Supporting Information: Bridge to Physical Formalism

Section 5 Addendum D: Functorial Irreducibility and the Categorical Structure of Emergence

5D.1 Introduction

Jonathan Gorard has developed a categorical perspective on computational irreducibility, demonstrating that irreducibility—the impossibility of shortcutting a computation—corresponds precisely to the failure of functoriality in maps between categories. This provides rigorous mathematical language for discussing when systems exhibit genuine emergence versus reducible behavior.

This addendum establishes the connection between Gorard's framework and our constraint structure, showing that:

1. The $N = 2 \rightarrow N \geq 3$ transition is precisely a **functoriality transition**
2. **Circulation**, **indivisibility**, and **non-diagonalizability** are all manifestations of functoriality failure
3. The categorical perspective **unifies** the different mathematical languages used in SI sections 5A-5C
4. Computational irreducibility emerges geometrically at $N \geq 3$

The Central Claim: The constraint framework naturally forms a category, and the transition from $N = 2$ to $N \geq 3$ marks the transition from functorial (reducible) to non-functorial (irreducible) structure. This provides categorical grounding for the emergence of complexity, time, and physical law.

5D.2 Review: Categories, Functors, and Irreducibility

5D.2.1 Categories

A **category** C consists of:

- **Objects:** A collection Ob(C)
- **Morphisms:** For each pair of objects A, B, a set Hom(A, B) of morphisms from A to B
- **Composition:** For morphisms $f: A \rightarrow B$ and $g: B \rightarrow C$, a composite $g \circ f: A \rightarrow C$
- **Identity:** For each object A, an identity morphism $id_A: A \rightarrow A$

Axioms:

- Associativity: $h \circ (g \circ f) = (h \circ g) \circ f$

- Identity: $f \circ id_A = f = id_B \circ f$ for $f: A \to B$

5D.2.2 Functors

A **functor** $F: C \to D$ between categories consists of:

- An object map: $A \mapsto F(A)$ for each object A in C
- A morphism map: $f \mapsto F(f)$ for each morphism f in C

Functoriality axioms:

- Preserves composition: $F(g \circ f) = F(g) \circ F(f)$
- Preserves identity: $F(id_A) = id_{F(A)}$

The composition preservation is the key property: **a functor cannot "shortcut" through intermediate steps.**

5D.2.3 Gorard's Irreducibility Criterion

Gorard defines computational irreducibility categorically:

Definition (Functorial Reducibility): A computational process is **reducible** if there exists a functor from the computation category to a simpler category that preserves the relevant structure.

Definition (Computational Irreducibility): A process is **irreducible** if no such structure-preserving functor exists—equivalently, if any candidate functor fails to preserve composition.

Key insight: Irreducibility is not about computational resources or practical limitations. It is a **structural property**—the non-existence of a shortcut functor.

5D.2.4 The Obstruction to Functoriality

For a candidate functor F, the obstruction to functoriality is measured by:

$$\mathcal{O}(f, g) = F(g \circ f) - F(g) \circ F(f)$$

- If $O(f, g) = 0$ for all composable pairs, F is a functor
- If $O(f, g) \neq 0$ for some pair, F fails to be a functor

The **total obstruction** can be defined as:

$$\mathcal{O}_{total} = \sum_{\text{composable pairs}} \|\mathcal{O}(f, g)\|$$

Non-zero total obstruction implies irreducibility.

5D.3 The Constraint Category

5D.3.1 Definition

We define the **constraint category** C_N for N features:

Objects: The N features {1, 2, ..., N}, each associated with a constraint configuration C^(α) ∈ V (the viable region).

Morphisms: For each ordered pair (α, β), a coupling morphism:

$$M^{(\alpha\beta)} : \alpha \to \beta$$

represented by the 5×5 coupling matrix M^(αβ) characterizing how features α and β relate.

Composition: For morphisms M^(αβ): α → β and M^(βγ): β → γ, the composite is:

$$M^{(\beta\gamma)} \circ M^{(\alpha\beta)} := M^{(\beta\gamma)} \cdot M^{(\alpha\beta)}$$

(matrix multiplication).

Identity: The identity morphism id_α: α → α is the identity matrix I_5.

5D.3.2 Verification of Category Axioms

Associativity: Matrix multiplication is associative:

$$\left(M^{(\gamma\delta)} \cdot M^{(\beta\gamma)}\right) \cdot M^{(\alpha\beta)} = M^{(\gamma\delta)} \cdot \left(M^{(\beta\gamma)} \cdot M^{(\alpha\beta)}\right)$$

Identity: The identity matrix satisfies:

$$M^{(\alpha\beta)} \cdot I_5 = M^{(\alpha\beta)} = I_5 \cdot M^{(\alpha\beta)}$$

Thus C_N is a well-defined category.

5D.3.3 The Category at Different N

N = 1: Single object, only identity morphism. Trivial category.

N = 2: Two objects A, B. Non-identity morphisms M^(AB) and M^(BA). No non-trivial compositions possible (would require a third object).

N = 3: Three objects A, B, C. Six non-identity morphisms. Compositions possible:

- M^(BC) ∘ M^(AB): A → C via B
- M^(AC): A → C directly
- And cyclic permutations

General N: N objects, N(N-1) non-identity morphisms, rich composition structure.

5D.4 The Target Category

5D.4.1 The Category of Linear Maps

The natural target category is **Vect**, the category of vector spaces and linear maps:

Objects: Copies of $\mathbb{R}^5$ (or the tangent space at each configuration)

Morphisms: Linear maps between these spaces

Composition: Standard composition of linear maps

5D.4.2 The Candidate Functor

Define a candidate functor F: C_N → Vect by:

- F(α) = $\mathbb{R}^5$ for each feature α
- F(M^(αβ)) = M^(αβ) (the coupling matrix itself, viewed as a linear map)

The question: Is F a functor? Does it preserve composition?

5D.4.3 What Preservation Would Mean

If F preserved composition:

$$F(M^{(\beta\gamma)} \circ M^{(\alpha\beta)}) = F(M^{(\beta\gamma)}) \circ F(M^{(\alpha\beta)})$$

This would say: the "direct" coupling from α to γ (via β) equals the matrix product of the individual couplings.

But what is the "direct" coupling M^(αγ)? In general, M^(αγ) (computed from the interaction potential) need not equal M^(βγ) · M^(αβ) (the matrix product).

5D.5 N = 2: Trivial Functoriality

5D.5.1 The Structure at N = 2

At N = 2, the constraint category C_2 has:

- Objects: {A, B}
- Morphisms: id_A, id_B, M^(AB), M^(BA)

Composition table:

∘	id_A	id_B	M^(AB)	M^(BA)
id_A	id_A	—	M^(AB)	—
id_B	—	id_B	—	M^(BA)
M^(AB)	—	M^(AB)	—	M^(AB)∘M^(BA)
M^(BA)	M^(BA)	—	M^(BA)∘M^(AB)	—

The only non-trivial compositions are:

- M^(AB) ∘ M^(BA): B → A → B (returns to B)
- M^(BA) ∘ M^(AB): A → B → A (returns to A)

These are **endomorphisms** (self-maps), not morphisms to a third object.

5D.5.2 Why Functoriality Holds

The functoriality condition F(g ∘ f) = F(g) ∘ F(f) is non-trivial only when the composition connects three distinct objects: f: A → B, g: B → C, giving g ∘ f: A → C.

At N = 2, there is no third object C. The only compositions are endomorphisms, and these trivially satisfy:

$$F(M^{(AB)} \circ M^{(BA)}) = M^{(AB)} \cdot M^{(BA)} = F(M^{(AB)}) \circ F(M^{(BA)})$$

Conclusion: At N = 2, functoriality holds vacuously. There are no three-object compositions to test. The system is **trivially reducible**.

5D.5.3 The Pre-Functorial Regime

N = 2 is not "functorial" in a meaningful sense—it is **pre-functorial**. The structure needed to *test* functoriality (three-object compositions) does not exist.

This parallels:

- Pre-boundary (SI 5C): Can't define inside/outside
- Pre-divisibility (SI 5B): Can't test factorization through intermediate
- Pre-temporal (SI Circulation): τ = 0 necessarily

All these "pre-" conditions reflect the same structural limitation at N = 2.

5D.6 N ≥ 3: Functoriality Failure

5D.6.1 The Structure at N = 3

At N = 3, the constraint category C_3 has:

- Objects: {A, B, C}
- Morphisms: Three identities plus six couplings M^(AB), M^(BA), M^(BC), M^(CB), M^(CA), M^(AC)

Key compositions connecting distinct objects:

- M^(BC) ∘ M^(AB): A → B → C
- M^(AC): A → C directly (if defined independently)

The functoriality test: Does M^(BC) ∘ M^(AB) = M^(AC)?

5D.6.2 The Obstruction

Define the **composition obstruction:**

$$\mathcal{O}_{ABC} = M^{(AC)} - M^{(BC)} \cdot M^{(AB)}$$

This measures how much the "direct" coupling A → C differs from the "via B" coupling.

Theorem (Generic Non-Functoriality at N ≥ 3):

For generic coupling structures, O_ABC ≠ 0. The candidate functor F fails to preserve composition.

Proof sketch:

The coupling matrices M^(αβ) are determined by the interaction potential Φ_int:

$$M_{ij}^{(\alpha\beta)} = \frac{\partial^2 \Phi_{int}}{\partial C_i^{(\alpha)} \partial C_j^{(\beta)}}$$

For independent features, the "direct" coupling M^(AC) is computed from the A-C interaction term, while M^(BC) · M^(AB) involves the A-B and B-C terms.

These are generically different unless the interaction potential has special factorization properties. For generic Φ_int, they differ. ■

5D.6.3 The Commutator Connection

A closely related obstruction is the **commutator:**

$$[M^{(AB)}, M^{(BC)}] = M^{(AB)} \cdot M^{(BC)} - M^{(BC)} \cdot M^{(AB)}$$

This measures whether the order of composition matters.

Proposition: The commutator [M^(AB), M^(BC)] is non-zero if and only if M^(AB) and M^(BC) are not simultaneously diagonalizable.

Connection to earlier results:

- SI Circulation Proof (Theorem 3): Three symmetric matrices are generically not simultaneously diagonalizable
- Therefore: [M^(AB), M^(BC)] ≠ 0 generically at N ≥ 3

The commutator obstruction and the composition obstruction are related but distinct. Both measure aspects of functoriality failure.

5D.6.4 The Total Non-Functoriality

Definition (Non-Functoriality Measure):

$$\mathcal{N}_F = \sum_{\text{triangles } \alpha\beta\gamma} \|\mathcal{O}_{\alpha\beta\gamma}\|_F + \|[M^{(\alpha\beta)}, M^{(\beta\gamma)}]\|_F$$

Properties:

- N_F = 0 at N = 2 (no triangles)
- N_F = 0 implies perfect functoriality
- N_F > 0 generically for N ≥ 3

5D.7 Circulation as Functoriality Measure

5D.7.1 Path-Dependence and Functoriality

The circulation integral (from SI: Circulation Proof):

$$\mathcal{C}(\gamma_{ABC}) = \oint_{\gamma_{ABC}} \nabla\Phi \cdot d\ell$$

measures **path-dependence**: how much the result depends on the path taken.

Functoriality is equivalent to path-independence:

- Path-independent → same result regardless of route → functorial
- Path-dependent → result depends on route → non-functorial

5D.7.2 Connecting Circulation to Commutators

Theorem (Circulation-Commutator Correspondence):

The circulation around a triangular loop is related to the commutator of coupling matrices:

$$|\mathcal{C}(\gamma_{ABC})| \sim \|[M^{(AB)}, M^{(BC)}]\|_F$$

More precisely, both are zero if and only if the coupling structure is simultaneously diagonalizable.

Proof sketch:

The circulation integral involves the curl of the gradient field:

$$\mathcal{C}(\gamma) = \iint_{\Sigma} (\nabla \times \nabla\Phi) \cdot dS$$

In the product space of feature configurations, this curl depends on the cross-derivatives of Φ_int, which are encoded in the coupling matrices. The non-commutativity of the coupling matrices directly contributes to the curl.

When [M^(AB), M^(BC)] = 0 (simultaneously diagonalizable), the coupling structure is "integrable" and the curl vanishes. When [M^(AB), M^(BC)] ≠ 0, there is geometric "twisting" that produces non-zero curl. ■

5D.7.3 The Unified Measure

We can now see that several quantities all measure the same phenomenon:

Definition (Irreducibility Index):

$$\mathcal{R}_N = \frac{1}{L_N} \sum_{\text{triangles}} \left(|\mathcal{C}(\gamma_{\alpha\beta\gamma})| + \|[M^{(\alpha\beta)}, M^{(\beta\gamma)}]\|_F + \|\mathcal{O}_{\alpha\beta\gamma}\|_F \right)$$

All three terms vanish together and are non-zero together.

R_N = 0: Reducible (functorial, divisible, zero circulation) **R_N > 0:** Irreducible (non-functorial, indivisible, non-zero circulation)

5D.8 Unification: Four Languages, One Phenomenon

5D.8.1 The Correspondence Table

The N = 2 → N ≥ 3 transition can be described in multiple equivalent languages:

Language	N = 2	N ≥ 3
Linear Algebra	Simultaneously diagonalizable	Not simultaneously diagonalizable
Stochastic (Barandes)	Divisible	Indivisible
Categorical (Gorard)	Functorial	Non-functorial
Geometric	Zero circulation	Non-zero circulation
Topological	Trivial holonomy	Non-trivial holonomy
Computational	Reducible	Irreducible

5D.8.2 Why They're Equivalent

All these descriptions capture the same geometric fact:

At N = 2: The coupling structure is "simple" — it can be decomposed into independent one-dimensional modes. All paths give the same answer. Shortcuts exist.

At N ≥ 3: The coupling structure is "complex" — it cannot be fully decomposed. Different paths give different answers. No shortcuts exist.

The different languages emphasize different aspects:

- Linear algebra: eigenstructure
- Stochastic: factorization of transitions
- Categorical: composition preservation
- Geometric: path-dependence
- Topological: holonomy around loops
- Computational: shortcut existence

But they describe one phenomenon: **the emergence of irreducibility at N ≥ 3**.

5D.8.3 Cross-References

This unification connects the SI sections:

SI Section	Key Result	Unified Interpretation
Circulation Proof	τ emerges at N ≥ 3	Ordering requires irreducibility
GA Foundations	Trivectors have chirality	Grade-3 = irreducible structure
Barandes	κ measures indivisibility	κ = degree of non-functoriality
Jacobson	G_N consistency constraints	Consistency = functorial requirement
This Section	Functoriality fails at N ≥ 3	The categorical meta-statement

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5D.9 Cobordism Interpretation

5D.9.1 Cobordism Categories

A **cobordism** between manifolds M_1 and M_2 is a manifold W whose boundary is $M_1 \sqcup M_2$:

$$\partial W = M_1 \sqcup M_2$$

Cobordisms form a category:

- Objects: (n-1)-dimensional manifolds
- Morphisms: n-dimensional cobordisms between them
- Composition: Gluing cobordisms along shared boundary

5D.9.2 Configurations as Manifolds

In our framework:

- A feature configuration C^(α) can be viewed as a "point" in constraint space
- An N-feature configuration is a collection of N such points
- The **convex hull** or **simplex** spanned by these points is a geometric object

Dimensional correspondence:

- N = 2: A line segment (1-simplex)
- N = 3: A triangle (2-simplex)

- N = 4: A tetrahedron (3-simplex)
- General N: An (N-1)-simplex

5D.9.3 Couplings as Cobordisms

The coupling $M^{(\alpha\beta)}$ between features α and β can be viewed as specifying a "transition" — a cobordism-like structure connecting the configurations.

At N = 2: The coupling specifies a 1-dimensional transition (a path).

At N = 3: The couplings together specify a 2-dimensional structure (a surface). The triangle ABC with couplings on each edge is like a 2-cobordism.

At N ≥ 3: Higher-dimensional cobordism structure emerges.

5D.9.4 Why N = 3 Enables Topology

The crucial dimensional jump:

- **1D (N = 2):** No "interior" — topology is trivial
- **2D (N = 3):** First "interior" — surface topology becomes non-trivial
- **3D+ (N ≥ 4):** Rich topological structure

The non-zero circulation at N ≥ 3 corresponds to **non-trivial holonomy** around the boundary of the 2-simplex. This is the first topological obstruction that can appear.

5D.9.5 Connection to TQFT

Topological Quantum Field Theory (TQFT) is a functor from the cobordism category to Vect (vector spaces).

Speculation: Our constraint framework might be related to a TQFT-like structure where:

- Configurations are boundary manifolds
- Couplings specify the cobordism/transition
- The functor's failure at N ≥ 3 indicates "non-topological" (i.e., physical) content

This remains speculative but suggests deep connections to quantum gravity approaches.

5D.10 Multicomputation and Branching

5D.10.1 Wolfram's Multiway Systems

In the Wolfram Physics Project, **multiway systems** describe computations that branch:

- A state can evolve to multiple successor states
- Different branches may yield different outcomes
- The **multiway graph** tracks all branches

Reducibility: If all branches converge to the same result, the computation is reducible.

Irreducibility: If branches genuinely diverge, the computation is irreducible — you cannot shortcut by following a single branch.

5D.10.2 N = 2 as Single-Thread

At N = 2, there is only one "path" between configurations: A → B. There is no branching because there is no third point to branch through.

Metaphor: N = 2 is like a computation with a single thread — deterministic evolution from A to B.

5D.10.3 N ≥ 3 as Genuine Branching

At N ≥ 3, multiple paths exist:

- A → B → C (via B)
- A → C directly
- A → C → B → C (longer paths)

These paths generically give **different results** (non-zero circulation). This is genuine multicomputational branching.

Metaphor: N ≥ 3 is like a computation with multiple threads that don't synchronize — the outcome depends on which path is taken.

5D.10.4 The Multiway Graph of Constraint Space

For N features, we can construct a multiway graph:

- Nodes: Feature configurations (points in V^N)
- Edges: Couplings/transitions between configurations

- Branch points: Any configuration with multiple outgoing couplings

The **causal invariance** (convergence of branches) holds at $N = 2$ but fails at $N \geq 3$.

5D.11 Computational Irreducibility and Complexity

5D.11.1 Wolfram's Original Concept

Stephen Wolfram introduced **computational irreducibility** (1984):

Some computations cannot be shortcut — there is no faster way to determine the outcome than running the computation step by step.

This is distinct from computational complexity (hard vs easy problems). Even "simple" rules (like Rule 110) can be computationally irreducible.

5D.11.2 Gorard's Categorical Precision

Gorard's contribution is making this precise:

Computational irreducibility ≡ Non-existence of a structure-preserving functor

This transforms an intuitive notion into a mathematical theorem: a process is irreducible if and only if the relevant functoriality condition fails.

5D.11.3 Irreducibility in Our Framework

Theorem (Emergence of Irreducibility):

The constraint category C_N is computationally reducible for $N \leq 2$ and generically irreducible for $N \geq 3$.

Proof:

- $N \leq 2$: Functoriality holds vacuously (no three-object compositions). Reducible.
- $N \geq 3$: Functoriality fails generically (Theorem in 5D.6.2). Irreducible. ■

5D.11.4 Physical Implications

If physics is described by constraint structure:

N = 2 systems are predictable: Their evolution can be "shortcut" — you can compute the outcome without simulating every step.

N ≥ 3 systems are generically unpredictable: No shortcut exists. You must "run the universe" to see what happens.

This provides a geometric foundation for:

- Quantum unpredictability (N = 2 embedded in N ≥ 3)
- Chaotic sensitivity (path-dependence)
- Emergent complexity (irreducible at macro scale)

5D.12 Implications and Synthesis

5D.12.1 Why N = 3 is the Complexity Threshold

We now have a complete picture of why N = 3 is special:

Property	**N = 2**	**N = 3+**
Boundaries	None	Exist
Triangles	None	Exist
Circulation	Zero	Non-zero possible
Commutators	Trivial	Non-trivial
Functoriality	Vacuous	Fails generically
Irreducibility	None	Generic
Ordering (τ)	Zero	Non-zero possible
Chirality	None	Intrinsic

N = 3 is the **threshold of complexity** — the minimum structure for genuine emergence.

5D.12.2 The Hierarchy of Emergence

```
N = 1: Trivial (single feature, no relations)
   ↓
N = 2: Pre-emergence (relations but no complexity)
   ↓ [THRESHOLD]
N = 3: Emergence begins (first irreducibility)
   ↓
N > 3: Rich emergence (compounding irreducibility)
   ↓
N → ∞: Thermodynamic/classical behavior (averaging)
```

5D.12.3 Connection to Consciousness and Observation

Speculatively, the N = 3 threshold might relate to:

- **Observation:** Requires observer-system-environment (minimum 3)
- **Self-reference:** Requires self-other-context (minimum 3)
- **Measurement:** Creates new correlations (adds features, can cross N = 3)

The emergence of irreducibility at N = 3 might be foundational to the "hard problems" of physics and consciousness.

5D.12.4 The Categorical Meta-Language

The functorial perspective provides a **meta-language** that unifies the other SI sections:

- **Finster (5A):** CFS operator products form a category; causal action = functorial consistency
- **Barandes (5B):** Stochastic processes form a category; indivisibility = non-functoriality
- **Jacobson (5C):** Consistency constraints = functorial requirements on the geometric factor
- **Circulation Proof:** Circulation = non-functoriality measure
- **GA Foundations:** Grade structure reflects category structure (grades ↔ simplex dimensions)

The categorical viewpoint is not "another framework" but the **structural skeleton** underlying all the others.

5D.13 Summary

5D.13.1 Main Results

1. **The constraint category C_N** is well-defined, with features as objects and coupling matrices as morphisms.
2. **Functoriality holds at N = 2** vacuously — no non-trivial compositions exist to test.
3. **Functoriality fails at N ≥ 3** generically — the commutator $[M^{(AB)}, M^{(BC)}] \neq 0$ obstructs composition preservation.
4. **Circulation = Functoriality failure** — both measure the same geometric fact: path-dependence.
5. **Irreducibility emerges at N = 3** — this is the categorical content of the N-transition.

5D.13.2 Unification

Phenomenon	Mathematical Expression	SI Section
Ordering emergence	$\tau > 0$ at $N \geq 3$	Circulation Proof
Chirality	Trivector structure	GA Foundations
Indivisibility	$\kappa > 0$	Barandes (5B)
Non-functoriality	$O \neq 0$, $[M,M'] \neq 0$	This section (5D)
Non-diagonalizability	No common eigenbasis	Circulation Proof
Proto-thermodynamics	G_N varies	Jacobson (5C)

All are manifestations of the **same N ≥ 3 transition**.

5D.13.3 The Upshot

Irreducibility is not a complication but a feature. It is what makes:

- Time possible (ordering requires irreducibility)
- Physics non-trivial (reducible systems are "simple")
- Emergence real (not just epistemic limitation)

The categorical perspective reveals that **complexity emerges necessarily** at N ≥ 3, not contingently. The irreducibility is geometric, not computational; structural, not practical.

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Appendix 5D.A: Category Theory Glossary

Category: A collection of objects and morphisms with composition and identity satisfying associativity and identity laws.

Functor: A structure-preserving map between categories (preserves composition and identity).

Natural transformation: A morphism between functors (preserves "naturality").

Commutative diagram: A diagram where all paths between the same endpoints give the same composite morphism.

Cobordism: A manifold whose boundary consists of two given manifolds; morphism in the cobordism category.

TQFT (Topological Quantum Field Theory): A functor from the cobordism category to the category of vector spaces.

Holonomy: The transformation acquired by parallel transport around a closed loop; measures curvature.

Appendix 5D.B: The Commutator-Circulation Relationship

Formal Statement

Let M^(AB), M^(BC), M^(CA) be the coupling matrices for a three-feature configuration, and let C(γ_ABC) be the circulation around the corresponding triangular loop.

Theorem: There exist constants c_1, $c_2 > 0$ depending on the constraint space geometry such that:

$$c_1 \|[M^{(AB)}, M^{(BC)}]\|_F \leq |\mathcal{C}(\gamma_{ABC})| \leq c_2 \|[M^{(AB)}, M^{(BC)}]\|_F$$

Corollary: $C(\gamma_{ABC}) = 0$ if and only if $[M^{(AB)}, M^{(BC)}] = 0$.

Proof Sketch

The circulation integral depends on the curl of the gradient field:

$$\mathcal{C}(\gamma) = \iint_{\Sigma} (\nabla \times \nabla \Phi_{total}) \cdot dS$$

In local coordinates on the product space $M_3 = V^3$, the components of the curl involve cross-partial derivatives of Φ_{total}. These cross-derivatives are controlled by the coupling matrices and their commutators.

The lower bound follows because non-commuting matrices create "twisting" in the gradient field that cannot cancel. The upper bound follows from the smoothness of Φ and boundedness of the coupling matrices in the viable region.

A complete proof requires specifying the metric structure and carrying out the integral explicitly for the triangular loop. ■