Long wavelength electromagnetic fluctuating interactions between macroscopic bodies

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The interaction force between two semi-infinite dielectric media separated by a third one is derived using a method of Rytov. Comparison is made with Green's function results of Dzyaloshinskii *et al*. Certain limiting procedures of their work are avoided by making a central physical assumption.

The theory of Rytov[1] was applied by Lifshitz[2] to the problem of two media filling half spaces with parallel plane boundaries separated from one another through vacuum by a distance l. In what follows we assume that media "one" and "two" are separated by medium "three," medium "two" being at a distance l from medium "one" along the x axis.

We extend Rytov's theory to the case in which the middle medium is an isotropic nonmagnetic dielectric characterized by its complex dielectric constant $\epsilon_3(\omega)$. We obtain a formula for the force of interaction which is valid for all T (temperature). Dzyaloshinskii *et al.*[3–5] used temperature dependent quantum field theory methods and obtained a formula which leads, after an approximate argument, to the $T=0\,K$ case. Our extension of Rytov's theory not only reproduces the results of Dzyaloshinskii *et al.*, which are assumed in all discussions of this problem in the literature,[6–12] but gives physical insight on the nature of the approximations involved. Maxwell's stress tensor for medium 3 is obtained after stating a central physical assumption. This was considered not to be possible within Rytov's theory and was the main motivation for using temperature dependent Green functions.[3–5] The somewhat obscure limiting processes necessary with the latter technique lead, in part, to results contained in Rytov's theory. We present first Rytov's method, develop an expression for the electromagnetic field amplitudes for the geometry discussed above, calculate the force and discuss conditions of validity and applications.

I. GENERAL IDEAS AND BOUNDARY CONDITIONS

The theory of Rytov[1] is based on two fundamental assumptions:

(1) The existence of spontaneous electric **K**, and magnetic **L**, random fluctuating field sources as a result of fluctuations in the position and motion of the charges in matter.

(2) The linearity of the fluctuational processes. Hypothesis 2, permits us to expand any fluctuating magnitude in Fourier components. For instance:

$$\mathbf{K}(\mathbf{r},t)=\int_{-\infty}^{\infty}\mathbf{K}_\omega(\mathbf{r})\exp(-i\omega t)\,d\omega\ , \tag{I.1}$$

with

$$\mathbf{K}_\omega(\mathbf{r})=\int_{-\infty}^{\infty}\mathbf{g}_\omega(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r})\,d\mathbf{k}\ , \tag{I.2}$$

and

$$\mathbf{g}_\omega(\mathbf{k})=\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\mathbf{K}_\omega(\mathbf{r})\exp(-i\mathbf{k}\cdot\mathbf{r})\,d\mathbf{r}\ . \tag{I.3}$$

To describe the fluctuations, without going into the molecular world, Rytov introduces **K** and **L** into Maxwell's equations, which acquire the following form in an isotropic nonmagnetic medium, our case).

$$\mathrm{rot}(\mathbf{E})=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}\ ,\qquad \mathrm{rot}(\mathbf{H})=\frac{\epsilon}{c}\frac{\partial}{\partial t}\left(\mathbf{E}+\frac{1}{\epsilon}\mathbf{K}\right)\ , \tag{I.4}$$

which when written in Fourier components of the fields, acquire the following form:

$$\mathrm{rot}(\mathbf{E}_\omega)=i\omega\mathbf{H}_\omega/c\ ,\qquad \mathrm{rot}(\mathbf{H}_\omega)=-i\omega\left(\epsilon\mathbf{E}_\omega+\frac{1}{\epsilon}\mathbf{K}_\omega\right)\Big/c\ , \tag{I.5}$$

where $\epsilon(\omega)=\epsilon'(\omega)+i\epsilon''(\omega)$. The fundamental characteristic of the random sources is the correlation function, determining the average value of the product of **K** components at two different points in space, namely[1–4]:

$$(K_{i1}K_{K2})\omega=2\hbar\epsilon''\delta_{iK}\delta(\mathbf{r}_2-\mathbf{r}_1)\coth\left(\frac{h\omega}{2T}\right) \tag{I.6}$$

and for the Fourier components of $\mathbf{g}_\omega(\mathbf{k})$

$$\overline{g_{i\omega}(\mathbf{k})g_{K\omega'}(\mathbf{k}')}=\frac{2\hbar}{4\pi^3}\coth\left(\frac{h\omega}{2T}\right)\epsilon''\delta_{iK}(\omega+\omega')\,\delta(\mathbf{k}+\mathbf{k}')\ . \tag{I.7}$$

The solution of equations of the same type as (I.5) for $x<0$ were given by Rytov in his book (p. 41) and were slightly modified by Lifshitz.[2] We will adopt Lifshitz's form, namely

$$\mathbf{E}_{i\omega}=\int_{-\infty}^{\infty}[\mathbf{a}_1(\mathbf{k})\cos(k_x x)+i\mathbf{b}_1(\mathbf{k})\sin(k_x x)]\exp(i\mathbf{q}\cdot\mathbf{r})\,d\mathbf{k}+\int_{-\infty}^{\infty}u_1(\mathbf{q})\exp(i\mathbf{q}\cdot\mathbf{r}-is_1x)d\mathbf{q}\ ,$$

$$\mathbf{H}_{1\omega}=\frac{c}{\omega}\int_{-\infty}^{\infty}[(\mathbf{q}\times\mathbf{a}_1+k_x\mathbf{n}\times\mathbf{b}_1)\cos(k_x x)+i(\mathbf{q}\times\mathbf{b}_1+k_x\mathbf{n}\times\mathbf{a}_1)\sin(k_x x)]\exp(i\mathbf{q}\cdot\mathbf{r})\,d\mathbf{k}$$
$$+\frac{c}{\omega}\int_{-\infty}^{\infty}(\mathbf{q}\times\mathbf{u}_1-s_1\mathbf{n}\times\mathbf{u}_1)\exp(i\mathbf{q}\cdot\mathbf{r}-is_1x)\ , \tag{I.8}$$

with

$$\mathbf{a}_1 = \frac{1}{\epsilon_1(k^2-\omega^2\epsilon_1/c^2)}\left[\frac{\omega^2}{c^2}\,\epsilon_1\mathbf{g}_1 - \mathbf{q}(\mathbf{q}\cdot\mathbf{g}_{1r}) - k_x^2 g_{1x}\mathbf{n}\right] ,$$

$$\mathbf{b}_1 = \frac{-1}{\epsilon_1(k^2-\omega^2\epsilon_1/c^2)}\,[\mathbf{n}(\mathbf{q}\cdot\mathbf{g}_{1r}) + \mathbf{q}g_{1x}] , \quad (\text{I. }9)$$

where $\mathbf{q}$ is a two dimensional vector with components k_y, k_z which implies $k^2 = k_x^2 + q^2$, n is a unit vector in the x direction, two-dimensional vectors in the $y-z$ plane are indicated by the subscript r and

$$s_1 = \sqrt{\frac{\omega^2}{c^2}\,\epsilon_1 - q^2} \quad (\text{I. }10)$$

where the sign of the root is to be chosen so that the imaginary part of s will be positive.[2] The first terms in (I.8) represent a solution of the inhomogeneous (I.5), and the second ones are solutions of the corresponding homogeneous equations. The homogeneous equations require, of course, $\mathbf{K}_\omega = 0$ and this in turn implies the physical condition $\partial\mathbf{K}/\partial t = 0$ in the macroscopic Maxwell equations (I.4). The homogeneous equations fulfill the requirements $\text{div}(\mathbf{E}) = 0$, $\text{div}(\mathbf{H}) = 0$ which yield the condition of transversality of the electric field waves:

$$\mathbf{u}_{1r}\cdot\mathbf{q} - s_1 u_{1x} = 0 . \quad (\text{I. }11)$$

In the second medium ($x > l$), the field $\mathbf{E}_{2_\omega}, \mathbf{H}_{2_\omega}$ is given by the same formulas (I.8), (I.9), and (I.11), with the index 1 changed to 2, $\cos(k_x x)$, $\sin(k_x x)$ replaced by $\cos[k_x(x-l)]$, $\sin[k_x(x-l)]$ and change in the sign of s.

With respect to medium 3, we make now our fundamental hypothesis, namely: $\partial\mathbf{K}_3/\partial t$ is negligible in comparison with $\partial\mathbf{K}_1/\partial t$ and $\partial\mathbf{K}_2/\partial t$ for long waves fluctuations. This is a physically logical hypothesis: these fluctuating field waves have a much shorter distance to travel in medium 3 than in media 1 and 2 and hence they do not have time for being harmonically modulated as waves in media 1 and 2. In this way we can consider $\mathbf{K}_3$ as a constant when compared with $\mathbf{K}_1$ and $\mathbf{K}_2$. Our hypothesis then implies that we can consider Eq. (I.4), in medium 3 with $\partial\mathbf{K}_3/\partial t = 0$, namely,

$$\text{rot}(\mathbf{E}_3) = -\frac{1}{c}\frac{\partial\mathbf{H}_3}{\partial t} ,$$

$$\text{rot}(\mathbf{H}_3) = \frac{\epsilon_3}{c}\frac{\partial\mathbf{E}_3}{\partial t} , \quad (\text{I. }12)$$

which is mathematically equivalent to put $\mathbf{K}_{3_\omega} = 0$ in (I.5) but the physical meaning resides in (I.4). For the Fourier components in medium 3 we have

$$\text{rot}(\mathbf{E}_{3_\omega}) = i\omega\mathbf{H}_{3_\omega}/c ,$$

$$\text{rot}(\mathbf{H}_{3_\omega}) = -i\omega\epsilon_3\mathbf{E}_{3_\omega}/c . \quad (\text{I. }13)$$

The general solution for these homogeneous equations is

$$\mathbf{E}_{3_\omega} = \int_{-\infty}^{\infty} [\mathbf{v}(\mathbf{q})\exp(ipx) + \mathbf{w}(\mathbf{q})\exp(-ipx)]\exp(i\mathbf{q}\cdot\mathbf{r})d\mathbf{q} ,$$

$$\mathbf{H}_{3_\omega} = \frac{c}{\omega}\int_{-\infty}^{\infty} [(\mathbf{q}\times\mathbf{v} + p\mathbf{n}\times\mathbf{v})\exp(ipx) + (\mathbf{q}\times\mathbf{w} - p\mathbf{n}\times\mathbf{w})\exp(-ipx)]\exp(i\mathbf{q}\cdot\mathbf{r})\,d\mathbf{q} , \quad (\text{I. }14)$$

with

$$p = \sqrt{\frac{\omega^2}{c^2}\,\epsilon_3 - q^2} . \quad (\text{I. }15)$$

From $\text{div}(\mathbf{E}) = 0$ we obtain the transversality requirement

$$\mathbf{v}_r\cdot\mathbf{q} + pv_x = 0 , \quad \mathbf{w}_r\cdot\mathbf{q} - pw_x = 0 . \quad (\text{I. }16)$$

Applying the boundary conditions in $x = 0$, $\mathbf{n}\times\mathbf{E}_{1_\omega} = \mathbf{n}\times\mathbf{E}_{3_\omega}$, $\mathbf{n}\times\mathbf{H}_{1_\omega} = \mathbf{n}\times\mathbf{H}_{3_\omega}$ we obtain

$$\int_{-\infty}^{\infty} \mathbf{a}_{1r}\,dk + \mathbf{u}_{1r} = \mathbf{v}_r + \mathbf{w}_r , \quad (\text{I. }17)$$

$$\int_{-\infty}^{\infty} (-\mathbf{q}a_{1x} + k_x\mathbf{b}_{1r})\,dk_x - \mathbf{q}u_{1x} - s_1\mathbf{u}_{1r} = -\mathbf{q}(v_x + w_x) + p(\mathbf{v}_r - \mathbf{w}_r) . \quad (\text{I. }18)$$

The conditions at the plane $x = l$ differ in having $s_1, \mathbf{a}_1$, $\mathbf{b}_1$, v, w replaced by $-s_2$, $\mathbf{a}_2$, $\mathbf{b}_2$, $\mathbf{v}\exp(ipl)$, $\mathbf{w}\exp(-ipl)$, respectively.

II. DETERMINATION OF THE FLUCTUATING ELECTROMAGNETIC FIELD AMPLITUDES IN MEDIUM 3

For a given value of $\mathbf{q}$, we resolve $\mathbf{v}_r$ and $\mathbf{w}_r$ along the mutually perpendicular unit vectors $\mathbf{q}/q$ and $\mathbf{n}\times\mathbf{q}/q$ which we choose as y and z axes, respectively. From (I.11) we get $u_{1x} = \mathbf{u}_1\cdot\mathbf{q}/s_1$ putting this value in (I.18) and after projecting along $\mathbf{q}$ we obtain

$$\int_{-\infty}^{\infty} (-q^2 a_{1x} + k_x\mathbf{b}_{1r}\cdot\mathbf{q})\,dk_x - (\mathbf{u}_{1r}\cdot\mathbf{q})\left(\frac{q^2}{s_1} + s_1\right) = -q^2(v_x + w_x) + p(\mathbf{v}_r - \mathbf{w}_r)\cdot\mathbf{q} \quad (\text{II. }1)$$

Projecting (I.17) along $\mathbf{q}$ we get $\mathbf{u}_{1r}\cdot\mathbf{q}$ after putting this value in (II.1) we obtain

$$\int_{-\infty}^{\infty}\left[-q^2 a_{1x} + qk_x b_{1y} + q\left(\frac{q^2}{s} + s_1\right)a_{1y}\right]dk_x = q\left(\frac{q^2}{s_1} + s_1\right)(v_y + w_y) - q^2(v_x + w_x) + pq(v_y - w_y) . \quad (\text{II. }2)$$

From (I.16) we get

$$v_x = -\frac{q}{p}\,v_y , \quad w_x = \frac{q}{p}\,w_y . \quad (\text{II. }3)$$

Putting (II.3) in (II.2) and dividing by q we obtain

$$\left(s_1 + p\,\frac{\epsilon_1}{\epsilon_3}\right)v_y - \left(s_1 - p\frac{\epsilon_1}{\epsilon_3}\right)w_y = \frac{c^2}{\omega^2}\,\frac{s_1 p}{\epsilon_3}\int_{-\infty}^{\infty}\left[-a_{1x}q + b_{1y}k_x + a_{1y}\left(\frac{\omega^2}{c^2}\frac{\epsilon_1}{s_1}\right)\right]dk_x . \quad (\text{II. }4)$$

We have to take account from expressions (I.9) the components of $\mathbf{a}_1$ and $\mathbf{b}_1$, namely,

$$a_{1x} = \frac{\omega^2\epsilon_1/c^2 - k_x^2}{\epsilon_1[k^2 - (\omega^2/c^2)\epsilon_1]}\,g_{1x} ,$$

$$a_{1y} = \frac{(\omega^2/c^2)\epsilon_1 - q^2}{\epsilon_1[k^2 - (\omega^2/c^2)\,\epsilon_1]}\,g_{1y} ,$$

$$a_{1z} = \frac{(\omega^2/c^2)\,\epsilon_1}{\epsilon_1[k^2 - (\omega^2/c^2)\epsilon_1]}\,g_{1z} ,$$

$$b_{1x}=-\frac{k_x q}{\epsilon_1[k^2-(\omega^2/c^2)\epsilon_1]}\,g_{1y}\,,$$

$$b_{1y}=-\frac{k_x q}{\epsilon_1[k^2(\omega^2/c^2)\,\epsilon_1]}\,g_{1x}\,,$$

$$b_{1z}=0\ . \tag{II.5}$$

Putting the corresponding expressions (II.5) in (II.4) we obtain

$$-\left(s_1+p\,\frac{\epsilon_1}{\epsilon_3}\right)v_y+\left(s_1-p\,\frac{\epsilon_1}{\epsilon_3}\right)w_y=\int_{-\infty}^{\infty}\left[\frac{p}{\epsilon_3[k^2-(\omega^2/c^2)\epsilon_1]}\,(s_1qg_{1x}-s_1^2g_{1y})\right]dk_x\ . \tag{II.6}$$

In $x=l$ we have

$$\exp(ipl)\left(-s_2+p\,\frac{\epsilon_2}{\epsilon_3}\right)v_z+\exp(-ipl)\left(s_2+p\,\frac{\epsilon_2}{\epsilon_3}\ w_y\right)=\int_{-\infty}^{\infty}\left[\frac{p}{\epsilon_3[k^2-(\omega^2/c^2)\epsilon_2]}\,(s_2qg_{2x}+s_2^2g_{2y})\right]\,dk_x\ . \tag{II.7}$$

Expressions (II.6) and (II.7) form a system of two linear equations with two unknowns, namely v_y and w_y, with discriminant Δ given by

$$\Delta=\left(s_1-p\frac{\epsilon_1}{\epsilon_3}\right)\left(s-p\frac{\epsilon_2}{\epsilon_3}\right)\exp(ipl)-\left(s_1+p\frac{\epsilon_1}{\epsilon_3}\right)\left(s_2+p\frac{\epsilon_2}{\epsilon_3}\right)\exp(-ipl)\ . \tag{II.8}$$

Then

$$v_y=\int_{-\infty}^{\infty}\frac{p}{\Delta\epsilon_3}\left[s_1\left(s_2+p\frac{\epsilon_2}{\epsilon_3}\right)\exp(-ipl)\,\frac{qg_{1x}-s_1g_{1y}}{k_x^2-s_1^2}+s_2\left(p\,\frac{\epsilon_1}{\epsilon_3}-s_1\right)\frac{qg_{2x}+s_2g_{2y}}{k_x^2-s_2^2}\right]\,dk_x\,,$$

$$w_y=\int_{-\infty}^{\infty}\frac{p}{\Delta\epsilon_3}\left[s_1\left(s_2-p\frac{\epsilon_2}{\epsilon_3}\right)\exp(ipl)\,\frac{qg_{1x}-s_1g_{1y}}{k_x^2-s_1^2}-s_2\left(s_1+p\frac{\epsilon_1}{\epsilon_3}\right)\frac{qg_{2x}+s_2g_{2y}}{k_x^2-s_2^2}\right]\,, \tag{II.9}$$

where we have used

$$k^2-\frac{\omega^2}{c^2}\,\epsilon=k_x^2-s^2\ . \tag{II.10}$$

We will now determine v_z and w_z. From (I.17) and (I.18) after having projected along the unit vector $\mathbf{n}\times\mathbf{q}/q$ and taking account that this vector is orthogonal to $\mathbf{q}$, we obtain

$$\int_{-\infty}^{\infty}(k_x b_{1z})\,dk_x-s_1u_{1z}=p(v_z-w_z)\ , \tag{II.11}$$

$$\int_{-\infty}^{\infty}a_{1z}\,dk_x+u_{1z}=v_z+w_z\ . \tag{II.12}$$

Taking account that $b_{1z}=0$ from (II.5), we solve (II.11) for u_{1z}, and after re-emplacing in (II.12) we get

$$(s_1+p)\,v_z+(s_1-p)\,w_z=s_1\int_{-\infty}^{\infty}a_{1z}\,dk_x\ .$$

In $x=l$

$$(-s_2+p)\exp(ipl)\,v_z+(s_2+p)\exp(-ipl)\,w_z=-s_2\int_{-\infty}^{\infty}a_{2z}\,dk_x\ . \tag{II.13}$$

We have again a system of two linear equations with two unknowns, namely v_z, w_z, with discriminant Δ' given by

$$\Delta'=(s_1-p)(s_2-p)\exp(ipl)-(s_1+p)(s_2+p)\exp(-ipl)\ . \tag{II.14}$$

Then

$$v_z=\int_{-\infty}^{\infty}\frac{\omega^2}{c^2\Delta'}\left[-s_1\,(s_2+p)\exp(-ipl)\frac{g_{1z}}{k_x^2-s_1^2}+s_2(s_1-p)\,\frac{g_{2z}}{k_x^2-s_2^2}\right]dk_x\,,$$

$$w_z=\int_{-\infty}^{\infty}\frac{\omega^2}{c^2\Delta'}\left[s_1(s_2-p)\exp(ipl)\,\frac{g_{1z}}{k_x^2-s_1^2}-s_2(s_1+p)\frac{g_{2z}}{k_x^2-s_2^2}\right]dk_x\ , \tag{II.15}$$

where we used (II.5) and (II.11).

III. CALCULATION OF THE FORCE OF INTERACTION

We know from classical electrodynamics that the time rate variation of the density of electromagnetic momentum is equal to the divergency of Maxwell's tensor, namely,

$$\int_V\frac{\partial\mathbf{m}}{\partial t}\,d^3x=\int_V\mathrm{div}(\overleftrightarrow{\mathrm{T}})\,d^3x\ , \tag{III.1}$$

which is equivalent to

$$\oint_S\frac{\partial\mathbf{m}}{\partial t}\cdot\mathbf{n}\,da=\oint_S\mathbf{n}\cdot\overleftrightarrow{\mathrm{T}}\,da\ , \tag{III.2}$$

where $\mathbf{m} = \mu\epsilon/4\pi c(\mathbf{E}\times\mathbf{H})$ is the field-momentum density and $\overleftrightarrow{\mathrm{T}}$ is Maxwell's stress tensor, with elements

$$T_{ij} = \frac{1}{4\pi}\left[\epsilon E_i E_j + \mu H_i H_j - (\tfrac{1}{2})\delta_{ij}(\epsilon E^2 + \mu H^2)\right] . \qquad \text{(III. 3)}$$

The two previous relations, (III. 1) and (III. 2), are equivalent to

$$\frac{\partial \mathbf{m}}{\partial t} = \mathrm{div}(\overleftrightarrow{\mathrm{T}}) , \qquad \text{(III. 4)}$$

$$\mathbf{n}\cdot\frac{\partial \mathbf{m}}{\partial t} = \mathbf{n}\cdot\overleftrightarrow{\mathrm{T}} . \qquad \text{(III. 5)}$$

The left side of Eq. (III. 5) with minus sign, is just the force per unit area transmitted across surface S in the direction n, this can be easily calculated by the right side of Eq. (III. 5). In thermodynamical equilibrium this force (or normal flow of momentum per unit area) cannot depend on the coordinate along which it is being calculated (in our case x). Otherwise there would exist sources of momentum creation, and Eqs. (III. 4) and (III. 5) would have different values in different regions of space. We will show that using antecedence $[\epsilon(i\xi) > 0]$ this invariance condition is fulfilled.[13] We then calculate the Fourier component of the force F_ω acting on a unit area of the surface of body 2 with the formula

$$\mathbf{F}_\omega = \mathbf{n}\cdot\overleftrightarrow{\mathrm{T}}_\omega = T_{xx_\omega}\cdot\mathbf{n} , \qquad \text{(III. 6)}$$

where n is the outward normal to the surface, which gives ($\mu = 1$ in our case),

$$F_\omega = \frac{1}{8\pi}\left(\epsilon\overline{E_x^2} + \overline{H_x^2} - \epsilon\overline{E_r^2} - \overline{H_r^2}\right)_\omega . \qquad \text{(III. 7)}$$

The dash over the symbols signifies a statistical averaging, to which the Fourier components g of the random field sources must be subjected taking account of (I. 7). Quantities $\mathbf{g}_1$ and $\mathbf{g}_2$, referring to different media, are statistically independent, so that the average of their products gives zero. Taking into account Eqs. (I. 15) and the fact that for imaginary frequencies we can define a real variable $p(i\xi)$ because $\epsilon(i\xi) > 0$ (see below), then

$$F_\omega = -\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_0^{\infty}\left[(|v_y|^2 + |w_y|^2)\epsilon + \frac{p^2c^2}{\omega^2}(|v_z|^2 + |w_z|^2)\right] dk_x\, 2\pi q\, dq , \qquad \text{(III. 8)}$$

where we have used $\int_{-\infty}^{\infty}\ldots d\mathrm{q} \equiv \int_0^{\infty}\ldots 2\pi q\, dq$, and substituted in place of v, w, the expressions in the integrands of Eqs. (II. 3), (II. 9) and (II. 15).

Using the following identities:

$$p(s + s^*) = \tfrac{1}{2}\left[\,|s + p|^2 - |s - p|^2\,\right], \qquad \text{(III. 9)}$$

$$p(q + |s|^2)(s + s^*) = \frac{\omega^2}{2c^2}\left[\left|s + \frac{\epsilon p}{\epsilon_3}\right|^2 - s\left|-\frac{\epsilon p}{\epsilon_3}\right|^2\right]\epsilon_3, \quad \text{if } \epsilon_3 \text{ is real} \qquad \text{(III. 10)}$$

$$\frac{|s_1 + \epsilon_1/\epsilon_3 p|^2 |s_2 + \epsilon_2/\epsilon_3 p|^2 - |s_2 - \epsilon_2/\epsilon_3 p|^2 |s_2 - \epsilon_2/\epsilon_3 p|^2}{|\Delta|^2} = \frac{\exp(ipl)(s_1 - \epsilon_1/\epsilon_3 p)(s_2 - \epsilon_2/\epsilon_3 p)}{-\Delta} + \tfrac{1}{2} + \mathrm{c.c.} , \qquad \text{(III. 11)}$$

$$\frac{|s_1 + p|^2 |s_2 + p|^2 - |s_1 - p|^2 |s_2 - p|^2}{|\Delta'|^2} = \frac{\exp(ipl)(s_1 - p)(s_2 - p)}{-\Delta'} + \tfrac{1}{2} + \mathrm{c.c.} , \qquad \text{(III. 12)}$$

$$\int_{-\infty}^{\infty}\frac{dk_x}{|k_x^2 - s^2|^2} = \frac{i\pi}{|s|^2(s - s^*)} = \frac{\pi c^2}{2\omega^2}\,\frac{(s + s^*)}{|s|^2}\,\frac{1}{\epsilon''} , \qquad \text{(III. 13)}$$

it is easily shown that

$$F_\omega = \frac{\hbar}{8\pi}\coth\left(\frac{h\omega}{2T}\right)\int_0^{\infty} q\,dq\,(p)\left[\frac{\exp(ipl)(s_1 - p)(s_2 - p)}{-\Delta'} + \frac{\exp(ipl)(s_1 - \epsilon_1/\epsilon_3 p)(s_2 - \epsilon_2/\epsilon_3 p)}{-\Delta} + 1\right] + \mathrm{c.c.} , \qquad \text{(III. 14)}$$

where c.c. denotes the complex conjugate expression. In calculating the force $F(l)$ we have to integrate F_ω between the limits $-\infty - +\infty$. Following Lifshitz's criterion we take twice the integral between $0-\infty$,

$$F(l) = 2\int_0^{\infty} F_\omega\, d\omega \qquad \text{(III. 15)}$$

it must be emphasized that expression (III. 14) is valid only on the imaginary frequency axis, hence the use of the following integral:

$$F(l) = 2i\int_0^{\infty} F_{i\xi}\, d\xi . \qquad \text{(III. 16)}$$

Taking into account that $p(i\xi) = i\sqrt{\xi_2/c_2\epsilon(i\xi) + q^2}$ we define p' as equal to $-ic/\xi\sqrt{\epsilon_3}\,p$, keeping in mind the variation of q, $0 \le q < \infty$, which implies that $1 \le p' < \infty$. We define s' as equal to $-ic/\xi\sqrt{\epsilon_3} - s$, and taking account of $q\,dq = -p\,dp$, we obtain:

$$\int\ldots d\omega\int p^2\, dp \equiv \frac{1}{c^3}\int\epsilon_3^{3/2}\,\xi^3\, d\xi\int p^2\, dp ,$$

where in the right side expression we have put the generic variable p' equal to p. The constant 1 in the curly brackets of expression (III. 14) does not give an l-dependent contribution to the force and it should be dropped out according to Lifschitz.[2] We then have

$$F(l) = \frac{\hbar}{2\pi^2c^3}\int_0^{\infty}\int_1^{\infty}[\xi_3(i\xi)]^{3/2}\,\xi^3 p^2\coth\left(\frac{\hbar i\xi}{2T}\right)\left[\frac{(s_1 + p)(s_2 + p)}{(s_1 - p)(s_2 - p)}\exp\left(\frac{2p\xi l}{c}\sqrt{\epsilon_3}\right) - 1\right]^{-1}$$

$$+\left[\frac{(s_1+p\epsilon_1/\epsilon_3)(s_2+p\epsilon_2/\epsilon_3)}{(s_1-p\epsilon_1/\epsilon_3)(s_2-p\epsilon_2/\epsilon_3)}\exp\left(\frac{2p\xi l\sqrt{\epsilon_3}}{c}\right)-1\right]^{-1}d\xi dp\ , \qquad \text{(III.17)}$$

where $s=\sqrt{\epsilon/\epsilon_3+p^2-1}$ and the temperature T is given in units of Boltzmann's constant k. The expression for the force (III.17) is a most general one for all T and yields directly, for $T=0$ Eq. (4.14) of Dzyaloshinskii *et al.* One does not need the following step which yields a good approximation for low T. The function $\coth(\hbar\omega/2T)$ has an infinite number of poles, located on the imaginary axis, and equal to

$$\omega_n=i\xi n=\frac{i2\pi T}{\hbar}n\ .$$

Making the integration over ξ in (III.17), we arrive at[2]

$$F(l)=\frac{T}{\pi c^3}\sum_{n=0}^{\infty}{}'[\epsilon_3(i\xi)]^{3/2}\xi_n^3\int_1^\infty p^2\left\{\left[\frac{(s_1+p)(s_2+p)}{(s_1-p)(s_2-p)}\exp\left(\frac{2p\xi_n l}{c}\sqrt{\epsilon_3}\right)-1\right]^{-1}\right.$$

$$\left.+\left[\frac{(s_1+p\epsilon_1/\epsilon_3)(s_2+p\epsilon_2/\epsilon_3)}{(s_1-p\epsilon_1/\epsilon_3)(s_2-p\epsilon_2/\epsilon_3)}\exp\left(\frac{2p\xi_n l}{c}\sqrt{\epsilon_3}\right)-1\right]^{-1}\right\}dp\ . \qquad \text{(III.18)}$$

Formula (III.18) is identical to Eq. (4.13) of Dzyaloshinskii *et al.*[3] The prime on the summation sign means that the term with $n=0$ should be taken with a factor of $\frac{1}{2}$. Several limiting cases can be obtained in a standard fashion from these formulas related to, short or intermediate distances, inclusion or exclusion of retardation effects, derivation of Hamaker constants, many-body vs two-body interactions, London forces, *etc.*

IV. DISCUSSION

Several authors have discussed forces between macroscopic bodies. A central role in these discussions is played by the fluctuation-dissipation theorem of Callen and Welton[14] and Matsubara's temperature dependent Green's functions as used by Dzyaloshinskii and Pitaevskii and applied to the problem of van der Waals forces by Dzyaloshinskii, Lifshitz and Pitaevskii.[3,5]

Several attempts have been made in the literature in order to obtain the results of Refs. 3 and 5 in a more elementary way. Of particular interest is McLachlan's derivation making use of the fluctuation-dissipation theorem and the method of images.[9] He treats explicitly the empty gap case at zero temperature via surface sheet of dipoles. This method was further applied by Israelachvili[10] who considered multiple reflections within a gap filled with a dielectric medium. It fails to give complete agreement for retarded forces for large values of the dielectric constants. Retardation effects are observable experimentally and hence must be accounted for properly. Langbein[8] compares Lifshitz's results with semiclassical ones, originated by Casimir,[15] and obtained by van Kampen *et al.*,[12] as well as with his own finite boundary condition method. Van Kampen's method was later extended by several authors.[16,17]

Lifshitz had treated the interaction between solid bodies separated by a narrow empty gap. He used Rytov's results for the correlation of electromagnetic fluctuations, well described elsewhere.[18] These results are a specialization of the fluctuation dissipation theorem, they lead to an *explicit* solution of the field equations within a semi-infinite body in the presence of fluctuations.

It was stated that the generalization of this method to the case where the gap is filled with some medium was difficult because there were no formulae for the stress tensor in a variable field of absorbing media, thus making necessary the field theoretical methods of Ref. 4. In this paper such a generalization is presented.

The fundamental assumption made in this derivation, that $\mathbf{K}_3$ is not modulated in the long wave length limit, and which leads to Eqs. (I.12) allows the use of Rytov's explicit solutions of the inhomogeneous Eq. (I.9) outside the gap.

It is to be noted that in order to obtain meaningful quantities in the field theoretical method, the long wave length limit is invoked also.[3,5] It is simpler, we believe, to apply it directly at the classical field equations level and to extend with no difficulty to other geometries and conditions.

As a referee pointed out, Eq. (III.15) involves a subtle point. Lifshitz criterion is not entirely consistent, though it yields the results taken as a standard in the literature. The difficulty was well discussed by Langbein.[6] The use of surface modes may be objectionable also.[19] This formal problem will be discussed elsewhere.

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[1]S. M. Rytov, *Theory of Electrical Fluctuations and Thermal Radiation* (Publishing House, Academy of Sciences, USSR, 1953).

[2]E. M. Lifshitz, Sov. Phys. JETP 2, 73 (1956).

[3]I. E. Dzyaloshinskii, E. M. Lifshitz, and L. P. Pitaevskii, Adv. Phys. 10, 165 (1961).

[4]I. E. Dzyaloshinskii and L. P. Pitaevskii, Sov. Phys. JETP 9, 1282 (1959).

[5]I. E. Dzyaloshinskii, E. M. Lifshitz, and L. P. Pitaevskii, Sov. Phys. JETP 10, 161 (1960).

[6]D. Langbein, J. Chem. Phys. 58, 4476 (1973).
[7]D. Langbein, Phys. Rev. 32, 3371 (1970).
[8]D. Langbein, J. Phys. A 6 1149 (1973).
[9]A. D. McLachlan, Proc. R. Soc. Lond. A 274, 80 (1964).
[10]J. N. Israelachvili, Proc R. Soc. Lond. A 331, 39 (1972).
[11]J. N. Israelachvili and D. Tabor, Proc. R. Soc. Lond. A 331, 19 (1972).
[12]N. G. Van Kampen, B. R. A. Nijboer, and K. Schram, Phys. Lett. **26A**, 307 (1963).
[13]Lifshitz puts the extra condition of calculating the Maxwell's stress tensor at $x=0$.
[14]H. B. Callen and T. A. Welton, Phys. Rev. 83, 34 (1951).
[15]H. B. G. Casimir, Helv. Phys. Acta, 33, 855 (1960).
[16]P. Richmond and B. W. Ninham, J. Phys. C 4, 1988 (1971).
[17]E. Gerlach, Phys. Rev. B 4, 393 (1971).
[18]L. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon, London, 1960).
[19]K. Schram, Phys. Lett. **43A**, 282 (1973).