Representation-Theoretic Structure of the K_N Grade Tower

A Schur-Weyl Perspective on the N-Hierarchy

In-progress research note, Goleudy.ai, April 2026

A note on terminology

This paper uses the labels SU(2), SU(3), SU(4), and SO(3) strictly in their abstract representation-theoretic sense — as organising symmetries of mathematical structures associated with the graph K_N. These labels identify specific irreducible representations of the corresponding Lie groups. Where the physics literature uses the word "spin" for irreducible representations of SU(2), we use "irreducible representation" throughout to keep the rep-theoretic sense separate from the angular-momentum meaning that "spin" carries in quantum dynamics. Wherever structural analogies to objects in physics are noted, these are reported as analogies between mathematical labels, not as claims that the BfN framework produces the physical objects carrying those labels.

Abstract

This note examines the representation-theoretic content of the exterior-algebra tower $\Lambda^0(V)$, $\Lambda^1(V)$, $\Lambda^2(V)$, $\Lambda^3(V)$, …, where V is the natural permutation representation of S_N acting on the vertices of the complete graph K_N. For each grade k, the signed-subset space $\Lambda^k(V)$ decomposes under S_N into exactly two irreducible Specht modules, whose Young diagrams have k+1 and k rows. Under Schur-Weyl duality, each summand pairs with a determined Weyl module of GL(k+1) respectively GL(k), and the dimensions of these partners are forced by the row counts. Applied grade by grade, this gives a precise rep-theoretic backbone to the question of "what structural content each N carries" — the question underlying the regime hierarchy developed in the *Regimes of N* position paper.

Five concrete consequences are reported. First, a previously inconsistent point in the framework literature — the value 2.19 for the K_5 "roundness residual" — is identified as a star-basis Gram computation rather than a full Gram computation, resolving the convention question. Second, the cycle space of K_N is shown to be the Schur-Weyl partner of the (N−3)-fold symmetric power of the SU(3) fundamental, restricting under SO(3) ⊂ SU(3) to a sequence of SO(3) irreducibles V_{N−3} ⊕ V_{N−5} ⊕ … whose K_4 entry is exactly the 3-dimensional SO(3) irrep identified by Bridge 1 as "3D space." Third, the full triangle space of K_N splits as SU(3)-content ⊕ SU(4)-content, with the SU(4) piece being the space of tetrahedron-boundary relations; the canonical S_5 isomorphism between the grade-3 and grade-4 realisations of this SU(4) content is the simplicial chain-complex boundary operator ∂_4, which is shown to match the *Regimes of N* paper's "coherence tension" reading of grade-4 as a unique forced correspondence. Fourth, at N=5 the entire Clifford algebra

Cl(5) decomposes under S_5 into a clean palindromic tower: five S_5 irreps spanning six grades, each occupying two consecutive grades, with Hodge duality verified algebraically to act as a sign-twisted isomorphism between paired grades. Fifth, and importantly for honest scope: this palindromic tower structure is not unique to N=5. It generalises to Cl(N) at every $N \geq 3$ as a consequence of Pascal-triangle symmetry in the hook-irrep content. What is specific to N=5 is that the hook dimensions at this Pascal row happen to match low-rank Lie-algebra reps (the SU(3) and SU(4) content noted above) — a dimensional coincidence that weakens at higher N. The final section collects honest caveats about what these results are and are not, and lists the concrete questions they leave open.

The claim of this note is limited to mathematical structure. Where the produced structures carry the same rep-theoretic labels as objects in physics (SU(3), SU(4), SO(3) irreps), this is reported as a structural analogy to be investigated, not as a derivation.

---

1. Setup and context

The Being-from-Nothingness (BfN) programme takes a single modal-logic axiom $\Diamond N \to \neg N$ ("if nothingness is possible, then nothingness does not obtain") as its sole axiomatic input. Structural content is then developed by asking what the axiom permits — what kinds of relations, graded structures, and configurations can arise — and tracking the progression of this content as the relational graph grows in vertex count N.

Several existing programme documents establish that specific N values correspond to qualitative transitions:

- At N = 3, the first cycle becomes possible (K_3 cycle rank 1), associated in the framework with the emergence of temporal ordering.
- At N = 4, every edge of K_N participates in exactly two triangles; Bridge 1 identifies this as the emergence of 3D space through an SO(3) rotation structure on the cycle space.
- At N = 5, Cl(5) closes: the five constraint types (β, κ, ρ, λ, τ) have exactly enough room for a pseudoscalar, and the framework's derivations of α and $\sin^2\theta$_W via the monogamy polytope become calculable.

The *Regimes of N* position paper consolidates these into an N-hierarchy: each N range is characterised by which mathematical vocabulary has well-defined reference. Below, the question is approached from a different angle. Rather than asking which vocabulary applies at each N, we ask what the *representation-theoretic content* of each graded space over K_N is, when K_N is viewed through its natural S_N symmetry.

The approach yields a clean backbone: each grade of Λ•(V) has exactly two irreducible S_N summands with Schur-Weyl partners of determined structure. This backbone gives a structural reason for several specific features of existing framework results (why 3D space and not some other dimension appears at K_4; why SU(3) content arises naturally in Cl(5) grade-3) without itself constituting a physics derivation.

2. The 2.19 convention question

Before the main analysis, a convention issue in the framework literature needs resolving. The paper *Three-Dimensional Geometry from the Distinguishability Axiom* reports a table of "roundness residuals" for K_N cycle spaces, with K_5 giving 2.19 and K_6 giving 4.47. The paper's prose defines the residual as $\|PGP - \alpha P\|_F$ where G is the full L×L Gram matrix on the C(N,3) triangles, P projects out the uniform direction, and α minimises the Frobenius norm.

Direct computation gives a different value. The full L×L unoriented Gram at K_5 has residual $\sqrt{20} \approx 4.47$, which is what the paper's Table 1 labels as the K_6 entry. The full Gram at K_6 has residual 7.17. The values 2.19 and 4.47 *do* appear, but on a different matrix: the **cycle-rank × cycle-rank basis Gram** constructed from the triangles containing a fixed vertex (the "star at vertex 0" basis). At K_5 this is a 6×6 matrix with residual 2.1909. At K_6 it is a 10×10 matrix with residual 4.4721. The match is exact to four decimal places for both.

So Table 1 is internally consistent once the prose is corrected: the load-bearing object is the star-basis Gram, not the full triangle Gram. This is also the natural object — it corresponds to what Bridge 3 does for K_4 (a 3×3 basis Gram), and the star basis gives a direct cycle-rank-dimensional picture rather than the redundant L-dimensional picture.

Given this clarification, the 6×6 oriented star-basis Gram for K_5 is:

```
+3  +1  +1  -1  -1   0
+1  +3  +1  +1   0  -1
+1  +1  +3   0  +1  +1
-1  +1   0  +3  +1  -1
-1   0  +1  +1  +3  +1
 0  -1  +1  -1  +1  +3
```

(Basis order: (0,1,2), (0,1,3), (0,1,4), (0,2,3), (0,2,4), (0,3,4), with standard orientation a→b→c→a for a<b<c.) Its spectrum is {5, 5, 5, 1, 1, 1}.

The "anisotropy 2.19" has a cleaner rep-theoretic reading, developed below: the 5:1 eigenvalue ratio reflects the split of the 6D cycle space into two 3-dimensional S_4 irreps under restriction from S_5 to the stabiliser of vertex 0. The 3+3 split, not the 2.19 residual, is the structurally load-bearing feature.

3. The grade-by-grade backbone

Let $V = \mathbb{C}^N$ be the permutation representation of S_N acting by permuting the N basis vectors. Then V decomposes as trivial ⊕ std, where std is the (N−1)-dimensional standard representation (the orthogonal complement of the all-ones vector). For each $k \geq 0$:

$$\Lambda^k(V) \;=\; \Lambda^k(trivial \oplus std) \;=\; \Lambda^k(std) \;\oplus\; \Lambda^{k-1}(std)$$

Each term on the right is an irreducible S_N representation, labelled by a Young diagram with exactly k+1 (respectively k) rows:

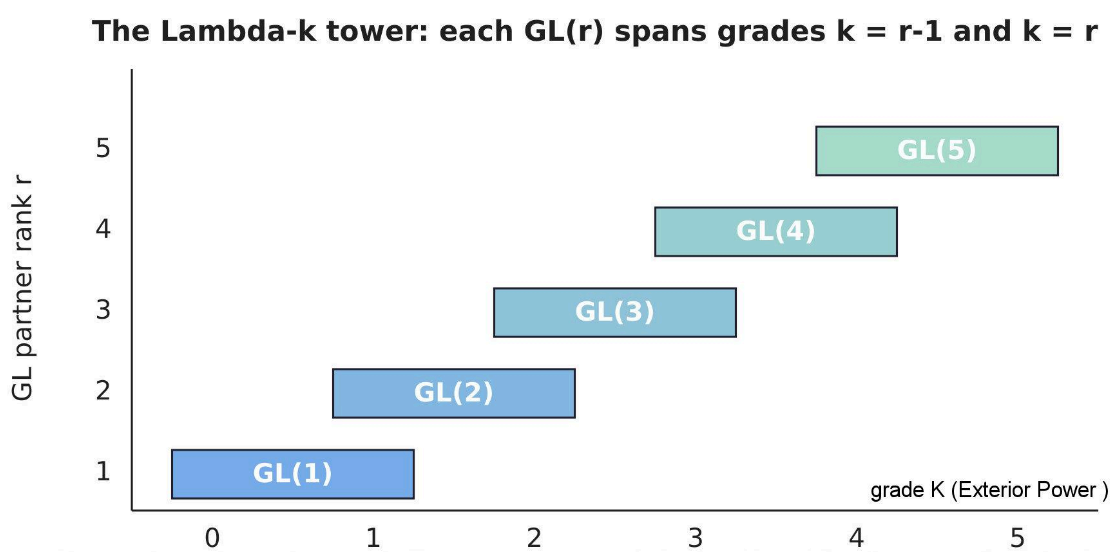
$$\Lambda^{k}(std) \;=\; S^{(N-k,\,1^{k})}, \qquad \Lambda^{k-1}(std) \;=\; S^{(N-k+1,\,1^{k-1})}$$

Under Schur-Weyl duality on $(\mathbb{C}^{r})^{\otimes N}$, a Specht module S^λ of S_N pairs with the Weyl module W_λ of GL(r) for r equal to the number of rows of λ, with dim W_λ equalling the number of semistandard Young tableaux of shape λ with entries in {1,…,r}. Applied here:

grade k	space	S_N content	GL partners
0	$\Lambda^0(V)$	trivial	GL(1)
1	$\Lambda^1(V) = V$	std ⊕ trivial	GL(2) ⊕ GL(1)
2	**$\Lambda^2(V)$** (signed edges)	S^(N−2,1,1) ⊕ S^(N−1,1)	**GL(3)** ⊕ **GL(2)**
3	$\Lambda^3(V)$ (signed triangles)	S^(N−3,1,1,1) ⊕ S^(N−2,1,1)	GL(4) ⊕ GL(3)
4	$\Lambda^4(V)$	$S^{(N-4,1^4)}$ ⊕ S^(N−3,1,1,1)	GL(5) ⊕ GL(4)

The ranks of the GL partners grow by one with each grade, with the previous grade's lower partner reappearing as the new grade's upper partner. At grade 2 the GL(3) partner appears for the first time. At grade 3 the GL(4) partner appears for the first time. And so on.

The structure is easier to see visually. Each GL(r) appears at exactly two grades — its "entrance" at k = r−1 (as the upper partner) and its "exit" at k = r (as the lower partner):



Diagonal reading: at grade k, the upper partner is GL(k+1) and the lower partner is GL(k).

Two structural consequences follow immediately. First, the "3" that appears in the SU(3) content of the framework's K_5 constants derivations is not chosen — it is the number of rows in the Young diagram of Λ^2(std), and this row count is 3 for every $N \geq 3$. Second, the

GL(k) content at each grade is canonically determined; it is not a modelling choice but a theorem of representation theory.

Numerical verification at N = 3–7 confirms the decomposition at every grade, with no errors.

---

4. Grade 2: the signed-edge space

4.1 Rep-theoretic content

At grade 2, the decomposition gives

$$\Lambda^2(V) \;=\; S^{(N-1,1)} \;\oplus\; S^{(N-2,1,1)}$$

The first summand is the standard representation itself (dimension N−1). Under Schur-Weyl it pairs with the 2-row Weyl module of GL(2), which restricts to SU(2) as the irreducible representation of dimension N−1. In the conventional physics labelling, this is the SU(2) representation with label j = (N−2)/2 (where "j" is the rep-theoretic index, not an angular momentum quantum number); we refer to it below by its dimension to keep the separation between the abstract rep-theoretic label and the physical meaning "spin" carries in quantum dynamics.

The second summand (dimension (N−1)(N−2)/2) is the cycle space of K_N. Under Schur-Weyl it pairs with the 3-row Weyl module of GL(3), which restricts to SU(3) as the (N−3)-fold symmetric power of the fundamental:

$$cycle\ space\ of\ K_N \;\leftrightarrow\; Sym^{N-3}\left(C^3\right)$$

The combined SU(2) ⊕ SU(3) rep-theoretic content across N:

N	SU(2) piece	SU(3) piece
3	2-dim irrep	trivial (dim 1)
4	3-dim irrep	fundamental (dim 3)
5	4-dim irrep	symmetric diquark (dim 6)
6	5-dim irrep	decuplet (dim 10)
7	6-dim irrep	Sym^4 (dim 15)

The SU(2) rep-theoretic index j = (N−2)/2 alternates between integer and half-integer with N parity: odd N gives half-integer index (the rep does not descend to SO(3); it is a projective representation of SO(3) or equivalently a genuine representation of SU(2) only), even N gives integer index (the rep descends to SO(3)). This is a clean Z/2 structural grading indexed by N within the SU(2) sector, with a direct rep-theoretic meaning — whether the rep is also a well-defined SO(3) rep or not.

4.2 The SO(3) content of the cycle space

Restricting the SU(3) symmetric-tensor content to SO(3) ⊂ SU(3) (the real subgroup of SU(3), which acts on $\mathbb{C}^3 = \mathbb{R}^3 \oplus i\mathbb{R}^3$), the standard branching rule gives

$$Sym^p(C^3)\downarrow_{SO(3)} \;=\; V_p \oplus V_{p-2} \oplus V_{p-4} \oplus ...$$

where V_ℓ is the (2ℓ+1)-dimensional irreducible representation of SO(3). (In physics applications V_ℓ is called the “angular momentum ℓ” or “spherical harmonic ℓ” representation, because it arises as the space of degree-ℓ harmonic polynomials; here the label ℓ is rep-theoretic — an index on irreducible SO(3) representations — with no physical meaning imported.) Applied to the cycle space of K_N:

N	cycle space under SO(3)	dim of first new SO(3) irrep
3	V_0	1 (scalar)
4	V_1	3 (vector)
5	V_0 ⊕ V_2	5 (quadrupole)
6	V_1 ⊕ V_3	7 (octupole)
7	V_0 ⊕ V_2 ⊕ V_4	9 (hexadecapole)

Two observations. First, the K_4 cycle space is exactly V_1 — the 3-dimensional SO(3) irrep. This is the same SO(3) content that Bridge 1 derives from the round-S^2 geometry of the K_4 cycle space through a different construction; the Schur-Weyl perspective shows it is structurally forced, appearing as the unique three-row-Young-diagram rep-theoretic content at grade 2 restricted under SO(3). Bridge 1’s construction and the Schur-Weyl perspective give the same answer by different routes.

Second, the parity label of the SO(3) content alternates cleanly: the label (−1)^(N−3) on each V_ℓ appearing. Odd-N cycle spaces carry only even-ℓ content. Even-N cycle spaces carry only odd-ℓ content. Whether this rep-theoretic alternation has physical content or is purely a mathematical artefact is an open question.

4.3 Honest scope

The grade-2 content at K_4 has SU(2) rep-theoretic dimension 3 and SU(3) rep-theoretic content equal to the fundamental. These rep labels match the labels of Standard Model objects — the SU(2) adjoint (W-triplet) and the SU(3) fundamental (quark triplet) — but the match is of rep-theoretic labels on mathematical objects, not of the objects themselves. The Standard Model’s SU(2)_L carries 2-dimensional (doublet) irreps on matter, and the full SM gauge group SU(3) × SU(2) × U(1) has a U(1) factor that does not appear in the grade-2 edge space (only in grades 0 and N, as the scalar and pseudoscalar). What the mathematics says, cleanly, is that the same sequence of rep-theoretic labels appears in both contexts; what would be needed for a stronger claim is a mapping from graph-grade content to physical fields, which is not established here.

5. Grade 3: the signed-triangle space

5.1 Rep-theoretic content

At grade 3:

$$\Lambda^3(V) \quad = \quad S^{(N-3,1,1,1)} \quad \oplus \quad S^{(N-2,1,1)}$$

The second summand is the cycle space again. The first is new: under Schur-Weyl it pairs with a 4-row Weyl module of GL(4), restricting to SU(4) as the (N−4)-fold symmetric power of the fundamental.

The combined rep content across N:

N	cycle piece (SU(3))	new piece (SU(4))	dim
3	dim 1	—	1
4	fundamental (dim 3)	trivial (dim 1)	4
5	symmetric diquark (dim 6)	**fundamental (dim 4)**	10
6	decuplet (dim 10)	sextet (dim 10)	20
7	Sym^4 (dim 15)	decuplet (dim 20)	35

5.2 Chain-complex interpretation

The triangle space of K_N fits into the simplicial chain complex

$$\ldots \to C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

in which C_k is the signed-k-subset space. The grade-3 decomposition has a clean interpretation in these terms:

- **Cycle piece S^(N−2,1,1)** = orthogonal complement of ker ∂_2 inside C_2 = the part of the triangle space that maps nontrivially into edge space. This is the content that carries graph cycles.
- **Kernel piece S^(N−3,1,1,1)** = ker ∂_2 = image of ∂_3 (since the complete complex is contractible). This is the space of triangle-combinations whose edge boundary cancels — spanned by tetrahedron-boundary relations

$$\partial_3(abcd) \quad = \quad t_{bcd} - t_{acd} + t_{abd} - t_{abc}$$

for each 4-subset of vertices.

At N = 5 there are C(5,4) = 5 tetrahedra. Their boundaries satisfy one linear dependency (the pentatope relation), leaving 4 independent — which is exactly the dimension of the kernel piece.

5.3 Cl(5) grade-3 specifically

At N = 5, the triangle space of K_N has dimension C(5,3) = 10, which equals the dimension of grade-3 of Cl(5). This is the unique N at which these two spaces coincide: five features exactly fill the five basis directions of Cl(5), and the ten trivectors of Cl(5) correspond to the ten triangles of K_5.

Under the S_5 relabelling symmetry of the five constraint types (β, κ, ρ, λ, τ), Cl(5) grade-3 therefore carries the decomposition

$$Cl(5)\ grade - 3 \ = \left[6D\ cycle\ piece:\ Sym^2\ of\ SU(3)\right] \oplus [4D\ kernel\ piece: fundamental\ of\ SU(4)]$$

The 6D cycle piece has a standard framework reading: it is the τ_circ-carrying content, associated with temporal circulation. The 4D kernel piece has not been identified with a specific framework structure in the existing documents. Its structural content is algebraic: the four independent tetrahedron-boundary relations among the ten trivectors of Cl(5), modulo the single pentatope dependency.

One reading of the 4D piece that is consistent with its chain-complex role: it is "closed-at-the-next-grade-up" content, trivector combinations whose bivector boundary cancels. This is a topologically closed 2-surface-analog content, distinct from the graph-cycle content.

---

6. Connection to the confinement conjecture (partial)

The initial research conjecture that motivated this work proposes a structural confinement theorem: "in Cl(5) at $N \geq 3$, any configuration with a grade-3 component whose cycle is not closed has $\Phi \to -\infty$." The proposed proof skeleton rests on K_3 cycle-rank = 1 — a trivector requires three bivectors forming a closed loop, and there is no configuration of "isolated trivector plus dangling grade-2 bridge."

The grade-3 analysis developed above does not prove this theorem, but gives a structurally adjacent statement worth stating carefully:

Adjacent structural fact. The kernel of the triangle-to-edge operator T is exactly the Λ^3(std) piece (the SU(4) summand). Elements of this kernel are annihilated by T by construction — their edge-boundary is zero. If Φ on triangle space is constructed from a quadratic form involving T (and the framework's edge-incidence picture looks compatible with this), then kernel elements contribute zero to the T-part of Φ.

What this is not. The initial conjecture concerns configurations in a *subgraph* where one of the three bivectors of a trivector is absent. The kernel analysis above treats the full K_N triangle space, where all edges are present; it cannot directly say anything about subgraph states where edges are missing. A proper proof of the conjecture would require (a) a framework-native construction of Φ on triangle-subspace states that can include subgraphs, and (b) an argument that subgraph-states with incomplete triangles sit at $\Phi = -\infty$ in that construction.

What the terminology should be. An earlier version of this work referred to the kernel piece as "non-closed trivectors," matching the language of the initial conjecture. This was a misreading. In the chain-complex sense, the kernel is the subspace of triangle-combinations that are *topologically more closed* than cycle-space elements, not less — their edge boundary cancels, making them boundaries of 3-simplices. The conjecture's "non-closed" refers to a graph-state condition about which bivectors are present in a subgraph, not about the chain-complex kernel.

So the grade-3 analysis gives the rep-theoretic skeleton but not the confinement theorem itself. The theorem as stated remains open and would require an explicit construction of Φ on subgraph states.

---

7. The complete Cl(5) tower under S_5

Grades 4 and 5 complete the exterior-algebra tower $\Lambda^0(V)$ through $\Lambda^5(V)$ at N=5. Because each Λ^k(V) has dimension C(5,k) and these sum to $2^5 = 32$, the six grades exhaust the Clifford algebra Cl(5) viewed as a vector space. The grade-4 and grade-5 pieces can be computed directly from the backbone theorem:

$$\Lambda^4(V) = S^{(1,1,1,1,1)} \oplus S^{(2,1,1,1)} = sign \oplus S^{(2,1,1,1)} \qquad (dim\ 1 + 4 = 5)$$

$$\Lambda^5(V) = S^{(1,1,1,1,1)} = sign \qquad (dim\ 1)$$

The 4-dimensional piece S^(2,1,1,1) at grade 4 is the *same* S_5 irreducible representation that appeared as the kernel of the triangle-to-edge incidence operator at grade 3. The 1-dimensional sign representation at grade 5 is the Cl(5) pseudoscalar.

7.1 The palindromic tower

Collecting grades 0 through 5, the full S_5 decomposition of Cl(5) is:

grade k	dim	S_5 decomposition	Schur-Weyl partners
0	1	trivial	GL(1)
1	5	trivial ⊕ std	GL(1) ⊕ GL(2)
2	10	std ⊕ S^(3,1,1)	GL(2) ⊕ GL(3)
3	10	S^(3,1,1) ⊕ S^(2,1,1,1)	GL(3) ⊕ GL(4)
4	5	S^(2,1,1,1) ⊕ sign	GL(4) ⊕ GL(5)
5	1	sign	GL(5)

The structure is clean: five S_5 irreducible representations span the entire 32-dimensional algebra, each occupying two consecutive grades — an "entrance" grade where it appears as the lower partner and an "exit" grade where it appears as the upper partner. Every S_5 irrep has this pattern, with no irrep appearing at non-consecutive grades and no grade containing the same irrep twice.

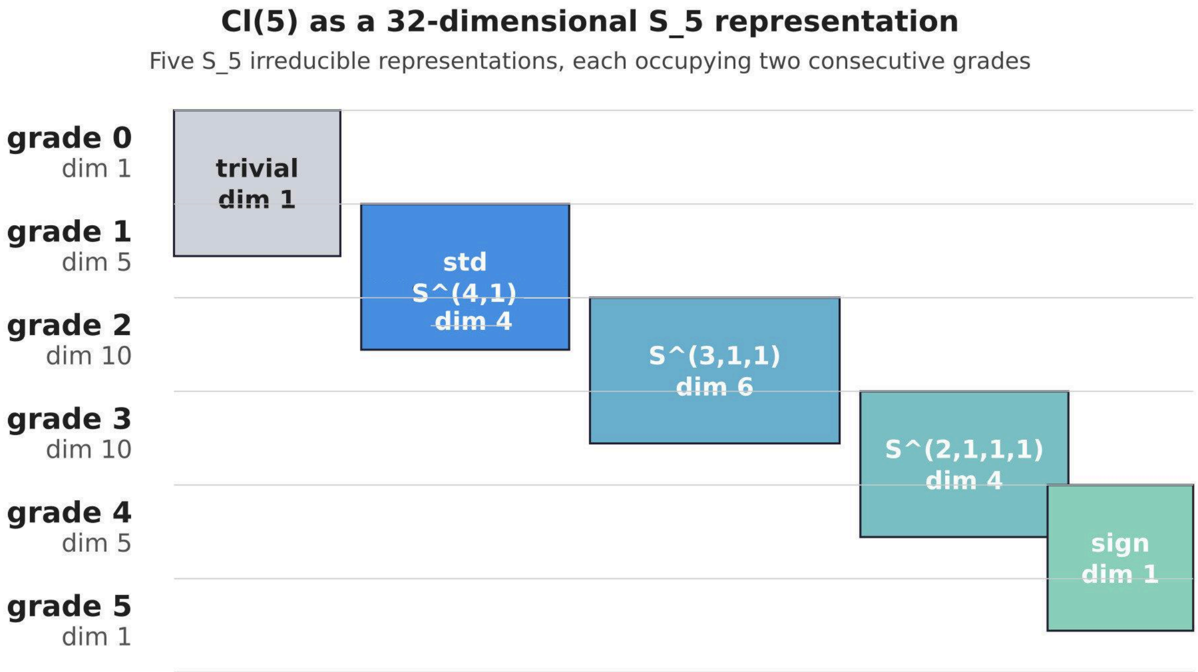


Figure: Cl(5) as a 32-dimensional S_5 representation. Five S_5 irreducible representations, each occupying two consecutive grades. The staircase encodes the backbone theorem Λ^k = S^(N−k, 1^k) □ S^(N−k+1, 1^(k−1)).

The dimensions of the irreps (1, 4, 6, 4, 1) themselves form a palindrome — this is the row of Pascal's triangle at N=5 (since S_5 has irreducibles of these dimensions with this structure), and the palindrome reflects the Hodge duality examined next.

7.2 Hodge duality as a rep-theoretic statement

At N=5, the Hodge star ★: Λ^k(V) → Λ^(5−k)(V) is a linear isomorphism that maps each grade to its complement. The rep-theoretic question is: how does ★ interact with the S_5 action?

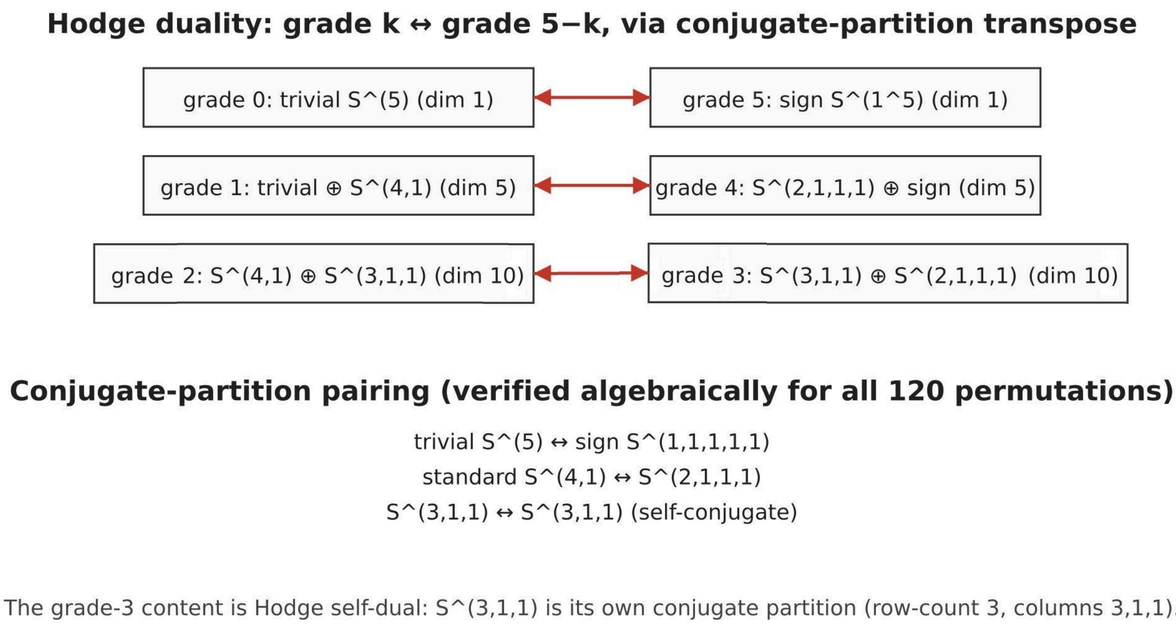
The answer is that ★ is not S_5-equivariant in the ordinary sense, but satisfies the twisted transformation law

$$\sigma \cdot\star\,(v) \;=\; \varepsilon(\sigma)\cdot\star\,(\sigma\cdot v)$$

for every σ ∈ S_5, where ε(σ) is the sign of the permutation. This is the rep-theoretic content of the statement "Hodge at odd dimension picks up an orientation sign." Equivalently, ★ is an isomorphism of S_5 representations

$$\Lambda^{5-k}(V) \cong \Lambda^{k}(V) \otimes\, sign.$$

Using the classical rep-theoretic identity that tensoring a Specht module S^λ with the sign representation gives the Specht module of the *conjugate partition*, S^(λ'), this predicts the grade-k and grade-(5−k) decompositions to be related by conjugate-partition transpose. The prediction can be read directly from the tower:



The three pairings visible in the figure:

- **grade 0 ↔ grade 5.** Trivial rep (partition (5)) and sign rep (partition (1,1,1,1,1)) are conjugate partitions.
- **grade 1 ↔ grade 4.** The pair trivial ⊕ std at grade 1 becomes sign ⊕ S^(2,1,1,1) at grade 4. Partitions (5) and (4,1) conjugate to (1^5) and (2,1,1,1).
- **grade 2 ↔ grade 3.** The pair std ⊕ S^(3,1,1) at grade 2 becomes S^(3,1,1) ⊕ S^(2,1,1,1) at grade 3. Partition (4,1) conjugates to (2,1,1,1), and S^(3,1,1) is self-conjugate (its Young diagram is its own transpose).

The transformation law σ·★(v) = ε(σ)·★(σ ·v) was verified numerically for every element of S_5 (all 120 permutations), with zero error to machine precision. The character comparison also matches on all seven conjugacy classes. This is a non-trivial self-consistency check: if the Schur-Weyl identification had been wrong at any grade, the Hodge transformation law would have failed for some permutation.

7.3 Observations

Three things are worth flagging from the complete tower that were not visible at the individual-grade level:

The S^(3,1,1) self-duality. The Young diagram of S^(3,1,1) is equal to its own transpose. This makes S^(3,1,1) the *unique* S_5 irreducible representation that is isomorphic to its sign-twist — the only S_5 irrep that appears in both positions of a Hodge pair (at grade 2 and at grade 3). This is a real rep-theoretic feature of the N=5 tower, but as Section 8 will show, the general pattern “odd N always has a self-conjugate middle irrep” is not unique to N=5 — it’s a consequence of Pascal-row-parity for every odd N. What is N=5-specific is the *dimension* 6 of this self-conjugate irrep, not its existence.

The sign rep as orientation. The 1-dimensional sign rep at grade 5 is the Cl(5) pseudoscalar. Under S_5 relabelling of the five constraint types, it picks up a factor of $\varepsilon(\sigma)$ — it registers whether the relabelling is even or odd. This gives a rep-theoretic reading of "orientation" on the Cl(5) structure: orientation is the feature that distinguishes even from odd permutations of constraint labels. Whether this rep-theoretic orientation has framework-native content (for example, as a marker of chirality or time-direction in framework applications) is an open question.

SU(4) content threads through grades 3 and 4. S^(2,1,1,1) appears as the 4D kernel at grade 3 and as the 4D non-sign piece at grade 4. Both are the same S_5 irreducible representation, so there is a canonical isomorphism between them. Section 7.4 identifies this isomorphism explicitly: it is the simplicial chain-complex boundary operator ∂_4, and it matches the *Regimes of N* paper's "coherence tension" reading of grade-4 as a unique forced correspondence.

7.4 The SU(4) thread: ∂_4 as the canonical grade-4 ↔ grade-3 map

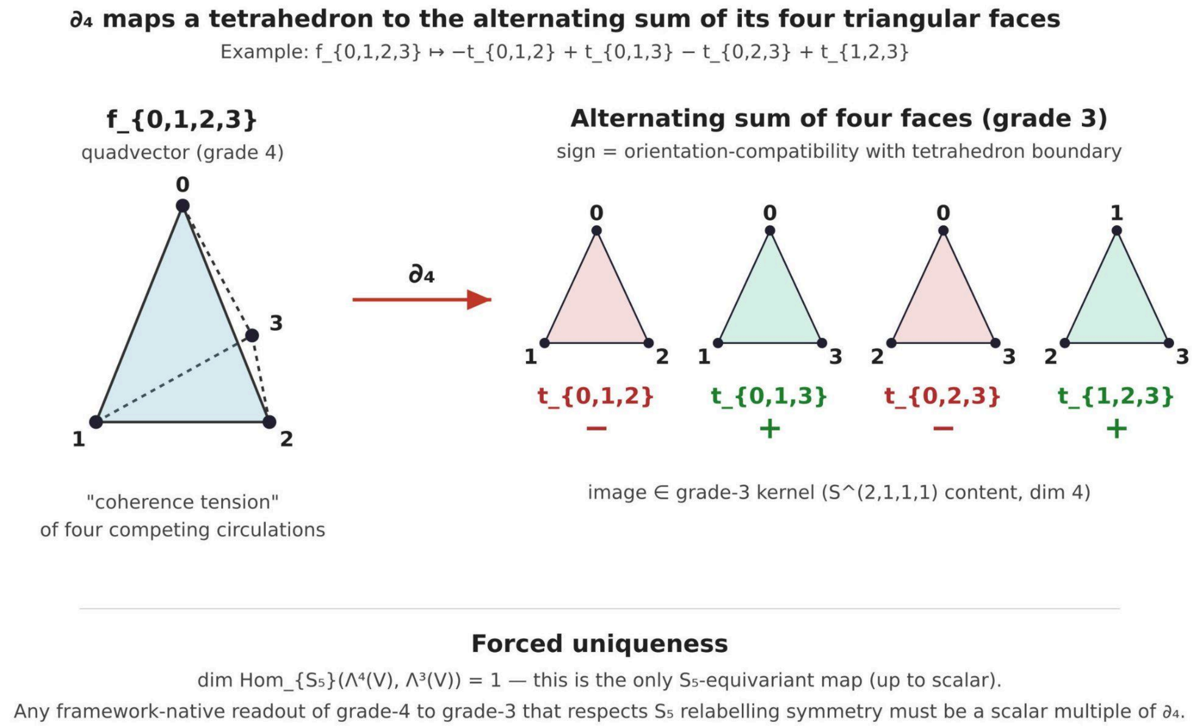
The S^(2,1,1,1) irrep appearing at both grade 3 (as the triangle-to-edge incidence kernel) and grade 4 (as the non-sign piece of the quadvector space) raises a structural question: what specific linear map realises the canonical isomorphism between these two realisations?

By Schur's lemma, the space of S_5-equivariant linear maps Hom_{S_5}(Λ^4(V), Λ^3(V)) is 1-dimensional — there is exactly one such map up to scalar. Direct computation (dim of Hom space = 1 via character inner product, confirmed by group averaging) identifies this map explicitly. It is the simplicial chain-complex boundary operator

$$\partial_4\left(f_{abcd}\right) \;=\; t_{bcd} - t_{acd} + t_{abd} - t_{abc}$$

which sends each grade-4 basis element f_{abcd} (the quadvector corresponding to the tetrahedron on vertices {a, b, c, d}) to the alternating sum of its four triangular faces in grade 3.

The boundary operator d4 maps a tetrahedron to the alternating sum of its four triangular faces.



The map ∂_4 has the expected structural properties: its 1D kernel is exactly the sign rep at grade 4 (the pseudoscalar direction), and its 4D image is exactly the S^(2,1,1,1) kernel piece at grade 3. So ∂_4 induces a genuine isomorphism of S_5 irreps

$$\partial_4 : S^{(2,1,1,1)}_{grade\ 4} \xrightarrow{\cong} S^{(2,1,1,1)}_{grade\ 3}$$

once restricted to the non-sign piece of its domain.

Match with the *Regimes of N* paper. The *Regimes of N* V3 paper describes grade-4 quadvectors in these terms: "A grade-4 quadvector captures *coherence tension* — the structural phenomenon that arises when four features form a K_4 tetrahedron and each edge participates in two triangles. Those triangles have independent grade-3 circulations, and the shared edges force these circulations to compete for finite relational capacity. The quadvector measures the degree to which competing circulations are mutually compatible or in conflict."

The map ∂_4 is exactly the formalisation of this prose. Each quadvector f_{abcd} is mapped to the alternating combination of the four face-triangles — a specific signed readout of the four competing circulations around the tetrahedron. The alternating signs encode orientation-compatibility: when the four circulations are perfectly compatible the combination cancels, when they conflict it does not. Because the Hom space has dimension 1, this reading is forced: any S_5-equivariant definition of "coherence tension as a grade-4 → grade-3 readout" must be a scalar multiple of ∂_4. There is no alternative construction at the rep-theoretic level.

The pentatope and the pseudoscalar's "global closure" reading. *Regimes of N* V3 also describes the grade-5 pseudoscalar I_5 as "global closure." Applied to the unique K_5 pentatope, the boundary operator ∂_5 lands in grade 4 as the vector (+1, −1, +1, −1, +1) in the basis ordering of 4-subsets of {0,1,2,3,4}. This is exactly the sign-rep direction at grade 4. So:

- The 1D sign-rep piece of grade 4 is the image of the pentatope under ∂_5.
- This image can be read as the "total coherence tension" across all 5 tetrahedra of K_5.
- The grade-5 pseudoscalar, viewed through ∂_5, does live in the "global closure" slot that *Regimes of N* describes qualitatively.

Both identifications are structural rather than accidental: the same rep-theoretic uniqueness argument (dim Hom = 1 for each pairing) forces them.

Net effect on the framework link. The identification is BfN-native in a precise sense: it matches existing framework prose exactly, is forced by Schur's lemma given S_5 symmetry, and works consistently across the pentatope / pseudoscalar / coherence-tension triangle. What the identification does *not* claim is that the framework produces ∂_4 — the simplicial chain boundary is a completely generic construction available for any complete graph K_N. What BfN supplies is the interpretation: what ∂_4 measures in a relational context, rather than why ∂_4 exists mathematically.

This is arguably the most satisfying kind of structural result for the framework: a previously qualitative BfN claim (coherence tension as readout of competing circulations) has a unique rep-theoretic formalisation that matches the framework's existing Hodge-pairing structure and its pseudoscalar-as-global-closure reading. The framework's interpretive vocabulary is now grounded in specific S_5-equivariant linear maps, and those maps are forced, not chosen.

---

8. The Cl(N) generalisation and what is specific to N=5

The structural features of Cl(5) in Section 7 invite the question: is N=5 privileged in the rep-theoretic sense, or is the palindromic tower structure a generic feature of Cl(N) at every N? This section reports the answer, which is sharper and more honest than the N=5-specific framing above would suggest.

8.1 The Cl(N) hook-irrep theorem

Explicit computation of the full tower $\Lambda^0(V)$ through $\Lambda^N(V)$ for N = 4, 5, 6, 7, 8 confirms the following theorem, which follows directly from the backbone theorem of Section 3:

Theorem. *For every N ≥ 3, the Clifford algebra Cl(N) viewed as a representation of S_N decomposes into exactly N distinct irreducible components, one for each hook-shaped partition (N−k, 1^k) with k = 0, 1, …, N−1. Each hook-irrep S^(N−k, 1^k) spans exactly two consecutive grades of the tower: its "entrance" at grade k (as the lower partner) and "exit" at grade k+1 (as the upper partner, except at the endpoints where it is the sole occupant).*

The dimension of the k-th hook-irrep is given by the hook-length formula as C(N−1, k). Consequently, the dimensions of the N hook-irreps in Cl(N) are exactly row (N−1) of Pascal's triangle. This row is palindromic by the binomial identity C(N−1, k) = C(N−1, N−1−k), which is precisely the rep-theoretic content of Hodge duality in Cl(N).

8.2 Consequences

The palindromic tower is generic. At every N tested (and by the theorem above, at every N ≥ 3), Cl(N) decomposes into exactly N hook-irreps whose dimensions form a palindromic Pascal row. Hodge duality verified as the sign-twist / conjugate-partition pairing works at every N — it is a consequence of the Pascal palindrome, not a special feature of N=5.

Odd N always carries a self-conjugate middle irrep. When N is odd, the Pascal row (N−1) has an odd number of entries with a unique central entry. The corresponding central hook-irrep is self-conjugate and appears at the two middle grades (N−1)/2 and (N+1)/2. When N is even, the Pascal row has an even number of entries with no central entry, and no hook in Cl(N) is self-conjugate. The "S^(3,1,1) self-dual middle content" at N=5 is therefore a feature of odd N generally — it appears at N=3 (S^(2,1) of dim 2), N=5 (S^(3,1,1) of dim 6), N=7 (S^(4,1,1,1) of dim 20), and so on.

Only hook-irreps appear. Of the p(N) total S_N irreps (5 for N=4, 7 for N=5, 11 for N=6, 15 for N=7, 22 for N=8), only N of them appear in Cl(N). The others — the "non-hook" irreps with Young diagrams having two or more columns of height ≥ 2 — are entirely absent from the Clifford algebra viewed through S_N. This is itself a structural fact about Λ•(V_perm).

8.3 What is specific to N=5

What is not generic — what N=5 specifically enjoys — is a dimensional coincidence:

N	Pascal row (N−1)	middle dim	Schur-Weyl partner(s)
3	(1, 2, 1)	2	SU(2) fundamental
5	(1, 4, 6, 4, 1)	6	**SU(3) symmetric diquark**; neighbour 4 = **SU(4) fundamental**
7	(1, 6, 15, 20, 15, 6, 1)	20	Schur-Weyl partner has rank and dim unrelated to familiar Lie algebras
9	(1, 8, 28, 56, 70, 56, 28, 8, 1)	70	high-rank partners, no standard physics-adjacent labels

At N=5, the dimensions of the hook-irreps in Cl(5) are (1, 4, 6, 4, 1). These happen to be the dimensions of irreducible representations of low-rank Lie algebras with labels that recur throughout physics (SU(2) trivial, SU(4) fundamental, SU(3) symmetric tensor). At larger odd N the palindromic structure remains, but the middle dimensions grow combinatorially (20, 70, 252, …) and no longer correspond to rep dimensions of low-rank familiar Lie algebras.

So the honest characterisation of N=5 is: the palindromic Hodge tower is a Pascal-triangle fact, not a framework-specific one, but at Pascal row 4 the hook dimensions are small enough that their Schur-Weyl partners connect by analogy to low-rank Lie-algebra content with physics-adjacent labels. This is the content of "N=5 is the scale at which the framework's relational structure has Schur-Weyl partners recognisable from physics." At other scales the Schur-Weyl partners still exist, but the analogy weakens.

8.4 What this means for the N-hierarchy

This reshapes rather than overturns the N=5 claim. The rep-theoretic skeleton developed in Sections 3–7 is a generic feature of Λ•(V_perm) as an S_N representation and holds at every N ≥ 3. What is specifically BfN-motivated is the *choice of N=5 as the working scale* — the claim that exactly five constraint types (β, κ, ρ, λ, τ) exhaust the relational content at full scope. Given that choice, Pascal row 4 is where the hook dimensions happen to land, which gives the SU(3) and SU(4) content that several framework results depend on.

The previous framing of this paper's contribution should be adjusted accordingly. The rep-theoretic consequences (GL(k+1) ⊕ GL(k) backbone, palindromic Hodge tower, SU(3) content of cycle spaces) are consequences of studying K_N under S_N, not of the BfN axiom directly. What is BfN-specific is the motivation for making that choice — the framework's claim that the five relational-constraint types and their S_N relabelling symmetry are the right mathematical setting for describing whatever the BfN axiom requires to obtain.

This is a more defensible position than "BfN produces the rep-theoretic structure": instead, BfN *picks out a setting* in which well-known rep-theoretic structure appears with dimensions that happen to land on physics-adjacent labels. Whether this is a deep alignment or a happy accident is not decided by the rep theory alone, but the alignment itself is a real feature of Pascal row 4 that any framework working at N=5 would inherit.

What the analysis in Section 7.4 adds to this picture is that framework interpretive claims can be tested against rep-theoretic uniqueness. When BfN reads grade-4 as "coherence tension" and grade-5 as "global closure," these are not decorative labels: they correspond to specific S_5-equivariant linear maps that are forced uniquely by Schur's lemma. This is the kind of cross-check that converts qualitative framework vocabulary into mathematically constrained content, and it is generalisable — any framework claim about a map between graded Cl(N) components can be checked for uniqueness against the Hom-space dimension, and either confirmed as the canonical intertwiner or identified as something else that needs separate justification.

---

9. What is established, and what is open

9.1 Established (mathematical theorems, numerically verified)

- For each grade k, $\Lambda^k(V)$ as an S_N representation decomposes as S^(N−k, 1^k) ⊕ S^(N−k+1, 1^(k−1)).
- Under Schur-Weyl duality, these summands pair with Weyl modules of GL(k+1) and GL(k), with dimensions forced by the Young-diagram row counts.

- At grade 2: cycle space of K_N is the (N−3)-fold symmetric power of the SU(3) fundamental; edge-complement carries the (N−1)-dimensional irreducible SU(2) representation.
- At grade 3: cycle piece carries SU(3) symmetric-power content; kernel piece carries SU(4) symmetric-power content.
- Under SO(3) ⊂ SU(3), the cycle space of K_N decomposes into SO(3) irreducibles V_{N−3} ⊕ V_{N−5} ⊕ … with alternation parity (−1)^(N−3) on the ℓ labels.
- At K_4 specifically, the cycle space is exactly the 3-dimensional SO(3) irrep V_1; this matches Bridge 1's SO(3) construction through a different route.
- At K_5 specifically, Cl(5) grade-3 decomposes under S_5 into SU(3)-sextet ⊕ SU(4)-fundamental.
- At N=5, the complete Cl(5) algebra decomposes under S_5 as a palindromic tower: five S_5 irreducible representations (trivial, std, S^(3,1,1), S^(2,1,1,1), sign) each spanning two consecutive grades. The Hodge transformation law σ·★(v) = ε(σ)·★(σ ·v) is verified algebraically for all 120 permutations of S_5.
- **Cl(N) hook theorem.** For every N ≥ 3, Cl(N) viewed as an S_N representation decomposes into exactly N hook-shaped irreps S^(N−k, 1^k) with dimensions given by row (N−1) of Pascal's triangle. Each hook spans two consecutive grades. The palindromic Hodge structure is a consequence of binomial symmetry, not a feature unique to N=5. Only N of the p(N) total S_N irreps appear in Cl(N); the non-hook irreps are absent.
- Odd N always carries a self-conjugate hook-irrep at its two middle grades; even N never does. This follows from Pascal-row parity.
- **Canonical intertwiner identification.** The unique (up to scalar) S_5-equivariant map between the grade-3 and grade-4 realisations of S^(2,1,1,1) in Cl(5) is the simplicial chain-complex boundary operator ∂_4. Confirmed by Schur's lemma counting (dim Hom = 1), explicit group averaging, and direct equivariance check on all 120 permutations of S_5. This map matches the *Regimes of N* V3 reading of grade-4 as "coherence tension readout of competing circulations around a tetrahedron." Further, the image of the K_5 pentatope under ∂_5 lands exactly in the sign-rep direction at grade 4, confirming the framework's reading of the pseudoscalar as "global closure."

9.2 Structural observations (real but not theorems)

- The sequence of rep-theoretic labels produced at each N looks like a sequence of SU(2) and SU(3) irreducible representations whose labels match some but not all Standard Model objects. The match is not systematic: the SU(2) content at K_4 has rep-theoretic dimension 3 (gauge-boson-like label), not 2 (matter-doublet label); the SU(3) content is only the totally-symmetric tower (fundamental, decuplet, …), not including the adjoint or antifundamental.
- Parity alternation in both the SU(2) and SO(3) sectors flips coherently with N parity. Whether this is a physical feature or a coincidence of Young-diagram labels is not decided by the rep theory alone.

- The "anisotropy" of the K_5 star-basis Gram (eigenvalues 5 and 1, ratio 5:1) is a basis-dependent quantity. The invariant operator TT^T on the cycle space is proportional to the identity with eigenvalue N. The invariant content of the K_5 "anisotropy" is not a scalar residual but a specific decomposition into two 3-dimensional S_4 irreps.
- S^(3,1,1) is the unique self-conjugate S_5 irrep appearing in Cl(5) (and more generally, every odd N has a self-conjugate hook at Pascal-row centre). The BfN cycle content sits at this Hodge-self-dual centre.
- What is specifically N=5 is a *dimensional coincidence*: Pascal row 4 has entries (1, 4, 6, 4, 1) whose middle values happen to be the dimensions of low-rank Lie-algebra reps (SU(4) fundamental, SU(3) symmetric tensor). This coincidence fades at larger odd N, where the middle Pascal entries grow combinatorially and no longer correspond to familiar low-rank Lie algebra reps.
- The framework's qualitative readings of grade-4 ("coherence tension") and grade-5 ("global closure") both correspond to specific, forced S_5-equivariant maps (∂_4 and ∂_5 respectively). This converts qualitative interpretive vocabulary into mathematically unique content that can be cross-checked against the rep theory.

9.3 Open questions this sharpens

- **Can the remaining SU(3) rep content be recovered from tensor products?** The sequence above gives only the symmetric-tensor tower. The adjoint 8, antifundamental $\bar{3}$, and mixed reps are missing. Tensor products of cycle spaces at different N (or between cycle and kernel pieces) might reach them; this is a concrete computation.
- **Is there a physical meaning to the N parity alternation?** Both the SU(2) rep-theoretic index and the SO(3) irrep label parity alternate in N. If the alternation corresponds to any empirical feature, it should produce falsifiable predictions. Currently it does not.
- **Do other BfN interpretive claims have unique rep-theoretic formalisations?** Section 7.4 showed that the framework's "coherence tension" reading of grade-4 and "global closure" reading of grade-5 both correspond to specific, Schur-forced S_5-equivariant maps. The same test can be applied to other framework constructions — for example, the promoter-structure readings at grades 2–3, or the proposed interpretations of specific bivector/trivector combinations. Each test either confirms the reading as the canonical intertwiner or reveals that the framework is using something non-equivariant (which itself would need accounting for).
- **What Φ construction gives the confinement theorem as stated?** Proving the confinement conjecture would require an explicit Φ on subgraph states. The rep-theoretic skeleton is compatible with such a construction but does not produce it. This is a well-defined framework task.
- **Why N=5 and not another working scale?** Given that the palindromic tower is generic to Cl(N) and only the dimensional coincidence is N=5-specific, the framework's justification for working at N=5 must rest on the axiomatic claim that exactly five constraint types (β, κ, ρ, λ, τ) exhaust the relational content. Whether this

claim can be supported independently of the Pascal-row-4 coincidence it produces is a framework-internal question that rep theory cannot resolve.

---

10. Position in the programme

This work sits alongside but does not replace existing programme documents. The *Regimes of N* position paper describes the N-hierarchy in terms of which mathematical vocabulary has well-defined reference at each scale. Bridge 1 derives 3D space at K_4 from a round-S^2 geometric construction. The constants derivations use the monogamy polytope at N=5 to compute α and $\sin^2\theta$_W. The Schur-Weyl perspective developed here gives a complementary rep-theoretic description of what structural content is carried at each grade and each N, and shows that several framework results (the "3" in SU(3) at K_5, the 3D vector at K_4) are rep-theoretic consequences that would arise in any framework using the K_N cycle-space structure, not features unique to the BfN axiomatic setup.

The honest standing of the perspective, given the Cl(N) generalisation in Section 8, is this: the rep-theoretic skeleton — backbone theorem, palindromic Hodge tower, hook-irrep content, GL(k+1) ⊕ GL(k) pairing — is generic to Λ•(V_perm) as an S_N representation and holds at every $N \geq 3$. It is not produced by the BfN axiom. What BfN supplies is the *motivation for studying K_N under S_N* in the first place, together with the framework-specific choice of N=5 as the working scale. Given that choice, Pascal row 4 is where the framework's structural content happens to land, and the hook-dimensions there coincide with low-rank Lie-algebra reps that recur in physics. This is more defensible than "BfN produces the structure": BfN picks out a setting in which independently-existing rep-theoretic structure appears with dimensions convenient for physics-adjacent analogy.

The SU(4)-thread finding in Section 7.4 adds a second, softer form of link to the axiom. It shows that two framework interpretive claims — "coherence tension" at grade 4 and "global closure" at grade 5 — correspond to unique, Schur-forced S_5-equivariant maps (∂_4 and ∂_5). BfN does not produce these maps (they are generic simplicial chain boundaries), but BfN's qualitative readings of them are confirmed as the *only* readings consistent with S_5 symmetry. This converts previously qualitative interpretive prose into mathematically constrained content. It is a pattern worth applying more broadly: each BfN reading of a graded Cl(N) component can in principle be tested against Hom-space uniqueness, and either vindicated as the canonical map or revealed to require framework-specific (non-equivariant) structure that would need its own justification.

Two natural next directions open up. The first is extending the uniqueness test to other BfN interpretive claims — in particular, the promoter-structure readings at grades 2–3, and any proposed interpretations of specific bivector/trivector couplings. The second is the longer-range framework-native construction of Φ on triangle-subspace and subgraph states, required to address the string-breaking and confinement calculations that the rep-theoretic skeleton developed here cannot itself carry out. The first is self-contained and can be done with the tools already in hand; the second requires genuine framework input that goes beyond rep theory.

A natural companion investigation, now that grade-4 and grade-5 at N=5 have been characterised concretely, is to extend the Schur-forced-map analysis to tensor products of Cl(5) components. Whether the framework's couplings between specific grade-k and grade-k' subspaces produce unique maps or multiplicity (Schur says the Hom space may be >1-dimensional for tensor products, depending on which irreps decompose into which), and whether the resulting maps match BfN descriptions of coupling dynamics, is a structurally well-posed question.

Correspondence: David Neale, Goleudy.ai, Rochester NY.