The SO(3) Gauge Freedom of the Trivector Complex Structure

Extending the Unitarity Result from N = 3 to N = 4

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Abstract

A companion paper [2] derives unitary structure at N = 3 from Ω conservation and the trivector complex structure $T^2 = -1$, with the key step being a proof that the normalised trivector is preserved by the dynamics. That proof relies on the one-dimensionality of $\Lambda^3 V_3$ — a property that fails at N = 4, where four face-trivectors span a 3-dimensional cycle space and individual trivectors can rotate into each other. This paper extends the unitarity result to N = 4 by showing that Gram-preserving evolution induces an SO(3) rotation on the cycle space, and that this rotation preserves unitarity up to gauge equivalence. The isometry group is SO(3) because any positive-definite inner product on a 3-dimensional space has O(3) as its isometry group — a classical result from linear algebra. Physical predictions (transition probabilities, expectation values) are invariant under this gauge group. The identification of the 3-parameter gauge group with the rotation group of three-dimensional space — which emerged at the same N = 4 threshold [1] — connects the gauge freedom of the complex structure to the symmetry of the cycle space. At N ⩾ 5 the cycle space is 6-dimensional and the argument requires additional structure; we state what is known and what remains open.

Note on scope. This paper explores the mathematical structures that emerge from the constraint framework's axiom. Connections to physical concepts (gauge symmetry, spatial rotation) are noted as structural parallels, not as claims that physics is being derived. The mathematics stands independently of any physical interpretation.

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1. Introduction

1.1 The Problem

The derivation of unitary structure in the companion paper [2] proceeds through three ingredients:

1. **Ω conservation** (from the axiom): the evolution preserves the total distinguishability measure.

2. **The trivector complex structure** $\hat{T}^2 = -1$ (from the Cl(5) algebra at $N \geq 3$).
3. **$\hat{T}$ commutation** $[U, \hat{T}] = 0$ (proved at $N = 3$ via confinement + one-dimensionality of $\wedge^3 V_3$).

Ingredients 1 and 2 hold at all $N \geq 3$. Ingredient 3 was proved only for $N = 3$, using the fact that $\wedge^3 V_3$ is one-dimensional: $L\_X(T) = \alpha T$, so $\hat{T}$ is automatically preserved.

At $N = 4$, four features generically span $V_4 \subset \mathbb{R}^5$ with $\dim(V_4) = 4$. The space $\wedge^3 V_4$ is 4-dimensional. Four face-trivectors exist (one per triangular face of K_4), three of which are independent. Individual trivectors can and do rotate into each other under the evolution. The $N = 3$ argument fails.

1.2 The Strategy

Rather than showing the complex structure is *preserved* (which it generically is not at $N = 4$), we show it is *rotated by an isometry* — and that this rotation is physically harmless because all isometrically-related complex structures give the same physical predictions.

The argument has three steps:

Step 1. Show that Gram-preserving evolution maps the cycle space to itself and acts by an element of the isometry group of the trivector Gram matrix.

Step 2. Identify this isometry group: it is SO(3), because any positive-definite inner product on $\mathbb{R}^3$ has O(3) as its isometry group.

Step 3. Show that SO(3) rotation of the complex structure preserves unitarity up to conjugation, so physical predictions are invariant.

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2. The Trivector Action at N = 4

2.1 Setup

Four features $a, b, c, d \in \mathbb{R}^5$, mutually distinguishable. Generically they span a 4-dimensional subspace $V_4 \subset \mathbb{R}^5$, with 1-dimensional orthogonal complement $V_4\perp$.

The four face-trivectors of K_4 are:

$$T_1 = a \wedge b \wedge c, \quad T_2 = a \wedge b \wedge d, \quad T_3 = a \wedge c \wedge d, \quad T_4 = b \wedge c \wedge d$$

These satisfy the dependency relation [1]:

$$T_4 = T_1 - T_2 + T_3 \qquad \text{... (1)}$$

(In the notation of [1]: T_1 = ABC, T_2 = ABD, T_3 = ACD, T_4 = BCD, and the relation reads BCD = (+1)·ABC + (−1)·ABD + (+1)·ACD.)

Three are linearly independent; we take $\{T_1, T_2, T_3\}$ as a basis for the 3-dimensional cycle space Z = $\mathrm{span}(T_1, T_2, T_3) \subset \wedge^3 V_4$.

2.2 Confinement of Feature Velocities

The evolution is generated by a vector field X on M_4. Since Φ and Ω depend only on the Gram matrix (relational quantities), the dynamics is invariant under reflections in $V_4\perp$ (which is 1-dimensional, so the only non-trivial transformation is $v\perp \to -v\perp$). If any feature velocity had a non-zero component in $V_4\perp$, the reflection would produce a different velocity at the same configuration, contradicting determinism of the flow. Therefore:

$$L\_X\, a,\ L\_X\, b,\ L\_X\, c,\ L\_X\, d \in V_4 \qquad \dots (2)$$

2.3 The Lie Derivative Maps Z to Z

Each feature velocity lies in V_4 = span(a, b, c, d), so we can write:

$$L\_X\, a = \alpha_1 a + \alpha_2 b + \alpha_3 c + \alpha_4 d$$

and similarly for b, c, d. By the Leibniz rule for the exterior algebra:

$$L\_X(T_1) = (L\_X\, a) \wedge b \wedge c + a \wedge (L\_X\, b) \wedge c + a \wedge b \wedge (L\_X\, c) \qquad \dots (3)$$

Substituting the expansions and using antisymmetry (terms with repeated factors vanish), each d-containing term can be expressed using the face-trivectors. For example, $d \wedge b \wedge c = T_4$, and $a \wedge d \wedge c = -T_3$, and $a \wedge b \wedge d = T_2$. Using the dependency (1) to eliminate T_4 where it appears, we obtain:

$$L\_X(T_1) = \Sigma\_j\ \Lambda_{1j}\ T_j$$

for coefficients Λ_{1j} determined by the feature velocity coefficients. Analogous calculations for T_2 and T_3 yield:

Proposition 1. The Lie derivative L_X maps the cycle space Z to itself. The action is represented by a 3×3 matrix Λ:

$$L\_X(T_i) = \Lambda_{ij}\ T_j$$

Proof. By (2), all feature velocities lie in V_4. By the Leibniz rule, $L\_X(T_i) \in \wedge^3 V_4$. The explicit computation shows each $L\_X(T_i)$ is a linear combination of face-trivectors. Since $\{T_1, T_2, T_3\}$ is a basis for the face-trivector subspace of $\wedge^3 V_4$ (with T_4 expressed via (1)), the image lies in Z. ■

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3. Gram Preservation Constrains Λ

3.1 The Trivector Gram Matrix

The face-trivectors carry a natural inner product from the Clifford algebra:

```
G^T_ij = ⟨Tᵢ† Tⱼ⟩₀
```

where $\langle\cdot\rangle_0$ is the scalar part and † is reversal. This is computed from the feature Gram matrix by the standard identity:

```
⟨u₁∧u₂∧u₃, v₁∧v₂∧v₃⟩ = det(uₐ · vᵦ)
```

Each entry of G^T is a 3×3 determinant of feature inner products.

3.2 Ω-Preserving Evolution Preserves the Gram Matrix

Proposition 2. If the evolution preserves Ω, and Ω depends only on the feature Gram matrix G^f, then G^f is preserved along the flow.

Proof. Since Φ and Ω depend only on G^f, the dynamics is SO(5)-equivariant: if (a, b, c, d) is a configuration and R ∈ SO(5), then (Ra, Rb, Rc, Rd) has the same Gram matrix and hence the same Φ, Ω, and dynamical evolution. The flow commutes with the SO(5) action.

The Gram matrix G^f is the complete SO(5)-invariant of the configuration: two configurations (a, b, c, d) and (a', b', c', d') have the same Gram matrix if and only if they are related by an element of SO(5) (this is a standard result — the Gram matrix characterises the configuration up to rigid rotation). Since the flow commutes with SO(5), it maps each SO(5)-orbit to itself. The Gram matrix, being constant on orbits, is therefore constant along the flow. ■

Corollary. The trivector Gram matrix G^T is preserved, since each entry is a determinant of entries of G^f.

3.3 The Constraint on Λ

Theorem 1. The matrix Λ lies in the Lie algebra of the isometry group of G^T:

```
Λ^T G^T + G^T Λ = 0                                          ... (4)
```

Proof. G^T is preserved along the flow:

```
(d/dλ) G^T_ij = ⟨L_X Tᵢ, Tⱼ⟩ + ⟨Tᵢ, L_X Tⱼ⟩ = 0
```

Substituting L_X $T_i = \Lambda_{ik} T_k$:

```
Λᵢₖ G^T_kj + Λⱼₖ G^T_ik = 0
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which is (ΛG^T)_ij + (G^T Λ^T)_ij = 0, giving equation (4). ■

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4. The Isometry Group Is SO(3)

4.1 A General Fact from Linear Algebra

Theorem 2. The isometry group of any positive-definite inner product on $\mathbb{R}^n$ is isomorphic to O(n). Its identity component is SO(n), with Lie algebra so(n) of dimension n(n−1)/2.

Proof. Let G be the positive-definite matrix defining the inner product. By Cholesky decomposition, G = LL^T for some invertible L. The map R → L^{-1}RL establishes an isomorphism between Isom(G) = {R : R^T G R = G} and O(n) = {S : S^T S = I}. ■

For n = 3, this gives: Isom(G^T) ≅ O(3), with identity component SO(3) and Lie algebra so(3) of dimension 3.

4.2 Explicit Computation for the Oriented Trivector Gram Matrix

The oriented Gram matrix (computed from the Clifford inner product of face-trivectors, using the orientations from [1]) is:

```
Go = |  3   1  -1 |
     |  1   3   1 |                                       ... (5)
     | -1   1   3 |
```

with eigenvalues {1, 4, 4} and eigenvectors:

- Eigenvalue 1: direction (1, −1, 1)/√3 — which corresponds to $T_4 = T_1 - T_2 + T_3$
- Eigenvalue 4: the 2-dimensional subspace perpendicular to (1, −1, 1)

The Lie algebra condition Λ^T Go + Go Λ = 0, written in the eigenbasis (D = diag(1, 4, 4)), becomes:

```
D_i Λ_ij + D_j Λ_ji = 0
```

This gives:

```
Λ₁₂ = -4Λ₂₁,    Λ₁₃ = -4Λ₃₁,    Λ₂₃ = -Λ₃₂
```

with Λ_{21}, Λ_{31}, Λ_{32} as three free parameters. The Lie algebra is 3-dimensional.

The three generators (in the eigenbasis) are:

```
E₁ = [[0, -4, 0], [1, 0, 0], [0, 0, 0]]
E₂ = [[0, 0, -4], [0, 0, 0], [1, 0, 0]]
E₃ = [[0, 0, 0], [0, 0, -1], [0, 1, 0]]
```

Their commutation relations:

```
[E₁, E₂] = 4E₃,   [E₂, E₃] = E₁,   [E₃, E₁] = E₂
```

This is so(3) with a non-standard normalisation reflecting the eigenvalue ratio. Rescaling $e_1 = E_1/2$, $e_2 = E_2/2$, $e_3 = E_3$ gives the standard so(3) algebra up to an overall scale.

All verified computationally: each generator satisfies Λ^T Go + Go Λ = 0 to machine precision.

4.3 Why SO(3) and Not SO(2)

One might expect the double eigenvalue {4, 4} to restrict the isometry group to SO(2) (rotations within the eigenvalue-4 subspace). This would be correct if we additionally required that the eigenspaces be preserved. But the Lie algebra condition (4) does not impose this — it allows generators that mix the eigenvalue-1 and eigenvalue-4 subspaces (generators E_1 and E_2 do exactly this). The full isometry group is SO(3), as Theorem 2 guarantees for any positive-definite form on $\mathbb{R}^3$.

Equivalently: via the Cholesky decomposition Go = LL^T, the change of basis L^{-1} transforms Isom(Go) to the standard O(3), which is manifestly 3-dimensional.

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5. Unitarity Up to Gauge

5.1 The Rotating Complex Structure

At any point along the evolution, the system has a current trivector in the cycle space:

```
T(λ) = Σᵢ cᵢ(λ) Tᵢ
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The coefficients evolve according to the SO(3) rotation: c(λ) = exp(λΛ) c(0).

The simplicity condition. For $\hat{T}(\lambda) = T(\lambda)/|T(\lambda)|$ to define a complex structure ($\hat{T}^2 = -1$), the trivector $T(\lambda)$ must be *simple* — expressible as a single wedge product $u \wedge v \wedge w$. A linear combination of simple trivectors is not automatically simple. In $\wedge^3 V_4$ (which is 4-dimensional), the simple trivectors form a quadric hypersurface (the Grassmannian Gr(3,4) embedded via the Plücker embedding).

The four face-trivectors $T_1, \ldots, T_4$ are each simple by construction (each is $a \wedge b \wedge c$ for three features). Their linear combinations within Z are simple if and only if they satisfy the Plücker relation. For the cycle space of K_4, the dependency relation $T_4 = T_1 - T_2 + T_3$ ensures that the face-trivectors span a 3-dimensional subspace of the 4-dimensional $\wedge^3 V_4$. A generic element of this 3-dimensional subspace is simple: the Plücker relation defines a quadric in $\wedge^3 V_4$, and the intersection of a generic 3-plane with this quadric is a 2-dimensional surface (not empty). The set of non-simple elements in Z has measure zero.

The SO(3) rotation preserves the Gram matrix G^T, which encodes the relative geometry of the face-trivectors. Since the simplicity condition is a polynomial relation on the coefficients (the Plücker relation), and the face-trivectors themselves are simple, the SO(3) orbit of any simple trivector in Z remains simple generically. (At special configurations where a rotation takes a simple element to the non-simple locus — a measure-zero event — the complex structure would degenerate, and the argument requires separate treatment.)

For the remainder of this section, we assume $T(\lambda)$ remains simple along the flow. At each such λ, $\hat{T}(\lambda)^2 = -1$.

5.2 The Gauge Decomposition

Fix a reference complex structure $J_0(v) = \hat{T}(0) \cdot v$. The evolution takes $\hat{T}(0)$ to $\hat{T}(\lambda)$ via an SO(3) rotation $R(\lambda) \in \mathrm{Isom}(Go)$:

$$\hat{T}(\lambda) = R(\lambda)\ \hat{T}(0)$$

where $R(\lambda)$ acts on the cycle space Z (the 3-dimensional space of face-trivectors). This SO(3) action on Z extends to the full Clifford algebra Cl(5) via the geometric product: $R(\lambda)$ acts as a rotor on the grade-3 sector of Cl(5), and this rotor action extends naturally to all grades. Concretely, if the SO(3) rotation on Z is generated by a bivector $B \in \wedge^2 Z$, the corresponding action on a general multivector $\psi \in Cl(5)$ is $\psi \to \exp(B/2)\ \psi \exp(-B/2)$. This is the standard rotor action in geometric algebra [7].

Decompose the full evolution operator as:

$$U(\lambda) = R(\lambda) \cdot U_0(\lambda) \qquad \ldots (6)$$

where $R(\lambda)$ is the gauge part (implementing the SO(3) rotation of the complex structure via the rotor action on Cl(5)) and $U_0(\lambda)$ is the physical part, defined by $U_0 = R^{-1}U$. By construction:

$$[U_0(\lambda),\ J_0] = 0 \quad \text{... (7)}$$

5.3 Physical Unitarity

Theorem 3 (Unitarity up to Gauge). The physical evolution $U_0(\lambda)$ is unitary with respect to the inner product defined by J_0.

Proof. The full evolution U preserves Ω, hence preserves all norms: $|U\psi|^2 = |\psi|^2$. The gauge rotation R is an isometry of the trivector Gram matrix (Theorem 1), hence preserves norms. Therefore $U_0 = R^{-1}U$ preserves norms.

U_0 commutes with J_0 (equation 7). By the polarisation identity ([2], Section 6.3):

$$\langle\psi|\phi\rangle\_J_0 = \tfrac{1}{4}[|\psi+\phi|^2 - |\psi-\phi|^2 + J_0(|\psi+J_0\phi|^2 - |\psi-J_0\phi|^2)]$$

Each term is preserved by U_0 (norm preservation + J_0 commutation). Therefore $\langle U_0\psi|U_0\varphi\rangle\_J_0 = \langle\psi|\varphi\rangle\_J_0$. This is unitarity: $U_0^\dagger U_0 = I$. ■

5.4 Gauge Invariance of Physical Predictions

Proposition 3. Transition probabilities are independent of the gauge choice.

Proof. Under a gauge rotation R ∈ Isom(Go):

$$|\langle R\psi|R\phi\rangle\_\{RJ_0R^{-1}\}|^2 = |\langle\psi|\phi\rangle\_J_0|^2$$

because R is an isometry. The squared inner product — and hence all physical predictions — is gauge-invariant. ■

5.5 The Main Result

Theorem 4 (N = 4 Unitarity). At N = 4, the Ω-preserving evolution on the constraint field is unitary up to an SO(3) gauge rotation of the complex structure. The gauge group is the isometry group of the K_4 trivector Gram matrix. Physical predictions are gauge-invariant.

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6. The SO(3) Gauge Group and Three-Dimensional Space

6.1 Three Gauge Parameters at the Three-Dimensional Threshold

The companion paper [1] proves that three spatial dimensions emerge at N = 4 — the cycle rank of K_4 is 3, and the cycle space has round S^2 directional geometry. This paper shows that three gauge parameters also emerge at N = 4 — the Lie algebra of Isom(Go) is so(3), with dimension 3.

This is not a coincidence. Both "threes" come from the same source: the 3-dimensionality of the K_4 cycle space. The spatial dimensions are the three independent directions of monogamy competition [1]. The gauge parameters are the three independent rotations of the complex structure within the same cycle space (this paper).

6.2 Structural Parallel with Spatial Rotation

In the constraint framework, a "spatial rotation" is a transformation that preserves all relational content (the Gram matrix) while changing the orientation of the feature configuration within $\mathbb{R}^5$. At N = 4, the cycle space Z is 3-dimensional, and the isometries of Z form SO(3). The gauge rotations of the complex structure are elements of this same SO(3).

This establishes a structural parallel — not claimed here as a physical derivation — between the gauge freedom of the complex structure and the rotational freedom of three-dimensional space. Whether this parallel extends to a physical identification (connecting the gauge symmetry of quantum mechanics to spatial rotational symmetry) is a question for further investigation.

6.3 The N-Hierarchy of Gauge Structure

N	Cycle rank	Cycle space dim	Gauge group	Lie algebra dim
3	1	1	{e} (trivial)	0
4	3	3	SO(3)	3
5	6	6	$\mathrm{Isom}(G^\wedge T_5) \subseteq O(6)$	$\leqslant 15$

At N = 3: no gauge freedom. The complex structure is unique and exactly preserved. Unitarity is exact.

At N = 4: SO(3) gauge freedom. The complex structure can rotate within the cycle space, but all rotated complex structures give equivalent predictions. Unitarity holds up to gauge.

At N = 5: the cycle space is 6-dimensional and the isometry group $\mathrm{Isom}(G^\wedge T_5)$ is a subgroup of O(6) whose dimension depends on the symmetry of $G^\wedge T_5$. For a generic positive-definite form on $\mathbb{R}^6$, the isometry group could be as small as {±I} (dimension 0) or as large as O(6) (dimension 15). The actual dimension for the K_5 trivector Gram matrix has not been computed and is an open problem.

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7. Observations at N ⩾ 5

7.1 What Changes

At N = 5, five features span all of $\mathbb{R}^5$. The cycle rank is 6. The trivector Gram matrix $G^\wedge T_5$ is 6×6.

The companion paper [1] shows the off-diagonal entries are NOT uniform: some K_5 triangle pairs share an edge (inner product ≠ 0) while others are edge-disjoint (inner product = 0). The unoriented Gram matrix does not have the form $\alpha I + \beta J$, and the roundness residual is 2.19.

The isometry group of any positive-definite form on $\mathbb{R}^6$ is O(6), with Lie algebra of dimension 15. This is the maximal gauge group. Whether the specific structure of $G\wedge T_5$ (its eigenvalue pattern, its connection to K_5 combinatorics) reduces this group to a smaller subgroup with physical significance is an open question.

7.2 The Pseudoscalar at N = 5

A new algebraic object appears at N = 5: the pseudoscalar $I_5 = e_1e_2e_3e_4e_5 \in \wedge^5\mathbb{R}^5$. Since $\dim(\wedge^5\mathbb{R}^5) = 1$, the one-dimensionality argument from the N = 3 proof applies:

```
L_X(I₅) = α₅ I₅   →   L_X(Î₅) = 0
```

The normalised pseudoscalar is exactly preserved. However, $I_5^2 = +1$ (not −1), so it is not a complex structure. It acts as a duality operator, mapping grade-k elements to grade-(5−k) elements. This duality links trivectors (grade 3) to bivectors (grade 2), creating additional constraints on the trivector evolution beyond Gram preservation alone.

Whether these additional constraints are sufficient to prove unitarity at N = 5 — possibly with a gauge group smaller than the full O(6) — remains open.

7.3 The Anisotropy of K_5

The non-roundness of the K_5 cycle space means the isometry group of $G\wedge T_5$ may have a non-trivial eigenvalue structure, with different eigenvalues corresponding to different "types" of cycle-space directions. In the K_4 case, the eigenvalues {1, 4, 4} distinguished the T_4-direction (eigenvalue 1) from the orthogonal plane (eigenvalue 4). In K_5, the eigenvalue pattern of the 6×6 Gram matrix would similarly partition the cycle space into subspaces with potentially different physical roles.

This is a well-defined mathematical problem: compute $G\wedge T_5$ for generic features in $\mathbb{R}^5$, find its eigenvalue structure, and determine the implications for the gauge group and unitarity argument. We leave it for future work.

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8. Summary

8.1 What Is Proved

#	Result	Method	Status
1	Feature velocities lie in V_4	Reflection argument in $V_4\perp$	Proven
2	L_X maps cycle space Z to itself	Leibniz rule + face-trivector basis	Proven
3	$\Lambda \in$ Lie(Isom(G^T))	Gram preservation (SO(5)-equivariance)	Proven
4	Isom(G^T) ≅ O(3), identity component SO(3)	Any pos. def. form on $\mathbb{R}^3$	Proven
5	Lie algebra is so(3), dimension 3	Explicit computation, verified numerically	Proven
6	U decomposes as $R \cdot U_0$ with $[U_0, J_0] = 0$	Construction	Proven
7	U_0 is unitary	Norm preservation + J_0 commutation + polarisation	Proven
8	Physical predictions are gauge-invariant	Isometry preserves	⟨·

8.2 What Remains Open

Item	Status
Simplicity of T(λ) along flow	Generically satisfied; measure-zero exceptions
N = 5 isometry group structure	Open (well-defined computation)
N = 5 unitarity	Open (pseudoscalar constraint is a lead)
Physical interpretation of SO(3) gauge = spatial SO(3)	Parallel noted, not claimed as derivation

8.3 How the Papers Connect

The companion paper [1] proves the K_4 cycle space is 3-dimensional with round S^2 directional geometry.

The companion paper [2] proves Ω conservation + trivector complex structure → unitarity at N = 3.

This paper proves that at N = 4, the evolution rotates the complex structure by an element of SO(3) = Isom(Go), and that this rotation is a gauge transformation preserving unitarity.

The three papers together establish, from the single axiom, the mathematical structures that parallel:

- Three spatial dimensions [1]
- Symplectic (Hamiltonian) structure at N = 2 [2]
- Unitary structure at N = 3 [2]
- Unitarity with SO(3) gauge symmetry at N = 4 (this paper)

Appendix A: Computational Verification

The oriented trivector Gram matrix:

```
Go = [[3, 1, -1], [1, 3, 1], [-1, 1, 3]]
```

Eigenvalues: {1.0, 4.0, 4.0}. Eigenvectors: eigenvalue 1 along (1, −1, 1)/√3 (= T_4 direction); eigenvalue 4 in the perpendicular plane.

Lie algebra generators (in eigenbasis):

```
E₁ = [[0,-4,0],[1,0,0],[0,0,0]]   E₂ = [[0,0,-4],[0,0,0],[1,0,0]]   E₃ = [[0,0,0],[0,0,-1],[0,1,0]]
```

Verification: E_k^T D + D E_k = 0 for D = diag(1,4,4) — confirmed to machine precision for all three generators.

Structure constants: $[E_1,E_2] = 4E_3$, $[E_2,E_3] = E_1$, $[E_3,E_1] = E_2$. This is so(3) (rescale to standard form via $e_1 = E_1/2$, $e_2 = E_2/2$, $e_3 = E_3$).

The Lie algebra dimension is 3, confirming the identity component of Isom(Go) is SO(3).

References

[1] D. Neale, "Three-Dimensional Geometry from a Distinguishability Axiom: The Cycle Space of K_4 and Frame-Invariant Directional Distributions." Preprint available at goleudy.ai.

[2] D. Neale, "Mathematical Structures from a Distinguishability Axiom: Symplectic and Unitary Geometry as Consequences of Relational Topology." Preprint available at goleudy.ai.

[3] G. Carcassi and C.A. Aidala, "Reverse Physics: From Laws to Physical Assumptions," *Foundations of Physics* **52**, 40 (2022).

[4] G. Carcassi and C.A. Aidala, *Assumptions of Physics*, Version 2.0 (Michigan Publishing, 2023).

[5] G. Carcassi and C.A. Aidala, "Hamiltonian mechanics is conservation of information entropy," *Stud. Hist. Phil. Sci. B* **70**, 19 (2020).

[6] V.I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer, 1989).

[7] C. Doran and A. Lasenby, *Geometric Algebra for Physicists* (Cambridge University Press, 2003).

[8] D. Neale, "Being from Nothingness: Deriving the Structure of Existence from a Single Axiom." Preprint available at goleudy.ai.

Companion to [1] and [2]. Preprint available at goleudy.ai.