Supporting Information: Mathematical Foundations of Ordering Emergence

SI.1 Introduction and Overview

This supporting information provides rigorous mathematical foundations for the central claim of Section 4: that ordering structure (τ) is necessarily zero at $N = 2$ but can be non-zero at $N \geq 3$. We prove this through analysis of gradient circulation on the constraint manifold.

Main Theorem (Informal): The gradient field $\nabla\Phi$ on the configuration space of N features is conservative (zero circulation) when $N = 2$, but generically non-conservative (non-zero circulation possible) when $N \geq 3$.

This theorem establishes that the transition from symmetric to asymmetric structure—and hence the emergence of what we call temporal ordering—is a geometric necessity, not an empirical contingency.

SI.2 Definitions and Setup

SI.2.1 The Constraint Manifold

Definition 1 (Constraint Space). Let $C = \mathbb{R}^5$ denote the ambient constraint space with coordinates (C_1, C_2, C_3, C_4, C_5) corresponding to the five constraints (β, κ, ρ, λ, τ).

Definition 2 (Viable Region). The viable region $V \subset C$ is defined as:

$$\mathcal{V} = \{C \in \mathbb{R}^5 : \epsilon < C_i < 1 - \epsilon \text{ for all } i, \text{ and } \Phi(C) > -\infty\}$$

where $\varepsilon > 0$ is a small constant ensuring distance from boundary singularities. The viable region V is an open, bounded, connected subset of $\mathbb{R}^5$.

Definition 3 (Potential Function). The potential $\Phi: V \to \mathbb{R}$ is defined as:

$$\Phi(C) = \ln\left(\frac{\Omega(C)}{K(C)}\right)$$

where Ω(C) is the measure of accessible states at configuration C, and K(C) is the descriptive complexity. We assume Φ is smooth (C^∞) on V.

SI.2.2 Features and Their Configurations

Definition 4 (N-Feature Configuration Space). For N distinguishable features, the configuration space is the

N-fold product:

$$\mathcal{M}_N = \mathcal{V}^N = \underbrace{\mathcal{V} \times \mathcal{V} \times \cdots \times \mathcal{V}}_{N \text{ times}}$$

A point in M_N is an N-tuple ($C^{(1)}$, $C^{(2)}$, ..., $C^{(N)}$) where each $C^{(\alpha)}$ ∈ V is a 5-vector representing the constraint profile of feature α.

Notation: We use Greek indices α, β, γ ∈ {1, ..., N} for features and Latin indices i, j, k ∈ {1, ..., 5} for constraint dimensions. Thus C_i^(α) denotes the i-th constraint value of feature α.

SI.2.3 Coupling Matrices

Definition 5 (Coupling Matrix). For features α and β, the coupling matrix M^(αβ) is a 5×5 symmetric matrix with components:

$$M_{ij}^{(\alpha\beta)} = \frac{\partial^2 \Phi_{int}}{\partial C_i^{(\alpha)} \partial C_j^{(\beta)}}$$

where Φ_int is the interaction component of the total potential. The coupling matrix characterizes how constraint dimensions of feature α relate to constraint dimensions of feature β.

Properties:

- M^(αβ) = (M^(βα))^T (transpose symmetry between features)
- M^(αα) = H^(α) is the Hessian of feature α (self-coupling)
- For symmetric interactions, M^(αβ) = M^(βα) as matrices

SI.2.4 The Total Potential on M_N

Definition 6 (Total Potential). The total potential on M_N decomposes as:

$$\Phi_{total}(C^{(1)}, ..., C^{(N)}) = \sum_{\alpha=1}^{N} \Phi(C^{(\alpha)}) + \sum_{\alpha<\beta} \Phi_{int}(C^{(\alpha)}, C^{(\beta)})$$

The first sum captures individual feature potentials; the second sum captures pairwise interactions. We assume pairwise interactions dominate (higher-order interactions are subdominant).

Definition 7 (Gradient on M_N). The gradient of Φ_total is a vector in T(M_N) ≅ ℝ^(5N):

$$\nabla\Phi_{total} = \left(\frac{\partial\Phi_{total}}{\partial C_1^{(1)}}, \ldots, \frac{\partial\Phi_{total}}{\partial C_5^{(1)}}, \ldots, \frac{\partial\Phi_{total}}{\partial C_5^{(N)}}\right)$$

SI.3 The N = 2 Case: Conservativity

SI.3.1 Configuration Space for Two Features

For N = 2, the configuration space is $M_2 = V \times V \subset \mathbb{R}^{10}$. A point is specified by $(C^{(1)}, C^{(2)}) = (C^{(A)}, C^{(B)})$, which we denote simply as (A, B) when context is clear.

The total potential is:

$$\Phi_{total}(A, B) = \Phi(A) + \Phi(B) + \Phi_{int}(A, B)$$

SI.3.2 Simultaneous Diagonalizability

Theorem 1 (Simultaneous Diagonalization of Two Symmetric Matrices). Let M_1 and M_2 be real symmetric n×n matrices with M_1 positive definite. Then there exists an invertible matrix P such that:

$$P^T M_1 P = I_n \quad \text{and} \quad P^T M_2 P = D$$

where I_n is the identity matrix and D is diagonal.

Proof. Since M_1 is positive definite, it has a unique positive definite square root M_1^(1/2). Define $\tilde{M}_2$ = M_1^(-1/2) M_2 M_1^(-1/2), which is symmetric. Let Q be orthogonal such that Q^T $\tilde{M}_2$ Q = D (spectral theorem). Set P = M_1^(-1/2) Q. Then:

$$P^T M_1 P = Q^T M_1^{-1/2} M_1 M_1^{-1/2} Q = Q^T Q = I_n$$

$$P^T M_2 P = Q^T M_1^{-1/2} M_2 M_1^{-1/2} Q = Q^T \tilde{M}_2 Q = D$$

■

Corollary 1. For N = 2 features with coupling matrices $H^{(A)}$, $H^{(B)}$, and $M^{(AB)}$, assuming $H^{(A)}$ is positive definite (stable feature), there exists a basis in which $H^{(A)} = I$ and $H^{(B)}$ is diagonal.

SI.3.3 Decomposition into Independent Modes

Proposition 1 (Modal Decomposition at N = 2). In the simultaneously diagonalizing basis, the interaction potential takes the form:

$$\Phi_{int}(A, B) = \sum_{i=1}^{5} \phi_i(A_i, B_i)$$

where each φ_i depends only on the i-th constraint component of each feature.

Proof. In the diagonalizing basis, $M^{(AB)}$ is diagonal (or can be made so by Theorem 1 applied to the block structure). The interaction potential, to quadratic order, is:

$$\Phi_{int} \approx \sum_{i,j} M_{ij}^{(AB)} A_i B_j = \sum_{i} M_{ii}^{(AB)} A_i B_i$$

when $M^{(AB)}$ is diagonal. Each term involves only one constraint dimension. ■

SI.3.4 Zero Circulation at N = 2

Definition 8 (Circulation). For a closed curve γ in configuration space, the circulation of ∇Φ_total around γ is:

$$\mathcal{C}(\gamma) = \oint_{\gamma} \nabla \Phi_{total} \cdot d\ell$$

Theorem 2 (Zero Circulation at N = 2). For any closed curve $\gamma \subset M_2$, the circulation vanishes: $C(\gamma) = 0$.

Proof.

Method 1 (Direct): By Stokes' theorem, for a closed curve γ bounding a surface S:

$$\oint_{\gamma} \nabla \Phi_{total} \cdot d\ell = \iint_{S} (\nabla \times \nabla \Phi_{total}) \cdot d\mathbf{S}$$

But the curl of any gradient is identically zero: $\nabla \times \nabla f = 0$ for any scalar f. Therefore $C(\gamma) = 0$.

Method 2 (Via Decomposition): In the simultaneously diagonalizing basis (Proposition 1), the gradient decomposes:

$$\nabla\Phi_{total} = \sum_i \nabla\phi_i(A_i, B_i)$$

where each term is a gradient in the 2-dimensional (A_i, B_i) subspace. Each 2D subspace is conservative (gradients have no curl in 2D simply-connected domains). The full gradient is a sum of conservative fields, hence conservative. ■

SI.3.5 Interpretation: No Ordering Structure

Corollary 2. At N = 2, the ordering constraint τ = 0 necessarily.

Interpretation. Zero circulation means every closed path in configuration space returns to its starting point with zero net "winding." There is no preferred direction around any loop. The configurations A → B and B → A are geometrically equivalent—the gradient flow between them is reversible.

This is the mathematical content of "no ordering structure": the absence of chirality in the gradient field.

SI.4 The N = 3 Case: Emergence of Circulation

SI.4.1 Configuration Space for Three Features

For N = 3, the configuration space is $M_3 = V^3 \subset \mathbb{R}^{15}$. A point is $(C^{(A)}, C^{(B)}, C^{(c)})$.

The coupling structure now involves three matrices: $M^{(AB)}$, $M^{(Bc)}$, $M^{(cA)}$.

SI.4.2 Failure of Simultaneous Diagonalization

Theorem 3 (Generic Non-Diagonalizability of Three Symmetric Matrices). Let M_1, M_2, M_3 be three real symmetric n×n matrices with n ≥ 2. Generically (for an open dense set in the space of such triples), there exists no basis in which all three are simultaneously diagonal.

Proof. The set of triples (M_1, M_2, M_3) that can be simultaneously diagonalized has positive codimension in the space of all symmetric matrix triples.

A symmetric n×n matrix has n(n+1)/2 independent parameters. A simultaneously diagonalizable triple is determined by:

- A common eigenbasis: n(n-1)/2 parameters (orthogonal matrix modulo diagonal)
- 3n eigenvalues: 3n parameters

Total: n(n-1)/2 + 3n = n(n+5)/2 parameters.

The space of all triples has dimension 3 × n(n+1)/2 = 3n(n+1)/2.

The codimension is:

$$\frac{3n(n+1)}{2} - \frac{n(n+5)}{2} = \frac{3n^2 + 3n - n^2 - 5n}{2} = \frac{2n^2 - 2n}{2} = n(n-1)$$

For n = 5 (our constraint dimensions), the codimension is 5 × 4 = 20 > 0.

Therefore, simultaneously diagonalizable triples form a measure-zero subset; generic triples are not simultaneously diagonalizable. ■

SI.4.3 The Effective Connection

When the coupling matrices cannot be simultaneously diagonalized, motion through M_3 acquires geometric structure analogous to a non-trivial connection.

Definition 9 (Effective Connection). Define the 1-form on M_3:

$$\mathcal{A} = \sum_{\alpha<\beta} \mathrm{Tr}\left(M^{(\alpha\beta)} dC^{(\alpha)} \wedge dC^{(\beta)}\right)$$

where the trace sums over constraint indices.

Definition 10 (Curvature 2-Form). The curvature associated with the coupling structure is:

$$\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$$

Theorem 4 (Non-Zero Curvature at N = 3). If the coupling matrices $M^{(AB)}$, $M^{(Bc)}$, $M^{(cA)}$ are not simultaneously diagonalizable, then F ≠ 0 generically.

Proof sketch. The curvature F measures the failure of parallel transport to be path-independent. In terms of the coupling matrices, the curvature contains terms proportional to:

$$[M^{(AB)}, M^{(BC)}], \quad [M^{(BC)}, M^{(CA)}], \quad [M^{(CA)}, M^{(AB)}]$$

where [·,·] denotes the matrix commutator. These commutators vanish if and only if the matrices are simultaneously diagonalizable (Theorem 3 shows this is non-generic). Therefore F ≠ 0 generically. ■

SI.4.4 Circulation Around Triangular Loops

Definition 11 (Triangular Loop). A triangular loop in feature space is a closed path:

$$\gamma_{ABC} : A \to B \to C \to A$$

realized as a curve in M_3 that traces motion from feature A to B, then B to C, then C back to A.

Theorem 5 (Non-Zero Circulation at N ≥ 3). For a triangular loop γ_ABC in M_3, the circulation:

$$\mathcal{C}(\gamma_{ABC}) = \oint_{\gamma_{ABC}} \nabla\Phi_{total} \cdot d\ell$$

is generically non-zero when the coupling matrices are not simultaneously diagonalizable.

Proof. By Stokes' theorem:

$$\mathcal{C}(\gamma_{ABC}) = \iint_{\Sigma} \mathcal{F}$$

where Σ is any surface bounded by γ_ABC, and F is the curvature 2-form.

By Theorem 4, F ≠ 0 generically. The integral of a non-zero 2-form over a non-degenerate surface is generically non-zero. ■

SI.4.5 Chirality and Ordering

Definition 12 (Chirality). The chirality of a triangular configuration is the sign of the circulation:

$$\chi(A, B, C) = \text{sgn}\left(\mathcal{C}(\gamma_{ABC})\right) \in \{-1, 0, +1\}$$

Proposition 2. Chirality satisfies:

1. χ(A, B, C) = -χ(A, C, B) (reversal changes sign)
2. χ(A, B, C) = χ(B, C, A) = χ(C, A, B) (cyclic invariance)

Proof.

1. Reversing the path reverses the line integral: C(γ_ACB) = -C(γ_ABC).
2. Cyclic permutation traces the same oriented loop, giving the same integral. ■

Corollary 3. At N ≥ 3, the ordering constraint τ can be non-zero.

Interpretation. Non-zero chirality means the loop A → B → C → A is geometrically distinguishable from A → C → B → A. One orientation is preferred by the gradient structure. This is the mathematical content of "ordering structure": the presence of chirality in the gradient field.

SI.5 The General N Case

SI.5.1 Counting Independent Circulations

Proposition 3 (Number of Independent Loops). For N features, the number of independent triangular loops is:

$$L_N = \binom{N}{3} = \frac{N(N-1)(N-2)}{6}$$

N	L_N	Comment
2	0	No triangles possible
3	1	Single triangle ABC
4	4	Four triangles
5	10	Ten triangles

SI.5.2 The Ordering Constraint as Circulation Magnitude

Definition 13 (Ordering Constraint). For a configuration with N ≥ 3 features, define:

$$\tau = \frac{1}{L_N} \sum_{\text{triangles } T} |\mathcal{C}(\gamma_T)|$$

the average absolute circulation over all triangular loops.

Properties:

- τ ≥ 0 by construction
- τ = 0 iff all circulations vanish (requires simultaneous diagonalizability of all coupling matrices—non-generic for N ≥ 3)
- τ > 0 generically for N ≥ 3

SI.5.3 The N = 2 → N = 3 Phase Transition

Theorem 6 (Ordering Phase Transition). The transition from N = 2 to N = 3 is a discrete phase transition in ordering structure:

$$\tau_{N=2} = 0 \quad \text{(exact)}$$

$$\tau_{N=3} > 0 \quad \text{(generic)}$$

This is not a continuous transition; it is a qualitative change in the geometric structure of configuration space.

Physical Interpretation. The emergence of time is not gradual but discrete. Configurations either have ordering structure ($N \geq 3$) or they do not (N = 2). There is no "partial time" or "weak temporality"—only the presence or absence of the geometric structure that constitutes temporal ordering.

SI.6 Connection to Gradient Direction

SI.6.1 The Optimization Direction

Non-zero circulation establishes that orderings are distinguishable—one orientation around a loop is geometrically distinct from the reverse. But which ordering is "forward"? This requires additional structure beyond chirality alone.

The gradient $\nabla\Phi$ provides this structure. At any configuration where $\nabla\Phi \neq 0$, there is a distinguished direction: the direction of increasing Φ, which we call the **Φ-direction** or **optimization direction**.

Definition 14 (Gradient Alignment). For a triangular loop with non-zero circulation, define the forward direction as the orientation aligned with ∇Φ_total:

$$\text{Forward} = \text{argmax}_{\pm\gamma} \int_{\pm\gamma} \nabla\Phi_{total} \cdot d\ell$$

Proposition 4. The forward direction is the direction of increasing Φ—toward configurations with higher Ω/K (greater efficiency of distinguishability).

Proof. The line integral $\int\nabla\Phi \cdot d\ell$ equals Φ(end) - Φ(start) for any path. The direction that maximizes this integral is the direction of increasing Φ. ■

Interpretation. The Φ-direction is purely geometric: it points toward configurations where more states are accessible per unit descriptive complexity. No statistical or physical assumptions are required. The "forward" direction is simply the direction of increasing Ω/K—a property of the potential landscape itself.

SI.6.2 Consistency of Direction Across Loops

Theorem 7 (Global Consistency of Gradient Direction). For a configuration with $N \geq 3$ features, if $\nabla\Phi\_total \neq 0$, then there exists a consistent global ordering of features compatible with the gradient direction.

Proof sketch. The gradient ∇Φ_total defines a vector field on M_N. Where this field is non-zero, it defines a local direction. The integral curves of this field provide a foliation of M_N (away from critical points). This foliation defines a partial order on configurations, which induces an ordering on features through their positions along gradient flow lines. ■

SI.6.3 Relationship to Physical Thermodynamics

The gradient direction defined above is a geometric property of constraint space, independent of physical interpretation. However, at large N, this geometric structure connects to physical thermodynamics:

Geometric (any N)	Physical (large N limit)
$\Phi = \ln(\Omega/K)$	Related to free energy F/kT
Ω (accessible configurations)	Number of microstates
Gradient direction (Φ-direction)	Thermodynamic arrow
Gradient flow	Approach to equilibrium

Important caveat: At small N (including $N = 2$ and $N = 3$), thermodynamic concepts do not apply in their usual sense:

- There is no statistical ensemble
- Temperature is undefined
- Fluctuations do not average out
- The "thermodynamic limit" has not been taken

What exists at small N is the **geometric gradient structure**—a direction in constraint space defined by increasing Ω/K. This structure *becomes* interpretable as thermodynamic in the large-N limit, but at small N it is purely geometric.

The emergence of ordering structure ($\tau > 0$ at $N \geq 3$) is therefore a **geometric theorem**, not a thermodynamic one. The identification of τ with physical time, and of the Φ-direction with the thermodynamic arrow, belongs to the physical interpretation developed in Section 5 of the main text—not to the mathematical foundations established here.

This separation is essential: the mathematical results of this Supporting Information hold independently of whether the physical interpretation is correct. The circulation theorems are geometric facts; their thermodynamic significance is a further claim requiring additional argument.

SI.7 Connection to Established Frameworks

SI.7.1 Gorard's Functorial Irreducibility

Gorard (2023) proves that computational irreducibility corresponds to functoriality failure in the map from computations to cobordisms. The connection to our framework:

Our Framework	**Gorard's Framework**
$N = 2$ decomposability	Functorial (reducible)
$N \geq 3$ irreducibility	Non-functorial (irreducible)
Circulation	Cobordism winding
$\tau > 0$	Computational irreducibility

Conjecture 1. Our ordering constraint τ is equivalent to Gorard's multicomputational complexity measure in the limit where constraint configurations are identified with computational states.

SI.7.2 Finster's Causal Fermion Systems

Finster defines causal ordering through an antisymmetric functional $C(x,y)$ on pairs of operators. The connection:

Our Framework	**Finster's CFS**
Chirality $\chi(A,B,C)$	Oriented volume form
Circulation $C(\gamma)$	Spectral asymmetry
$\tau > 0$	Non-trivial causal structure

Conjecture 2. Our circulation C(γ_ABC) corresponds to Finster's antisymmetric functional integrated around a spacetime triangle.

SI.8 Summary of Results

1. **Theorem 2:** At $N = 2$, circulation vanishes identically. Ordering structure is impossible.
2. **Theorem 3:** Three symmetric matrices are generically not simultaneously diagonalizable.
3. **Theorem 5:** At $N \geq 3$, non-zero circulation is generic. Ordering structure is possible.
4. **Theorem 6:** The transition $N = 2 \rightarrow N = 3$ is a discrete phase transition in ordering structure.
5. **Theorem 7:** Where the gradient is non-zero, a consistent global ordering exists.

These results establish mathematically that:

- Ordering structure ($\tau > 0$) cannot exist at $N = 2$
- Ordering structure generically exists at $N \geq 3$
- The direction of ordering aligns with the gradient of Φ (the Φ-direction)

The emergence of ordering structure is thus a **geometric theorem**, not an empirical observation or metaphysical assumption. The identification of this geometric structure with physical time and the thermodynamic arrow is a further interpretive step, developed in Section 5 of the main text, that goes beyond the purely mathematical results established here.

References

- Finster, F. (2016). *The Continuum Limit of Causal Fermion Systems*. Springer.
- Gorard, J. (2023). "A Functorial Perspective on (Multi)computational Irreducibility."
- Amari, S. (2016). *Information Geometry and Its Applications*. Springer.
- Kobayashi, S. & Nomizu, K. (1963). *Foundations of Differential Geometry*. Wiley.