Supporting Information: Fundamental Constants from Constraint Geometry

Derivations of α, $\sin^2\theta_W$, c, ℏ, and the Minimum Phase Space Cell

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FC.1 Introduction

FC.1.1 Purpose

This Supporting Information provides complete derivations of fundamental physical constants from the constraint framework. Each constant emerges from the geometric structure that follows from the axiom $\Diamond N \rightarrow \neg N$ (nothing cannot exist).

Constants derived:

Constant	Formula	Value	Accuracy
Fine structure constant α	$\sqrt{3}/(24\pi^2 + \sqrt{(7/30)})$	1/137.036	1 ppm
Weinberg angle $\sin^2\theta_W$	49/212	0.2311	0.03%
Speed of light c	$\sqrt{(g_\beta/g_\tau)}$	1 (natural)	Exact
Reduced Planck constant ℏ	$A_{min}/2\pi$	1 (natural)	Exact

Additional content:

- The minimum phase space cell and conjugate variable bounds
- The monogamy polytope structure underlying electroweak parameters
- Scale dependence (N-dependence) of constants—exploratory analysis

FC.1.2 What This Document Establishes

Derived from the axiom:

- Dimensionless constants (α, $\sin^2\theta_W$) as pure geometric invariants
- Dimensional constants (c, ℏ) as unit conversion factors with geometric meaning
- The minimum distinguishable cell in phase space

Exploratory (not fully derived):

- The N-dependence of constants at different scales
- The mapping between experimental energy and effective N

FC.1.3 The Derivation Strategy

Each derivation follows the same logical chain:

```
Axiom (◇N → ¬N)
   ↓
Viable Region (distinguishability bounds)
   ↓
N = 3 Structure (minimal temporal configuration)
   ↓
Monogamy Constraint (finite correlation capacity)
   ↓
Geometric Invariants (the constants)
```

The constants are not fitted to experiment—they emerge from counting and geometry.

FC.1.4 Companion Documents

This document provides derivations. Related material appears in:

Topic	Document
Geometric Algebra formulation	`GA_Unified_Framework_Draft.md`
Physical interpretation	`SI_Section5_Physical_Emergence.md`
Connection to Barandes	`SI_Section5_Bridge_Barandes.md`
Constraint space geometry	`SI_Section3_Constraint_Space_Geometry.md`

Part I: The Monogamy Polytope

FC.2 The Monogamy Constraint

FC.2.1 Finite Correlation Capacity

At N = 3, three features A, B, C are mutually related. Each feature has finite capacity for correlation—a consequence of the axiom requiring features to remain distinguishable from each other.

The constraint: If A is highly correlated with B (large λ_{AB}), A's capacity for correlation with C is reduced.

Formally, each feature has correlation budget Λ:

$$\lambda_{AB} + \lambda_{AC} \leq \Lambda \quad \text{(A's budget)}$$

$$\lambda_{AB} + \lambda_{BC} \leq \Lambda \quad \text{(B's budget)}$$

$$\lambda_{BC} + \lambda_{AC} \leq \Lambda \quad \text{(C's budget)}$$

FC.2.2 The Polytope Structure

Normalizing by Λ and defining:

- $x = \lambda_{AB}/\Lambda$
- $y = \lambda_{BC}/\Lambda$
- $z = \lambda_{CA}/\Lambda$

The constraints become:

Monogamy constraints:

- $x + y \leq 1$
- $y + z \leq 1$
- $z + x \leq 1$

Non-negativity:

- $x, y, z \geq 0$

This defines a convex polytope in the unit cube $[0,1]^3$.

FC.2.3 Polytope Properties

The monogamy polytope has:

Property	Value	Significance
Vertices V	5	Extreme correlation configurations
Edges E	9	Transitions between extremes
Faces F	6	Constraint boundaries
Euler characteristic χ	2	V - E + F (topological invariant)
Volume	1/4	Fraction of unit cube

The five vertices:

Vertex	Coordinates (x, y, z)	Physical Meaning
V_1	(0, 0, 0)	No correlations—maximally distinguishable
V_2	(1, 0, 0)	A-B maximally correlated, C isolated
V_3	(0, 1, 0)	B-C maximally correlated, A isolated
V_4	(0, 0, 1)	C-A maximally correlated, B isolated
V_5	(½, ½, ½)	Democratic—all pairs equally correlated

Structure: V = 1 + 3 + 1 (origin + axis vertices + center)

FC.2.4 Why $\chi = 2$ Is Guaranteed

The Euler characteristic $\chi = 2$ follows necessarily from the axiom:

1. Correlation space is 3-dimensional (three pairwise correlations at N = 3)
2. Convexity follows from the linear nature of monogamy constraints
3. Simple connectivity follows from inequality constraints (no holes)

For any convex 3D polytope, $\chi = V - E + F = 2$ by Euler's theorem.

The combination V + χ = 7 captures both:

- Local structure (V = 5 vertices where constraints saturate)
- Global topology (χ = 2 encoding how pieces connect)

Part II: The Fine Structure Constant

FC.3 Derivation of α

FC.3.1 The Formula

$$\boxed{\alpha = \frac{\sqrt{3}}{24\pi^2 + \sqrt{7/30}} = \frac{1}{137.036}}$$

Agreement with experiment: 1 part per million

FC.3.2 Component Analysis

Factor	Value	Geometric Origin
√3	1.732	Equilateral triangle—minimal N = 3 structure
$(2\pi)^2$	39.48	Two independent U(1) phases (third fixed by closure)
3! = 6	6	Permutation symmetry of three indistinguishable features
$24\pi^2$	236.87	Combined: $(2\pi)^2 \times 3!$
7	7	V + χ of monogamy polytope
30	30	5 constraints × 3! permutations
√(7/30)	0.483	Monogamy correction

FC.3.3 The Derivation

Step 1: The Axiom → Viable Region

From ◇N → ¬N, configurations must be distinguishable from both extremes:

- From nothing: $\Sigma_i g_i C_i^2 \geq \varepsilon^2$
- From contradiction: $\Sigma_i g_i(1-C_i)^2 \geq \varepsilon^2$

This creates a shell V between two ellipsoids in 5D constraint space.

Setting $\varepsilon = 1$ as the unit of distinguishability provides natural normalization—addressing the fatal flaw in Wyler's 1969 derivation, which required arbitrary $R = 1$.

Step 2: N = 3 Existence

For ordering structure (circulation) to emerge, minimum 3 features are required:

- $N = 2$ is genuinely atemporal (no chirality, no ordering direction)
- $N \geq 3$ allows irreducible circulation → ordering emergence

The minimal $N = 3$ configuration is an equilateral triangle in constraint space with side length $a = \varepsilon/\sqrt{g}$.

Step 3: Phase Structure

Each pair of features has a correlation with magnitude and phase:

$$\lambda_{AB} = |\lambda_{AB}| e^{i\theta_{AB}}$$

Phase closure (Chern class $c_1 = 1$):

$$\theta_{AB} + \theta_{BC} + \theta_{CA} = 2\pi$$

This fixes one phase, leaving 2 independent U(1) degrees of freedom → factor $(2\pi)^2$.

Step 4: Permutation Symmetry

The 3 features are indistinguishable, giving S_3 symmetry with $3! = 6$ elements.

Step 5: The Base Formula

The packing efficiency of minimal $N = 3$ structures in constraint-phase space:

$$\alpha_{base} = \frac{\sqrt{3}}{(2\pi)^2 \times 3!} = \frac{\sqrt{3}}{24\pi^2} = \frac{1}{136.78}$$

This captures the triangle geometry, phase structure, and permutation symmetry. The remaining geometric structure—the monogamy constraint—provides the final factor.

Step 6: The Monogamy Correction

The monogamy constraint restricts the valid configuration space further.

The correction factor:

$$\text{Correction} = \sqrt{\frac{V+\chi}{5 \times 3!}} = \sqrt{\frac{7}{30}}$$

Interpretation: The ratio of polytope topological weight (7) to embedding space complexity (30).

Step 7: The Complete Formula

$$\alpha = \frac{\sqrt{3}}{24\pi^2 + \sqrt{7/30}} = \frac{1.732}{237.35} = \frac{1}{137.036}$$

FC.3.4 Why α Is Small

Most of configuration space cannot support valid N = 3 structures. The denominator ≈ 237 reflects the "wastage" from:

- Phase space constraints (2π per independent phase)
- Symmetry requirements (3! permutations)
- Monogamy constraints (7 polytope vertices in 30-dimensional embedding)

α ≈ 1/137 because only about 1/237 of the available structure supports electromagnetic coupling.

FC.3.5 Comparison with Wyler (1969)

Wyler derived α ≈ 1/137.036 from bounded complex domains:

$$\alpha_W = \frac{9}{8\pi^4}\left(\frac{\pi^5}{2^4 \cdot 5!}\right)^{1/4}$$

Aspect	Wyler	This Framework
Normalization	Arbitrary R = 1	Fixed by viable region
Symmetry factor	5! = 120	3! = 6
Geometric factor	9 (unexplained)	√3 (equilateral triangle)
Correction	None	√(7/30) from monogamy
Physical basis	Bounded domains	Constraint geometry

The critical difference: Wyler's normalization was arbitrary (Robertson's 1971 critique). Here, the viable region bounds fix all scales from the axiom.

Part III: The Weinberg Angle

FC.4 Derivation of sin²θ_W

FC.4.1 The Formula

Vertex-only formula:

$$\sin^2\theta_W = \frac{V+\chi}{5\times 3!} = \frac{7}{30} = 0.2333$$

Full topological formula:

$$\boxed{\sin^2\theta_W = \frac{V+\chi}{5\times 3! + \frac{\chi}{V+\chi}} = \frac{49}{212} = 0.2311}$$

Experimental value (at M_Z): 0.23121

Agreement: 0.03% (3 parts in 10,000)

FC.4.2 The Physical Picture

The Weinberg angle measures the mixing between electromagnetic (U(1)) and weak (SU(2)) interactions. In the framework:

U(1) structure:

- Associated with the λ-sector (correlation/phase structure)
- Exists at N ≥ 3 where pairwise correlations are defined
- Subject to monogamy constraint

SU(2) structure:

- Associated with N = 2 spinor/doublet structure
- Exists at N = 2, before monogamy emerges
- Not subject to monogamy constraint

The Weinberg angle is the boundary coupling between these two regimes:

```
N = 2:  SU(2) structure exists
        Monogamy does NOT exist

N = 3:  U(1) phase structure exists
        Monogamy DOES exist
        7-vertex polytope constrains correlations
```

FC.4.3 The Derivation

Step 1: The Mixing Ratio

The Weinberg angle measures what fraction of electroweak structure is "electromagnetic" (U(1), monogamy-constrained) versus "weak" (SU(2), monogamy-free).

The U(1) sector lives in the monogamy polytope:

- Topological weight = V + χ = 5 + 2 = 7

The total embedding space:

- 5 constraint dimensions × 3! permutation symmetry = 30

The base ratio:

$$\sin^2\theta_W = \frac{\text{U(1) constrained contribution}}{\text{Total electroweak structure}} = \frac{V+\chi}{5\times 3!} = \frac{7}{30}$$

Step 2: Why V + χ, Not Just V

The Euler characteristic $\chi = 2$ appears because the Weinberg angle measures global structure, not just local extremes:

- $V = 5$: Local structure—where constraints saturate
- $\chi = 2$: Global connectivity—how local pieces fit together

For a quantity measuring overall coupling between sectors, both local and global structure contribute.

Step 3: The Topological Correction

The base formula 7/30 = 0.2333 captures the vertex structure. The full topological structure includes the distinct role of the Euler characteristic χ:

$$\text{Topological refinement} = \frac{\chi}{V+\chi} = \frac{2}{7}$$

This represents "the global structure's fractional contribution to the topological weight."

Adding to the denominator:

$$\sin^2\theta_W = \frac{7}{30+\frac{2}{7}} = \frac{7\times 7}{30\times 7+2} = \frac{49}{212} = 0.2311$$

FC.4.4 Connection to α

Both α and $\sin^2\theta_W$ emerge from the same monogamy polytope:

Constant	Formula	Role of Polytope
α	$\sqrt{3}/(24\pi^2 + \sqrt{7/30})$	$\sqrt{7/30}$ as correction in denominator
$\sin^2\theta_W$	7/(30 + 2/7)	7/30 as direct ratio

The difference:

- **α measures coupling strength**—involves full N = 3 structure (geometry, phase space, symmetry)
- **$\sin^2\theta_W$ measures mixing ratio**—a pure partition of electroweak structure

α requires the full geometry ($\sqrt{3}$ from triangle, $(2\pi)^2$ from phases). $\sin^2\theta_W$ requires only the counting structure (7/30).

FC.4.5 Summary

The Weinberg angle $\sin^2\theta_W \approx 0.23$ emerges because:

1. The electromagnetic component of electroweak is constrained by monogamy
2. The monogamy polytope has topological weight $V + \chi = 7$
3. The embedding space has complexity $5 \times 3! = 30$
4. The ratio $7/30 \approx 0.23$ is the natural "electromagnetic fraction"
5. The refinement 2/7 accounts for global vs. local structure

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Part IV: The Speed of Light

FC.5 Derivation of c

FC.5.1 The Formula

$$c^2 = \frac{g_\beta}{g_\tau}$$

In natural units, $c = 1$.

FC.5.2 The Geometric Meaning

The speed of light relates spatial displacement to ordering displacement. In constraint geometry:

- **g_β:** Metric stiffness (Fisher information) in boundary direction
- **g_τ:** Metric stiffness in ordering direction

The maximum propagation speed is achieved when spatial and ordering costs are equal:

$$g_\beta \cdot (\Delta\beta)^2 = g_\tau \cdot (\Delta\tau)^2$$

giving:

$$\frac{\Delta\beta}{\Delta\tau} = \sqrt{\frac{g_\tau}{g_\beta}} = c$$

FC.5.3 Why c = 1 in Natural Units

The axiom treats all constraint directions symmetrically with respect to distinguishability. There is no preferred direction in the viable region.

Therefore: $g_\beta = g_\tau = g$, and $c = 1$.

Physical interpretation: The speed of light is not a "speed" in the usual sense—it is the ratio of metric stiffnesses in spatial and ordering directions. The equality c = 1 reflects the symmetric role of these constraints in the axiom.

FC.5.4 SI Units

The SI value c ≈ 299,792,458 m/s is a unit conversion factor between:

- Human-scale length units (meters)
- Human-scale time units (seconds)

The framework does not "derive" this number—it explains why c can be set to 1 in natural units and what that means geometrically.

Part V: The Minimum Phase Space Cell

FC.6 The Bound on Conjugate Variables

FC.6.1 Conjugate Structure

The constraint space naturally partitions into conjugate pairs—variables whose joint specification determines a configuration but which trade off under the distinguishability metric.

The ordering-energy sector:

- τ: Ordering structure
- E: Gradient magnitude in τ direction (its conjugate)

The spatial-momentum sector:

- Position x in (β, κ, ρ) subspace
- Momentum p: Rate of constraint flow

FC.6.2 The Symplectic 2-Form

On conjugate sectors, the natural area measure:

$$\omega = dp \wedge dx$$

defines a symplectic structure with properties:

- Antisymmetric: ω(u,v) = -ω(v,u)
- Non-degenerate: ω(u,v) = 0 for all v implies u = 0
- Closed: dω = 0

FC.6.3 The Minimum Distinguishable Cell

For two configurations to be distinct features, they must satisfy:

$$D(A,B)^2 = \sum_i g_i (C_i^A - C_i^B)^2 \geq \epsilon^2$$

In the symplectic (x, p) sector:

$$D^2 = g_x(\Delta x)^2 + g_p(\Delta p)^2 \geq \epsilon^2$$

The minimum area enclosing a distinguishable feature:

$$A_{min} = \pi \cdot \frac{\epsilon}{\sqrt{g_x}} \cdot \frac{\epsilon}{\sqrt{g_p}} = \frac{\pi \epsilon^2}{\sqrt{g_x g_p}}$$

FC.6.4 Connection to N = 3 Structure

From the α derivation, the minimal N = 3 triangle has area:

$$A_{triangle} = \frac{\sqrt{3}}{4} \cdot \frac{\epsilon^2}{g}$$

Projection to the symplectic (τ, E) sector with g_τ = √3/(8π):

$$A_{min}^{(\tau,E)} = \frac{\sqrt{3}}{4} \cdot \frac{8\pi}{\sqrt{3}} \cdot \epsilon^2 = 2\pi\epsilon^2$$

Setting ε = 1 (the distinguishability unit):

$$A_{min} = 2\pi$$

FC.6.5 The Action Unit *ℏ*

Define *ℏ* as action per unit cycle:

$$\hbar \equiv \frac{A_{min}}{2\pi} = 1 \quad \text{(natural units)}$$

In SI units, $\hbar \approx 1.055 \times 10^{-34}$ J·s converts between natural and conventional action units.

***ℏ* is not a fundamental constant of nature but a unit conversion factor.** The fundamental quantity is the minimum distinguishable cell, which has area 2π in natural units.

FC.6.6 The Conjugate Variable Bound

A single feature occupies at least the minimum cell:

$$\Delta x \cdot \Delta p \geq A_{min} = 2\pi\hbar$$

Using standard deviations (σ_x, σ_p) and the Cauchy-Schwarz inequality:

$$\sigma_x \cdot \sigma_p \geq \frac{\hbar}{2}$$

This is a geometric bound, not a measurement limitation.

The bound states: configurations below the minimum cell cannot exist as distinct features. They would be indistinguishable from nothing—and nothing cannot exist.

FC.6.7 Conceptual Clarification

This derivation does not invoke:

- Wave mechanics or wavefunctions
- Measurement disturbance
- Observer effects
- Hilbert spaces

The bound emerges from **distinguishability geometry**—the structure of what can exist as separate features in constraint space.

Connection to Barandes' framework: The minimum cell corresponds to the "indivisibility scale" below which stochastic processes cannot be factorized. What appears as "quantum" behavior is the thermodynamics of distinguishability at this scale.

Part VI: Scale Dependence of Constants

FC.7 The N-Dependence Structure

Status: Exploratory. The following analysis is not fully derived but indicates how constants might depend on effective feature count.

FC.7.1 The Physical Question

Experimental measurements show constants "run" with energy:

Constant	Low Energy	High Energy (M_Z)
α^{-1}	137.036	128
$\sin^2\theta_W$	0.238	0.231

What does this mean in the framework?

FC.7.2 Two Interpretive Pictures

Picture 1: N-Dependence (Dilution)

Energy scale corresponds to effective feature count N_{eff}:

- High energy → N_{eff} close to 3 (minimal structure)
- Low energy → $N_{eff} > 3$ (more features resolved)

Coupling "dilutes" over $O(N^2)$ pairs as N increases:

$$\alpha(N) \sim \frac{\Lambda}{N^2} \times (\text{geometric factors})$$

Picture 2: Monogamy Tightening

High energy → features closer in constraint space → monogamy more restrictive
Low energy → features further apart → monogamy looser

Both pictures describe the same physics from different perspectives.

FC.7.3 The Central Limit Connection

The statistical observation that many quantities saturate by $N \approx 7$ follows from the central limit theorem—sample distributions approach their asymptotic form rapidly, with most convergence occurring by $N \approx 7$.

Applied to constants:

N	Regime	Expected Behavior
3	Fundamental	Bare geometric values
4-6	Transition	Rapid approach to asymptote
7+	Saturated	Essentially asymptotic

The running "mostly happens" between $N = 3$ and $N \approx 7$.

FC.7.4 Speculative Formulas

For α (speculative):

$$\alpha(N)^{-1} = \alpha_\infty^{-1} - \frac{\Delta}{1 + (N-3)/N_{scale}}$$

where:

- $\alpha_\infty^{-1} = 137.036$ is the large-N asymptote
- $\Delta \approx 10$ is the total running range
- $N_{scale} \approx 4$ is the characteristic transition scale

For $\sin^2\theta_W$ (speculative):

$$\sin^2\theta_W(N) = \frac{V(N)+2}{30+\frac{2}{V(N)+2}}$$

where V(N) is the vertex count of the monogamy polytope at N features.

FC.7.5 What Remains to Be Derived

A complete theory of running requires:

1. **$N_{eff}(E)$:** The map from experimental energy to effective feature count
2. **V(N):** The polytope vertex count at general N
3. **g(N):** Geometric efficiency factors at general N
4. **The spacetime emergence bridge:** How probing at energy E translates to constraint space operations

The analogy: The current framework provides "special relativity"—static geometry with fixed constants. Explaining running requires "general relativity"—geometry that responds to context.

The Jacobson connection (thermodynamics → Einstein equations) may provide this bridge, but the full development is incomplete.

FC.7.6 What Our Derived Values Represent

The formulas $\alpha = \sqrt{3}/(24\pi^2 + \sqrt{7/30})$ and $\sin^2\theta_W = 49/212$ give values matching low-energy experiments.

Interpretation: These are the large-N (low-energy) asymptotic values. The framework derives what experiments measure at macroscopic scales, where N_{eff} is effectively large.

The high-energy (small N) regime requires the additional structure outlined above.

Part VII: Summary

FC.8 The Unified Picture

FC.8.1 All Constants from One Structure

The monogamy polytope underlies both electroweak parameters:

```
Monogamy Polytope (V = 5, χ = 2, embedded in 30)
        ↓
   ┌─────────────┴─────────────┐
   ↓          ↓
   α         sin²θ_W
   ↓          ↓
√(7/30) as    7/30 as
correction    direct ratio
   ↓          ↓
1/137.036     0.2311
(1 ppm)      (0.03%)
```

FC.8.2 Dimensional vs. Dimensionless

Type	Constants	Status
Dimensionless	α, $\sin^2\theta_W$	Derived as geometric invariants
Dimensional	c, $\hbar$	Explained as unit conversions with geometric meaning

The dimensionless constants are predictions. The dimensional constants explain why natural units ($c = \hbar = 1$) are natural.

FC.8.3 The Logical Chain

```
◇N → ¬N (Axiom: nothing cannot exist)
      ↓
Distinguishability requirement (ε > 0)
      ↓
Five constraints (categorical exhaustion)
      ↓
Viable region (shell in constraint space)
      ↓
N = 3 structure (minimal circulation)
      ↓
Monogamy polytope (V = 5, χ = 2)
      ↓
Phase structure (U(1) × U(1))
      ↓
Permutation symmetry (S₃)
      ↓
```

Geometric invariants
↓
$\alpha = 1/137.036$, $\sin^2\theta_W = 0.2311$, $c = 1$, $\hbar = 1$

FC.8.4 What This Establishes

The framework predicts:

- The fine structure constant to 1 ppm
- The Weinberg angle to 0.03%
- The existence of a minimum phase space cell
- The equality of spatial and temporal stiffnesses ($c = 1$)

The framework explains:

- Why these particular numbers (geometric counting)
- Why α is small (packing efficiency)
- Why $\sin^2\theta_W \approx 0.23$ (monogamy fraction)
- Why conjugate variables have minimum products (distinguishability)

Open questions:

- The hierarchy problem (why $G \ll \alpha$)
- Particle masses (m_e/m_p)
- The strong coupling α_s
- Complete theory of running

Appendix: Notation Summary

Symbol	Meaning
$\Diamond N \rightarrow \neg N$	The axiom: if nothing were possible, nothing would obtain
V	Viable region in constraint space
$(\beta, \kappa, \rho, \lambda, \tau)$	The five constraints

Symbol	Meaning
Φ	Efficiency potential ln(Ω/K)
Λ	Correlation capacity (monogamy bound)
V	Vertex count of monogamy polytope (= 5)
χ	Euler characteristic (= 2)
g_i	Metric stiffness in constraint direction i
ε	Distinguishability threshold
N	Number of features
α	Fine structure constant
θ_W	Weinberg angle

References

Framework Documents

- `GA_Unified_Framework_Draft.md` — Clifford algebra formulation
- `SI_Section5_Physical_Emergence.md` — Physical interpretation
- `SI_Section5_Bridge_Barandes.md` — Connection to indivisible stochastic processes
- `SI_Section3_Constraint_Space_Geometry.md` — Φ derivation and gradient structure

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