

Three-Dimensional Geometry from a Distinguishability Axiom

The Cycle Space of K_4 and Frame-Invariant Directional Distributions

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Abstract

We investigate the discrete geometry that emerges from the cycle space of complete graphs within a constraint framework built from the axiom $\diamond N \rightarrow \neg N$ ("if nothingness is possible, then nothingness does not obtain"). The framework requires mutually distinguishable elements whose relational structure is encoded in a five-dimensional inner product space (the Clifford algebra $Cl(5)$). At the participation scope $N = 4$, four elements produce the complete graph K_4 , whose cycle space — the space of independent closed paths — has dimension exactly three.

We prove that this cycle space, equipped with the natural inner product induced by edge-sharing, has round S^2 directional geometry. The proof rests on a single algebraic identity: $PG_3P = 2P$, where $G_3 = 2I + J$ is the Gram matrix of the edge-sharing inner product and P is the projection perpendicular to the uniform direction. The identity follows from $PJ = 0$ (the all-ones matrix annihilates vectors perpendicular to the uniform direction). We further prove that K_4 is the only complete graph with this property: for K_5 and beyond, the off-diagonal entries of the Gram matrix are non-uniform, breaking the $\alpha I + \beta J$ structure and destroying roundness.

Computationally, the roundness residual is 10^{-16} for K_4 versus 2.19 for K_5 .

These results establish that the combinatorial structure of K_4 encodes, at the discrete algebraic level, the Lie-algebraic content of three-dimensional Euclidean geometry: a 3D positive-definite inner product space with round S^2 , cross product, and $SO(3)$ symmetry. This has a structural parallel to Carcassi and Aidala's theorem [1, 2] that three-dimensional spaces are the unique spaces admitting frame-invariant distributions over directions — the same $so(3) \cong \mathbb{R}^3$ condition, arrived at from the opposite direction. The passage from this discrete algebraic structure to continuous three-dimensional geometry is not addressed here; the result characterises the algebraic skeleton, not the manifold.

This paper explores the mathematical structures that emerge from a relational constraint framework. The structural parallel to Carcassi's theorem on spatial dimensionality is developed as a correspondence between mathematical forms, not as a claim that three-dimensional space is being derived from logic. The mathematical results — the Gram matrix computation, the roundness proof, and the uniqueness theorem — stand independently of any physical interpretation.

Keywords: cycle space, complete graphs, Gram matrix, S^2 geometry, frame-invariance, Clifford algebra, distinguishability, three dimensions

1. Introduction

1.1 Motivation

Why does the mathematical structure of three-dimensional Euclidean geometry — with its round S^2 , cross product, and $SO(3)$ symmetry — occupy a privileged position among geometries of arbitrary dimension? This question has a long mathematical history, from Hurwitz's theorem [3] on cross products (possible only in dimensions 1, 3, and 7) to the classification of normed division algebras ($\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$). In physics, Ehrenfest [4] observed in 1917 that three spatial dimensions are required for stable orbits; Tegmark [5] extended this to the structure of partial differential equations. More recently, Callender [6] has examined the question from the philosophy of physics.

Carcassi and Aidala's Reverse Physics programme [1, 2, 7] provides a particularly clean result: three-dimensional spaces are the *only* spaces for which distributions over directions are frame-invariant. Their proof identifies a self-duality condition — the Hodge dual maps vectors to bivectors only when the dimensions of these spaces coincide, i.e., $C(n,1) = C(n,2)$, giving $n = 3$ — as the mathematical core of the uniqueness.

This paper approaches the same question from a different direction. Within a constraint framework built from a single axiom about distinguishability [8], we find that the cycle space of K_4 — the complete graph on four vertices — has dimension three with round S^2 directional geometry and a cross product. K_4 is the unique complete graph with this property. The result is combinatorial and algebraic; it does not invoke physics.

1.2 This Paper

We prove three theorems:

Theorem A (Roundness). The cycle space of K_4 , with the edge-sharing inner product, has Gram matrix $G_3 = 2I + J$. Its directional subspace (perpendicular to the uniform direction) carries the round metric on S^2 : $PG_3P = 2P$.

Theorem B (Uniqueness). K_4 is the only complete graph whose cycle-space Gram matrix has the form $\alpha I + \beta J$. For all K_N with $N \geq 5$, some pairs of triangles share no edges, producing non-uniform off-diagonal entries and breaking roundness.

Theorem C (Arithmetic). The equation $(N-1)(N-2)/2 = 3$ has unique positive integer solution $N = 4$.

We then develop the structural parallel to Carcassi's frame-invariance theorem and discuss the oriented cycle space structure (cross product, Euclidean isomorphism).

1.3 Scope and Epistemological Position

This paper establishes mathematical facts about the cycle space of K_4 and identifies a structural parallel to Carcassi's result on three-dimensional geometry. The parallel is a correspondence between mathematical forms: the round S^2 that Carcassi derives analytically (from frame-invariance of directional distributions) and the round S^2 that we derive combinatorially (from the edge-sharing inner product on K_4 's cycle space). Showing that two mathematical structures are isomorphic is a mathematical result. Claiming that either one "explains" the dimensionality of physical space would be a physical claim requiring additional interpretive assumptions. This paper makes the mathematical claim only.

1.4 Relation to Other Work

The constraint framework [8] produces structures that parallel those of several existing programmes. A companion paper [9] shows that the same framework produces symplectic geometry at $N = 2$ and unitary geometry at $N \geq 3$, with structural parallels to Carcassi's identification of the fork between classical and quantum mechanics. The present paper is independent of [9]; the only shared element is the framework's axiom and $Cl(5)$ structure, which provide the context for why K_4 appears at $N = 4$.

The question of why three spatial dimensions has been addressed from many angles. String theory compactifies extra dimensions; causal set theory and loop quantum gravity attempt to derive dimensionality from discrete structures. Our result is closer in spirit to the algebraic approaches — Hurwitz's restriction of cross products to dimensions 1, 3, 7 [3], and Carcassi's restriction of frame-invariant directional distributions to dimension 3 [1] — in that the number 3 emerges from a structural constraint rather than from dynamical evolution.

2. The Framework

2.1 The Axiom

The axiom $\diamond N \rightarrow \neg N$ ("if nothingness is possible, then nothingness does not obtain") requires that something exists and is distinguishable from nothing. For multiple elements to exist, they must be mutually distinguishable, with a minimum separation $\varepsilon > 0$ in the configuration space.

2.2 Five Constraint Dimensions and $Cl(5)$

The claim that distinguishability requires exactly five independent constraint dimensions is established in [8] through a categorical exhaustion argument, which we summarise here for self-containedness.

For one thing to be robustly distinguishable from another, five independent conditions must be satisfied:

(i) Boundary (β): there must be a boundary separating what is "this" from what is "not-this." Without a boundary, the thing merges with its background and ceases to be distinguishable. Formally: a partition of the relational neighbourhood into interior and exterior.

(ii) Pattern (κ): there must be internal structure that characterises the thing — something that could in principle be re-identified. Without pattern, two bounded regions are interchangeable. Formally: a statistical regularity in the constraint profile.

(iii) Resource (ρ): there must be capacity to sustain the boundary and pattern against perturbation. A boundary with no resource to maintain it degrades; a pattern with no substrate dissolves. Formally: the energetic or material capacity available within the boundary.

(iv) Integration (λ): the parts must cohere into a whole rather than being merely co-located. A pile of sand has boundary, pattern, and resource but no integration. Formally: the degree to which relational structure is shared across the boundary.

(v) Ordering (τ): there must be an asymmetric structure distinguishing "before" from "after" or "cause" from "effect." Without ordering, the thing has no temporal identity — it cannot persist. Formally: a cyclic asymmetry in the relational graph (τ requires closed loops; it is identically zero at $N = 2$).

The argument that these five are *complete* (no sixth category is independent of the first five) and *independent* (no subset derives the others) is the categorical exhaustion [8]. The five categories are represented as orthonormal basis vectors e_1, \dots, e_5 of the Clifford algebra $Cl(5)$, satisfying $e_i e_j + e_j e_i = 2\delta_{ij}$. The $Cl(5)$ structure is not assumed — it is the minimal algebra that encodes five independent, mutually interacting dimensions with a well-defined inner product.

An element (relational profile) is a grade-1 vector $\mathbf{a} = \sum a^i e_i$. The relation between two elements is the grade-2 bivector $B_{ab} = \mathbf{a} \wedge \mathbf{b}$. For N elements, the relational structure is encoded in the complete graph K_N , with elements as vertices and relations as edges. The cycle rank — the number of independent closed paths — is:

$$\text{cycle_rank}(K_N) = E - V + 1 = \frac{(N - 1)(N - 2)}{2} \quad (1)$$

Remark for the reader evaluating the premise. The results of this paper (Theorems 1-4) depend on the $Cl(5)$ structure only through two facts: (a) elements are vectors in \mathbb{R}^5 , and (b) relations are bivectors in $\wedge^2 \mathbb{R}^5$. The specific physical interpretation of the five constraint dimensions does not affect the algebraic and combinatorial results. A reader who prefers to take "five-dimensional inner product space" as an assumption rather than a derived consequence will find that the mathematical content of Sections 3-7 is unchanged.

2.3 The Monogamy Constraint and the Sharing Threshold

Each element has finite capacity to participate in relations: the total bivector magnitude through

any vector is bounded ($\sum |\mathbf{v} \wedge \mathbf{x}_i| \leq \Lambda$). When edges are shared between cycles, this finite capacity forces *competition* — an edge serving two triangles cannot provide full capacity to both.

The key structural question is: at what N does edge-sharing first occur?

- K_2 (one edge): no triangles, no cycles, no sharing.
- K_3 (three edges, one triangle): each edge belongs to exactly one triangle. No competition.
- K_4 (six edges, four triangles): each edge belongs to exactly two triangles. **Sharing begins.**
- K_5 and beyond: sharing continues but is non-uniform (some pairs of triangles share an edge, others do not).

K_4 is therefore not selected by fiat but identified by a mathematical criterion: it is the *minimal* complete graph where the monogamy constraint creates competition among cycles. We study its cycle space for this reason. Theorem 2 (Section 6) will confirm that K_4 is also the *unique* complete graph whose cycle space has round directional geometry — the sharing threshold is not merely the first instance but the only instance with this property.

3. The Cycle Space of K_4

3.1 Structure

K_4 has vertices $\{A, B, C, D\}$, six edges, and four triangular faces: ABC, ABD, ACD, BCD.

Represent each triangle as a binary vector over the six edges, giving the incidence matrix T :

	AB	AC	AD	BC	BD	CD
ABC:	[1	1	0	1	0	0]
ABD:	[1	0	1	0	1	0]
ACD:	[0	1	1	0	0	1]
BCD:	[0	0	0	1	1	1]

The rank of T is 3, confirming the cycle rank. Any three triangles form a basis; the fourth satisfies $BCD = ABC \oplus ABD \oplus ACD \pmod{2}$. This is the dependency relation over $GF(2)$ (the unoriented cycle space). Over \mathbb{Z} (the oriented cycle space, Section 7), the signs distinguish orientations: $BCD = ABC - ABD + ACD$.

3.2 The Edge-Sharing Pattern

Every pair of distinct triangles shares exactly one edge:

Pair	Shared edge
$ABC \cap ABD$	AB
$ABC \cap ACD$	AC
$ABC \cap BCD$	BC
$ABD \cap ACD$	AD
$ABD \cap BCD$	BD
$ACD \cap BCD$	CD

This perfect uniformity — every pair sharing exactly 1 — is the crucial structural property.

4. The Gram Matrix

4.1 Definition and Computation

The edge-sharing inner product is $\langle C_i, C_j \rangle =$ number of edges common to C_i and C_j , computed as $G = TT^T$.

The 4×4 Gram matrix:

$$G_4 = \begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{pmatrix} = 2I_4 + J_4 \quad (2)$$

where J_4 is the 4×4 all-ones matrix. Diagonal entries are 3 (each triangle has 3 edges); off-diagonal entries are 1 (each pair shares exactly 1 edge).

The 3×3 basis Gram matrix (choosing $\{ABC, ABD, ACD\}$):

$$G_3 = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix} = 2I + J \quad (3)$$

4.2 Eigenvalues

Since $G_3 = 2I + J$, and J has eigenvalue 3 for the uniform vector $(1,1,1)/\sqrt{3}$ and eigenvalue 0 for all

perpendicular vectors:

- **Eigenvalue 5** (multiplicity 1): eigenvector $(1,1,1)/\sqrt{3}$
- **Eigenvalue 2** (multiplicity 2): all vectors perpendicular to $(1,1,1)$

All eigenvalues are positive; G_3 is positive definite.

For the full $G_4 = 2I_4 + J_4$: eigenvalue 6 (multiplicity 1) and eigenvalue 2 (multiplicity 3). The **triple degeneracy** at eigenvalue 2 is the signature of isotropy — all three directions in the directional subspace are exactly equivalent.

5. The Central Result: Round S^2 Geometry

5.1 The Projection

The uniform eigenvector $\mathbf{u} = (1,1,1)/\sqrt{3}$ represents the scalar "total cyclic content." The directional content lives in the 2-dimensional subspace perpendicular to \mathbf{u} . The projection onto this subspace is:

$$P = I - \mathbf{u}\mathbf{u}^T = I - \frac{1}{3}J \quad (4)$$

5.2 The Central Identity

Theorem 1 (Round S^2). The projected Gram matrix on the directional subspace is proportional to the identity:

$$PG_3P = 2P \quad (5)$$

Proof. Expanding:

$$PG_3P = P(2I + J)P = 2PIP + PJP = 2P^2 + PJP$$

Since P is a projection: $P^2 = P$.

Since $PJ = (I - (1/3)J)J = J - (1/3)J^2 = J - (1/3) \cdot 3J = J - J = 0$.

Therefore: $PG_3P = 2P + 0 = 2P$. ■

The step $PJ = 0$ is the heart of the proof: the all-ones matrix J maps every vector to a multiple of $(1,1,1)$, and P annihilates the $(1,1,1)$ direction.

Computational verification: residual $\|PG_3P - 2P\| = 8.4 \times 10^{-16}$ (machine precision).

5.3 Interpretation

The metric $PG_3P = 2P$ is isotropic on the directional subspace: all directions are metrically equivalent. This is the defining property of a round sphere — no direction is preferred, and the metric looks the same from every point on the sphere. The proportionality constant 2 is a scale factor with no geometric significance.

The directional space of the K_4 cycle space has round S^2 geometry.

6. Uniqueness

6.1 Computational Comparison

Graph	Triangles	Off-diagonal sharing values	Uniform?	$\ PG_{sP} - \alpha P\ $
K_3	1	—	trivial	—
K_4	4	{1}	yes	8.4×10^{-16}
K_5	10	{0, 1}	no	2.19
K_6	20	{0, 1}	no	4.47

Table 1. Roundness of the directional subspace for K_3 through K_6 .

6.2 The Structural Reason

Theorem 2 (Uniqueness of K_4). Among complete graphs, K_4 is the only one whose cycle-space Gram matrix has the form $\alpha I + \beta J$.

Proof. The form $\alpha I + \beta J$ requires all off-diagonal entries of the Gram matrix to be equal — meaning every pair of triangles shares the same number of edges. In K_4 : every pair of triangles shares exactly one edge (Table in Section 3.2). This is a balanced combinatorial design: 4 triangles, 6 edges, each edge in exactly 2 triangles, each pair sharing exactly 1 edge.

In K_5 : there exist triangle pairs that share no edge. For example, triangles $\{0,1,2\}$ and $\{0,3,4\}$ share vertex 0 but no edge — their edge sets $\{(01),(02),(12)\}$ and $\{(03),(04),(34)\}$ are disjoint. Other pairs share one edge. The off-diagonal entries include both 0 and 1, breaking the $\alpha I + \beta J$ form. To confirm this is not an artefact of the labelling: K_5 has $C(5,3) = 10$ triangles and $C(10,2) = 45$ triangle pairs. Each of the 10 edges belongs to $C(3,1) = 3$ triangles, giving $C(3,2) = 3$ edge-sharing triangle pairs per edge, for $10 \times 3 = 30$ sharing pairs. The remaining $45 - 30 = 15$ pairs share no edge. The non-uniformity is pervasive, not exceptional.

In K_N for $N \geq 5$: edge-disjoint triangle pairs always exist (any two triangles sharing at most one vertex have disjoint edges). The sharing is non-uniform for all $N \geq 5$. ■

Corollary. The $PG_3P = \alpha P$ identity, and hence round S^2 directional geometry, holds only for K_4 among complete graphs.

6.3 Arithmetic Uniqueness

Theorem 3. The equation $(N-1)(N-2)/2 = 3$ has unique positive integer solution $N = 4$.

Proof. The equation gives $N^2 - 3N - 4 = 0$, factoring as $(N-4)(N+1) = 0$. The positive root is $N = 4$. ■

The equation $(N-1)(N-2)/2 = 7$ (the other dimension admitting a cross product, via the octonions [3]) gives $N^2 - 3N - 12 = 0$ with discriminant 57. Since $\sqrt{57}$ is irrational, there is no integer solution. The constraint framework independently excludes 7-dimensional geometry through arithmetic, complementing Hurwitz's exclusion via non-associativity of the octonions [3] and the framework's own $Cl(5)$ structure (whose highest independent grade is 5).

7. The Oriented Cycle Space

7.1 Oriented Gram Matrix

The preceding results use the unoriented (edge-sharing) Gram matrix G_3 , which encodes the round S^2 of the directional subspace. For the cross product, orientation is required. Assigning consistent orientations to the triangles ($A \rightarrow B \rightarrow C \rightarrow A$, etc.) and representing each as a signed edge vector yields the oriented Gram matrix:

$$G_o = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix} \quad (6)$$

with eigenvalues $\{1, 4, 4\}$, all positive. The determinant is 16. The unoriented metric establishes roundness (the structural parallel to Carcassi's theorem). The oriented metric furnishes the cross product — additional algebraic structure consistent with that roundness. Orientation refines the unoriented geometry; it does not conflict with it.

7.2 Euclidean Isomorphism

Since G_o is positive definite, the Cholesky decomposition $G_o = LL^T$ exists. L^{-1} provides an explicit isomorphism from the oriented cycle space to standard Euclidean \mathbb{R}^3 (verified computationally: $L^{-1} G_o L^{-T} = I$ to machine precision).

Theorem 4. The oriented cycle space of K_4 is isomorphic to Euclidean \mathbb{R}^3 .

Proof. Any 3-dimensional real inner product space with positive-definite metric is isomorphic to \mathbb{R}^3 with the standard inner product. The Cholesky factor provides the explicit isomorphism. ■

In this 3-dimensional Euclidean space, the cross product is defined via the Levi-Civita tensor and metric. Computational verification confirms orthogonality ($\langle e_1 \times e_2, e_1 \rangle = 0$), antisymmetry ($e_1 \times e_2 = -e_2 \times e_1$), and the Jacobi identity — all to machine precision. The symmetry group $SO(3)$ acts naturally.

7.3 The Dependency Relation

Over the rationals, the fourth triangle satisfies $BCD = (+1) \cdot ABC + (-1) \cdot ABD + (+1) \cdot ACD$. The alternating signs encode orientation; the integer coefficients reflect the integrality of the combinatorial structure.

8. Structural Parallel to Carcassi's Theorem

8.1 The Correspondence

Carcassi proves [1] that frame-invariant distributions over directions on $S^{(n-1)}$ exist if and only if the rotation algebra satisfies $so(n) \cong \mathbb{R}^n$, which holds uniquely for $n = 3$.

We prove that the cycle space of K_4 is a 3-dimensional Euclidean space with round S^2 directional geometry and a cross product — the structure in which $so(3) \cong \mathbb{R}^3$ is realised.

The correspondence has four components:

- (i) A *direction* in the cycle space is a normalised cycle — a unit vector representing a particular distribution of relational competition among the independent axes.
- (ii) *Frame-invariance* requires that the description of directional distributions be independent of the choice of basis. Since $G_4 = 2I + J$ is invariant under all permutations of the four triangles (the S_4 symmetry of K_4), any choice of three basis triangles gives the same metric structure.
- (iii) A change of basis in the cycle space is a rotation in the 3D inner product space — an element of $SO(3)$. Invariance under $SO(3)$ acting on S^2 is Carcassi's condition.
- (iv) The condition is satisfiable because the directional metric is round (Theorem 1: $PG_3P = 2P$).

8.2 The Nature of the Parallel

Two programmes — one working analytically from the mathematical content of physical theories (Carcassi), the other working synthetically from a single axiom about distinguishability (this paper) — arrive at the same mathematical structure: a 3-dimensional space with round S^2 and cross product. Neither programme was designed to engage with the other.

The parallel is mathematical: the same Lie-algebraic condition ($\mathfrak{so}(3) \cong \mathbb{R}^3$) appears in both contexts. Whether this mathematical coincidence reflects a deeper connection — whether the cycle space of K_4 has anything to do with the dimensionality of physical space — is an interpretive question that this paper does not attempt to resolve. The mathematical content (Theorems 1-4, the uniqueness results, the computational verifications) stands independently.

8.3 Co-Emergence

A companion paper [9] shows that the same constraint framework produces, at $N \geq 3$, a complex structure $T^2 = -1$ and unitary geometry — structures with parallels to quantum mechanics. The mechanisms of [9] (non-decomposability, trivector complex structure) and the mechanism of this paper (edge-sharing, round cycle space) share a common cause: the formation of closed loops in the relational graph.

Edge-sharing simultaneously produces:

- The 3-dimensional cycle space with round S^2 (this paper)
- Non-decomposable coupling matrices (companion paper [9])
- Sub-additive relational capacity (the monogamy constraint)

These mathematical structures emerge together at the same threshold for the same structural reason. In Carcassi's programme, three spatial dimensions and the classical/quantum fork are treated as separate assumptions. In the constraint framework, they are consequences of the same combinatorial event.

9. Discussion

9.1 Summary of Results

#	Result	Method	Status
1	$\dim(\text{cycle space of } K_4) = 3$	Cycle rank formula	Theorem (standard)
2	Gram matrix $G_3 = 2I + J$	Direct computation	Theorem
3	G_3 positive definite (eigenvalues 2, 2, 5)	Eigendecomposition	Theorem
4	Round S^2 : $PG_3P = 2P$	Algebraic identity $PJ = 0$	Theorem 1
5	K_4 uniquely round among K_N	Combinatorial (sharing uniformity)	Theorem 2

#	Result	Method	Status
6	$N = 4$ unique for cycle rank 3	Quadratic formula	Theorem 3
7	7D excluded by arithmetic	Discriminant analysis	Theorem 3 corollary
8	Go positive definite (eigenvalues 1, 4, 4)	Eigendecomposition	Theorem
9	Cycle space \cong Euclidean \mathbb{R}^3	Cholesky / positive-definite 3D	Theorem 4
10	Cross product with correct properties	Levi-Civita + metric	Computational verification

Table 2. Summary of results. All have been verified computationally.

9.2 Scope Limitations

The continuum limit. This paper establishes that K_4 encodes the *algebraic skeleton* of 3D Euclidean geometry: the Lie algebra $\mathfrak{so}(3) \cong \mathbb{R}^3$, the round S^2 of directions, the cross product. The passage from this discrete algebraic structure to a continuous three-dimensional manifold — how many overlapping K_4 substructures in a large relational network might collectively produce smooth geometry — is a separate mathematical problem. The result here characterises what dimensionality the continuum limit *must* have if it exists, but does not establish that the limit exists.

The physical identification. The mathematical correspondence between cycle-space directions and physical spatial directions is exact at the level of Lie algebra isomorphism ($\mathfrak{so}(3) \cong \mathbb{R}^3$). Whether this correspondence reflects a physical fact about the dimensionality of space, or is merely a mathematical coincidence between two contexts where the number 3 appears, is an interpretive question. This paper establishes the mathematical correspondence and leaves the interpretation open.

Beyond K_4 . At $N = 5$, the cycle rank is 6 and the directional geometry is anisotropic (roundness residual 2.19). Understanding the structure of the K_5 cycle space — which has connections to the full $\text{Cl}(5)$ bivector space — and why the round geometry of K_4 does not extend is an open question.

9.3 Relation to the Broader Programme

This paper and its companion [9] address different aspects of Carcassi's programme from the same axiom. This paper addresses the dimensionality question (why 3); the companion addresses the geometry question (why symplectic or unitary). Together they suggest that the constraint framework's axiom produces a family of mathematical structures — three-

dimensional round geometry, symplectic structure, unitary structure, cross products — that collectively parallel the mathematical infrastructure of physical theories.

Whether this family of parallels constitutes evidence for a deeper connection between the axiom and physics, or is an elaborate mathematical coincidence, remains to be determined. A further companion paper [10] extends the unitarity result of [9] from $N = 3$ to $N = 4$, using the $SO(3)$ isometry group of the oriented Gram matrix G_o (Section 7.1) to show that the complex structure rotates by a gauge transformation that preserves physical predictions. The mathematical content is presented here for its own interest.

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