Notes on PHC efficiency

Let's define our ODE

$$(m + m_A)\ddot{z} = mg - m_b g - k[z + z_0 + H\cos(\omega t)]$$

Where:

m	–	Payload mass
m_A	–	Payload added mass
$\ddot{z}$	–	Payload vertical acceleration
g	–	Acceleration of gravity
m_b	–	Payload buoyancy mass
k	–	PHC stiffness
z_0	–	Equilibrium elongation
H	–	Wave amplitude
ω	–	Angular wave frequency
t	–	Time

We have included added mass and buoyancy. Hydrodynamic drag, PHC damping (i.e. hydraulic restriction) and seal friction are not included (because it would make it impossible to find an analytical solution), but it would not make a big difference for most cases.

Cleaning up

$$\ddot{z} + \frac{k}{m + m_A} z = \frac{mg - m_b g - kz_0 - kH\cos(\omega t)}{m + m_A}$$

Defining:

$$\omega_0^2 = \frac{k}{m + m_A}$$

$$z_0 = \frac{(m - m_b)g}{k}$$

The ODE simplifies to

$$\ddot{z} + \omega_0^2 z = -\omega_0^2 H\cos(\omega t)$$

Try solution

$$z = R\cos(\omega t)$$

Differentiate

$$\ddot{z} = -R\omega^2\cos(\omega t)$$

Substitute

$$-R\omega^2\cos(\omega t) + \omega_0^2 R\cos(\omega t) = -\omega_0^2 H\cos(\omega t)$$

Simplify

$$R = \frac{\omega_0^2}{\omega^2 - \omega_0^2} H$$

Final expression

$$z = \frac{\omega_0^2}{\omega^2 - \omega_0^2} H\cos(\omega t)$$

We can from this see that the ratio of the payload amplitude R to the wave amplitude H is:

$$\frac{R}{H} = \frac{\omega_0^2}{\omega^2 - \omega_0^2}$$

Let us expand on this to make it more relatable, for the wave we have:

$$\omega = \frac{2\pi}{T_p}$$

Where:

T_p – Wave period

And for the compensator we have:

$$\omega_0 = \sqrt{\frac{k}{m + m_A}}$$

We can define average compensator stiffness by calculating the force at max stroke and at minimum stroke, subtracting them and dividing by the stroke length.

The force at mid stroke it is at equilibrium with the forces in play so that:

$$F_{mid} = mg - m_b g = p_0 A_0$$

Where:

F_{mid}	–	Mid stroke force
p_0	–	Equilibrium pressure
A_0	–	Pressure area

At zero stroke we will have the following force

$$F_{-} = p_0 A_0 \left(\frac{V_0}{V_{-}}\right)^{\gamma} = p_0 A_0 \left(\frac{R_{go} A_0 S - \frac{1}{2} A_0 S}{R_{go} A_0 S}\right)^{\gamma} = p_0 A_0 \left(\frac{R_{go} - \frac{1}{2}}{R_{go}}\right)^{\gamma}$$

Where:

F_{-}	–	Zero stroke force
V_0	–	Initial volume (mid stroke)
V_{-}	–	Zero stroke volume
R_{go}	–	Gas to oil ratio
S	–	Compensator stroke
γ	–	Adiabatic compression exponent

And at full stroke we will have the following force

$$F_{+} = p_0 A_0 \left(\frac{V_0}{V_{+}}\right)^{\gamma} = p_0 A_0 \left(\frac{R_{go} A_0 S - \frac{1}{2} A_0 S}{R_{go} A_0 S - A_0 S}\right)^{\gamma} = p_0 A_0 \left(\frac{R_{go} - \frac{1}{2}}{R_{go} - 1}\right)^{\gamma}$$

Where:

F_{+}	–	Full stroke force
V_{+}	–	Full stroke volume

The average stiffness is then given by:

$$k = \frac{p_0 A_0 \left(\frac{R_{go} - \frac{1}{2}}{R_{go} - 1} \right)^{\gamma} - p_0 A_0 \left(\frac{R_{go} - \frac{1}{2}}{R_{go}} \right)^{\gamma}}{S} = \frac{p_0 A_0}{S} \left[\left(\frac{R_{go} - \frac{1}{2}}{R_{go} - 1} \right)^{\gamma} - \left(\frac{R_{go} - \frac{1}{2}}{R_{go}} \right)^{\gamma} \right]$$

Cleaned up to

$$k = \frac{(m - m_b)g}{S} \left(R_{go} - \frac{1}{2} \right)^{\gamma} \left[\frac{1}{(R_{go} - 1)^{\gamma}} - \frac{1}{R_{go}^{\gamma}} \right]$$

We then have the following expression for ω_0

$$\omega_0 = \sqrt{\frac{(m - m_b)g}{S(m + m_A)} \left(R_{go} - \frac{1}{2} \right)^{\gamma} \left[\frac{1}{(R_{go} - 1)^{\gamma}} - \frac{1}{R_{go}^{\gamma}} \right]}$$

We could also define the natural period of the compensator as:

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{S(m + m_A)}{(m - m_b)g \left(R_{go} - \frac{1}{2} \right)^{\gamma} \left[\frac{1}{(R_{go} - 1)^{\gamma}} - \frac{1}{R_{go}^{\gamma}} \right]}}$$

Let's define the following ratio to simplify the expression:

$$\lambda = \frac{m + m_A}{m - m_b}$$

We then get

$$T_0 = 2\pi \sqrt{\frac{S\lambda}{g \left(R_{go} - \frac{1}{2} \right)^{\gamma} \left[\frac{1}{(R_{go} - 1)^{\gamma}} - \frac{1}{R_{go}^{\gamma}} \right]}}$$

The efficiency of a PHC can further be described as:

$$\eta = 1 - \left| \frac{R}{H} \right| = 1 - \left| \frac{\omega_0^2}{\omega^2 - \omega_0^2} \right| = 1 - \left| \frac{1}{\left(\frac{\omega}{\omega_0} \right)^2 - 1} \right|$$

Or equivalently

$$\eta = 1 - \left| \frac{1}{\left(\frac{T_0}{T_p}\right)^2 - 1} \right|$$

Now, as an example, let’s calculate the efficiency of a PHC with $S = 4.5\text{m}$, for $\lambda = 2$. Assume $\gamma = 1.9$.

R_{GO}	T_0
3	9.2
4	10.4
5	11.4
6	12.4
7	13.4
8	14.2
9	15.0
10	15.8
11	16.5
12	17.2
13	17.9
14	18.6
15	19.2
16	19.8
17	20.4
18	21.0
19	21.5
20	22.1