve cuts the line x = 1 is :

   (a) $y + ex = 1$ (b) $e(x + y) = 1$ (c) $x + y = e$ (d) none of these

4. Let $f(x) = (x - 1)^4 . (x - 2)^n$, $n \in N$. Then f(x) has :

   (a) a maximum at x = 1 if n is even (b) a maximum at x = 2 if n is odd

   (c) a maximum at x = 1 if n is odd (d) None of these

5. The two curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2 y - y^3 - 2 = 0$:

   (a) cut at an angle $\frac{\pi}{4}$ (b) cut at right angles (c) touch each other (d) cut at an angle $\frac{\pi}{3}$

6. Let $f(x) = 2 \sin^3 x - 3 \sin^2 x + 12 \sin x + 5$, $0 \le x \le \frac{\pi}{2}$. Then f(x) is :

   (a) increasing in $\left[0, \frac{\pi}{2}\right]$

   (b) increasing in $\left[0, \frac{\pi}{4}\right]$ and decreasing in $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

   (c) decreasing in $\left[0, \frac{\pi}{2}\right]$

   (d) None of these

7. The function $f(x) = \sin^4 x + \cos^4 x$ increases if :

   (a) $\frac{5\pi}{8} < x < \frac{3\pi}{4}$ (b) $\frac{\pi}{4} < x < \frac{3\pi}{8}$ (c) $0 < x < \frac{\pi}{8}$ (d) $\frac{3\pi}{4} < x < \frac{5\pi}{8}$

8. If $4a + 2b + c = 0$ then the equation $3ax^2 + 2bx + c = 0$ has at least one real root lying between :

   (a) 0 and 2 (b) 1 and 2 (c) 0 and 1 (d) None of these

9. Let $f(x) = 2x^2 - \log|x|, x \neq 0$, then f(x) is :

(a) monotonically increasing in $\left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$

(b) monotonically increasing in $\left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, +\infty\right)$

(c) monotonically increasing in $\left(-\infty, \frac{-1}{2}\right) \cup \left(0, \frac{1}{2}\right)$

(d) None of these

10. If $y = 4x - 5$ is a tangent to the curve $y^2 = px^3 + q$ at (2, 3), then :

(a) p = 2, q = - 7 (b) p = - 2, q = 7 (c) p = - 2, q = -7 (4) p = 2, q = 7

11. Tangents to the curve $y = \cos(x + y)$, $-2\pi \leq x \leq 2\pi$ and parallel to the line $x + 2y = 0$ are :

(a) $2x + 4y + 3\pi = 0$, $2x + 4y - \pi = 0$ (b) $2x + 4y - 3\pi = 0$, $2x + 4y + \pi = 0$

(c) $x + y - \pi = 0$, $x + 2y + \pi = 0$ (d) None of these

12. The angle formed by the y-axis and the tangent to the parabola $y = x^2 + 4x - 17$ at the point $P\left(\frac{5}{2}, -\frac{3}{4}\right)$:

(a) $\frac{\pi}{2} - \tan^{-1} 9$ (b) $\pi - \tan^{-1} 9$ (c) $\frac{\pi}{4} - \tan^{-1} 8$ (d) None of these

13. The portion of the tangent to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ which is intercepted between the axes is :

(a) of constant length (b) of variable length (c) a = 2 + 3i (d) None of these

14. The function $f(x) = \int_{-1}^{x} t(e^t - 1)(t-1)(t-2)^3(t-3)^5\,dt$ has a local maximum at x = :

(a) 0 (b) 1 (c) 2 (d) 4

15. A cone is circumscribed to a sphere of radius r, if the volume of cone is minimum, then the altitude is :

(a) 2r (b) r (c) 3r (d) 4r

More than one option correct

1. If $f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \leq x \leq 2 \\ 37 - x, & 2 < x \leq 3 \end{cases}$

Then

(a) $f(x)$ is increasing on [–1, 2] (b) $f(x)$ is continuous on [–1, 3]

(c) $f'(2)$ doesn't exist (d) $f(x)$ has the maximum value at $x = 2$

2. The critical points of the function $f(x) = (x - 2)^{2/3}(2x + 1)$ are

(a) –1 and 2 (b) 1 (c) 1 and –1/2 (d) 1 and 2

6. An extremum value of the function $f(x) = (\sin^{-1} x)^3 + (\cos^{-1} x)^3$ $(-1 < x < 1)$ is

(a) $7\pi^3/8$ (b) $\pi^3/8$ (c) $\pi^3/32$ (d) $\lambda\pi^3/16$

4. Let $f(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$ then

(a) f increases on $[1, \infty)$ (b) f decreases on $[1, \infty)$

(c) f has a minimum at $x = 1$ (d) f has neither maximum nor minimum

5. The critical points of the function $f(x) = \dfrac{|x-1|}{x^2}$ are

(a) 0 (b) 1 (c) 2 (d) –1

6. Let $f(x) = (x^2 - 1)^n (x^2 + x - 1)$ then $f(x)$ has local extremum at $x = 1$ when

(a) $n = 2$ (b) $n = 3$ (c) $n = 4$ (d) $n = 5$

7. If $\phi(x) = f(x) + f(2a - x)$ and $f''(x) > 0$, $a > 0$, $0 \le x \le 2a$ then

(a) $\phi(x)$ increases in $(a, 2a)$ (b) $\phi(x)$ increasing in $(0, a)$

(c) $\phi(x)$ decreases in $(a, 2a)$ (d) $\phi(x)$ decreases in $(0, a)$

8. The minimum value of the function defined by $f(x) = \text{maximum } \{x, x + 1, 2 - x\}$ is

(a) 0 (b) ½ (c) 1 (d) 3/2

9. On $[1, e]$, the least and greatest values of $f(x) = x^2 \ln x$ is

(a) e, 1 (b) 1, e (c) 0, e^2 (d) none of these

10. The maximum area of the rectangle that can be inscribed in a circle of radius r is

(a) πr^2 (b) r^2 (c) $\dfrac{\pi r^2}{4}$ (d) $2r^2$

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Exercise – II

GENERAL TYPE (Assertion & Reason / Passage Based / Matching Type Questions)

Assertion & Reason Type

In the following question, a statement of Assertion (A) is given which is followed by a corresponding statement of reason (R). Mark the correct answer out of the following options/codes.

(a) If both (A) and (R) are true and (R) is the correct explanation of (A).

(b) If both (A) and (R) are true but (R) is not correct explanation of (A).

(c) If (A) is true but (R) is false.

(d) If (A) is false but (R) is true.

1. A: The function $f(x) = |x^3|$ is differentiable at $x = 0$

   R: At $x = 0$, $f'(x) = 0$

2. A: $x^3 + 4x - 2 = 0$ has exactly one root in the interval (0, 1)

   R: $f(x) = x^3 + 4x - 2$ is increasing in (0, 1)

3. A: If $f(x) = e^{(x-1)(x-3)}$ then Rolle's theorem applies for $f(x)$ in [1, 3]

   R: Lagrange's Mean Value Theorem is applied for $f(x) = e^{(x-1)(x-3)}$ in any interval.

4. A: $f(x) = 1 + 3x^2 + 3^2x^4 + 3^3x^6 + ... + 3^{30}x^{60}$ has exactly one point of local minima.

   R: $f'(x)$ changes sign at $x = 0$ only.

5. A: Let $f(x) = 5 - 4(x-2)^{2/3}$, then at $x = 2$, the function $f(x)$ attains neither least value nor greatest value.

   R: $x = 2$ is the only critical point.

Passage Based Questions

Passage – 1

Let $y = a\sqrt{x} + bx$ be curve and $(2x - y) + \lambda(2x + y - 4) = 0$ be family of lines.

1. If the curve has slope $-\frac{1}{2}$ at (9, 0) then a tangent belonging to family of lines is

   (a) $x + 2y - 5 = 0$ (b) $x - 2y + 3 = 0$

   (c) $3x - y - 1 = 0$ (d) $3x + y - 5 = 0$

2. A line of the family cutting positive intercepts on axes and forming triangle with coordinate axes, then minimum length of the line segment between axes is

(a) $\left(2^{2/3}-1\right)^{3/2}$ (b) $\left(2^{2/3}+1\right)^{3/2}$ (c) $7^{3/2}$ (d) 27

3. Two perpendicular chords of the curve $y^2-4x-4y+4=0$ belonging to family of lines form diagonal of a quadrilateral, then minimum area of quadrilateral is

(a) 16 (b) 32 (c) 64 (d) 50

Passage – 2

Consider a function $f(x)=\left(\alpha-\frac{1}{\alpha}-x\right)(4-3x^2)$ where '□' is a positive parameter

1. Number of points of extrema of f(x) for a given value of □ is

(a) 0 (b) 1 (c) 2 (d) 3

2. Absolute difference between local maximum and local minimum values of f(x) in terms of □ is

(a) $\frac{4}{9}\left(\alpha+\frac{1}{\alpha}\right)^3$ (b) $\frac{2}{9}\left(\alpha+\frac{1}{\alpha}\right)^3$ (c) $\left(\alpha+\frac{1}{\alpha}\right)^3$ (d) Independent of □

3. Least possible value of the absolute difference between local maximum and local minimum values of f(x) is

(a) $\frac{32}{9}$ (b) $\frac{16}{9}$ (c) $\frac{8}{9}$ (d) $\frac{1}{9}$

Matching Type Questions

1 Match the following with 'c' of Rolle's theorem

Column I	**Column II**
(a) $f(x) = e^x \sin x$, $[0, \pi]$	(p) $\frac{\pi}{3}$
(b) $f(x) = e^x (\sin x - \cos x)$, $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$	(q) $\frac{3\pi}{4}$
(c) $f(x) = x + \sin 2x$, $[0, \pi]$	(r) π
	(s) $\frac{2\pi}{3}$

(a) (a-q), (b-r), (c-p, s), (b)(a-r), (b-q), (c-p, s),
(c) (a-p,s), (b-r), (c-q), (d)(a-r), (b-p, s), (c-q)

2. Match the greatest value of the function in column 1

Column I	Column II
(a) $y=\sqrt{100-x^2}$ on [–6, 8]	(p) no greatest value
(b) $y = 2\tan x - \tan^2 x$ on $[0, \pi/2)$	(q) 10
(c) $y=\tan^{-1}\dfrac{1-x}{1+x}$ [0, 1]	(r) $\pi/4$
(d) $y=\dfrac{a^2}{x}+\dfrac{b^2}{1-x}$ on (0, 1) $a > 0$, $b > 0$	(s) 1

(a) (a-s), (b-r), (c-p), (d-q)
(b)(a-s), (b-r), (c-q), (d-p)
(c) (a-p), (b-r), (c-s), (d-q)
(d)(a-q), (b-s), (c-r), (d-p)

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Exercise - III

Subjective type

1. Let $f(x)$ be a polynomial of degree 6, which satisfies $\lim_{x\to 0}\left(1+\dfrac{f(x)}{x^3}\right)^{1/x} = e^2$ and has local maximum at $x = 1$ and local minimum at $x = 0$ and 2 then $5f(3)$ is equal to.

2. A straight line L with negative slope passes through the point (8, 2) and cuts the positive coordinate axes at points P and Q. The absolute minimum value of $OP + OQ$ as L varies, where O is the origin is.

3. If $y = f(x)$ is represented as $x = g(t) = t^5 - 5t^3 - 20t + 7$ and $y = h(t) = 4t^3 - 3t^2 - 18t + 3$ $(-2 < t < 2)$ then max $f(x)$ is equal to

4. The cost of the fuel for running a bus is proportional to the square of the speed generated in *kmph*. It costs Rs. 48 per hour when the bus is moving with a speed of 20 *km*/*hr*. What is the most economical speed if the fixed charges are Rs. 108 for one four over and above the running charges.

5. A window in the form of a rectangle is surmounted by a semi-circular opening. The total perimeter of the window is 10 cm, find the dimensions of the rectangular part of the window to admit maximum light through the whose opening.

6. A square tank of capacity 250 cubic m. has to be dug out. The cost of land is Rs. 50 per sq. m. The cost of digging increases with the depth and for the whose tank is $400 \times (\text{depth})^3$ rupees. Find the dimensions of the tank for the least total cost.

7. If $f(x)=\left(\frac{a^2-1}{3}\right)x^3+(a-1)x^2+(a-1)x^2+2x+1$ is monotonic increasing for every $x \in$ R then find the range of values of 'a'.

8. Use the function $f(x)=x^{1/x}$, $x>0$ to determine the bigger of two numbers e^{π} & π^{e}.

9. Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6r\sqrt{3}$.

10. Depending on the values of $p \in$ R, find the value of 'a' for which the equation $x^3+2px^2+p=a$ has three distinct real roots.

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Exercise - IV

IIT – JEE PROBLEMS

A. Fill in the blanks

1. The function $y = 2x^2 - \log |x|$ is monotonically increasing for values of x satisfying the inequalities ______ and monotonically decreasing for values of x satisfying the inequalities ______.
2. Let P be a variable point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci F_1 and F_2. If A is the area of the triangle PF_1F_2, then the maximum value of A is ______ .
3. Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point (1, 2).
4. Find all the tangents the curve $y = \cos (x + y)$, $-2\pi \le x \le 2\pi$, that are parallel to the line $x + 2y = 0$.
5. A point P is given on the circumference of a circle of radius r. Chord QR is parallel to the tangent at ? Determine the maximum possible area of the triangle PQR.
6. A window of fixed perimeter (including the base of the arch) is n the form of a rectangular surmounted by a semi-circle. The semi-circular portion is fitted with coloured glass, while the rectangular portion is fitted with clear glass. The clear glass transmits three times as much light per square metre as tae colured glass. What is the ratio of the sides of the rectangle so that the window transmits the maximum light?
7. What normal to curve $y = x^2$ forms the shortest chord?
8. If $A > 0$, $B > 0$ and $A + B = \pi/3$, then the maximum value of $\tan A \tan B$ is ______ .
9. Tangent at a point P_1 (other than (0, 0)) on the curve $y = x^3$ meet the curve again at P_2. The tangent at P_2 meets the curve at P_3, and so on. Show that the abscissae of $P_1, P_2 \ldots P_n$ form a G.P. Also find the ratio

   [area ($\Delta P_1 P_2 P_3$)] / [area $\Delta P_2 P_3 P_4$].
10. Suppose $f(x)$ is a function satisfying the following conditions

    (a) $f(0) = 2$, $f(1) = 1$

    (b) f has a minimum value at $x = 5/2$ and

    (c) for all x

    $$f(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$$

    where a, b are some constants. Determine the constants a, b and the function $f(x)$.

B. Multiple Choice Questions with ONE correct answer

11. If $0 < b^2 < c$, then $f(x) = x^3 + bx^2 + cx + d$

(a) has no local maxima (b) has no local minima

(c) is strictly increasing on R (d) is strictly decreasing on R

12. Let $f(x)$ be twice differentiable function such that $f(x) = x^2$, $x = 1, 2, 3$. Then

(a) $f''(x) = 2 \; \forall \; x \in (1, 3)$ (b) $f''(x) = 2$ for some $x \in (1, 3)$

(c) $f''(x) = 3 \; \forall \; x \in (2, 3)$ (d) $f''(x) = f'(x)$ for some $x \in (2, 3)$

13. Let f and g be increasing and decreasing functions, respectively from $[0, \infty)$ to $[0, \infty)$. Let $h(x) = f(g(x))$. If $h(0) = 0$, then $h(x) - g(x)$ is

(a) always zero (b) always negative (c) always positive (d) none of these

14. The function f defined by $f(x) = (x + 2)e^{-x}$ is

(a) decreasing for all x (b) decreasing in $(-\infty, -1)$ and increasing in $(-1, \infty)$

(c) increasing for all x (d) decreasing in $(-1, \infty)$ and increasing $(-\infty, -1)$

15. On the interval [0, 1], the function $x^{25}(1 - x)^{75}$ takes its maximum value at the point

(a) 0 (b) 1/4 (c) 1/2 (d) 1/3

16. The function $f(x) = |px - q| + r|x|$, $x \in (-\infty, \infty)$, where $p > 0$, $q > 0$, $r > 0$ assumes its minimum value only of one point if

(a) $p \neq q$ (b) $r \neq q$ (c) $r \neq p$ (d) $p = q = r$

17. The function $f(x) = \dfrac{\log(\pi + x)}{\log(e + x)}$ is

(a) increasing on $[0, \infty)$

(b) decreasing on $[0, \infty)$

(c) increasing on $[0, \pi/e]$ and decreasing on $[\pi/e, \infty)$

(d) decreasing on $[0, \pi/e]$ and increasing on $[\pi/e, \infty)$

18. If $f(x) = \dfrac{x}{\sin x}$ and $g(x) = \dfrac{x}{\tan x}$, where $0 < x \leq 1$, then in this interval

(a) both $f(x)$ and $g(x)$ are increasing function

(b) both $f(x)$ and $g(x)$ are deceasing function

(c) $f(x)$ is in increasing function (d) $g(x)$ is an increasing function

19. If $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$ for every real number x, then the minimum value of f

(a) doesnot exist because f is unbounded (b) is not attained even though f is bounded

(c) is equal to 1 (d) is equal to -1

20. The number of values of x where the function $f(x) = \cos x + \cos\left(\sqrt{2}\,x\right)$ attains the maximum is

(a) 0 (b) 1 (c) 2 (d) infinite

21. The function $f(x) = \sin^4 x + \cos^4 x$ increases in

(a) $0 < \pi < \pi/8$ (b) $\pi/4 < x < 3\pi/8$ (c) $3\pi/8 < x < 5\pi/8$ (d) $5\pi/8 < x < 3\pi/4$

22. Let $f(x) = \begin{cases} |x| & \text{for } 0, |x| \le 2 \\ 1 & \text{for } \quad x = 0 \end{cases}$ Then at $x = 0$, f has

(a) a local maximum (b) no local maximum (c)a local minimum (d) no extremum

23. Let $f(x) = (1 + b^2) x^2 + 2bx + 1$ and let $m(b)$ be the minimum value of $f(x)$. As b varies, the range of $m(b)$ is

(a) [0, 1] (b) (0, 1/2] (c) [1/2, 1] (d) (0,1]

24. If $f(x) = xe^x (1 - x)$, then $f(x)$ is

(a) increasing on [– 1/2, 1] (b) decreasing on R

(c) increasing on R (d) decreasing on [–1/2, 1]

25. The point(s) on the curve $y^3 + 3x^2 = 12y$ where the tangent is vertical, is (are)

(a) $\left(\pm 4/\sqrt{3}, -2\right)$ (b) $\left(\pm \sqrt{113}, 1\right)$ (c) (0, 0) (d) $\left(\pm 4/\sqrt{3}, 2\right)$

C. Multiple Choice Questions with ONE or MORE THAN ONE correct answer

26. If $y = a \log x + bx^2 + c$ has its extremum values at $x = -1$ and $x = 2$, then

(a) $a = 2, b = -1$ (b) $a = 2, b = -1/2$ (c) $a = -2, b = ½$ (d) none of these

27. The normal to the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ at any point 'θ' is such that

(a) it makes a constant angle with the x-axis (b)it passes through the origin

(c) it is at a constant distance from the origin (d)none of these

28. If $y = f(x) = \dfrac{x+2}{x-1}$ then

(a) $x = f(y)$ (b) $f(1) = 3$

(c) y increases with x for $x < 1$

(d) f is a rational function x

29. If the line $ax + by + c = 0$ is a normal to the curve $xy = 1$, then

(a) $a > 0, b > 0$ (b) $a > 0, b < 0$ (c) $a < 0, b > 0$ (d) $a < 0, b < 0$

30. If $f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \le x \le 2 \\ 37 - x, & 2 < x \le 3 \end{cases}$ then

(a) $f(x)$ is increasing on [– 1, 2] (b)$f(x)$ is continuous on [–1, 3]

(c) $f'(2)$ does not exist (d)$f(x)$ has the maximum value at $x = 2$

31. Let $h(x) = f(x) - (f(x))^2 + (f(x))^3$ for every real number x. Then

(a) h is increasing whenever f is increasing

(b) h is increasing whenever f is decreasing

(c) h is decreasing whenever f is decreasing

(d) nothing can be said in general

32. On the ellipse $4x^2 + 9y^2 = 1$, the points at which the tangents area parallel to the line $8x = 9y$ are

(a) (2/5, 1/5) (b) (– 2/5, 1/5) (c) (– 2/5, –1/5) (d) (2/5, –1/5)

33. A tangent drawn to the curve $y = f(x)$ at $P(x, y)$ cuts the x-axis and y-axis at A and B respectively such that $BP : AP = 3 : 1$, given that $f(1) = 1$ then

(a) equation of the curve is $x\frac{dy}{dx} - 3y = 0$

(b) normal at (1, 1) is $x + 3y = 4$

(c) curve passes through (2, 1/8)

(d) equation of curve is $x\frac{dy}{dx} + 3y = 0$

34. $f(x)$ is a cubic polynomial which has local maximum at $x = -1$. If $f(2) = 18$, $f(1) = -1$ and f has local minimum at $x = 0$ then

(a) the distance between $(-1, 2)$ and $(a, f(a))$ where $x = a$ is the point of local minima is $2\sqrt{5}$

(b) $f(x)$ is increasing for $x \in \left[1, 2\sqrt{5}\right]$

(c) $f(x)$ has local minima at $x = 1$

(d) the value of (0) = 5

35. $f(x) = \begin{cases} e^x, & 0 \le x \le 1 \\ 2 - e^{x-1}, & 1 < x \le 2 \\ x - e & 2 < x \le 3 \end{cases}$, $g(x) = \int_0^x f(t)\, dt$ $x \in [1, 3]$ then $g(x)$ has

(a) local maxima at $x = 1 + \log 2$ and local minima at $x = e$

(b) local maxima at $x = 1$ and local minima at $x = 2$

(c) no local maxima

(d) no local minima

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Answers

Exercise - I

Only One Option is correct

1.	(d)	2.	(d)	3.	(d)	4.	(c)	5	(b)
6.	(a)	7.	(b)	8.	(a)	9.	(b)	10.	(a)
11.	(b)	12.	(a)	13.	(a)	14.	(b)	15.	(d)

More Than One Choice Correct

1.	(a, b, c, d)	2.	(d)	3.	(c)	4.	(a, c)	5.	(a, b, c)
6.	(a, c)	7.	(a, d)	8.	(d)	9.	(c)	10.	(d)

Exercise - II

Assertion and Reason

1.	(b)	2.	(b)	3.	(b)	4.	(a)	5.	(d)

Passage Based Questions

Passage – 1

1. (b) 2. (b) 3. (b)

Passage – 2

1. (c) 2. (a) 3. (a)

Matching Type Questions

1. (a) 2. (d)

Exercise - III

Subjective Type

1. 224 2. 18 3. 14

4. 30 km/hr 5. $l = \frac{20}{(\pi + 4)}, b = \frac{10}{(\pi + 4)}$ 6. $d = 2.5$, $l = 10$

7. $a \in (-\infty, -3] \cup [1, \infty)$ 8. $e^{\pi} > \pi^{e}$

10. $p < a < \frac{32p^3}{27} + p$, if $p > 0$; $\frac{37p^3}{27}$ $+ < a < p$, if $p < 0$

Exercise - IV

IIT-JEE Level Problem

Section - A

1. $\left(-\frac{1}{2} < x < 0 \text{ or } x > \frac{1}{2}\right); x < -\frac{1}{2} \text{ or } 0 < x < \frac{1}{2}$ 2. 3. $(x + y = 3)$

4.	$x + 2y - \frac{\pi}{2} = 0, x + 2y + \frac{3\pi}{2} = 0$			5.	$\frac{3\sqrt{3}}{4}r^2$				
6.	$6 : 6 + \pi$			7.	$\sqrt{2}x - 2y + 2 = 0$				
8.	$\frac{1}{3}$	9.	$\frac{1}{10}$	10.	$a = \frac{1}{4}, b = -\frac{5}{4}$				

Section – B

11.	(c)	12.	(b)	13.	(a)	14.	(d)	15.	(b)
16.	(c)	17.	(b)	18.	(c)	19.	(d)	20.	(a)
21.	(b)	22.	(d)	23.	(d)	24.	(a)	25.	(d)

Section - C

26.	(d)	27.	(c)	28.	(a, d)	29.	(b, c)	30.	(a, b, c, d)
31.	(a, c)	32.	(b, d)	33.	(c, d)	34.	(b, c)	35.	(a, b)

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