Correlation Boundaries at Finite N: Chirality Thresholds and the Emergence of Thermodynamic Structure

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Companion to "Correlation Boundaries and the First Law" [1]

Abstract

A companion paper [1] introduces the concept of correlation boundaries — surfaces across which pairwise correlations vanish — and proves that the thermodynamic deficit on any surface is $\Delta \propto P \cdot A$, where P is the cross-boundary correlation and A is the correlation anisotropy. That result operates in the large-N regime: many degrees of freedom, smooth surfaces, continuous integrals. This paper extends the analysis to finite N, where the continuum description breaks down and new structure becomes visible.

We work within a framework of pairwise correlations subject to a monogamy constraint, augmented by the graded structure of the exterior algebra $\Lambda^*\mathbb{R}^5$. The grade hierarchy provides a natural classification of relational structures: grade-2 bivectors encode pairwise correlations, while grade-3 trivectors encode an irreducible three-body structure — the oriented volume of a triangular loop in the five-dimensional correlation space — that we call *chirality*. This chirality (denoted τ_circ) is identically zero at $N \leqslant 2$ (no triangular loops exist) and generically nonzero at $N \geqslant 3$.

The central results are:

(i) At $N \leqslant 2$, any correlation boundary annihilates the system. At N = 3–5, every correlation boundary destroys chirality on at least one side of the partition.

(ii) **N = 6 is the minimum at which a correlation boundary preserves chirality on both sides** — both subsystems retain triangular loop structure. This is the first N at which thermodynamic quantities (temperature, entropy) are simultaneously well-defined on both sides of a boundary, making the first law conceptually applicable.

(iii) The thermodynamic deficit acquires a leading finite-N correction of order 1/N from the destruction of triangular chirality loops — a grade-3 contribution invisible to the pairwise correlation count P. This correction has no analog in the entanglement entropy description.

(iv) The interpolation from discrete (N ~ few) to continuum ($N \to \infty$) boundaries passes through three qualitative transitions: chirality onset (N = 3), bilateral thermodynamics (N = 6), and statistical emergence (N ~ 30).

This paper explores the mathematical structures that emerge when the correlation boundary formalism of [1] is extended to finite N using the grade hierarchy of the exterior algebra. Physical parallels — to gravitational thermodynamics, quantum gravity corrections, and decoherence — are noted as structural correspondences between mathematical forms. The mathematical results stand independently of any physical interpretation.

Keywords: correlation boundaries, finite-N corrections, chirality, trivector, exterior algebra, thermodynamic deficit, monogamy constraint

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I. INTRODUCTION

A. Context

The companion paper [1] establishes a framework for understanding which surfaces are thermodynamic. The answer: surfaces where the cross-boundary correlation P vanishes (correlation boundaries) or where the correlation anisotropy A vanishes (symmetry degeneracy). The thermodynamic deficit $\Delta \propto P \cdot A$ was derived under three assumptions: large N (many degrees of freedom), smooth surfaces (continuous integrals), and small perturbations (first-order expansion). These are the assumptions of semiclassical gravity, where Jacobson's [2] and Verlinde's [3, 4] programs operate.

But the formalism's starting point — pairwise correlations with a monogamy constraint — is not restricted to large N. It applies at any $N \geq 2$. This paper asks: what do correlation boundaries look like at small N?

B. What the large-N formalism misses

The analysis reveals three features visible only at finite N:

Boundaries are existential. At small N, creating a correlation boundary threatens the existence of features on one or both sides. A feature with zero correlations to anything else is, within the monogamy framework, indistinguishable from the absence of a feature. The cost of a boundary is not merely entropic but ontological.

Boundaries destroy chirality. The companion paper's formalism captures pairwise (grade-2) correlations threading through a surface. But at $N \geq 3$, triangular loops carry an irreducible grade-3 structure — the oriented volume of three edge vectors in the correlation space — that the pairwise count P does not register. Severing cross-boundary correlations destroys these loops, collapsing chirality locally. This is a cost beyond P.

There is a sharp threshold at N = 6. The first law $\delta Q = TdS$ requires thermodynamic quantities on *both* sides of a surface. These quantities require chirality (grade-3 structure) to be well-defined. Below N = 6, no partition preserves chirality on both sides. The first law is not merely inaccurate at small N — its terms are undefined.

C. Scope and epistemological position

This paper explores the mathematical structures that emerge when the correlation boundary formalism is extended to finite N. The extension requires tools from the exterior algebra $\Lambda^*\mathbb{R}^5$ — specifically, the distinction between grade-2 (bivector) and grade-3 (trivector) structures. The five-dimensional space is motivated by a constraint framework [5] in which five independent dimensions are derived from a distinguishability axiom; however, the results of this paper depend on the dimensionality d only through the requirement that $d \geq 3$ (so that trivectors exist and are generically nonzero). A reader who prefers to treat the five-dimensional correlation space as an assumption will find the mathematical content unchanged.

Physical parallels — to the minimum area for gravitational thermodynamics, to quantum gravity corrections in the Bekenstein-Hawking entropy, to decoherence thresholds — are noted as correspondences between mathematical forms arising in different contexts. They are not claims that quantum gravity is being derived from the monogamy constraint.

D. Plan

Section II establishes the framework: the monogamy constraint, the grade hierarchy of the exterior algebra, and the definition of chirality. Section III analyzes correlation boundaries at each N from 1 through 5. Section IV proves the N = 6 bilateral chirality threshold. Section V derives the finite-N correction to the thermodynamic deficit. Section VI describes the three qualitative transitions from discrete to continuum behavior. Section VII discusses structural parallels and open questions.

II. FRAMEWORK

A. Pairwise correlations with monogamy

The starting point is identical to the companion paper [1]. Consider N features with pairwise correlation strengths $\lambda_{ij} \in [0, \Lambda]$, subject to the monogamy constraint:

$$\sum_{j \neq i} \lambda_{ij} \leq \Lambda \quad \forall i. \tag{1}$$

A *surface* Σ partitions the features into Inside (I) and Outside (O). The *cross-boundary correlation* is $P(\Sigma) = \Sigma_{\{i \in I, j \in O\}} \lambda_{ij}$. A *correlation boundary* is a surface with P = 0.

The companion paper works in the continuum limit (large N, smooth surfaces). This paper works at finite N, where surfaces are discrete partitions and the features are countable.

B. The grade hierarchy of the exterior algebra

The companion paper treats each correlation λ_{ij} as a scalar — a single number measuring the strength of the (i, j) relationship. At finite N, this scalar description is insufficient. The *structure* of a correlation matters, not just its strength.

Each feature is characterised by a vector $\mathbf{a} \in \mathbb{R}^d$, where d is the dimensionality of the correlation space. The relation between features **a** and **b** is the bivector:

$$B_{ab} = \mathbf{a} \wedge \mathbf{b} \in \bigwedge^2 \mathbb{R}^d. \tag{2}$$

The scalar correlation strength $\lambda_{ij} = |B_{ab}|$ is the magnitude of this bivector. Different bivectors with the same magnitude represent different *kinds* of correlation — they point in different directions in the space of antisymmetric tensors.

For three features **a**, **b**, **c**, the triangular loop carries a trivector:

$$T_{abc} = \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \in \bigwedge^3 \mathbb{R}^d. \tag{3}$$

This is the *oriented volume* of the parallelepiped spanned by the three feature vectors. It is an irreducible grade-3 quantity: it cannot be reconstructed from the three pairwise bivectors B_{ab}, B_{bc}, B_{ca} alone (knowing the three edge lengths of a triangle does not determine its signed area in dimensions $d \geqslant 3$).

The exterior algebra $\wedge^* \mathbb{R}^d$ provides the natural hierarchy:

Grade	Structure	Dimension (d=5)	Relational content	N threshold
1	Vector	5	Individual feature profile	$N \geqslant 1$
2	Bivector	10	Pairwise correlation	$N \geqslant 2$
3	Trivector	10	Irreducible triple structure (chirality)	$N \geqslant 3$
4	Quadvector	5	Four-body coherence	$N \geqslant 4$
5	Pseudoscalar	1	Global orientation	$N \geqslant 5$

Each grade becomes nontrivial at a specific N: grade-k structures require at least k feature vectors to form a nonzero k-vector. Below this threshold, the structure is identically zero — not small, not approximate, but algebraically absent.

C. Why five dimensions

The companion paper [1] requires only that a correlation space exists with dimension $d \geqslant 1$ and that correlations are subject to a monogamy constraint. The present paper requires $d \geqslant 3$ (so that trivectors exist and are generically nonzero in the correlation space).

The specific choice $d = 5$ is motivated by a constraint framework [5] in which five independent aspects of distinguishability are derived through categorical exhaustion:

(i) Boundary (β): separation of interior from exterior — without which a feature merges with its background.

(ii) Pattern (κ): internal structure that could in principle be re-identified — without which bounded regions are interchangeable.

(iii) Resource (ρ): capacity to sustain structure against perturbation — without which boundaries and patterns degrade.

(iv) Integration (λ): coherence of parts into a whole — without which co-located components are merely a pile.

(v) Ordering (τ): asymmetric structure distinguishing orientations — without which the system has no directional character.

The independence of these five dimensions has been validated empirically across three substrates (cellular automata, Game of Life, Gray-Scott reaction-diffusion), with principal component analysis consistently requiring five components (maximum inter-constraint correlation $|r| < 0.30$ across all substrates) [6].

Remark for the reader evaluating the premise. The results of this paper (Propositions 1–4 and Theorems 2–3) depend on the exterior algebra structure only through three facts: (a) features are vectors in $\mathbb{R}^d$ with $d \geqslant 3$; (b) pairwise relations are grade-2 bivectors; (c) triangular loops carry grade-3 trivectors that are generically nonzero when $d \geqslant 3$. The specific value $d = 5$, the physical interpretation of the five constraint dimensions, and the categorical exhaustion argument do not affect the mathematical content of Sections III–VI. A reader who prefers to take "d-dimensional correlation space with $d \geqslant 3$" as an assumption will find the results unchanged.

D. Chirality as trivector magnitude

The ordering structure that this paper analyzes is the trivector magnitude of a triangular loop:

$$\tau_{circ}(a, b, c) = |T_{abc}| = |\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}|. \tag{4}$$

This quantity — which we call *chirality* — has two key properties:

It is irreducible. τ_{circ} cannot be computed from the three pairwise correlation strengths $|B_{ab}|$, $|B_{bc}|$, $|B_{ca}|$ alone. It encodes genuinely three-body information: the relative orientation of the three edges in d-dimensional space.

It has a sharp N threshold. At $N \leq 2$, no triangular loop exists, so $\tau\_circ = 0$ identically. At $N \geq 3$, τ_circ is generically nonzero (it vanishes only if the three feature vectors are coplanar — a measure-zero condition in $\mathbb{R}^d$ for $d \geq 3$).

A terminological note. In the companion paper [1], the thermodynamic deficit is expressed entirely in terms of pairwise quantities (P and A). Chirality is the leading structure that this pairwise language misses. The finite-N correction derived in Section V is precisely the cost of this omission.

E. The spatial/temporal distinction

Two distinct formulations of τ_circ must be distinguished [6]:

Spatial chirality (this paper's primary concern). Edge vectors v_XY measure asymmetric relational properties between coexisting features X and Y. The closed triangle $v\_\{AB\} + v\_\{BC\} + v\_\{CA\} \neq 0$ generically, because the edge measures are not differences of a scalar function. Spatial chirality is nonzero at $N \geq 3$ whenever three coexisting features have non-coplanar relational profiles.

Temporal chirality (trajectory torsion). Edge vectors measure differences between consecutive temporal states: $v\_\{AB\} = f(B) - f(A)$. The closed triangle telescopes: $v\_\{AB\} + v\_\{BC\} + v\_\{CA\} = 0$ identically, giving rank ≤ 2 and zero trivector. The resolution is to use three consecutive edges from four states (A→B, B→C, C→D), which detects genuine helicity of the system's trajectory through correlation space. This requires four temporal snapshots and is relevant to dynamical questions not addressed here.

Throughout this paper, "chirality" refers to spatial τ_circ — the irreducible oriented volume of triangular relations among coexisting features. The question we address is: what happens to this structure when a correlation boundary is imposed?

F. The thermodynamic potential

The companion paper [1] works with a general potential Φ satisfying additivity under independence (P1), smoothness (P2), and monotonicity (P3). We adopt the same framework. Φ_cross — the contribution to Φ from cross-boundary correlations — is the quantity whose perturbation response generates the thermodynamic deficit (Appendix A of [1]).

At finite N, Φ_cross receives contributions from both grade-2 (pairwise) and grade-3 (chirality) structures. The companion paper's deficit theorem captures the grade-2 contribution. The finite-N correction derived in Section V captures the leading grade-3 contribution.

III. CORRELATION BOUNDARIES AT SMALL N

We now analyze what happens when a correlation boundary (P = 0) is imposed on a system of N features, for each N from 1 through 5. The analysis reveals a staircase of increasingly permissive structure, culminating in the N = 6 threshold of Section IV.

A. N = 1: No relations, no existence

A single feature has no pairwise correlations — there are no other features to correlate with. Within the monogamy framework, where a feature's identity is constituted by its relational profile (the totality of its correlations), a feature with zero correlations is indistinguishable from the absence of a feature. N = 1 is not a degenerate case but a vacuous one: there is nothing to partition, no boundary to define, and no thermodynamic structure to discuss.

B. N = 2: Boundaries annihilate the system

Two features A and B with a single correlation λ_AB ∈ (0, Λ]. The sole nontrivial partition is {A} vs {B}, with cross-boundary correlation:

$$\mathcal{P} = \lambda_{AB}. \tag{5}$$

This is the *entirety* of the system's correlation structure. A correlation boundary (P = 0) requires λ_AB = 0, leaving each feature as an isolated N = 1 configuration — relationally vacuous.

No triangular loop exists at N = 2, so the trivector (Eq. 4) is identically zero: τ_circ = 0. The system has no chirality regardless of whether a boundary is present.

Proposition 1. *At N = 2, a correlation boundary reduces the system to two isolated features with zero correlations. The system's entire relational content is destroyed.*

C. N = 3: Chirality appears but cannot survive a boundary

Three features A, B, C with correlations (λ_AB, λ_BC, λ_CA) constrained by the monogamy polytope:

$$\lambda_{AB} + \lambda_{AC} \leq \Lambda, \quad \lambda_{AB} + \lambda_{BC} \leq \Lambda, \quad \lambda_{BC} + \lambda_{CA} \leq \Lambda. \tag{6}$$

Chirality onset. The single triangular loop A–B–C supports τ_circ = |**a** ∧ **b** ∧ **c**| > 0 generically (Eq. 4). This is a topological transition: at N = 2, grade-3 structure is algebraically absent; at N = 3, it is generically present. The exterior algebra acquires a nontrivial $\wedge^3$ component for the first time.

Boundary structure. At N = 3, every nontrivial partition isolates one feature:

Partition	P(Σ)	Inside	Outside
{A} vs {B,C}	$\lambda\_AB + \lambda\_AC$	N = 1	N = 2, $\tau\_circ = 0$
{B} vs {A,C}	$\lambda\_AB + \lambda\_BC$	N = 1	N = 2, $\tau\_circ = 0$
{C} vs {A,B}	$\lambda\_BC + \lambda\_CA$	N = 1	N = 2, $\tau\_circ = 0$

A correlation boundary (P = 0) sets all correlations of the isolated feature to zero, leaving it as N = 1 (vacuous). The remaining pair has N = 2 with $\tau\_circ = 0$ — the only triangular loop has been destroyed.

Proposition 2. *At N = 3, every correlation boundary destroys one feature and removes all chirality from the remainder.*

Remark. The potential difference $\Delta^{(3)} = \Phi(A,B,C) - [\Phi(\{A\}) + \Phi(\{B,C\})]$ is the cross-boundary contribution Φ_cross. At N = 3, this includes not just the pairwise correlation cost (captured by P) but also the *chirality cost*: the full system has $\tau\_circ > 0$ while neither partition subsystem does. This grade-3 contribution to Φ_cross is invisible to the pairwise count P — a point we quantify in Section V.

D. N = 4: Boundaries that preserve existence but destroy chirality

Four features with six correlations and four triangular loops: ABC, ABD, ACD, BCD.

First survivable partition. The balanced partition {A,B} vs {C,D} leaves both sides as N = 2 systems, each retaining one internal correlation. Neither side is vacuous.

Proposition 3. *N = 4 is the minimum at which a correlation boundary produces two subsystems that both retain nonzero correlations.*

Proof. At N ⩽ 3, every partition isolates at least one feature completely (Propositions 1–2). At N = 4, the partition {A,B} vs {C,D} gives both sides a retained correlation (λ_AB and λ_CD respectively). ■

Chirality is still destroyed. Each of the four triangular loops (ABC, ABD, ACD, BCD) has at least two of its three edges crossing the {A,B}|{C,D} boundary. A triangle with two cross-boundary edges has P > 0 on those edges; setting P = 0 destroys the triangle. All four loops are killed. Both sides have $\tau\_circ = 0$.

More generally: any balanced (2|2) partition at N = 4 kills all triangles (each triangle involves at least one vertex from each side, hence at least two cross-boundary edges). And the unbalanced partition {A} vs {B,C,D} preserves one triangle (BCD) on the outside but destroys the isolated feature — one side is vacuous, the other has chirality.

Discrete anisotropy. At N = 4, the first discrete analog of the continuum anisotropy A(x) from [1] appears. For partition {A,B} vs {C,D}, define:

$$\mathcal{A}_{discrete} = (\lambda_{AC} + \lambda_{BC}) - (\lambda_{AD} + \lambda_{BD}). \tag{7}$$

This measures whether cross-boundary correlations preferentially connect to C rather than D (or vice versa). The analog of the spherical symmetry degeneracy ([1], Sec. VI.C) is A_discrete = 0, where cross-boundary correlations are symmetric under exchange of C and D.

E. N = 5: One-sided chirality

Five features with ten correlations and ten triangular loops.

Balanced partition. The partition {A,B} vs {C,D,E} gives inside N = 2 (τ_circ = 0) and outside N = 3 (τ_circ > 0 generically). Chirality survives on one side only.

Near-balanced partition. The partition {A,B,C} vs {D,E} gives inside N = 3 (τ_circ > 0) and outside N = 2 (τ_circ = 0). Again, chirality on one side only.

Proposition 4. *No partition at N = 5 preserves chirality on both sides.*

Proof. A partition into sets of sizes n and N − n preserves chirality on both sides only if both n ⩾ 3 and N − n ⩾ 3, since τ_circ requires a triangular loop which requires at least three features. This gives N ⩾ 6. At N = 5, every partition has min(n, N − n) ⩽ 2, so at least one side has τ_circ = 0. ■

F. Summary of the staircase

N	Survivable boundary?	Chirality preserved?	Status
1	—	—	Vacuous
2	No (destroys both)	No (τ_circ = 0 always)	Boundary annihilates
3	No (destroys one)	No (kills the only triangle)	Boundary destroys chirality
4	**Yes** (2|2 partition)	No (kills all triangles)	Existentially viable, chirally destructive
5	Yes	**One side only**	Asymmetric proto-structure
6	**Yes**	**Both sides**	**Bilateral chirality (Theorem 2)**

The staircase has a clear logical structure. At each step, one restriction is lifted: N = 4 lifts the existential restriction (boundaries no longer destroy features); N = 6 lifts the chirality restriction (boundaries no longer destroy all grade-3 structure). The gap between N = 4 and N = 6 — where boundaries are survivable but chirally destructive — is the regime where the companion paper's formalism applies (P and A are defined) but the thermodynamic *interpretation* is problematic (temperature and entropy require chirality to be well-defined, as we now argue).

IV. THE N = 6 BILATERAL CHIRALITY THRESHOLD

A. The theorem

At N = 6, the balanced partition {A,B,C} vs {D,E,F} creates two N = 3 subsystems. Each retains one internal triangle (ABC and DEF respectively) and has $\tau\_circ > 0$ generically.

Theorem 2 (Bilateral Chirality Threshold). [Theorem 1 is in the companion paper [1].] *N = 6 is the minimum at which a correlation boundary can partition a system into two subsystems that both possess chirality ($\tau\_circ > 0$).*

Proof. Necessity: by Proposition 4, $N \leqslant 5$ is insufficient (no partition at $N \leqslant 5$ preserves chirality on both sides). Sufficiency: at N = 6, the partition {A,B,C} vs {D,E,F} gives both sides three features. Each side's three feature vectors generically span a 3-dimensional subspace of $\mathbb{R}$^d (for $d \geqslant 3$), giving nonzero trivector $|\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}| > 0$. ■

Remark. The theorem is tight: N = 6 is achieved by a 3|3 partition and no smaller N suffices. The result depends only on the fact that τ_circ requires three features to form a triangular loop — it is independent of the dimensionality d (provided $d \geqslant 3$), the specific values of the correlations, or the geometry of the correlation space.

B. Structural parallel: why chirality matters for thermodynamics

The companion paper [1] proves that the first law $\delta Q = TdS$ holds on surfaces where P = 0 or A = 0. But the first law is an equation whose *terms* must be defined before the equation can be evaluated. The quantities δQ, T, and dS each have a structural prerequisite that connects to chirality:

Directionality. The first law distinguishes energy flowing *into* a region from energy flowing *out*. This requires an orientation — a way to distinguish the two sides of the surface. In the exterior algebra, orientation is provided by the top-form (pseudoscalar) structure, which requires the full grade hierarchy to be nontrivial. More concretely: the notion of "inside" versus "outside" is well-defined for any partition, but the notion of a *directed flow* across the boundary — energy moving from inside to outside rather than merely redistributing — requires an asymmetric structure (chirality) on at least one side.

Conjugacy. Temperature T is thermodynamically conjugate to entropy S: $T = \partial E/\partial S$ at fixed external parameters. This conjugacy requires at least two independent thermodynamic variables (energy and entropy) that can be varied independently. In the correlation framework, these variables are associated with even-grade (symmetric) and odd-grade (antisymmetric) structures respectively. At N = 2, only even-grade (grade-2) structure exists — there is one "kind" of thermodynamic variable, not two. The conjugacy that defines temperature requires the odd-grade structure that appears at $N \geq 3$.

Bilateral requirement. The first law relates quantities on *both* sides of the surface: δQ crosses the boundary, T characterises the interior, dS measures the interior's response. If one side lacks chirality (and hence lacks well-defined T), the equation has an undefined term. Both sides must independently support thermodynamic structure.

These arguments establish a structural parallel between bilateral chirality and the applicability of the first law. We state this as:

Corollary (Structural). *The first law $\delta Q = TdS$ is structurally well-posed — in the sense that all terms have well-defined analogs in the correlation framework — only when both sides of the partition support chirality. By Theorem 2, this requires $N \geq 6$.*

Note on scope. This corollary identifies a structural prerequisite within the mathematical framework, not a physical claim about the minimum number of degrees of freedom required for thermodynamics in nature. Whether the framework's structural prerequisites map to physical conditions is an interpretive question beyond the scope of this paper.

C. The 3|3 partition and structured information transfer

An independent argument arrives at the same threshold. Consider the question: what is the minimum system that can support a *reference subsystem* and a *target subsystem*, where each has its own internal ordering structure?

Within the correlation framework, information transfer is a relational concept — subsystem I gains information about subsystem O through the cross-boundary correlations that connect them. For this information to be *temporally structured* (ordered in a sequence rather than simultaneous), the reference subsystem I must have its own chirality ($\tau\_circ > 0$), providing an internal ordering against which the sequence of correlations can be structured.

The minimum reference subsystem is $N = 3$ (the smallest system with chirality). The minimum target subsystem, if it is to have its own thermodynamic characterisation, is also $N = 3$. The minimum total is $N = 6$ with a 3|3 partition — the same threshold as Theorem 2.

The coincidence between the bilateral chirality threshold and the minimum structured information transfer threshold is not accidental: both require the same mathematical condition (two independently chiral subsystems separated by a boundary). This structural coincidence is noted as a mathematical fact about the framework, not as a physical claim about measurement or observation.

D. Counting cross-boundary structure at N = 6

At $N = 6$ with the 3|3 partition {A,B,C} vs {D,E,F}, the correlation structure partitions as follows:

Structure	Total	Internal to I	Internal to O	Cross-boundary
Edges (grade 2)	15	3	3	9
Triangles (grade 3)	20	1	1	18

Of the 20 triangular loops, 18 have at least one edge crossing the boundary. These 18 loops are destroyed by the correlation boundary (P = 0), leaving only the two internal triangles (ABC and DEF). The chirality cost of the boundary is the loss of 18/20 = 90% of the system's triangular structure.

This counting will be used in Section V to estimate the chirality correction to the thermodynamic deficit.

V. THE CHIRALITY CORRECTION TO THE THERMODYNAMIC DEFICIT

A. Where the companion paper's proof is pairwise

The companion paper [1] proves (Appendix A, Eq. A4):

$$\Delta_{FL}(\Sigma) = \int_{\Sigma} \mathcal{P}(x) \cdot \hat{\mathcal{A}}(x) \cdot h(x)\, d\mu_{\Sigma} + O(\varepsilon^2, 1/N). \tag{8}$$

The proof proceeds by expanding the cross-boundary potential Φ_cross to first order in the pairwise correlations λ_ij (Step A1 of [1]):

$$\Phi_{cross} = \sum_{i \in I, j \in O} \lambda_{ij} \cdot \phi^0_{ij} + O(\lambda^2). \tag{9}$$

This expansion is *pairwise*: it sums over edges (i, j) crossing the boundary, weighted by the correlation susceptibility φ⁰_ij. The O(λ²) error term, which the companion paper bounds but does not decompose, contains all multi-body contributions — including the grade-3 chirality contribution that is the subject of this section.

This is not a deficiency of the companion paper's proof — at large N the remainder is genuinely small. But at finite N, the grade-3 contribution can be comparable to the leading term, and its structure (irreducible, positive-definite, invisible to P) makes it qualitatively important even when it is quantitatively small.

B. The grade hierarchy of Φ_cross

The potential Φ, satisfying properties P1–P3 of [1], depends on the full correlation structure —

not just pairwise correlations but all higher-order structures derivable from the feature vectors. For features $\mathbf{a}_1, ..., \mathbf{a}_N \in \mathbb{R}^d$, Φ_{cross} admits a cluster decomposition:

$$\Phi_{cross} = \Phi^{(2)}_{cross} + \Phi^{(3)}_{cross} + \Phi^{(4)}_{cross} + \cdots \quad (10)$$

where $\Phi^{(k)}_{cross}$ depends on k-body structures that straddle the boundary:

- $\Phi^{(2)}_{cross} = \Sigma_{\{i\in I, j\in O\}} \varphi^{(2)}(|\mathbf{a}_i \wedge \mathbf{a}_j|)$ — the pairwise (grade-2) contribution. This is what the companion paper captures.
- $\Phi^{(3)}_{cross} = \Sigma_{\{\text{cross triangles}\}} \varphi^{(3)}(|\mathbf{a}_i \wedge \mathbf{a}_j \wedge \mathbf{a}_k|)$ — the chirality (grade-3) contribution, summing over all triangles with at least one vertex on each side.
- $\Phi^{(4)}_{cross}$ and higher: four-body and higher contributions, involving quadvectors and above.

The functions $\varphi^{(k)}$ encode how the potential responds to grade-k structure. By monotonicity (P3), $\varphi^{(k)}$ is increasing: richer structure means higher potential. By smoothness (P2), the first-order expansion around zero gives:

$$\phi^{(k)}(\tau) - \phi^{(k)}(0) \approx c_k \cdot \tau \quad (11)$$

where $c_k = (\varphi^{(k)})'(0)$ is the grade-k susceptibility.

C. The chirality cost of a boundary

When a correlation boundary is imposed, cross-boundary edges carry zero correlation. Any triangular loop with at least one cross-boundary edge has its trivector magnitude reduced: the trivector $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ depends continuously on the edge vectors, and shrinking a cross-boundary edge to zero collapses the loop.

The chirality cost of destroying a single loop (i, j, k) is, to first order:

$$\delta\Phi^{(3)}_{loop} = c_\tau \cdot \tau_{circ}(i, j, k) \quad (12)$$

where $\tau_{circ} = |\mathbf{a}_i \wedge \mathbf{a}_j \wedge \mathbf{a}_k|$ is the loop's trivector magnitude before the boundary is imposed, and c_τ is the grade-3 susceptibility.

D. Counting and scaling

Loop counting. Let $L_{total} = C(N, 3) = N(N-1)(N-2)/6$. For a balanced partition ($n = N/2$):

$$L_{cross} = L_{total} - 2 \cdot \binom{N/2}{3} \quad (13)$$

At large N: L_cross/L_total → 3/4. At N = 6: L_cross = 18, L_total = 20 (Section IV.D). Define the chirality loop fraction f_τ = L_cross/L_total.

Per-loop chirality. Each feature's correlation budget Λ is shared among O(N) partners by the monogamy constraint, giving individual edge correlations λ ~ Λ/N. The trivector magnitude of a loop with three edges of strength ~Λ/N scales as:

$$\bar{\tau}_{circ} \sim \Lambda^3/N^3 \tag{14}$$

since the trivector is trilinear in the edge vectors.

Total chirality correction. Summing the cost (Eq. 12) over all destroyed loops:

$$\delta\Delta^{(3)} = c_\tau \sum_{\text{cross loops}} \tau_{circ}(i,j,k) \approx c_\tau \cdot L_{cross} \cdot \bar{\tau}_{circ} \sim c_\tau \cdot \frac{3N^3}{24} \cdot \frac{\Lambda^3}{N^3} = \frac{c_\tau \Lambda^3}{8} \tag{15}$$

This is a constant — independent of N at leading order.

Comparison to pairwise deficit. The pairwise deficit (Eq. 8) scales as P · A. For a balanced partition: there are N²/4 cross-boundary pairs each with λ ~ Λ/N, giving P ~ NΛ/4. The anisotropy A is O(1). So P · A ~ NΛ/4, which grows linearly with N. The ratio of chirality correction to pairwise deficit:

$$\frac{\delta\Delta^{(3)}}{\mathcal{P} \cdot \mathcal{A}} \sim \frac{c_\tau \Lambda^3/8}{N\Lambda/4} = \frac{c_\tau \Lambda^2}{2N} = O(1/N) \tag{16}$$

The chirality correction is O(1/N) relative to the pairwise deficit. At N = 6, it is O(1) — comparable to the leading term. At N ~ 30, it is ~3%. At N ~ 100, it is ~1%.

E. The corrected deficit theorem

In the continuum limit, the companion paper expresses the deficit as an integral over the surface involving the anisotropy A(x) and perturbation profile h(x). In the discrete setting, these reduce to a sum over cross-boundary edges weighted by a sign factor sgn_ij ∈ {+1, −1}, which is the discrete analog of the directional anisotropy: sgn_ij = +1 if the (i, j) correlation is preferentially outward (partner j is on the "outside-heavy" side of the local correlation distribution at i), and sgn_ij = −1 if preferentially inward. At large N, the sum Σ λ_ij · φ⁰_ij · sgn_ij recovers the continuum integral ∫ P(x) · A(x) · h(x) dμ_Σ of [1, Eq. A4].

Theorem 3 (Finite-N corrected deficit). *For a system of N features in ℝ^d (d ≥ 3), partitioned by Σ into sets of sizes n and N − n with both n ≥ 3 and N − n ≥ 3, the thermodynamic deficit is:*

$$\Delta_{FL}^{(N)}(\Sigma) = \underbrace{\sum_{i\in I, j\in O} \lambda_{ij} \cdot \phi_{ij}^{0} \cdot \mathrm{sgn}_{ij}}_{\text{grade-2 (pairwise): [1], Eq. A4}} + \underbrace{c_\tau \sum_{\substack{(i,j,k)\\ \text{cross-boundary}}} \tau_{circ}(i,j,k)}_{\text{grade-3 (chirality): this paper}} + O(1/N^2) \quad (17)$$

where sgn_ij encodes the anisotropy, c_τ is the grade-3 susceptibility of Φ, and the second sum runs over all triangular loops with at least one vertex on each side of Σ.

Proof sketch. Expand Φ_cross in the cluster decomposition (Eq. 10). The grade-2 term reproduces the companion paper's result by the argument in [1, Appendix A]. The grade-3 term contributes Eq. (15). Grade-k terms for k ⩾ 4 involve k-vectors with magnitude ~Λ^k/N^k, summed over C(N,k) ~ N^k/k! clusters. Each contributes a total of ~c_k Λ^k/k! to the deficit — a constant independent of N, just like the grade-3 term. Relative to the growing pairwise deficit (~NΛ), each gives O(1/N). But the coefficients c_k Λ^k/k! decrease with k (each additional grade costs a factor Λ/N per edge, and the factorial suppresses the combinatorial growth), making the grade-3 term dominant among all k ⩾ 3. The next correction (grade-4) is O(Λ/N) smaller, i.e., $O(1/N^2)$ relative to the leading pairwise deficit. ■

F. Properties of the chirality correction

The grade-3 correction has four notable properties:

1. Definite sign. Destroying a triangular loop removes chirality: τ_circ goes from positive to zero. Since Φ is monotonically increasing in the richness of relational structure (P3), the chirality cost is always positive: δΔ^(3) > 0. The correction always *increases* the deficit — a boundary is more costly than the pairwise count P suggests.

2. Invisible to holographic entanglement entropy. The identification P ∝ S_EE in [1, Sec. IV] captures grade-2 contributions only — it is the pairwise-dominated approximation that underlies the Ryu-Takayanagi area law. The exact von Neumann entropy S_EE = −Tr(ρ ln ρ) does capture multipartite entanglement (including three-body correlations such as GHZ-type states). But the holographic, area-law approximation used in [1] and throughout the emergent gravity literature does not. The grade-3 correction is invisible to this area-law approximation — it is a contribution to the thermodynamic deficit that the pairwise (holographic) formalism cannot represent. A full quantum treatment would subsume it into the exact S_EE, but such a treatment goes beyond the pairwise framework of both this paper and its companion.

3. Exactly zero at correlation boundaries. When P = 0, all cross-boundary edges carry zero correlation. A trivector with one zero edge vanishes: $|\mathbf{a} \wedge \mathbf{0} \wedge \mathbf{c}| = 0$. Therefore δΔ^(3) = 0, and the companion paper's exact result (Δ = 0 at correlation boundaries) is unaffected. The correction matters for near-boundaries and ordinary surfaces, not for exact correlation boundaries.

4. Dominant at small N. At N = 6, the ratio δΔ^(3)/(P · A) is O(1). The chirality correction is comparable to the pairwise deficit — neither dominates. This is the regime where grade-3 structure fundamentally matters: the cost of a boundary is not just the pairwise correlations it severs but the triangular loops it destroys.

G. The hierarchy of finite-N corrections

Source	Scaling (relative to P · A)	Origin	Character
Chirality loops (grade 3)	**$O(1/N)$**	Destroyed triangular structure	Deterministic, positive
Four-body coherence (grade 4)	$O(1/N^2)$	Destroyed quadvector structure	Deterministic
Monogamy saturation	$O(\Lambda^2/N^2)$	Nonlinear expansion of $\Phi^{(2)}_{cross}$	Deterministic
Discreteness	$O(1/N^{3/2})$	Finite sums vs. continuous integrals	Systematic
Fluctuations	$O(1/\sqrt{N})$	Statistical variation in T, S, Δ	Stochastic, zero mean

The chirality correction is the leading *deterministic* finite-N effect. Fluctuations ($O(1/\sqrt{N})$) are larger in magnitude but average to zero over an ensemble; the chirality correction has a definite positive sign and does not average away. It is the systematic bias that makes the thermodynamic deficit larger than the pairwise approximation predicts.

H. Budget redistribution under boundary formation

The monogamy constraint creates a conservation law for boundary formation. Each feature's correlation budget satisfies $\lambda_{inside} + \lambda_{outside} \leq \Lambda$. As the boundary sharpens ($\lambda_{outside} \to 0$), freed budget becomes available for internal correlations. Surviving internal loops become stronger per edge, partially compensating the chirality cost.

Quantitatively: if a feature's cross-boundary budget redistributes to internal edges, each surviving internal edge gains $\delta\lambda \sim \Lambda/N^2$. The chirality of surviving loops increases by $\delta\tau_{circ} \sim \Lambda^2/N^2 \cdot \delta\lambda \sim \Lambda^3/N^4$ per loop. Summed over $L_{internal} \sim N^3/48$ surviving loops, the total compensation is $O(\Lambda^3/N)$. This is $O(1/N)$ relative to the chirality cost $O(\Lambda^3)$, making the redistribution a subleading correction to the correction. The $O(1/N)$ scaling of Theorem 3 is not affected.

VI. THREE TRANSITIONS FROM DISCRETE TO CONTINUUM

The results of Sections III–V describe a staircase of qualitative changes as N increases from the smallest possible systems to the thermodynamic limit. Three transitions stand out, each lifting a structural restriction on the applicability of thermodynamic concepts.

A. Transition 1: Chirality onset ($N = 2 \to 3$)

The trivector τ_circ jumps from algebraically absent (no triangular loop exists at $N = 2$) to generically nonzero (one triangle at $N = 3$). This is a discrete topological transition in the relational graph: the appearance of the first closed loop.

In the exterior algebra, grade-3 structure appears for the first time — a qualitatively new class of relational information that is irreducible to pairwise data (Section II.D). Correlation boundaries become definable as partitions of features, but remain existentially destructive (Propositions 1–2).

What becomes available: Chirality, directional asymmetry, irreducible three-body structure.

What remains unavailable: Survivable boundaries (both sides lose relational content), bilateral thermodynamic structure.

B. Transition 2: Bilateral chirality ($N = 5 \to 6$)

Both sides of a partition can independently support chirality (Theorem 2). The structural prerequisites for the first law — directed energy transfer, well-defined temperature, entropy counting — are simultaneously available on both sides of a boundary for the first time.

This transition is discrete: at $N = 5$, no partition preserves bilateral chirality; at $N = 6$, the 3|3 partition does. There is no intermediate state.

What becomes available: Bilateral thermodynamic structure, relational measurement (two independently ordered subsystems), the first law as a structurally well-posed equation.

What remains unavailable: Smooth surfaces, continuous integrals, the continuum description of the companion paper [1].

C. Transition 3: Statistical emergence ($N \sim 30$)

As N increases beyond 6, the chirality correction (Theorem 3) decreases as $1/N$ relative to the pairwise deficit, and statistical fluctuations in thermodynamic quantities decrease as $1/\sqrt{N}$. At $N \sim 30$, relative fluctuations drop below ~20%, and the continuum description — smooth surfaces, infinitesimal perturbations, continuous integrals — becomes a valid approximation to the discrete structure.

This transition is continuous. There is no sharp N at which the continuum description activates; the approximation improves smoothly. The companion paper's deficit theorem (Eq. 8) becomes quantitatively accurate, with the chirality correction contributing only ~3% at N = 30.

What becomes available: The full apparatus of the companion paper [1] — continuous surfaces, the deficit theorem $\Delta \propto P \cdot A$, the identification $P \propto S\_EE$, the Ryu-Takayanagi formula.

D. The interpolation

Between transitions, the key quantities evolve as follows:

Quantity	N ⩽ 2	N = 3–5	N = 6–30	N ≫ 30
Chirality (τ_circ)	Absent	Present, fragile	Present, bilateral	Dilute (~Λ^3/N^3 per loop)
Temperature (T)	Undefined	One-sided at best	Defined bilaterally, fluctuating	Smooth scalar field
Entropy (S)	Undefined	ln(few configurations)	Growing, discrete	Extensive, ∝ Area or Volume
Deficit (Δ)	Undefined	Discrete Φ_cross	P · A + O(1/N) chirality (Thm. 3)	Smooth S_EE · A
Anisotropy (A)	Undefined	Discrete asymmetry	Fluctuating	Smooth tensor field

The table summarises the mathematical structures available at each stage. Whether these mathematical structures correspond to physical quantities with the same names is an interpretive question addressed in Section VII.

VII. STRUCTURAL PARALLELS AND OPEN QUESTIONS

A. Structural parallel: lower bound for gravitational thermodynamics

Jacobson's derivation [2] of the Einstein equations assumes that δQ = TdS holds on local Rindler horizons. Within the present framework, this assumption has a structural prerequisite: the horizon must be embedded in a system with N ⩾ 6 degrees of freedom (Theorem 2 and the Corollary of Section IV.B), so that both sides of the horizon support chirality.

In the holographic context, the number of degrees of freedom associated with a surface scales as Area/ℓ^2_P. The N ⩾ 6 condition translates to a minimum area:

$$A_{min} \sim 6\,\ell_P^2. \quad (18)$$

This is a Planck-scale bound. For astrophysical horizons ($A \gg \ell^2\_P$), the condition is trivially satisfied. The bound is relevant only at the Planck scale, where it provides a structural reason why gravitational thermodynamics might break down: not because the first law is *violated* but because its terms are *undefined* when fewer than six degrees of freedom participate.

Note on scope. This is a structural parallel: within the mathematical framework, $N \geqslant 6$ is required for bilateral chirality; within gravitational thermodynamics, a minimum area is expected for the Jacobson derivation to apply. The parallel is suggestive but does not constitute a derivation of the Planck-scale bound from the monogamy constraint. The identification of "features" with Planck-scale degrees of freedom is an interpretive step that the mathematics does not compel.

B. Structural parallel: the chirality correction as a finite-N observable

The 1/N chirality correction to the thermodynamic deficit (Theorem 3, Eq. 17) has a structural parallel in the quantum gravity literature. For a black hole with Bekenstein-Hawking entropy S_BH = A/(4G_N), if the effective number of features is N_eff ~ S_BH, the chirality correction to the first law would scale as:

$$\frac{\delta\Delta^{(3)}}{\Delta^{(2)}} \sim \frac{c_\tau \Lambda^2}{2S_{BH}} \tag{19}$$

For astrophysical black holes (S_BH ~ 10^{77}), this ratio is negligible. For Planck-scale black holes (S_BH ~ O(1)), it is O(1) — the chirality correction would dominate the pairwise deficit. The correction is the leading deterministic deviation from smooth gravitational thermodynamics and has a definite positive sign (Section V.F), which distinguishes it from statistical fluctuations.

Whether this mathematical structure corresponds to an actual quantum gravity correction — and whether any candidate theory of quantum gravity produces a 1/S_BH correction to the first law with the specific properties of Theorem 3 (grade-3 origin, definite sign, invisible to entanglement entropy) — is a question for future work.

C. Structural parallel: decoherence thresholds

The three transitions (chirality onset, bilateral thermodynamics, statistical emergence) have structural parallels in the theory of decoherence. As a quantum system couples to an environment, the effective N of the combined system grows:

N = 3 threshold: The combined system acquires chirality — irreducible three-body structure that distinguishes it from a collection of independent pairs. This parallels the onset of non-Markovian dynamics in open quantum systems, where three-body correlations between system, environment, and memory become relevant.

N = 6 threshold: The system can support a partition into two independently ordered subsystems. This parallels the emergence of a structured reference-target partition — the point at which temporally ordered information transfer becomes structurally possible because both subsystems have independent internal ordering.

N ~ 30 threshold: Statistical behaviour emerges; individual correlations are too dilute to track. This parallels the classical limit — the point at which quantum coherence becomes operationally irrelevant because the per-relation coherence scales as $1/N$.

These parallels are noted as correspondences between mathematical forms. The decoherence programme operates within quantum mechanics; the present framework operates within the monogamy-constrained correlation formalism. Whether the two formalisms describe the same physics at different levels of description is an open question.

D. Relation to loop quantum gravity

The $N = 6$ threshold has a structural resonance with loop quantum gravity (LQG), where the area spectrum is discrete and a surface's thermodynamic properties depend on the number of spin-network punctures it carries [7]. The requirement that each side of a boundary carry at least three features to support chirality is analogous to the requirement that a surface carry a minimum number of punctures for the area operator to have a well-defined thermal interpretation.

We note this as a structural resonance, not a claimed correspondence. Establishing whether "features" in the present framework map to LQG punctures — and whether the chirality correction (Theorem 3) maps to specific loop quantum gravity corrections to the Bekenstein-Hawking entropy — requires detailed comparison that is beyond the scope of this paper.

E. Open questions

(i) The grade-3 susceptibility c_τ. Theorem 3 gives the chirality correction up to the constant c_τ, which depends on the specific potential Φ. Determining c_τ requires either specifying Φ (which would connect to a specific microscopic theory) or deriving bounds on c_τ from the structural properties P1–P3 alone.

(ii) Non-balanced partitions. Sections III–V focus primarily on balanced partitions. For unbalanced partitions ($n \ll N - n$), the chirality cost is asymmetric: the smaller side loses a larger fraction of its loops. The scaling of the chirality correction for highly unbalanced partitions — relevant to the physical scenario of a small system observed by a large environment — requires separate analysis.

(iii) Higher-grade corrections. The grade-4 (quadvector) correction to the deficit is $O(1/N^2)$ (Section V.E). At $N \sim 10$–20, both the grade-3 and grade-4 corrections may be simultaneously relevant. A systematic analysis of the full grade hierarchy of corrections, including cross-grade interference terms, is needed for quantitative accuracy in the mesoscopic regime.

(iv) Dynamical boundaries. This paper treats correlation boundaries as static partitions. In a dynamical setting, boundaries evolve: the partition changes as features move between I and O.

The chirality cost of a dynamical boundary includes contributions from the *rate* of loop destruction, not just the total number destroyed. This connects to the question of how fast a thermodynamic boundary can form — a question relevant to black hole formation and cosmological horizon dynamics.

(v) Connection to the constraint framework's axiom. The companion paper [1] takes the monogamy constraint as given and notes (Sec. VII.E.iv) that a deeper derivation from distinguishability principles is an open question. If such a derivation is achieved, it would ground both papers' results in a single axiom, connecting the large-N thermodynamic structure (companion paper) and the finite-N chirality structure (this paper) to a common origin.

VIII. CONCLUSION

We have extended the correlation boundary formalism of [1] to finite N, revealing structure invisible in the large-N regime.

The central results:

(i) The bilateral chirality threshold (Theorem 2). N = 6 is the minimum at which a correlation boundary preserves grade-3 (trivector) structure on both sides of the partition. This provides a structural lower bound for the applicability of the first law: below N = 6, the thermodynamic equation's terms are not all defined within the framework.

(ii) The chirality correction (Theorem 3). The thermodynamic deficit acquires a leading finite-N correction of order 1/N from the destruction of triangular chirality loops. This correction is deterministic, positive-definite, and invisible to the pairwise correlation count P — it is a grade-3 contribution that the entanglement entropy description does not capture.

(iii) Three qualitative transitions. The interpolation from discrete (N ~ few) to continuum ($N \to \infty$) boundary structure passes through chirality onset (N = 3), bilateral thermodynamics (N = 6), and statistical emergence (N ~ 30). Each transition lifts a structural restriction on the applicability of thermodynamic concepts.

These results connect the companion paper's resolution of the Wang-Braunstein problem — which operates in the large-N continuum regime — to the finite-N regime where the grade hierarchy of the exterior algebra becomes essential. The chirality correction (grade-3) is the leading deterministic effect that the companion paper's pairwise (grade-2) formalism misses. Together, the two papers provide a description of thermodynamic surfaces that is valid from the continuum limit down to the smallest systems capable of supporting bilateral thermodynamic structure.

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