Pascal's Arithmetic Triangle and Leibniz's Harmonic Triangle: An Exploration Of Two- And Three-Dimensional Relationships

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ABSTRACT

Pascal's Arithmetic Triangle is an ever-expanding, equilateral triangle of numbers having the value of 1 at the top and along the sides, and each subsequent term is the sum of its two corresponding diagonal numbers from above. Less commonly known is Leibniz's Harmonic Triangle. It is a triangular array of numbers having the unit fractions going down the left most diagonal and each subsequent term is the difference between the number above and the number adjacent to it. In this research we utilize known theorems of Pascal's Arithmetic Triangle to discover properties of Leibniz's Harmonic Triangle explainable by the commutative and associative properties of arithmetic. Results of our exploration include a comparison of alternating row sums, the hockey stick and reverse hockey stick theorems, the hexagonal property, and the reciprocal row sums of each triangle. Further, by noting these observations in two dimensions, we extend these properties to the construction of Pascal's Tetrahedron and Leibniz's Tetrahedron in three dimensions and postulate related relationships between the two tetrahedrons.

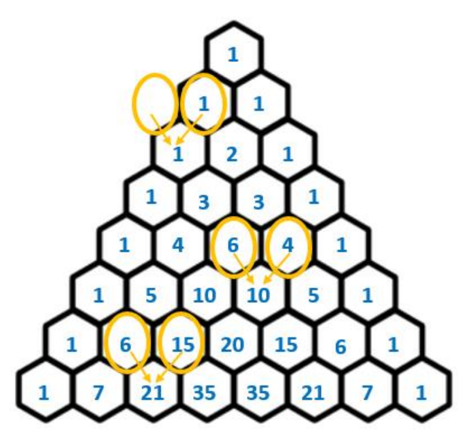
A Few Properties of Mathematics

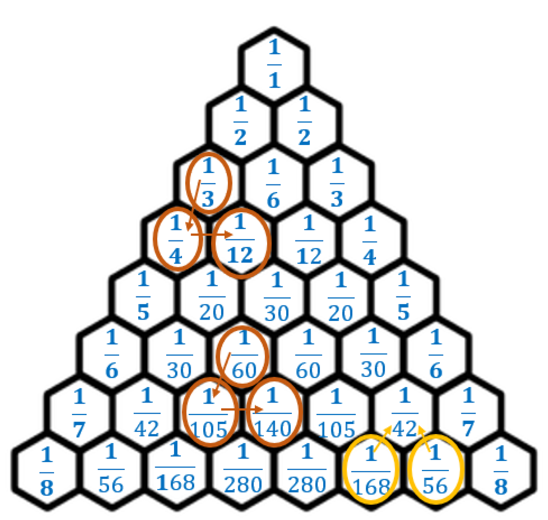
Commutative Property	Associative Property
Order of terms can change without changing the result	Order of multiple terms can change without changing the result
Addition	Addition
X + Y = Y + X	X + (Y + Z) = (X + Y) + Z
3+8 = 8+3	1 + (4 + 9) = (1 + 4) + 9
Multiplication	Multiplication
A * B = B * A	A * (B * C) = (A * B) * C
2 * 7 = 7 * 2	5 * (6 * 2) = (5 * 6) * 2

Constructions

Pascal's Arithmetic Triangle

Starting with the value of 1 at the top
Each subsequent term is the sum of its two corresponding diagonal numbers from above



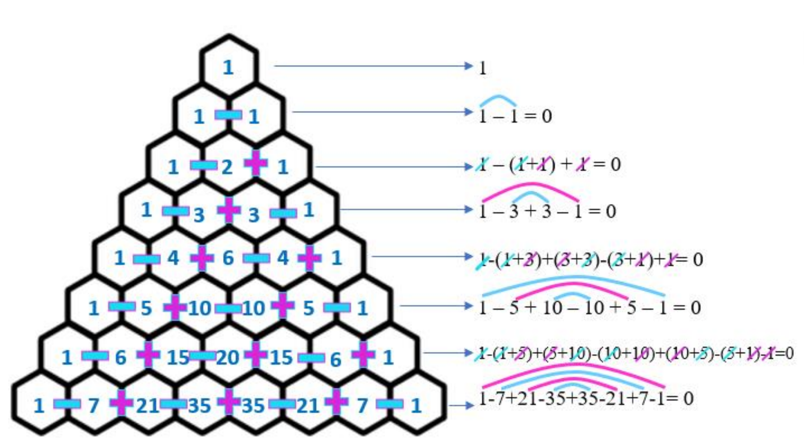


Leibniz's Harmonic Triangle

Starting with the unit fractions going down the left most diagonal
Each subsequent term is the difference between the number above and the number adjacent

Note: The sum of two corresponding diagonal numbers from below produces each term

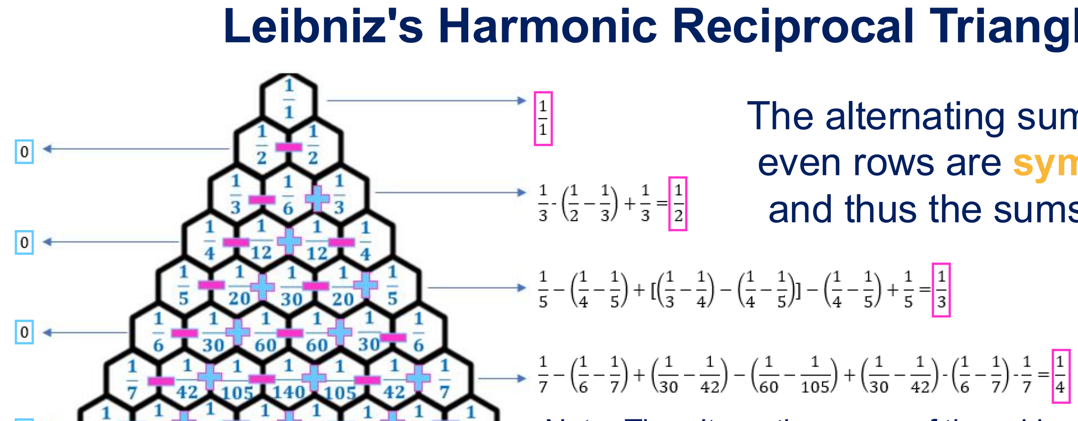
Alternating Sums



Pascal's Arithmetic Triangle

The alternating sums of Pascal's Arithmetic Triangle are symmetric and thus the sums are 0

Leibniz's Harmonic Reciprocal Triangle

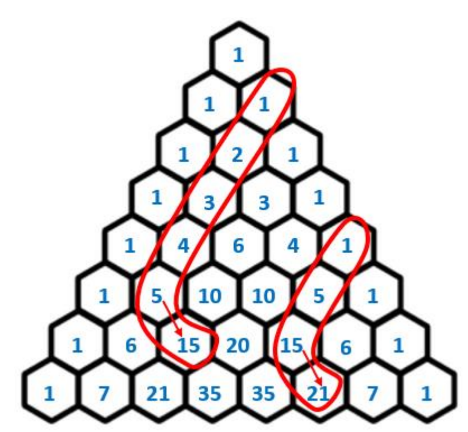


The alternating sums of the even rows are symmetric and thus the sums are 0

Note: The alternating sums of the odd rows are 1/n where n is the nth odd row

Hockey Stick and Reverse Hockey Stick Theorems

Pascal's Arithmetic Triangle

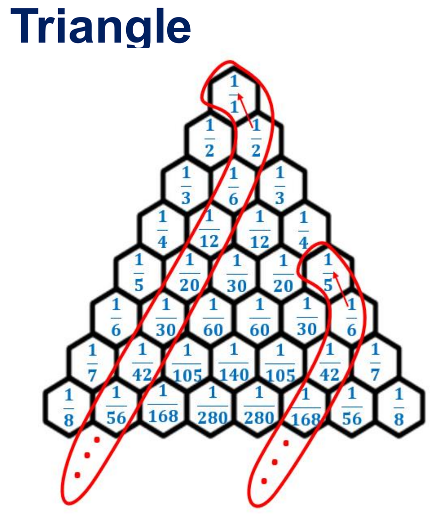


Hockey Stick Theorem
The sum of the terms going down across a finite diagonal is equal to the term below it in the opposite direction

Examples
1+2+3+4+5 =15
1+5+15=21

Leibniz's Harmonic Triangle

Reverse Hockey Stick Theorem
The sum of the terms going across an infinite diagonal is equal to the term above it in the opposite direction



Examples

$$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\ldots=\frac{1}{1}$$

$$\frac{1}{1*2}+\frac{1}{2*3}+\frac{1}{3*4}+\frac{1}{4*5}+\ldots=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\frac{1}{1}$$

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots=\sum_{n=1}^{\infty}\frac{1}{n}$$ → Reverse Hockey Stick Theorem → $\lim_{n\to 0}\frac{1}{n}=\infty$ → diverges (goes off to infinity)

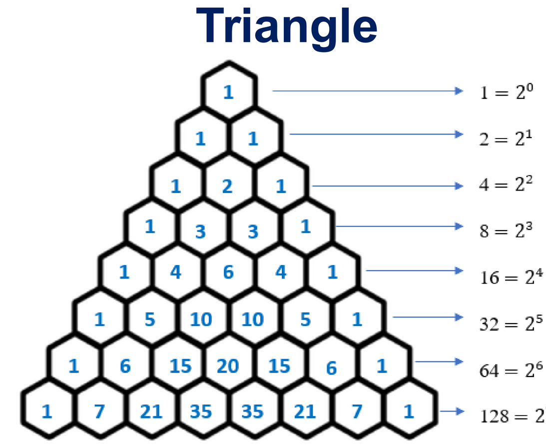
$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n}$ → diverges due to the p-series test

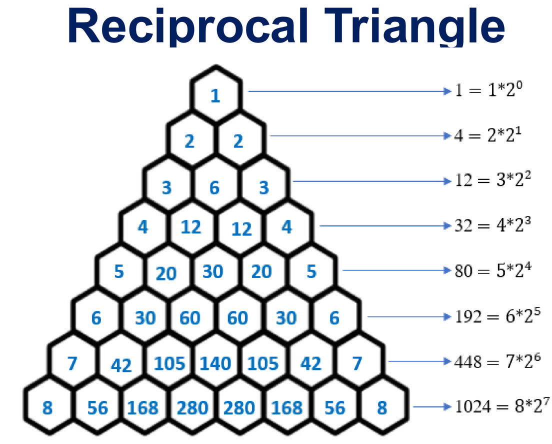
p-series test $\sum_{n=1}^{\infty}\frac{1}{n^p}$ Converges when $p>1$ Diverges when $p\le 1$ for all positive p-values

Reciprocal Row Sums

The row sums of Leibniz's Harmonic Reciprocal Triangle are scalar multiples of the row sums of Pascal's Arithmetic Triangle and their row numbers

Pascal's Arithmetic Triangle	Leibniz's Harmonic Reciprocal Triangle
$1 = 2^0$	$1 = 1*2^0$
$2 = 2^1$	$4 = 2*2^1$
$4 = 2^2$	$12 = 3*2^2$
$8 = 2^3$	$32 = 4*2^3$
$16 = 2^4$	$80 = 5*2^4$
$32 = 2^5$	$192 = 6*2^5$
$64 = 2^6$	$448 = 7*2^6$
$128 = 2^7$	$1024 = 8*2^7$

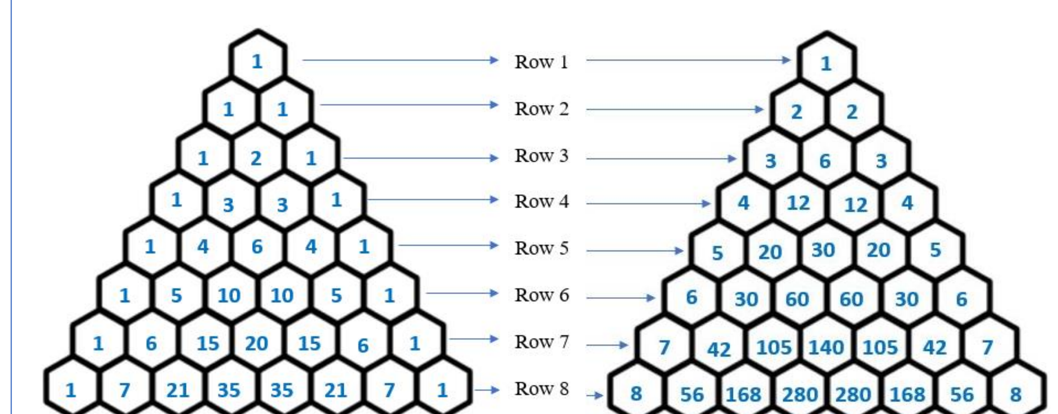




Reciprocal Scalar Multiples

The product of each term in Pascal's Triangle and its corresponding row number yield Leibniz's Reciprocal Triangle

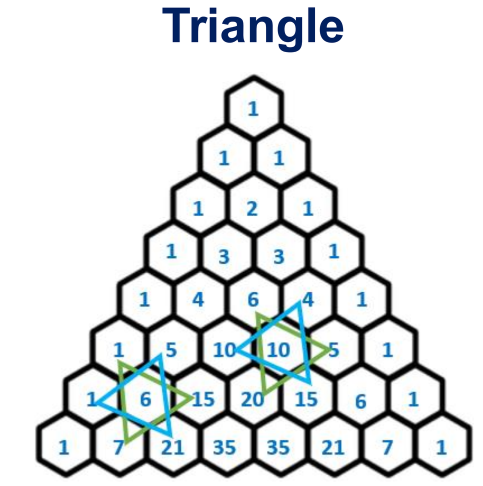
Pascal's Arithmetic Triangle → Leibniz's Reciprocal Harmonic Triangle



Hexagon Property

The product of any three non-adjacent values surrounding and individual term is equivalent to the product of the three other non-adjacent values

Pascal's Arithmetic Triangle

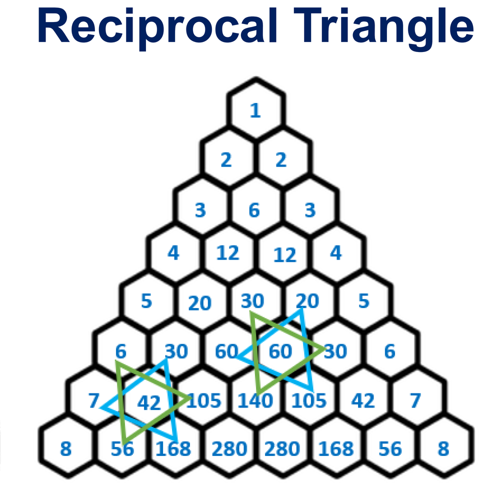


1*5*21 = 1*7*15
105 = 105

10*4*15 = 6*5*20
600 = 600

Additionally, the products in the hexagon property of Leibniz's Harmonic Triangle is directly proportionate to the scalar multiple of the three rows that construct each value to Pascal's Arithmetic Triangle

Leibniz's Harmonic Reciprocal Triangle



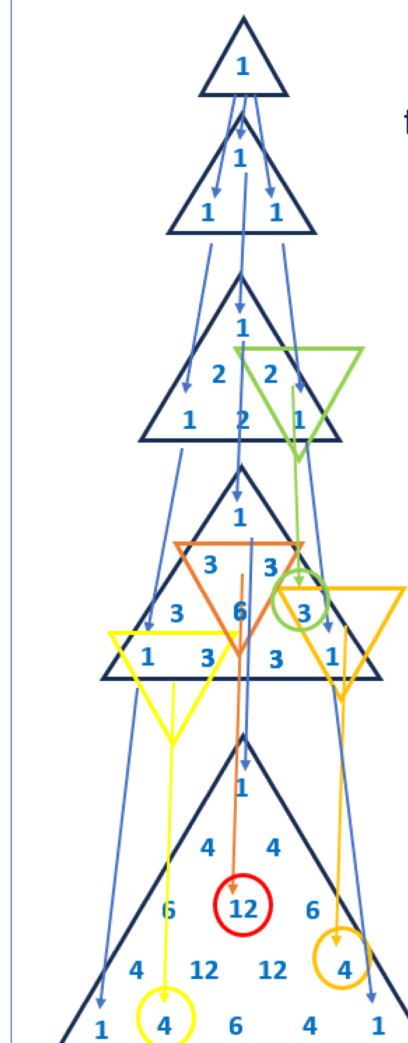
7*30*168 = 6*105*56
35280 = 35280
divide out product of scalar multiples
35280 = 35280
6*7*8 6*7*8
105 = 105

60*20*105 = 30*30*140
126000 = 126000
divide out product of scalar multiples
126000 = 126000
5*6*7 5*6*7
600 = 600

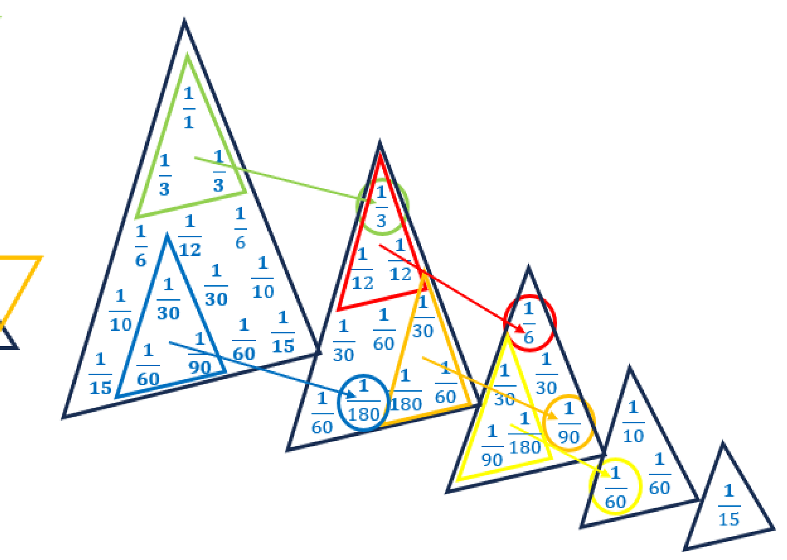
Constructions

Arithmetic Tetrahedron

Starting with the value of 1 at the top
Each subsequent row is the sum of its three corresponding diagonal numbers from above



Harmonic Tetrahedron



Starting with the unit triangular fractions going down the left most diagonal
Each subsequent term is the difference between the number above and the two numbers adjacent

Reflection

Pascal's Arithmetic Triangle utilizes the sum of two corresponding diagonal numbers from above, whereas Leibniz's Harmonic Triangle utilizes the sum of two corresponding diagonal numbers from below. It is natural the two triangles contain parallel relationships due to mathematical properties of equality. The **Hockey Stick Theorem** produces a sum equal to a value in the diagonal entry in the opposite direction, while the **Reverse Hockey Stick Theorem** produces an infinite sum equal to a value in the diagonal **entry** in the opposite direction. The rows of Pascal's Triangle and the even rows of Leibniz's Harmonic Triangle are symmetric, so their **Alternating Row Sums** are 0. Since Leibniz's Reciprocal Triangle and Pascal's Triangle are scalar multiples of one another, the **Hexagon Property** and **Reciprocal Row Sums** are scalar multiples of one another. Based on the properties of these two triangles, we can construct a **Harmonic Tetrahedron** from the **Arithmetic Tetrahedron** utilizing associated scalar multiples. **Future investigation** involves extending these known relationships to discover related properties of both three-dimensional figures.

Future Implications

Research explorations with the Harmonic Tetrahedron

- Recursive Form
- Row Sums
- Hexagon Property
- Hockey Stick Theorem
- Closed Form
- Alternating Row Sums
- Telescoping Series
- Reverse Hockey Stick Theorem
- Additional Dimensions

Acknowledgements

I would like to extend my sincere thanks to Dr. Richard Grassl and Dr. Robert Powers for your guidance as my thesis advisor and research mentor.