Supporting Information: Bridge to Physical Formalism

Section 5 Addendum B: Indivisible Stochastic Processes Correspondence

5B.1 Introduction

Section 5 of the main text identifies structural parallels between our constraint framework and Barandes' reformulation of quantum mechanics in terms of indivisible stochastic processes. This addendum develops those parallels mathematically, proposing a specific mapping between our N-dependence structure and Barandes' divisibility criterion.

The Central Claim: The transition from N = 2 to N ≥ 3 in our framework corresponds to the transition from structurally pre-divisible to divisibility-testable regimes in Barandes' framework. Furthermore, the pattern constraint κ quantifies the degree of indivisibility—the "quantumness"—of a configuration.

Significance: Both frameworks share a crucial philosophical commitment: quantum mechanical structure (Hilbert spaces, wavefunctions, superposition) is not fundamental but emergent from more primitive structure. If the correspondence holds:

1. Our geometric framework provides a spatial/relational grounding for Barandes' temporal/stochastic structure
2. The quantum-classical transition becomes a geometric phase transition in constraint space
3. Decoherence acquires a precise geometric meaning as emergence of divisibility

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5B.2 Review of Indivisible Stochastic Processes

5B.2.1 Classical Stochastic Processes

A classical (textbook) stochastic process describes a system evolving probabilistically through a configuration space. The dynamics are specified by transition matrices:

$$\Gamma_{ij}(t \leftarrow t_0) = P(X_t = i \,|\, X_{t_0} = j)$$

giving the probability of being in state i at time t given state j at time t_0.

Stochastic matrix properties:

- All entries non-negative: $\Gamma_{ij} \geq 0$
- Columns sum to unity: $\Sigma_i \, \Gamma_{ij} = 1$

5B.2.2 The Divisibility Condition

A stochastic process is **divisible** if for any intermediate time t' between t_0 and t:

$$\Gamma(t \leftarrow t_0) = \Gamma(t \leftarrow t') \cdot \Gamma(t' \leftarrow t_0)$$

This says: to get from t_0 to t, we can always factor through any intermediate time t'. The process has no "memory" of how it reached t'—only the state at t' matters.

Markov processes satisfy divisibility. They obey the Chapman-Kolmogorov equation and have well-defined instantaneous transition rates.

5B.2.3 Indivisible Stochastic Processes

Barandes defines an **indivisible stochastic process** as one that fails the divisibility condition. Given $\Gamma(t \leftarrow t_0)$, if we *attempt* to define an intermediate transition matrix:

$$\tilde{\Gamma}(t \leftarrow t') \equiv \Gamma(t \leftarrow t_0) \cdot \Gamma^{-1}(t' \leftarrow t_0)$$

the result generically fails to be a valid stochastic matrix. Specifically, $\tilde{\Gamma}$ will have **negative entries**—it becomes a "pseudo-stochastic matrix."

Why negative entries? The inverse of a stochastic matrix is only stochastic if both are permutation matrices (containing only 0s and 1s). For matrices with genuine probabilities (entries strictly between 0 and 1), the inverse necessarily has negative entries.

Physical interpretation: Negative "probabilities" signal that the intermediate factorization is unphysical. The process genuinely cannot be divided—it is indivisible.

5B.2.4 The Stochastic-Quantum Correspondence

Barandes' central result: **quantum systems are indivisible stochastic processes**.

The correspondence works as follows:

- Configuration space states ↔ measurement outcomes
- Transition matrices Γ ↔ constructed from Born rule probabilities
- Indivisibility ↔ quantum coherence/interference
- Division events ↔ measurements/decoherence

Hilbert space emergence: In this view, Hilbert spaces and wavefunctions are not fundamental ontology but convenient mathematical tools ("appurtenances") for computing the transition matrices of indivisible stochastic processes.

5B.2.5 Non-Markovian Realizers

A given indivisible stochastic process admits multiple "non-Markovian realizers"—ways of embedding the indivisible dynamics into a larger Markovian framework via hidden variables or environmental coupling. These realizers form an equivalence class; the physical predictions depend only on the equivalence class, not the specific realizer.

This parallels our framework's treatment of Hidden Markov Models: the observed non-Markovian behavior emerges from Markovian dynamics in a higher-dimensional space with restricted observer access.

5B.3 The N-Dependence Mapping

5B.3.1 N = 2: Pre-Divisibility Regime

In our framework, N = 2 configurations involve exactly two features A and B. The fundamental limitation: **there is no intermediate feature C through which to factor**.

The divisibility question asks: "Can transition A → B be factored through intermediate C?"

At N = 2, this question is **structurally meaningless**—not because the answer is "no," but because the question cannot be posed. There is no C.

Analogy: Asking whether a two-point set is "connected" or "disconnected" is meaningless—connectedness requires at least the possibility of intermediate points.

Claim: The N = 2 regime is *pre-divisibility*: prior to the structure needed to define divisibility at all.

5B.3.2 N ≥ 3: Divisibility Becomes Testable

At N ≥ 3, we have at least three features A, B, C. Now we can meaningfully ask:

"Can the coupling structure A ↔ B be factored through C?"

In matrix terms: given coupling matrices M^(AB), M^(BC), M^(CA), can we write:

$$M_{eff}^{(AB)} = f(M^{(AC)}, M^{(CB)})$$

for some composition function f that preserves the "stochastic" properties of couplings?

The generic answer is no. This is the content of Theorem 3 in SI: Circulation Proof—three symmetric matrices generically cannot be simultaneously diagonalized.

5B.3.3 The Correspondence Table

Our Framework	Barandes' Framework
N = 2 configuration	Pre-divisibility regime
N ≥ 3 configuration	Divisibility-testable regime
Coupling matrix M^(αβ)	Transition matrix Γ(β ← α)
Simultaneous diagonalizability	Divisibility
Generic non-diagonalizability	Indivisibility
Circulation C ≠ 0	Pseudo-stochastic intermediate
κ (pattern constraint)	Degree of indivisibility
τ (ordering constraint)	Temporal structure enabling division events

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5B.4 Mathematical Correspondence

5B.4.1 Coupling Matrices as Transition Structures

In our framework, the coupling matrix M^(αβ) characterizes the relationship between features α and β:

$$M_{ij}^{(\alpha\beta)} = \frac{\partial^2 \Phi_{int}}{\partial C_i^{(\alpha)} \partial C_j^{(\beta)}}$$

In Barandes' framework, the transition matrix Γ(β ← α) characterizes the stochastic evolution from state α to state β.

The structural parallel:

- Both are matrices encoding pairwise relationships
- Both satisfy certain positivity/normalization conditions in physical regimes
- Both compose via matrix multiplication for sequential relationships

Proposed identification: Under appropriate normalization and positivity constraints:

$$\Gamma(\beta \leftarrow \alpha) \sim \exp(-M^{(\alpha\beta)}/T)$$

where T is a "temperature" parameter controlling the sharpness of transitions. This exponential map converts coupling strengths to transition probabilities, analogous to Boltzmann factors.

5B.4.2 Divisibility and Simultaneous Diagonalization

Barandes' divisibility criterion:

$$\Gamma(t \leftarrow t_0) = \Gamma(t \leftarrow t') \cdot \Gamma(t' \leftarrow t_0)$$

is satisfiable with valid (non-negative) intermediate Γ(t ← t').

Our framework's analog:

For three features A, B, C, consider whether:

$$M^{(AC)} = g(M^{(AB)}, M^{(BC)})$$

for some "consistent" composition g.

Theorem (Divisibility-Diagonalization Correspondence):

If M^(AB), M^(BC), M^(CA) are simultaneously diagonalizable, then the coupling structure is divisible in the sense that transitions factor through any intermediate feature. If they are not simultaneously diagonalizable (generic case), the structure is indivisible.

Proof sketch:

In the simultaneously diagonalizing basis, each coupling matrix is diagonal:

$$M^{(\alpha\beta)} = \text{diag}(\mu_1^{(\alpha\beta)}, \ldots, \mu_5^{(\alpha\beta)})$$

Transitions in each constraint dimension decouple. The composed transition A → B → C factors:

$$\mu_i^{(AC)} = \mu_i^{(AB)} + \mu_i^{(BC)}$$

(or multiplicatively for the exponentiated transition matrices). Each dimension evolves independently—perfect divisibility.

When simultaneous diagonalization fails, the dimensions couple. Transition A → C through B generates "cross-terms" that cannot be absorbed into a simple A → C coupling. The intermediate $\tilde{\Gamma}$(C ← B) would require these cross-terms, which manifest as negative entries. ■

5B.4.3 Commutators and Indivisibility

The failure of simultaneous diagonalization is measured by commutators:

$$[M^{(AB)}, M^{(BC)}] = M^{(AB)}M^{(BC)} - M^{(BC)}M^{(AB)}$$

Non-zero commutator implies indivisibility.

Definition (Indivisibility Measure):

For an N-feature configuration with N ≥ 3, define:

$$\mathcal{I}_N = \frac{1}{L_N} \sum_{\text{triangles } \alpha\beta\gamma} ||[M^{(\alpha\beta)}, M^{(\beta\gamma)}]||_F$$

where:

- L_N = C(N,3) is the number of triangles
- ||·||_F is the Frobenius norm: ||A||_F = √(Tr(A†A))

Properties of I_N:

- I_N = 0 at N = 2 (vacuously—no triangles exist)
- I_N = 0 iff all coupling matrices commute (divisible case)
- I_N > 0 generically for N ≥ 3 (indivisible case)

5B.4.4 Relation to Circulation

The commutator structure connects to the circulation integral of SI: Circulation Proof.

Proposition: The indivisibility measure I_N is related to the average absolute circulation:

$$\mathcal{I}_N \sim \frac{1}{L_N} \sum_{\text{triangles}} |\mathcal{C}(\gamma_{\alpha\beta\gamma})|$$

Proof sketch: The circulation around a triangular loop involves the curl of the gradient field, which in turn depends on the non-commutativity of the coupling structure. The Frobenius norm of the commutator and the circulation magnitude both measure "how far" the structure is from being simultaneously diagonalizable/conservative. ■

This establishes: **Circulation (SI.1-8) and indivisibility (Barandes) are two manifestations of the same geometric structure.**

5B.5 The Pattern Constraint κ as Indivisibility

5B.5.1 Reinterpreting κ

The pattern constraint κ was introduced as measuring "structural complexity" or "non-factorizability" of a configuration. The Barandes correspondence gives this a precise meaning:

κ measures the degree of indivisibility of the configuration.

High κ configurations:

- Have complex, non-factorizable correlations
- Cannot be divided through intermediate features
- Exhibit quantum-like coherence
- Have non-commuting coupling matrices

Low κ configurations:

- Have simple, factorizable structure
- Can be approximately divided
- Exhibit classical-like behavior
- Have approximately commuting couplings

5B.5.2 Formal Definition

Definition (κ as Normalized Indivisibility):

$$\kappa = \frac{\mathcal{I}_N}{\mathcal{I}_N^{(max)}}$$

where I_N^(max) is the maximum indivisibility achievable for the given configuration geometry.

Properties:

- $\kappa \in [0, 1]$
- $\kappa = 0$ implies divisibility (classical limit)
- $\kappa = 1$ implies maximal indivisibility (maximally quantum)

5B.5.3 κ at Different N

N = 2: κ is not defined via indivisibility (no triangles). However, κ can still be non-zero, measuring internal structure of the two-feature coupling. This is the "pure quantum" regime—maximally non-factorizable because factorization isn't even structurally possible.

N = 3: First regime where κ-as-indivisibility is well-defined. Generic configurations have $\kappa > 0$.

Large N: κ can vary continuously. Environmental averaging tends to reduce κ toward zero (decoherence).

5B.5.4 Comparison with Barandes' Criteria

Barandes characterizes indivisibility via:

1. Negative entries in would-be intermediate transition matrices
2. Failure of the divisibility equation
3. Equivalence classes of non-Markovian realizers

Our κ characterizes indivisibility via:

1. Non-commutativity of coupling matrices
2. Failure of simultaneous diagonalization
3. Non-zero circulation around feature triangles

Claim: These are equivalent characterizations. The negative entries in Barandes' framework correspond to the "imaginary" components that arise when trying to factor non-commuting matrices.

5B.6 Decoherence as Emergence of Divisibility

5B.6.1 The Decoherence Problem

Quantum systems appear to "decohere" when interacting with environments—losing quantum coherence and becoming classical. In Barandes' framework, this corresponds to the emergence of (approximate) divisibility.

Our framework provides a geometric picture of this process.

5B.6.2 Environmental Coupling

Consider a system S with N_S features coupled to an environment E with N_E features. The total configuration has N = N_S + N_E features.

The full coupling structure involves:

- Internal system couplings: M^(αβ) for α, β ∈ S
- Internal environment couplings: M^(αβ) for α, β ∈ E
- System-environment couplings: M^(αβ) for α ∈ S, β ∈ E

5B.6.3 Tracing Out the Environment

The effective system dynamics involves "tracing out" the environment:

$$M^{(S)}_{eff} = \mathcal{T}_E \left[M^{(SE)}_{full} \right]$$

where T_E denotes an appropriate marginalization/averaging over environmental features.

Key result: As N_E → ∞, the effective system coupling M^(S)_eff becomes approximately diagonal.

5B.6.4 The Law of Large Numbers Argument

Theorem (Emergent Divisibility):

For a system S coupled to an environment E with N_E features, as N_E → ∞, the effective indivisibility measure approaches zero:

$$\lim_{N_E \to \infty} \mathcal{I}^{(S)}_{eff} = 0$$

almost surely, under generic coupling assumptions.

Proof sketch:

The off-diagonal elements of M^(S)_eff arise from system-environment couplings mediated through different environmental features. For large N_E, these contributions are:

1. Numerous (scaling as N_E)
2. Quasi-independent (different environmental features)
3. Randomly signed (no preferred alignment)

By the law of large numbers, the sum of many quasi-independent, randomly-signed terms averages to zero. The off-diagonal elements of M^(S)_eff vanish, leaving a diagonal (divisible) structure. ■

Physical interpretation: Decoherence is the emergence of divisibility through environmental averaging. The environment provides sufficiently many "intermediate features" that effective factorization becomes possible.

5B.6.5 Decoherence Rate

The rate of decoherence should scale with:

$$\gamma_{dec} \sim N_E \cdot \langle |M^{(SE)}|^2 \rangle$$

where the average is over system-environment couplings.

Prediction: Decoherence rate is proportional to the number of environmental features times the average coupling strength squared. This is consistent with standard decoherence theory but provides geometric grounding.

5B.6.6 The κ-Decoherence Relation

As the system decoheres:

$$\kappa_{eff}^{(S)}(t) \to 0 \quad \text{as} \quad t \to \infty$$

The effective pattern constraint decreases, signaling the quantum-to-classical transition.

Quantitative prediction: The decay of κ should follow:

$$\kappa(t) = \kappa_0 \cdot e^{-\gamma_{dec} \cdot t}$$

with the decoherence rate γ_dec determined by environmental coupling as above.

5B.7 Hilbert Space Emergence

5B.7.1 The Shared Commitment

Both frameworks reject Hilbert space as fundamental ontology:

Barandes: "Hilbert spaces of quantum theory and their ingredients, including wave functions, can be relegated to secondary roles as convenient mathematical appurtenances."

Our framework: Quantum structure emerges from the geometry of N = 2 configurations in constraint space. Hilbert space is a description of bivector structure, not fundamental reality.

5B.7.2 From Indivisibility to Hilbert Space

Barandes shows how Hilbert space emerges from indivisible stochastic structure:

1. Start with configuration space and transition matrices
2. Indivisibility prevents classical probabilistic description
3. The mathematical tool that captures indivisible dynamics is... Hilbert space
4. Wavefunctions encode the structure needed to compute transition probabilities

The Hilbert space is not "where the system lives" but a mathematical device for tracking indivisible correlations.

5B.7.3 From Constraint Geometry to Hilbert Space

In our framework:

1. Start with constraint space and coupling matrices
2. At N = 2, the configuration is a bivector in Cl(5)
3. The bivector space has dimension 10 (see SI: Geometric Algebra Foundations)
4. This space, with appropriate inner product, becomes a Hilbert space

The bivector space IS the emergent Hilbert space.

5B.7.4 Dimension Matching

The bivector space of Cl(5) has dimension C(5,2) = 10.

This suggests that the "natural" Hilbert space dimension arising from our 5-constraint structure is 10-dimensional—or factors thereof.

Speculative connection:

- Spin-1/2: 2-dimensional Hilbert space = dimension of a single constraint's bivector subspace?
- Spin-1: 3-dimensional = ...?
- The standard model's structure might reflect the geometry of Cl(5) bivectors.

This is highly speculative but suggests avenues for investigation.

5B.7.5 Hilbert Space Dilation

Barandes introduces "Hilbert space dilation"—extending the Hilbert space to accommodate additional structure like spin. In our framework, this corresponds to:

Moving from bivector (grade-2) to higher-grade structures.

The full 32-dimensional Cl(5) algebra provides ample room for dilation. Different grades might correspond to different types of quantum numbers.

5B.8 Emergeables and the Constraint Hierarchy

5B.8.1 Barandes' Emergeables

Barandes introduces **emergeables**: quantities that emerge from patterns in stochastic dynamics but don't exist at the configuration space level.

Examples:

- **Spin:** Not a configuration space property; emerges via Hilbert space dilation
- **Interference patterns:** Emerge from indivisible dynamics
- **Entanglement measures:** Emerge from multi-system correlations

These are "real" in the sense of being measurable, but "derivative" in the sense of not being fundamental.

5B.8.2 Constraints as Mid-Level Emergeables

In our framework, the five constraints might themselves be emergeables from a more fundamental level:

Most fundamental: Pure distinguishing relations (the axiom $\Diamond N \rightarrow \neg N$)

First emergence: The five constraints (β, κ, ρ, λ, τ) as necessary conditions for mutual distinguishability

Second emergence: Quantum observables (at $N = 2$) and classical observables (at large N)

The constraints are "more fundamental" than quantum mechanics but "less fundamental" than pure relational structure.

5B.8.3 The Full Emergence Hierarchy

```
Level 0: Axiom (◇N → ¬N)
    ↓
Level 1: Distinguishing relations
    ↓
Level 2: Five constraints (β, κ, ρ, λ, τ)
    ↓
Level 3a: Indivisible structure (N = 2, high κ) → Quantum observables
Level 3b: Divisible structure (large N, low κ) → Classical observables
    ↓
Level 4: Thermodynamic quantities (Φ in large-N limit)
    ↓
Level 5: Spacetime geometry (from β, τ structure)
```

Each level emerges from the one below but has genuine causal/explanatory power at its own level.

5B.8.4 Comparison of Emergence Structures

Barandes	Our Framework
Configuration space	Constraint space
Stochastic dynamics	Gradient flow dynamics
Indivisibility	High κ / non-diagonalizability
Hilbert space	Bivector structure of Cl(5)
Wavefunctions	Constraint configuration vectors
Emergeables	Higher-level constraints / observables

5B.9 Predictions and Open Questions

5B.9.1 Testable Predictions

Prediction 1: Indivisibility Threshold

There should exist a critical value κ_c such that:

- $\kappa > \kappa_c$: System exhibits quantum behavior (indivisible)
- $\kappa < \kappa_c$: System exhibits classical behavior (divisible)

This threshold should be calculable from constraint geometry, possibly related to the viable region boundary.

Prediction 2: N-Scaling of Decoherence

Decoherence rate should scale as:

$$\gamma_{dec} \propto N_{env}^{\alpha}$$

with exponent α determined by the geometry of system-environment coupling. Standard decoherence theory suggests $\alpha = 1$; our framework might modify this in specific regimes.

Prediction 3: Geometric Signature of Quantumness

A system's "quantumness" (amenability to quantum computation, interference visibility, etc.) should correlate with its indivisibility measure I_N, which is computable from coupling matrices.

Prediction 4: Division Events as N-Transitions

Barandes' "division events" (moments when new divisibility structure emerges) should correspond to effective changes in N—environmental features entering or leaving the relevant coupling structure.

Measurement is an N-transition: the apparatus features couple to the system, changing effective N and inducing divisibility.

5B.9.2 The Quantum-Classical Boundary

The correspondence suggests that the quantum-classical boundary is not:

- A matter of size (large ≠ classical)
- A matter of complexity (complex ≠ classical)
- Fundamentally fuzzy or conventional

Rather, it is:

- A geometric phase transition in constraint space
- Characterized by the divisibility/indivisibility threshold
- Sharp in principle, though practically gradual due to environmental averaging

5B.9.3 Open Questions

Q1: What is the precise mathematical map between constraint coupling matrices and Barandes' transition matrices?

Q2: Can we derive the Born rule from constraint geometry, given that Barandes derives it from indivisible stochastic structure?

Q3: How do specific quantum systems (hydrogen atom, harmonic oscillator, spin chains) map to specific configurations in constraint space?

Q4: Is there a constraint-geometric interpretation of quantum field theory, where field configurations replace particle configurations?

Q5: Can the correspondence be extended to Barandes' treatment of gauge symmetries and CPT?

5B.10 Summary

Main Result: The N-dependence structure of our framework corresponds to the divisibility structure of Barandes' indivisible stochastic processes:

Our Framework	Barandes
$N = 2$	Pre-divisibility
$N \geq 3$, generic	Indivisible (quantum)
Large N, averaged	Divisible (classical)
κ	Indivisibility measure
τ emergence ($N \geq 3$)	Division events possible

Significance:

1. **Shared ontology:** Both reject Hilbert space fundamentalism; quantum structure emerges from deeper relational/stochastic structure.
2. **Geometric grounding:** Our framework provides spatial/relational foundations for Barandes' temporal/stochastic structure.
3. **Unified decoherence:** Quantum-to-classical transition is geometrically characterized as emergence of divisibility through environmental coupling.
4. **Predictive power:** The correspondence suggests testable predictions about the quantum-classical boundary.

Status: The correspondence is structural and motivated but not rigorously proven. Establishing the precise mathematical mapping is future work.

References

- Barandes, J. A. (2025). "Quantum Systems as Indivisible Stochastic Processes." arXiv:2507.21192.
- Barandes, J. A. (2023). "The Stochastic-Quantum Correspondence." arXiv:2302.10778.
- Milz, S. & Modi, K. (2021). "Quantum Stochastic Processes and Quantum Non-Markovian Phenomena." *PRX Quantum* 2, 030201.
- Wolf, M. M. & Cirac, J. I. (2008). "Dividing Quantum Channels." *Commun. Math. Phys.* 279, 147-168.

Appendix 5B.A: Pseudo-Stochastic Matrices

Definition

A **pseudo-stochastic matrix** is a real matrix with:

- Columns summing to 1
- At least one negative entry

Pseudo-stochastic matrices arise when attempting to "invert" stochastic evolution.

The Inversion Problem

Theorem: Let Γ be a stochastic matrix that is not a permutation matrix. Then Γ^{-1} (if it exists) has at least one negative entry.

Proof: Let Γ and Γ^{-1} both have non-negative entries. Since $\Gamma\Gamma^{-1} = I$, each row of Γ must be orthogonal to all but one column of Γ^{-1}. For non-negative matrices, orthogonality requires disjoint support. This forces Γ to be a permutation matrix. ■

Physical Interpretation

Negative "probabilities" in pseudo-stochastic matrices signal that the physical process cannot be factored as assumed. The negativity is not a failure of mathematics but a feature indicating indivisibility.

Appendix 5B.B: Commutators and the Frobenius Norm

The Frobenius Norm

For a matrix A, the Frobenius norm is:

$$||A||_F = \sqrt{\mathrm{Tr}(A^\dagger A)} = \sqrt{\sum_{i,j} |A_{ij}|^2}$$

Properties:

- $||A||_F \geq 0$, with equality iff $A = 0$
- $||\alpha A||_F = |\alpha| \cdot ||A||_F$
- $||A + B||_F \leq ||A||_F + ||B||_F$ (triangle inequality)
- $||AB||_F \leq ||A||_F \cdot ||B||_F$ (submultiplicativity)

Commutator Norm as Non-Commutativity Measure

For matrices A and B:

$$||[A, B]||_F = ||AB - BA||_F$$

This is:

- Zero iff A and B commute
- Bounded by $2||A||_F \cdot ||B||_F$
- Basis-independent

The commutator norm measures "how non-commutative" two matrices are—equivalently, how far they are from being simultaneously diagonalizable.

Relation to Simultaneous Diagonalizability

Theorem: Symmetric matrices A and B are simultaneously diagonalizable if and only if $[A, B] = 0$.

Proof: (⇒) If P diagonalizes both, then PAP^{-1} and PBP^{-1} are diagonal, hence commute. Thus $AB = BA$.

(⇐) If $[A, B] = 0$, then B preserves eigenspaces of A. Restricted to each eigenspace of A, B can be diagonalized. The combined eigenbasis diagonalizes both. ■