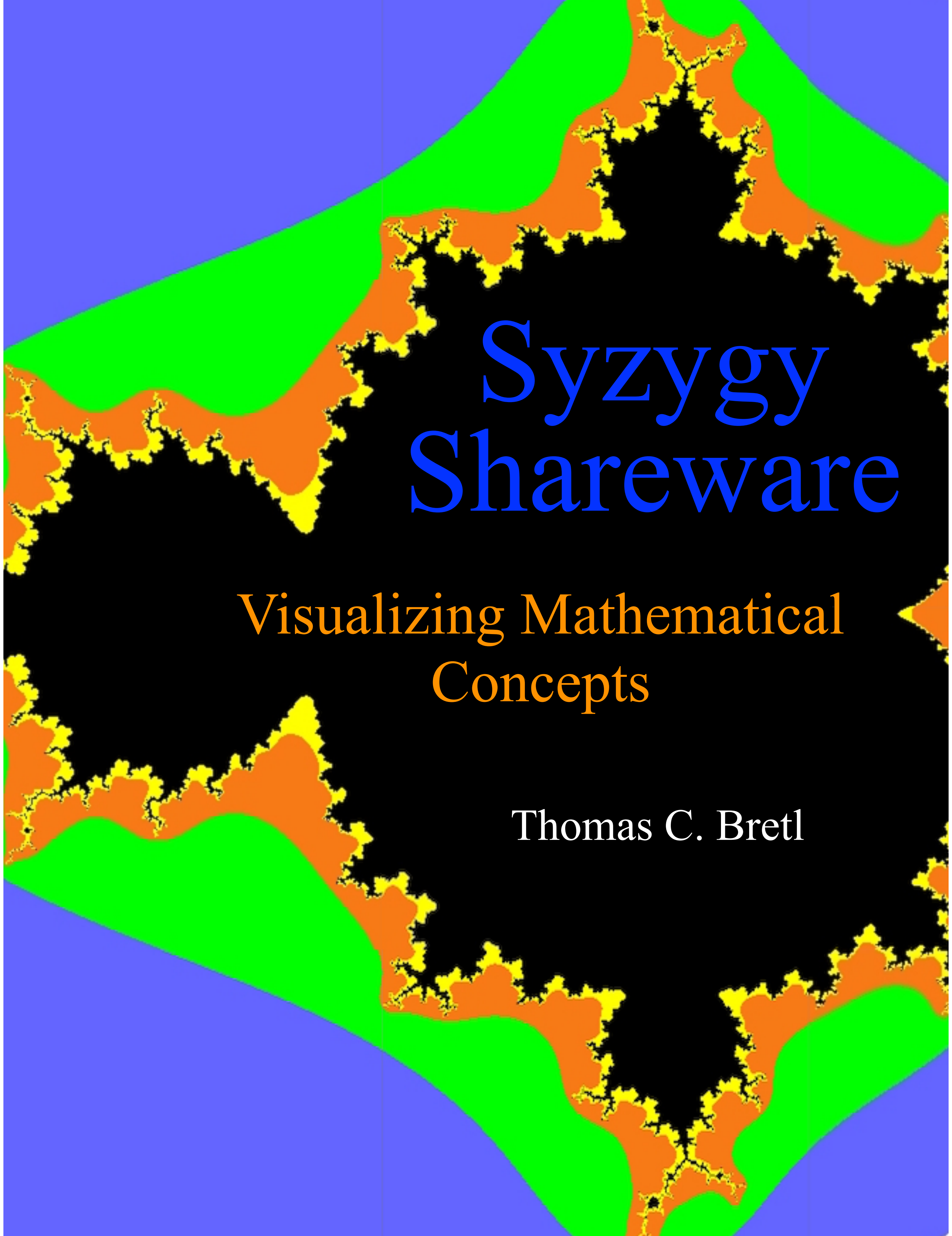
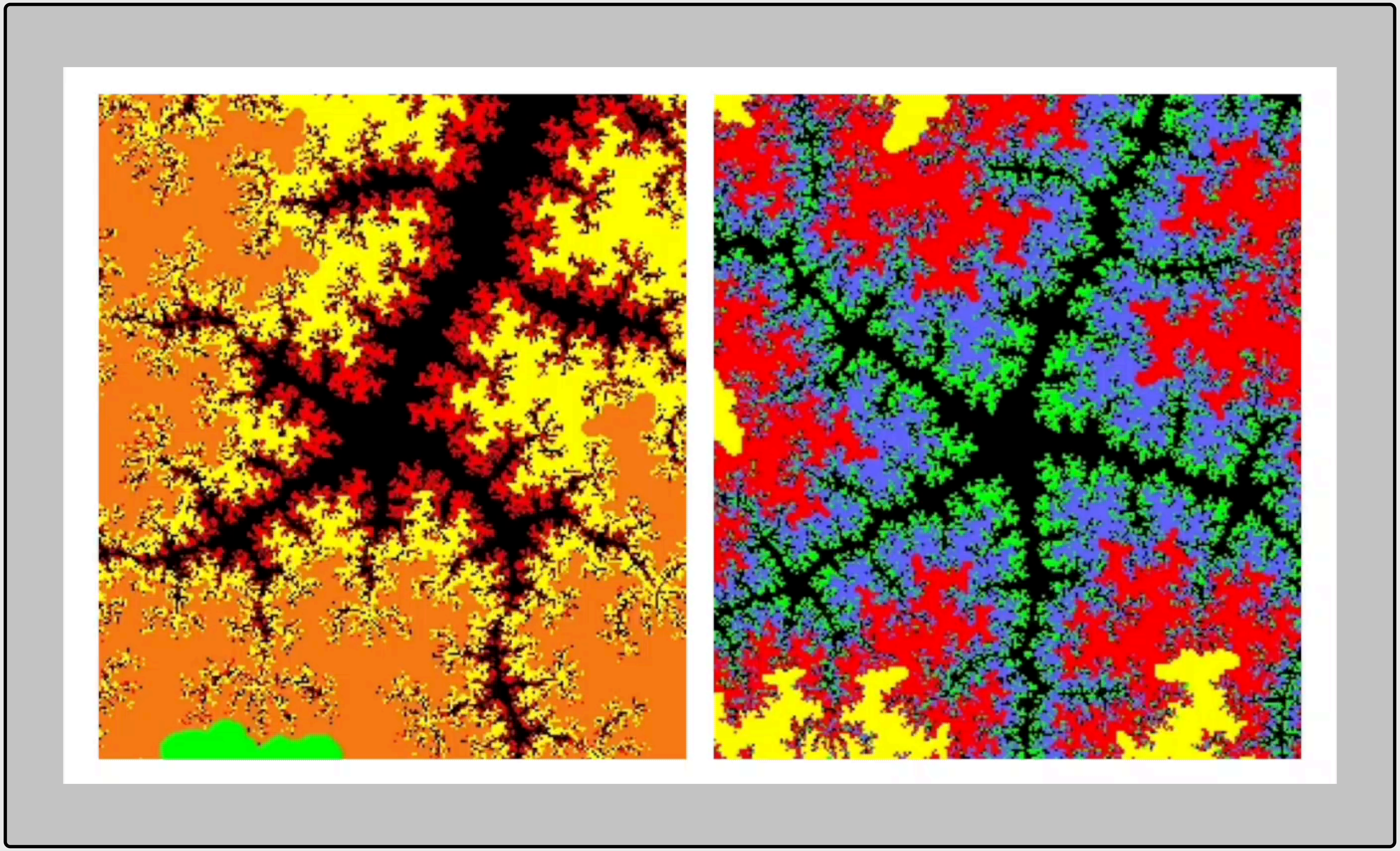


# Syzygy Shareware

## Visualizing Mathematical Concepts

Thomas C. Bretl

# 1. About this Book



# Copyright & Contact Info





For further information about both the programs discussed in this book and other programs I have written, visit my Syzygy Shareware blog at reckonsupport.blogspot.com. You can leave comments or suggestions either there or at tcbretl.weebly.com. Thanks to LiveCode for creating the software I used to write my programs, and thanks to iBooks Author for making it relatively easy to write and publish a book.

# Preface

Although it can certainly stand by itself, the purpose of this book is to serve as a resource for users of my math and science computer programs. The programs can be obtained for free from my **Syzygy Shareware** site at http://tcbretl.weebly.com.

My software is designed to help and encourage students to explore, see in new ways, discover patterns and solve problems. The target age group ranges from elementary school through college. I have tried to provide challenges that fall on the fine line between too easy (boring) and too hard (frustrating), at a level where the student can become deeply, completely, and actively involved.

In 1985 I attended a talk given by Dr. Paul MacCready at a national education conference. He told the story of how he and his team designed the Gossamer Condor and won the Kremer Prize for developing the first human-powered aircraft. He thought he was able to succeed partly because he did not know how to design airplanes! A human-powered aircraft required a new approach to aircraft design.

Near the end of his talk, Dr. MacCready asked himself how his experience might in some way relate to education. He answered by saying he thought we did a good job at teaching students how to solve problems that have already been solved, but perhaps could do a better job preparing students to solve problems that have yet to be solved. I hope that my shareware in some way helps in that preparation.

# Programs

**Arithmetic Quilts** - a program that creates mod arithmetic tables using colors instead of numbers. Beautiful and intricate patterns result.

**Calc Visualizer** - a program that allows you to explore basic calculus concepts and visualize what differential and integral calculus are all about.

**Probability Simulator** - a program that flips coins, rolls dice, spins spinners, and presents some challenging probability problems.

**Celestial Dances** - a program that simulates the relative motions of the sun, planets, and moons, revealing all sorts of intricate geometric patterns.

**SquArray** - a program that presents both numbers and symbols in square arrays, allowing you to play with patterns, and challenging you to create magic squares.

**Stepwising** - a program that makes recursive stepwise approximations of the motions of stars, planets, and satellites.

**Life** - a recursive simulation based on Conway's Game of Life.

**Mandelbrot Set** - a program that uses the recursive definition of the Mandelbrot Set to make color images of it.

**Dot-dots** - a program which creates Sierpinski's triangle and other fractals via a recursive point plotting process.

**Algernon** - a program that challenges young students to guide a mouse through a maze.

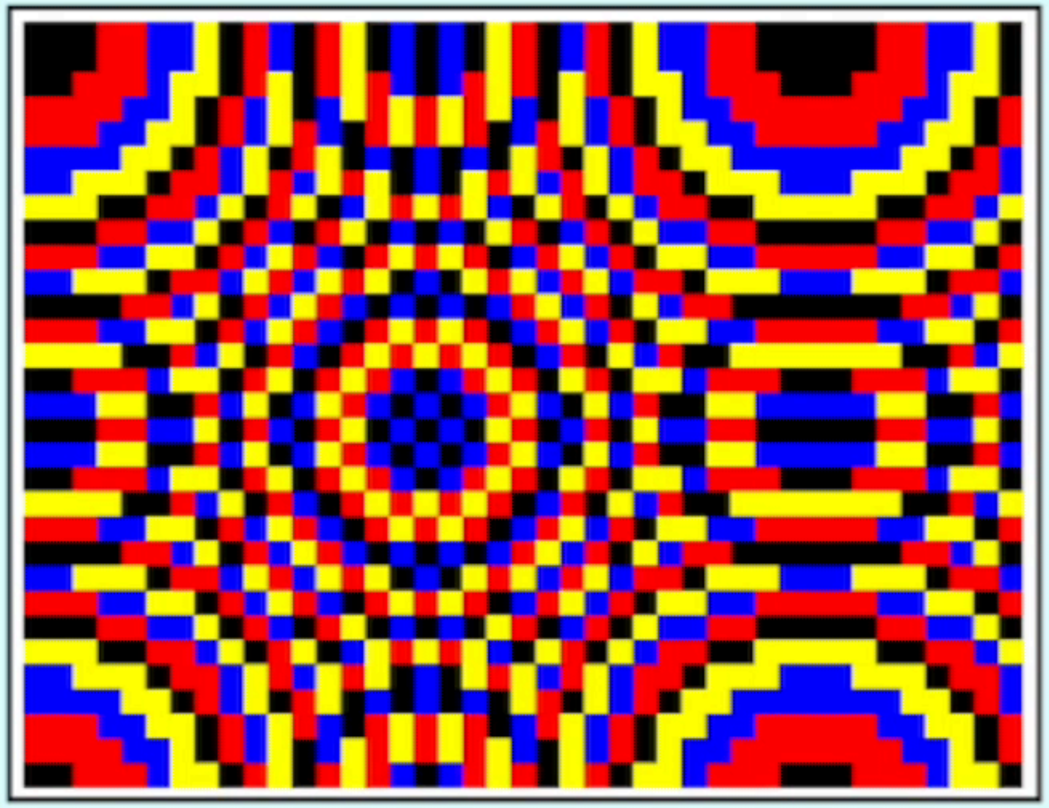
**ArithmeDarts** - a dart game in which you aim by estimating the coordinates of the target balloons.

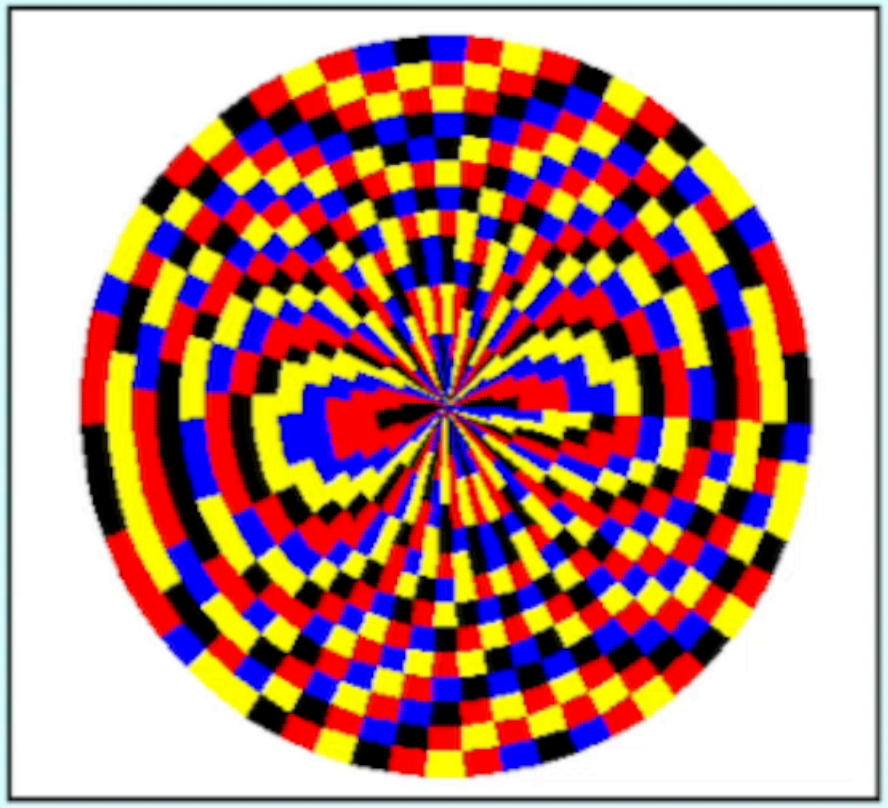
**Reckonings** - a mathematical equation game in which you combine four given numbers to equal one of nine target numbers.

**FactorMan** - a factoring game played against either FactorMan or another human being.

# Contents

# 2. Arithmetic Quilts





## 2.1 Introduction

In the summer of 1963 I attended an NSF Summer Science Training program at Stevens Institute of Technology in Hoboken, NJ. I learned to program an IBM mainframe computer (using punch cards) and attended lectures about linear algebra.

One day, during a class break, the student sitting next to me asked whether I had ever made mod arithmetic tables using colors or symbols in place of numbers. I had not, but I was interested.

Years later, as desktop computers became available, I wrote programs to create those same mod arithmetic tables. It was fun to vary the mod number, adjust the colors, change the rule for combining the numbers, and investigate the beautiful patterns that were produced as a result.

I have written and shared many versions of my **Arithmetic Quilts** program. Several of those versions are available for free from my **Syzygy Shareware** site at http://tcbretl.weebly.com.

This chapter gives some background information about mod arithmetic and how it relates to groups and rings. Included are many colorful examples of the patterns you can create using **Arithmetic Quilts**.

## 2.2 Mod Arithmetic Tables

Do you remember making arithmetic tables in elementary school? The boldface numbers in the first row are added to (or multiplied by) the boldface numbers in the first column. Each answer is placed in the cell that is in the same row and the same column as the numbers being added (or multiplied).

The first table to the right shows all of the addition facts for the integers 0 through 9, and the second table shows all of the multiplication facts for the integers 0 through 9. Each of these tables could, of course, include many more rows and columns, thus showing the arithmetic facts for much larger numbers.

Addition Table

| **+** | **0** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** |
|---|---|---|---|---|---|---|---|---|---|---|
| **0** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **1** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| **2** | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| **3** | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| **4** | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| **5** | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| **6** | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| **7** | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| **8** | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
| **9** | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |

Multiplication Table

| **x** | **0** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** |
|---|---|---|---|---|---|---|---|---|---|---|
| **0** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **1** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **2** | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| **3** | 0 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 |
| **4** | 0 | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 |
| **5** | 0 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 |
| **6** | 0 | 6 | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 |
| **7** | 0 | 7 | 14 | 21 | 28 | 35 | 42 | 49 | 56 | 63 |
| **8** | 0 | 8 | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 |
| **9** | 0 | 9 | 18 | 27 | 36 | 45 | 54 | 63 | 72 | 81 |

Mod arithmetic tables work in exactly the same way, but they are limited to a finite set of non-negative integers. Mod 5 arithmetic, for example, is limited to the integers 0 through 4. Both the numbers being added or multiplied, and the answers, must all be between 0 and 4. Numbers larger than 4 are "equivalent" to numbers between 0 and 4.

| | |
|---|---|
| 5 is the same as 0 | 5 = 0 mod 5 |
| 6 is the same as 1 | 6 = 1 mod 5 |
| 7 is the same as 2 | 7 = 2 mod 5 |
| ... | |

You can convert any given positive integer to its mod 5 equivalent by dividing that integer by 5 and finding the remainder. Since the remainder is an integer that is smaller than the divisor, this method always gives a result that is between 0 and 4.

**Examples:**

8 divided by 5 equals 1 with a remainder of 3, so 3 replaces any 8's in a mod 5 table.
14 divided by 5 equals 2 with a remainder of 4, so 4 replaces any 14's in a mod 5 table.

The tables shown below are the mod 5 equivalents to the arithmetic tables shown on the previous page.

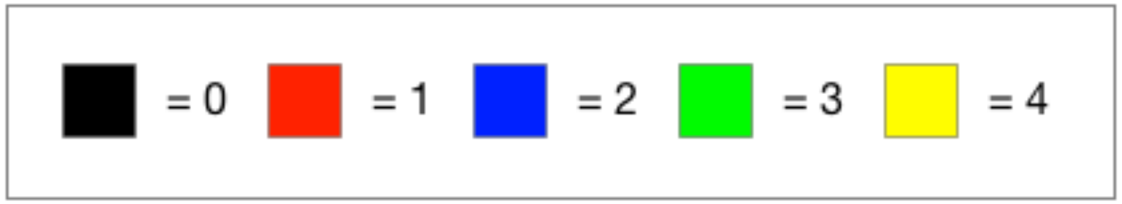
Addition Mod 5

| + | 0 | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 3 | 4 |
| 1 | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 3 | 4 | 0 |
| 2 | 2 | 3 | 4 | 0 | 1 | 2 | 3 | 4 | 0 | 1 |
| 3 | 3 | 4 | 0 | 1 | 2 | 3 | 4 | 0 | 1 | 2 |
| 4 | 4 | 0 | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 3 |
| 0 | 0 | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 3 | 4 |
| 1 | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 3 | 4 | 0 |
| 2 | 2 | 3 | 4 | 0 | 1 | 2 | 3 | 4 | 0 | 1 |
| 3 | 3 | 4 | 0 | 1 | 2 | 3 | 4 | 0 | 1 | 2 |
| 4 | 4 | 0 | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 3 |

Multiplication Mod 5

| x | 0 | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 3 | 4 |
| 2 | 0 | 2 | 4 | 1 | 3 | 0 | 2 | 4 | 1 | 3 |
| 3 | 0 | 3 | 1 | 4 | 2 | 0 | 3 | 1 | 4 | 2 |
| 4 | 0 | 4 | 3 | 2 | 1 | 0 | 4 | 3 | 2 | 1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 3 | 4 |
| 2 | 0 | 2 | 4 | 1 | 3 | 0 | 2 | 4 | 1 | 3 |
| 3 | 0 | 3 | 1 | 4 | 2 | 0 | 3 | 1 | 4 | 2 |
| 4 | 0 | 4 | 3 | 2 | 1 | 0 | 4 | 3 | 2 | 1 |

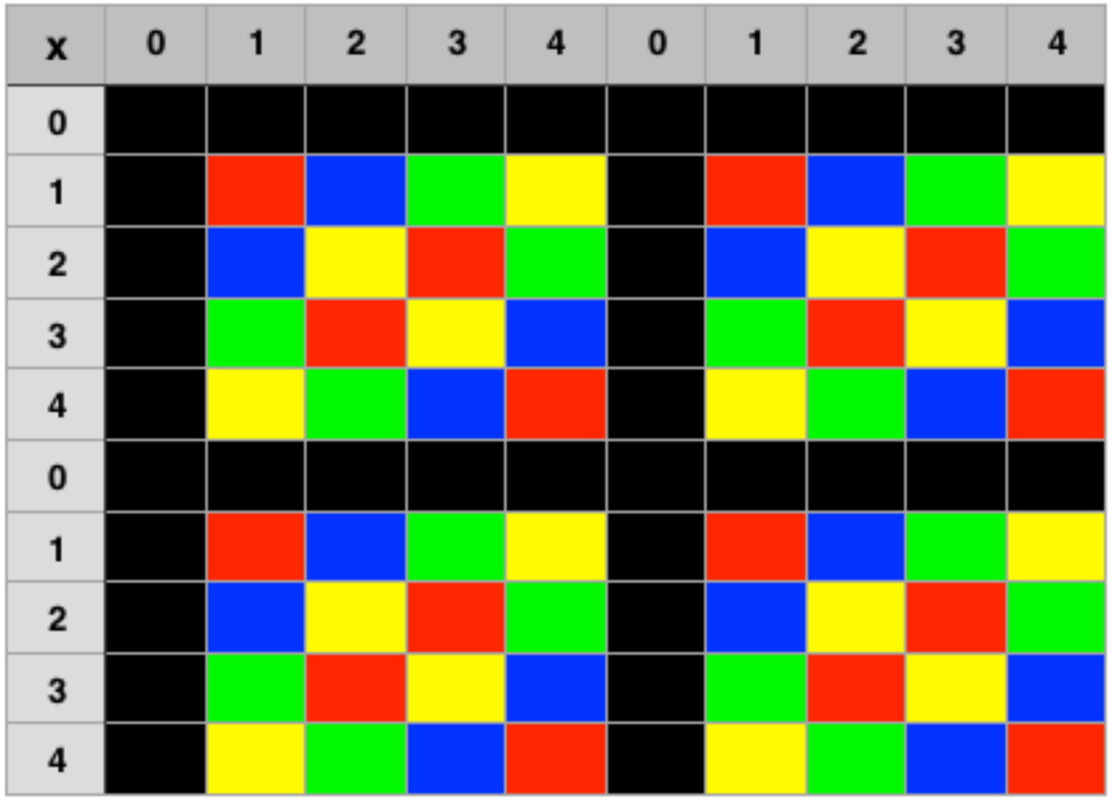
Now let's replace the numbers with colors, using this color code.




Using colors makes the patterns in the tables stand out more clearly. Addition is characterized by diagonals, and multiplication is characterized by square boxes bordered in black 0's.

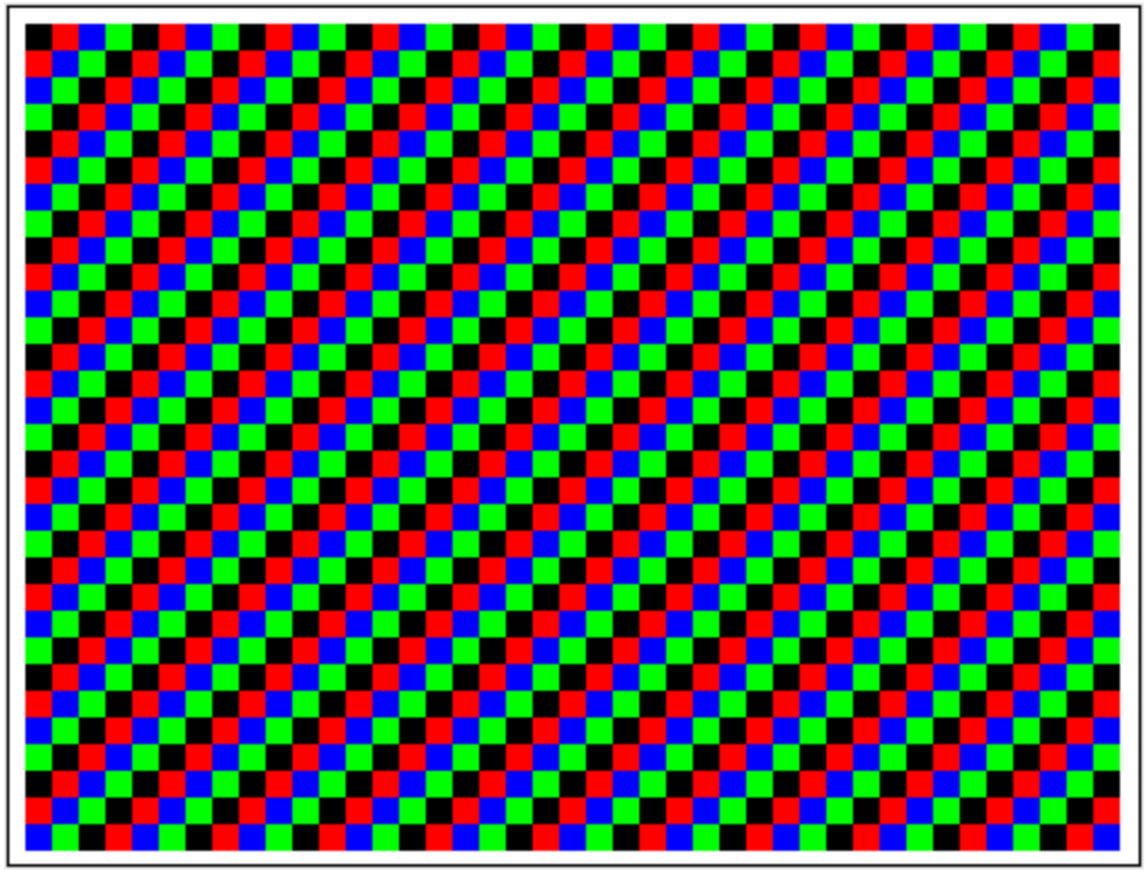
Addition Mod 5

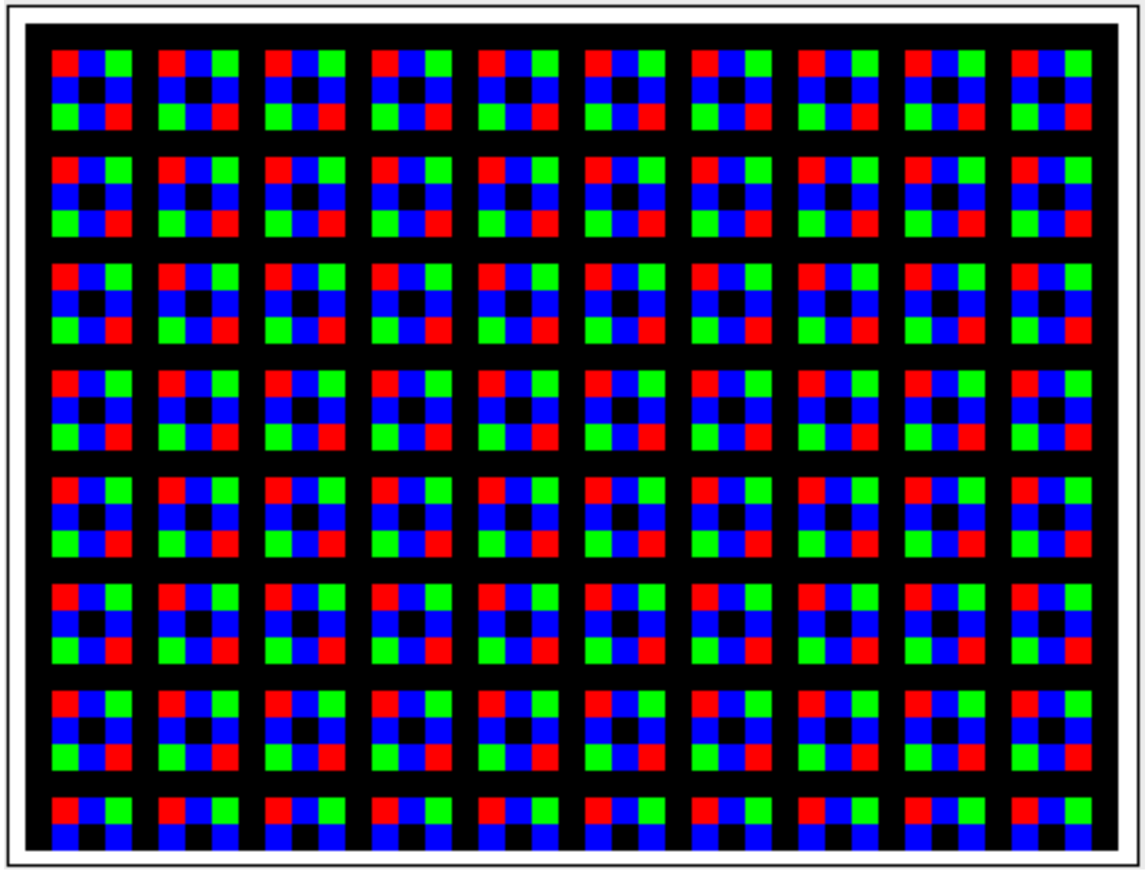

Multiplication Mod 5



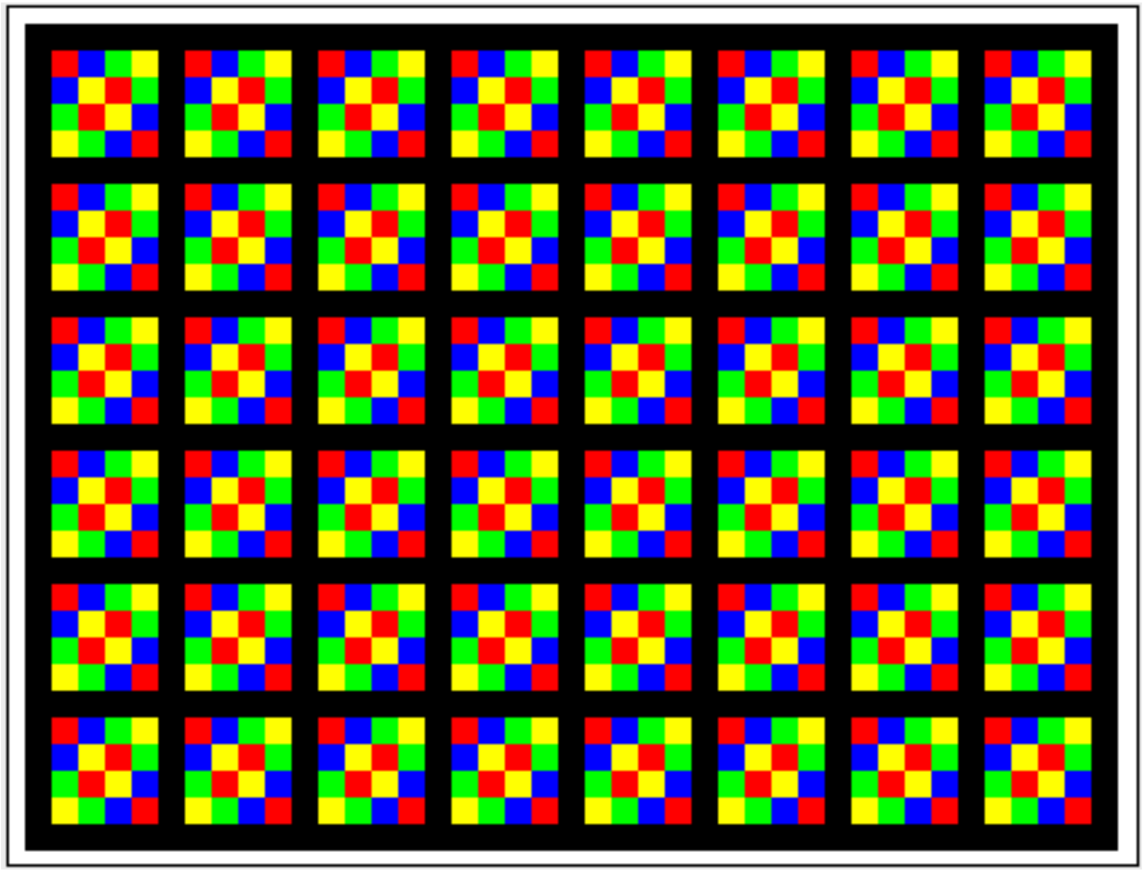

Extending the tables to include more rows and columns makes the patterns stand out even more. (see next page).

Addition and multiplication mod 4

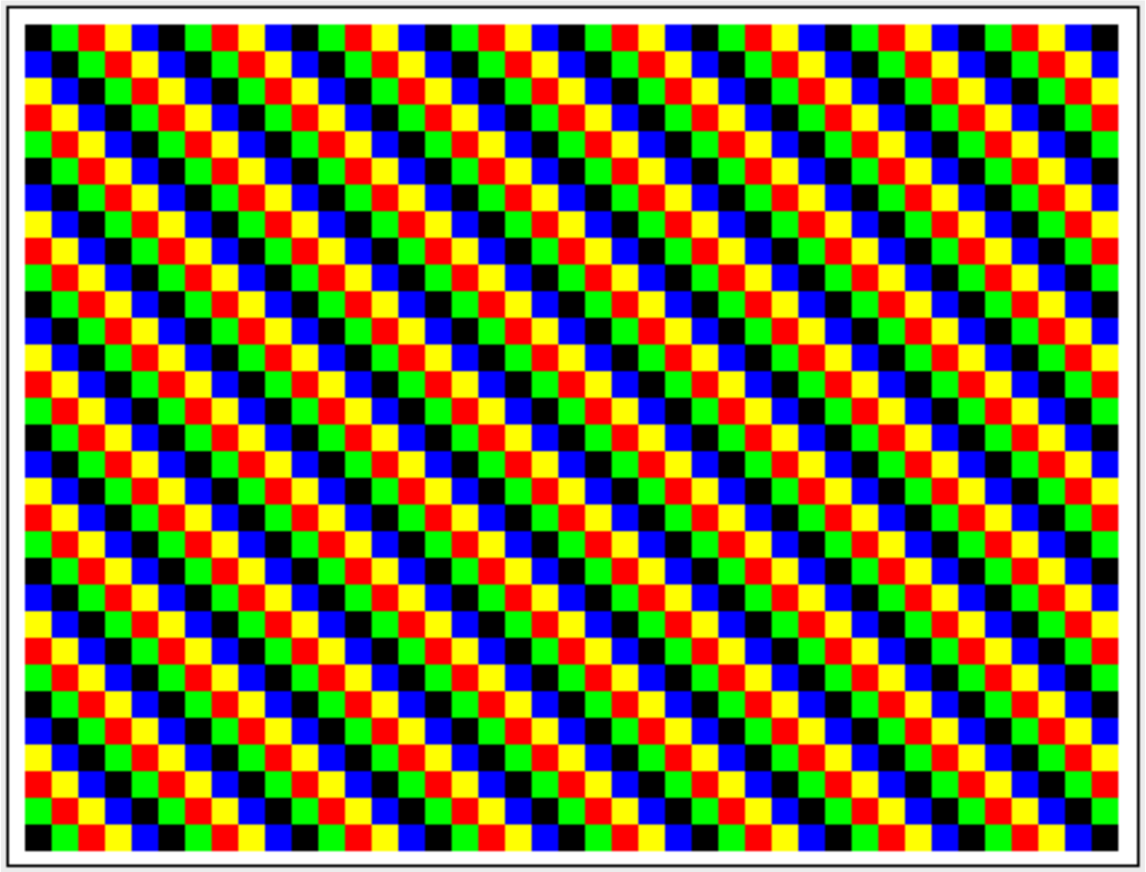




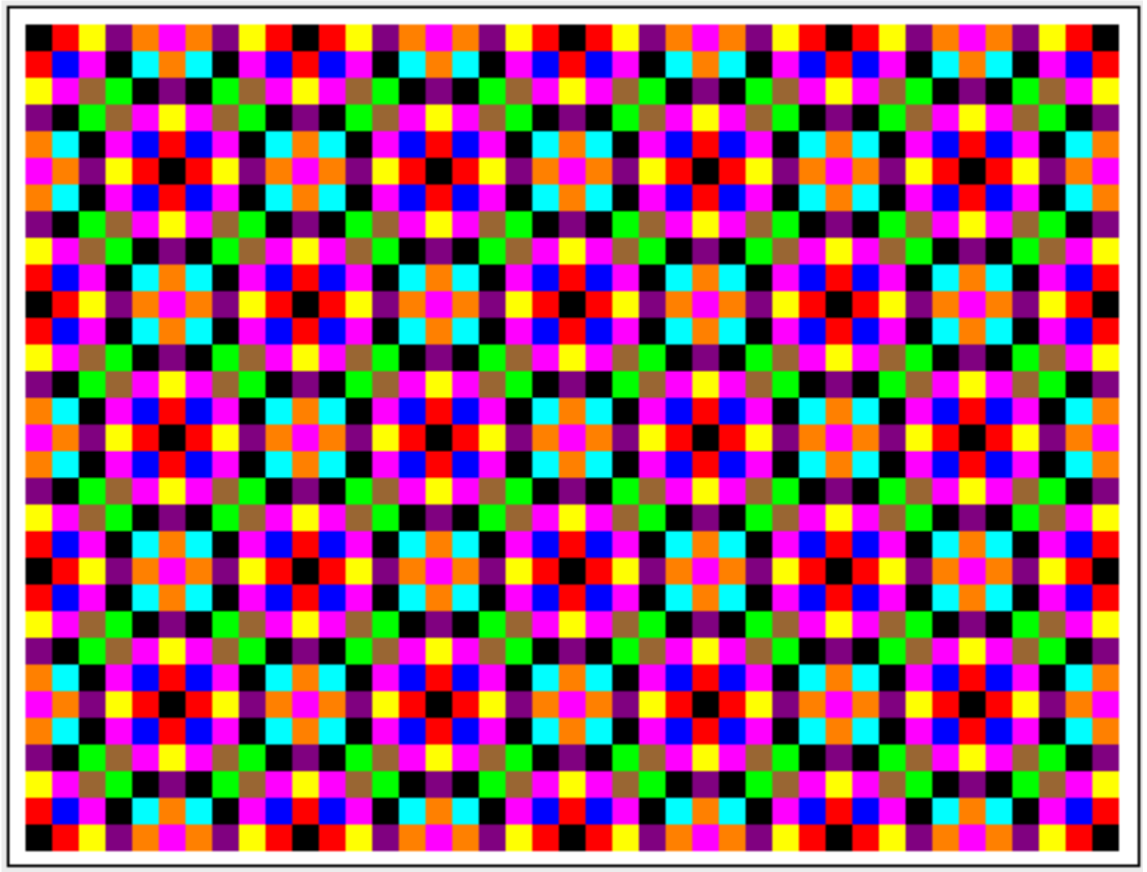
Addition and multiplication mod 5



Mod arithmetic tables are not limited to simple addition and multiplication. For example, you can define new “operations” that involve a combination of addition and multiplication. In these two examples, “•” is used to symbolize the operation.



**Example 1**: x • y = 3x + 2y in mod 5



**Example 2**: x • y = $x^2 + y^2$ in mod 10

## 2.3 Groups and Rings

A **group** is defined as a set of numbers together with an operation (symbolized here by +) for combining those numbers. The + operation can be, but certainly does not have to be, "ordinary addition." Four conditions must be satisfied:

1. If a and b are numbers in the set, then a+b must be a number in the set. This is called **closure**.
2. If a, b, and c are numbers in the set, then (a+b)+c = a+(b+c). This is called the **associative property** of the + operation.
3. There exists a number 0 in the set such that 0+n = n, for all numbers n in the set. The number 0 is called the i**dentity element** for the + operation.
4. For every number n in the set, there exists a unique number m in the set such that n+m = 0, where 0 is the identity element defined in condition #3. The number m is called the **inverse** of the number n.

The integers between 0 and 4, under the operation of mod 5 addition, form a group (usually designated as **Z5**).

When you add two numbers in **Z5**, you get another number in the same set (0, 1, 2, 3, or 4).

(a+b)+c equals a+(b+c), no matter what numbers in the set you use for a, b, and c.

0 is the identity element. When you add 0 to any of the numbers in the set, you simply get that other number.

Each number in the set has an inverse satisfying condition 4. 1+4=0, 2+3=0, 3+2=0, 4+1=0, and 0+0=0.

------------------------------------------------------------------

A group is called **abelian** if one more property, the commutative property, is also satisfied: If a and b are any two numbers in the set, then a+b=b+a. **Z5** is abelian.

A group is called **cyclic** if all of its elements can be generated from just one of its elements. **Z5** is cyclic; all of its elements can be generated from the number 1:

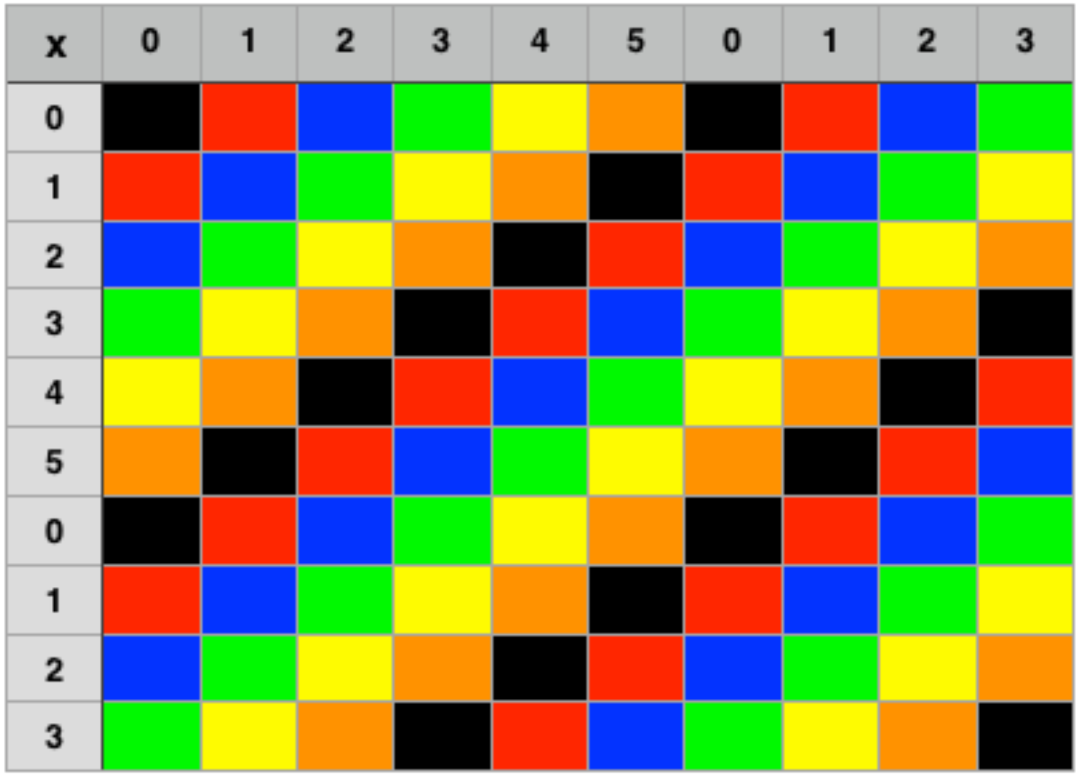
1+1=2, 2+1=3, 3+1=4, 4+1=0.

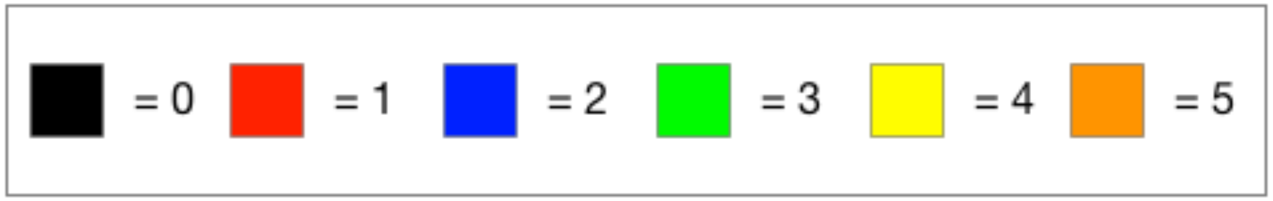
------------------------------------------------------------------

A subset S of a group G is called a **subgroup** of G if S itself is a group under the same operation as the group G. **Z5** has no subgroups except for the trivial one that consists of just one element, 0.

**Z6**, however, has two non-trivial subgroups: S1={0,2,4} and S2={0,3}.

## Addition Mod 6

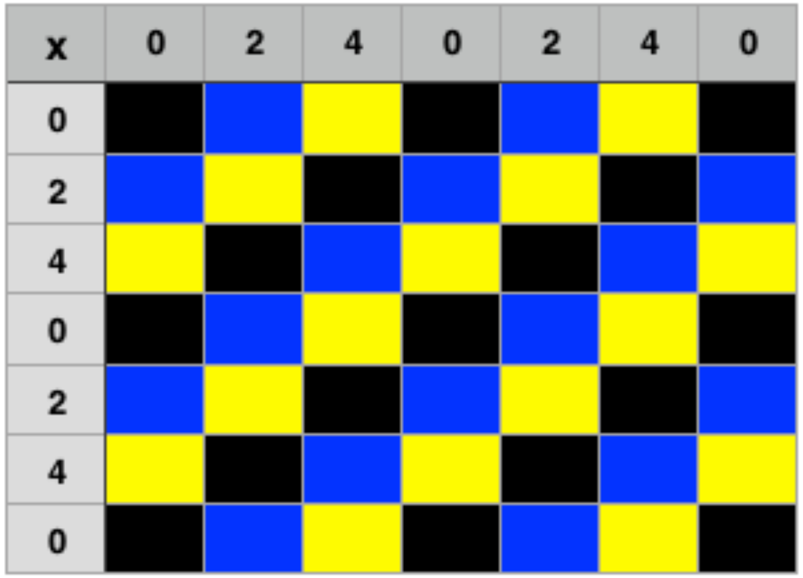



If you add two numbers that are in set S1, you always get a number that is also an element of S1. If you add two numbers that are in set S2, you always get a number that is also an element of S2.

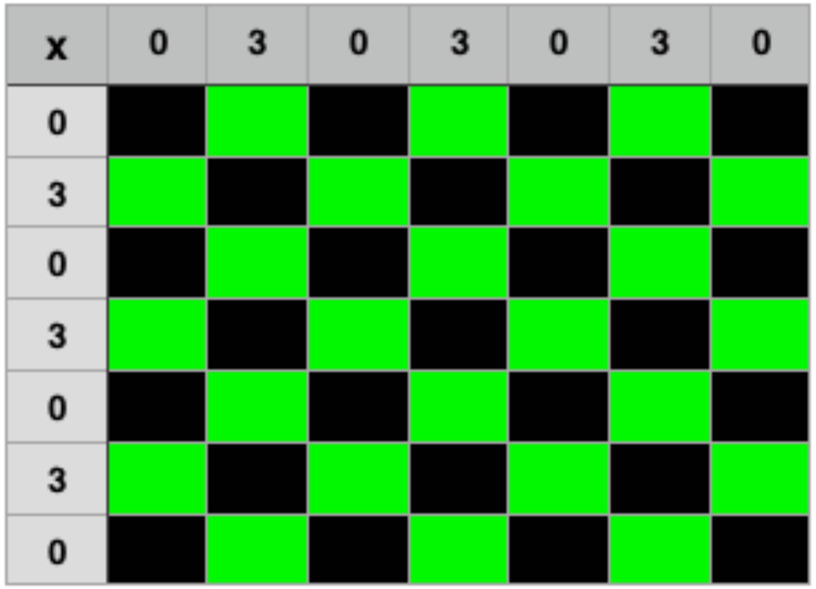
Note that both S1 and S2 are cyclic subgroups. S1 is generated by the number 2, and S2 is generated by the number 3.

## S1 Addition Mod 6




0 + 2 = 2, 2 + 2 = 4 and 4 + 2 = 0

## S2 Addition Mod 6




0 + 3 = 3 and 3 + 3 = 0

# Rings

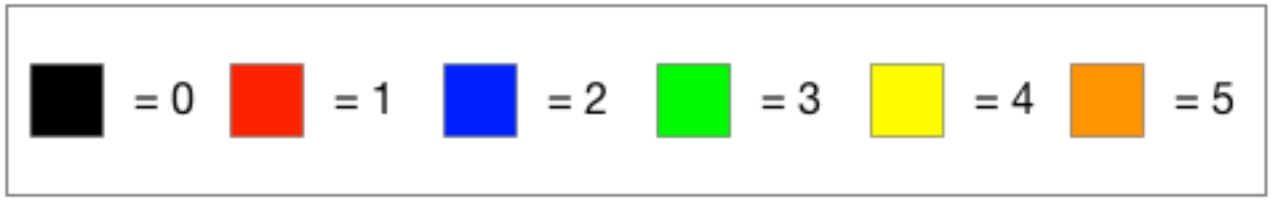
A **ring** is defined as a set of numbers together with two operations (here denoted by + and •), for combining those numbers. The operations can be, but do not have to be, ordinary addition and multiplication. Three conditions must be satisfied:

1. The set is an abelian group under the + operation.
2. If a, b, and c are numbers in the set, then a•(b•c)=(a•b)•c. This is the associative property for the • operation.
3. If a, b, and c are numbers in the set, then a•(b+c)=a•b+a•c and (b+c)•a=b•a+c•a. This is called the distributive property.

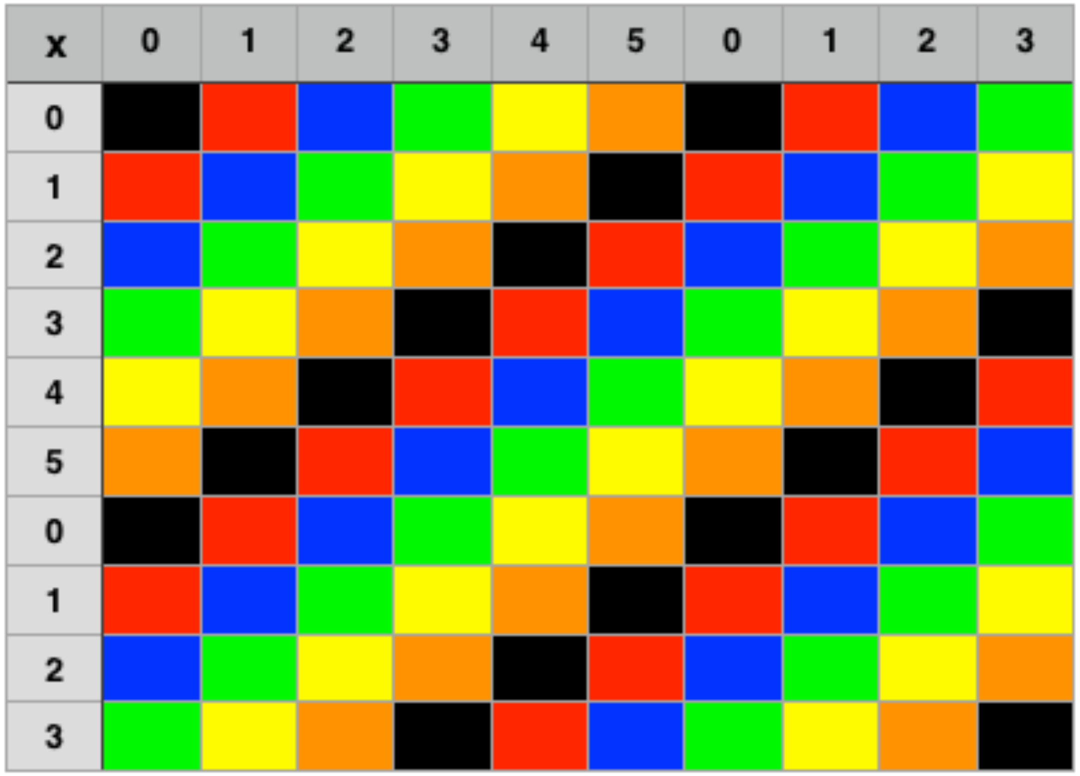
**Z6** is a nice example of a ring. The + operation is mod 6 addition, and the • operation is mod 6 multiplication.

Notice how the subgroups of Mod 6 addition show up in the multiplication table as the multiples of 2 (blue, yellow, and black), 3 (green and black), and 4 (blue, yellow, and black).

Also, notice that 2•3=0 mod 6, and 4•3=0 mod 6. The numbers 2, 3, and 4 are called zero divisors in the ring **Z6**.




Addition Mod 6



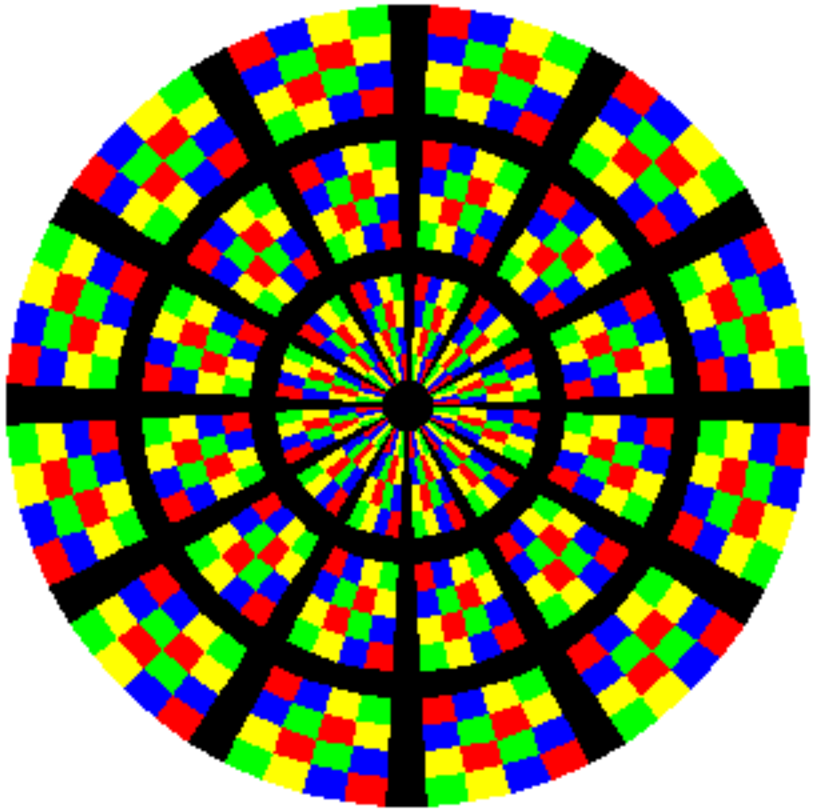

Multiplication Mod 6

## Quilts based on Polar Coordinates

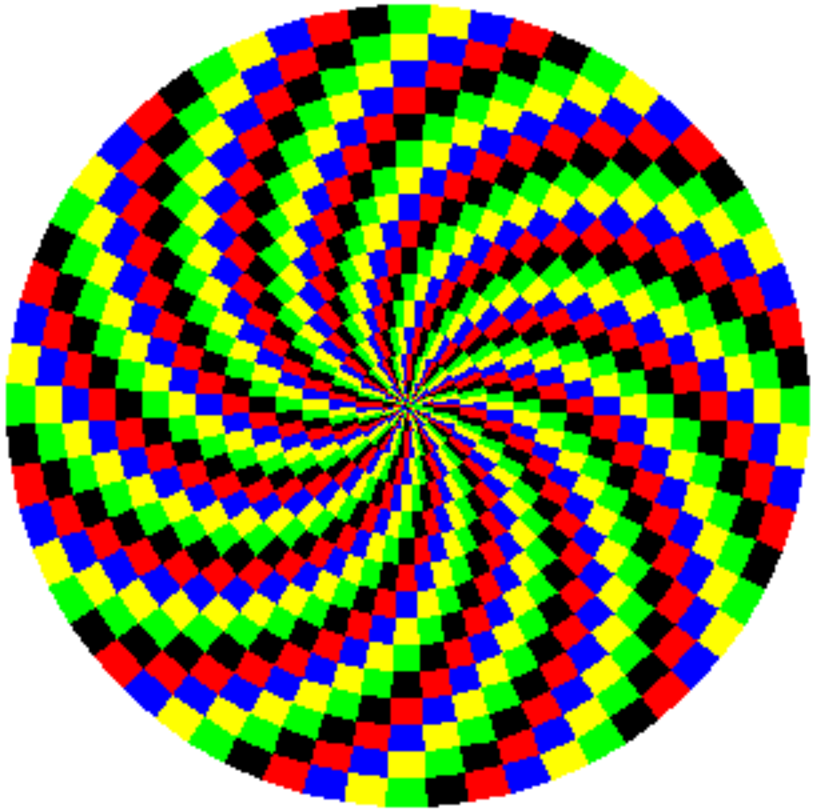
You can create arithmetic quilts using polar coordinates instead of rectangular coordinates. One variable is the distance out from the center, and the other variable is the angle, measured counter-clockwise, from the x-axis.

In these examples, the distances from the center range from 0 to 14, and the angles range from 0 to 360 in multiples of 6.

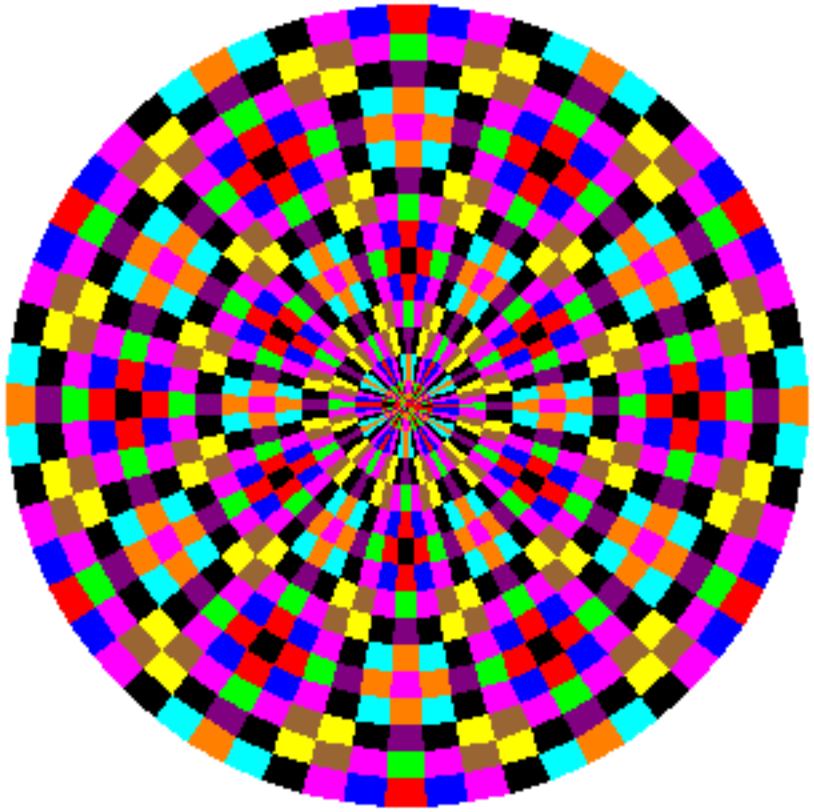
$x \cdot y$
mod 5



$x + y$
mod 5



$x^2 + y^2$
mod 10



## 2.4 Program Activities

Use the **Arithmetic Quilts** program to help answer the following questions:

1. Set the quilt operation to x+y. Try a variety of mod values. Regardless of the mod, what sort of pattern do you see?
2. Set the quilt operation to xy. Try a variety of mod values. Regardless of the mod, what sort of pattern do you see?
3. Compare the xy mod 5 pattern to the xy mod 6 pattern. In what ways are they similar? In what ways are they different?
4. Try mod 3, 4, 7, 8, 9, and 10. Which ones are more like mod 5 and which ones are more like mod 6?
5. Set the quilt operation to x+xy. Try different mods. In what ways are the resulting patterns similar to the patterns for xy? In what ways are they different?
6. Set the quilt operation to y+xy. Try different mods. In what ways are the resulting patterns similar to the patterns for xy? In what ways are they different?
7. Make the expression equal to $x^2+y^2$. Try different mods. In what ways are all of the resulting patterns similar to each other?

Sometimes it is easier to understand a quilt pattern if you first factor the expression that is being evaluated.

1. Set the quilt operation to 2x+2y. How does the resulting quilt compare to that for x+ y?
2. Set the quilt operation to 3x+3y instead. Now how does the resulting quilt compare to that for x+y?
3. Try 4x+4y, 5x+5y, and 6x+6y as expressions. In each case, how does the resulting quilt compare to that for x+y? To understand these results, it may help to think of the expressions in their factored form
4. Use what you have learned from problems 1-3 to predict how many different colors will appear in the following quilts:

   a. mod 8 with 2x+2y as the operation
   b. mod 9 with 3x+3y as the operation
   c. mod 10 with 4x+4y as the operation
   d. mod 7 with nx+ny as the operation (where n is any positive whole number)

5. Earlier you used x+xy as the operation and compared the resulting quilt to the quilt for xy. Thinking of x+xy as x(1+y) should make it easier to understand why those two quilts were so similar.

6. Use factoring to help make the following predictions:

a. Predict how the quilt for $3y+xy$ (in mod 5) compares to the quilt for $xy$.

b. Predict which squares of $x^2+xy$ will be black in mod 5, assuming black is the color for 0

c. Predict the quilt pattern for $x^2+y^2+2xy$ in mod 5.

d. Predict how the quilt for $xy+3x+2y$ will compare to the quilt for $xy$ in mod 6.

7. Without running the program, try to match the following operations with the quilts shown below.

$x \bullet y = x^2+y^2$ mod 5

$x \bullet y = x + y$ mod 6

$x \bullet y = xy$ mod 9

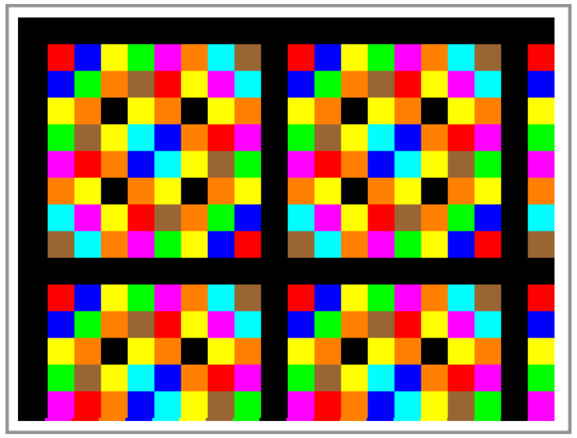
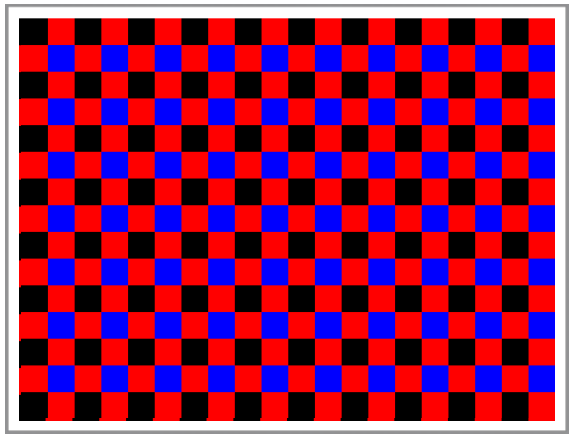
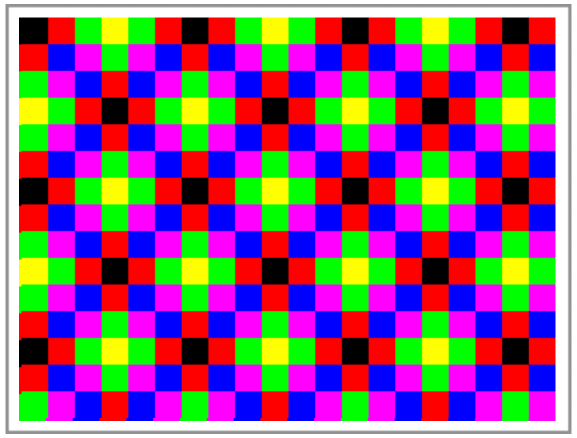
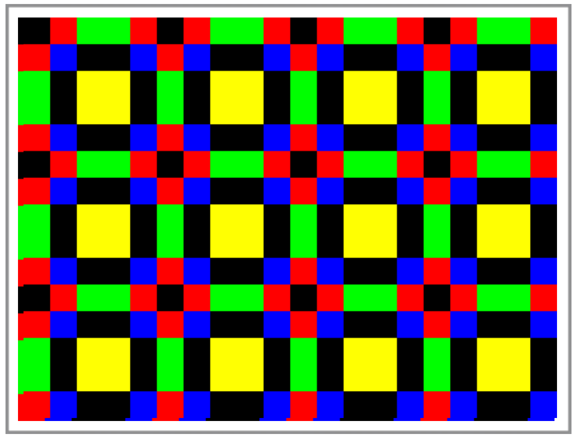
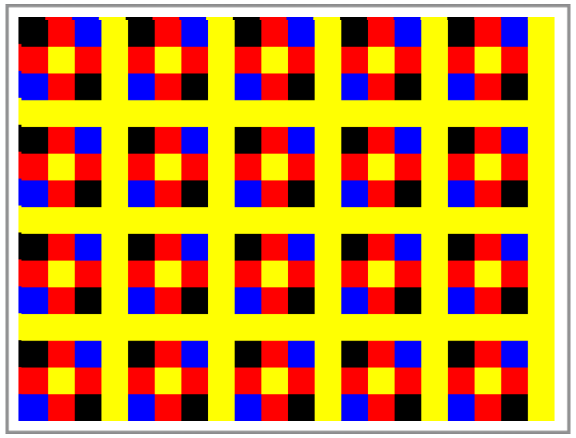
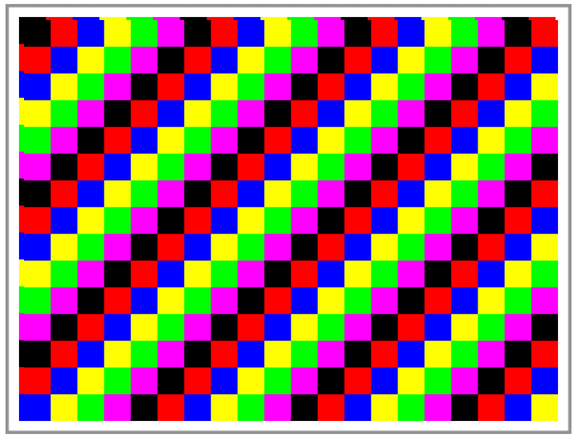
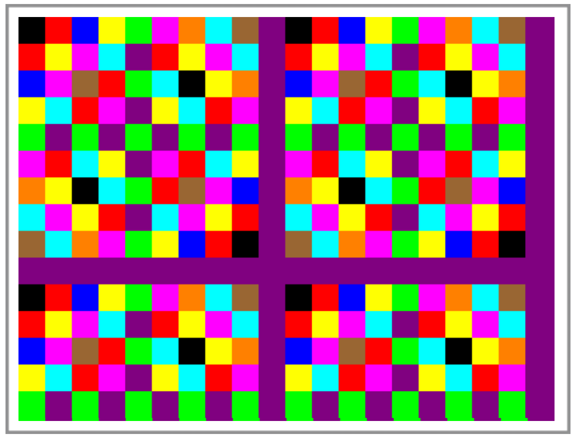
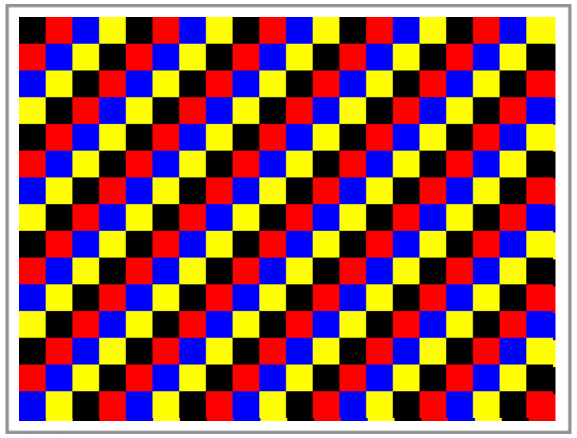
$x \bullet y = x + y + xy$ mod 4

$x \bullet y = x^2+y^2$ mod 6

$x \bullet y = x^2+y^2$ mod 4

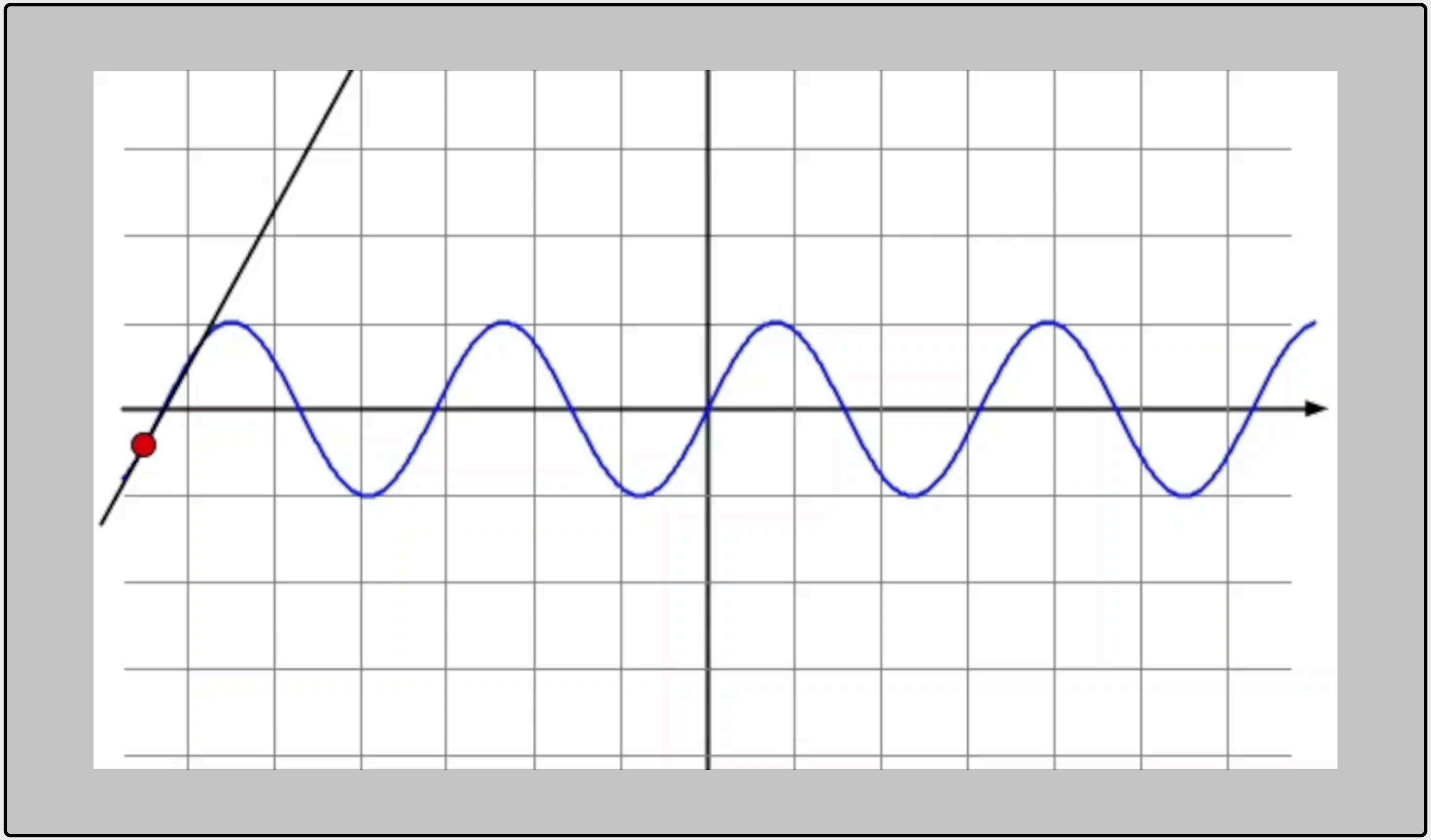
$x \bullet y = x + y + xy$ mod 10

$x \bullet y = x + y$ mod 4

The correct answers to problem 7 are revealed in Chapter 10.

# 3. Slopes and Tangents



## 3.1 Introduction

When I first learned about calculus, I thought it was magic. All of a sudden I could figure things out that were difficult or impossible to figure out before. Maxima and minima of curves? No problem. Areas of regions under or between curves? No problem. Related rates? No problem. Volumes and surface ares of geometric solids? No problem.

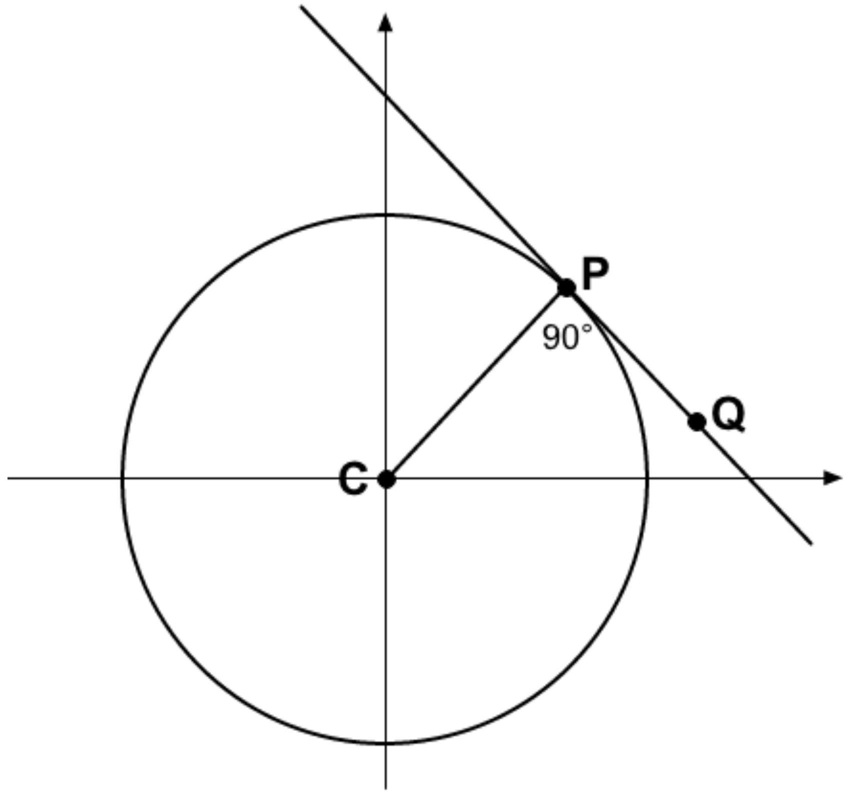
Years later, however, I became increasingly interested in investigating calculus problems that can actually be solved in other ways, using only geometry and algebra. This chapter presents some of those methods.

My **Calculus Visualizer** program is designed to provide a highly interactive and visual approach to both differential and integral calculus. It includes a section that involves finding the slopes of tangents to curves without using calculus. The program is available for free from my **Syzygy Shareware** site at http://tcbretl.weebly.com.

# 3.2 Geometric Constructions

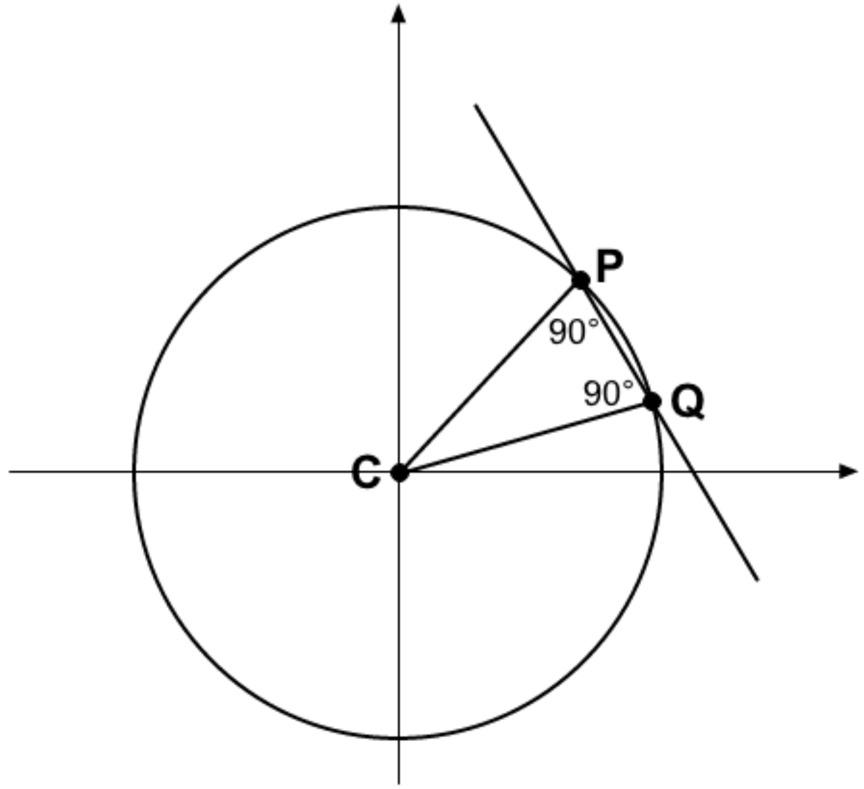
## Circles

The tangent to a circle must be perpendicular to the radius drawn to the point of tangency. Its slope is therefore the negative reciprocal of the slope of the radius.




Example: If $\overleftrightarrow{PQ}$ is tangent to circle C at point P, then it must be perpendicular to the radius $\overline{CP}$. If the slope of $\overleftrightarrow{CP}$ happens to be 3/2, then the slope of $\overleftrightarrow{PQ}$ must be -2/3.

Proof: Suppose that line $\overleftrightarrow{PQ}$ , drawn perpendicular to radius $\overline{CP}$, is <u>not</u> a tangent. It would then have to intersect the circle at another point. Suppose that point is point Q.




$\overline{CP}$ and $\overline{CQ}$ are both radii, so ΔCPQ is isosceles. Therefore ∠CPQ and ∠CQP must both be right angles, leaving us with a triangle with angles that total to more than 180 degrees! Therefore line PQ must be tangent to the circle at point P.

# Ellipses

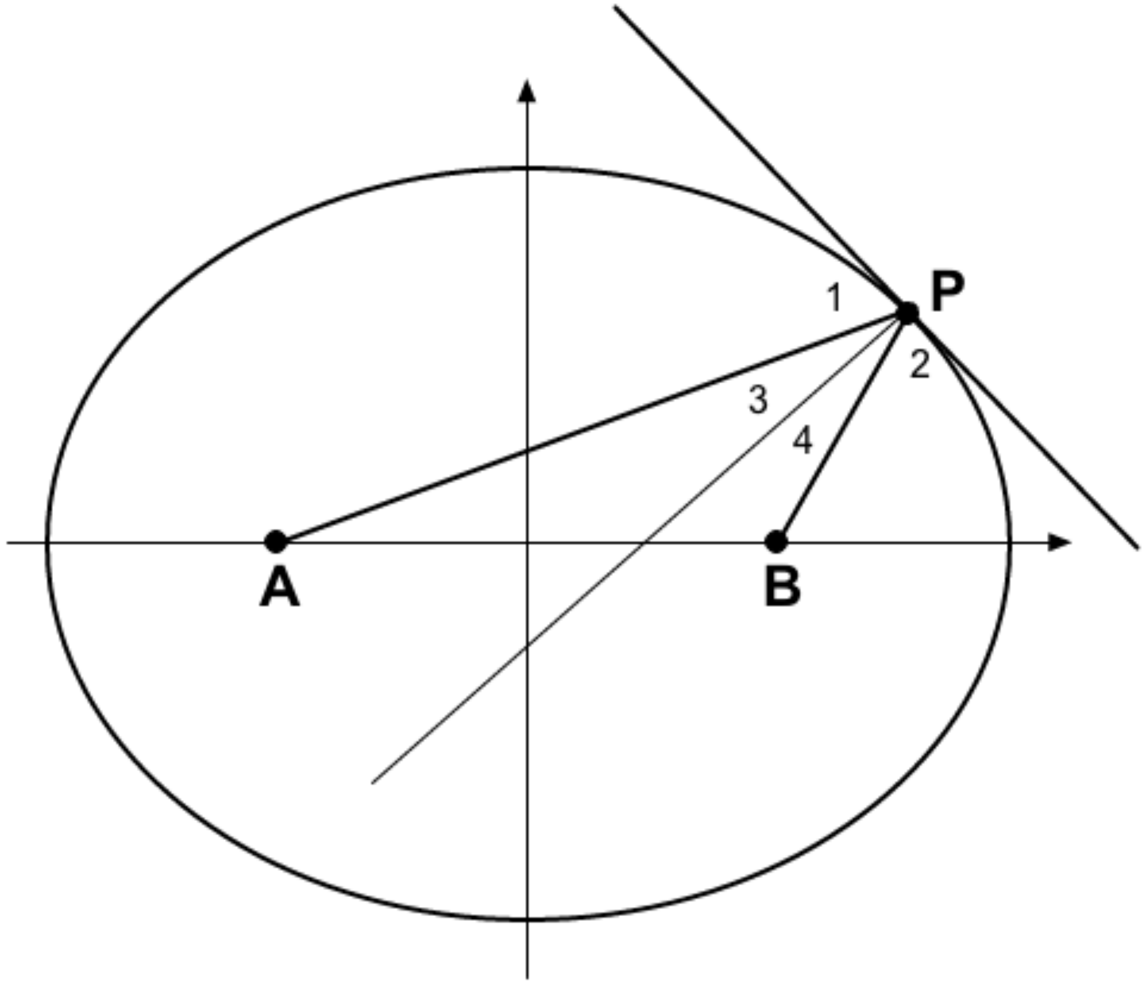
If you have ever been in an elliptically shaped room, you know that it is easy for a person standing at one focus to hear the whispers of a person standing at the other focus.

Sound originating at focus A reflects off the wall at point P and passes through focus B. This is true regardless of point P's specific location on the ellipse.

Sound bounces off the wall at the same angle as it hits the wall, so angles 1 and 2, measured between the tangent to the ellipse at point P and the line segments drawn from the two foci to point P must be equal.

If you bisect $\angle APB$ into angles 3 and 4, then $m\angle 1 + m\angle 3 = m\angle 2 + m\angle 4$, and, because the sum of all four angles must be 180°, $m\angle 1 + m\angle 3 = 90°$.

To construct the tangent, first construct the angle bisector of $\angle APB$, and then construct a line through point P which is perpendicular to that angle bisector.



To find the actual numerical value of the tangent's slope, given the coordinates of the point of tangency, you can start by finding the slope of the bisector of ∠ APB:

$\alpha = \theta + \gamma$ because $\alpha$ is an exterior angle of ΔPBC

$\theta = \beta + \gamma$ because $\theta$ is an exterior angle of ΔPCA

Eliminate $\gamma$ and solve for $\theta$:

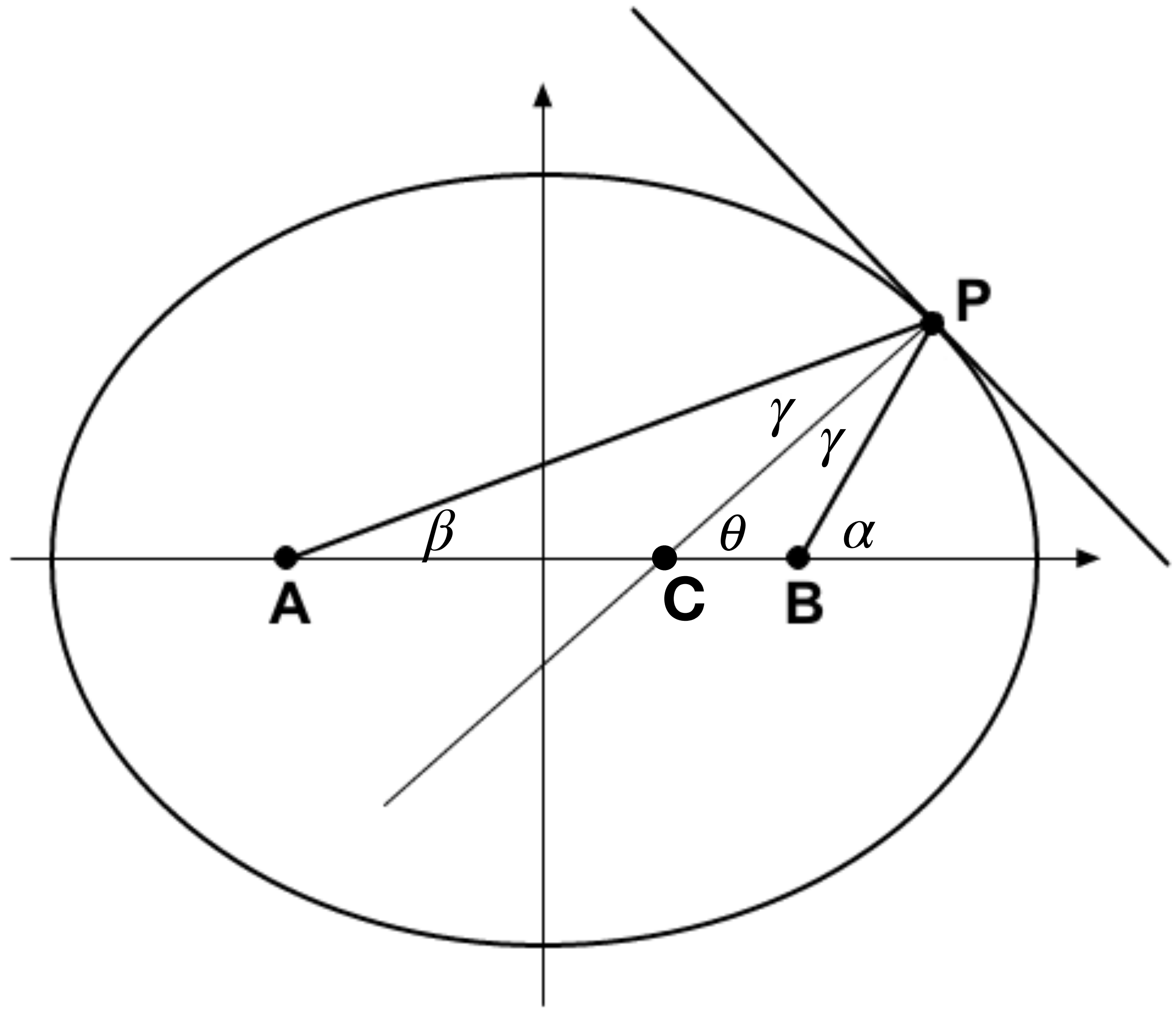
$$\theta = \beta + \alpha - \theta \qquad 2\theta = \beta + \alpha \qquad \theta = \frac{\beta + \alpha}{2}$$

You can easily calculate the slopes of $\overleftrightarrow{BP}$ and $\overleftrightarrow{AP}$ because you know the coordinates of points A, B, and P. (Remember, A and B are the foci of the ellipse.

You can then find $\alpha$ and $\beta$ by taking the arctan of the slopes of $\overleftrightarrow{BP}$ and $\overleftrightarrow{AP}$. As shown above, $\theta = \dfrac{\beta + \alpha}{2}$.

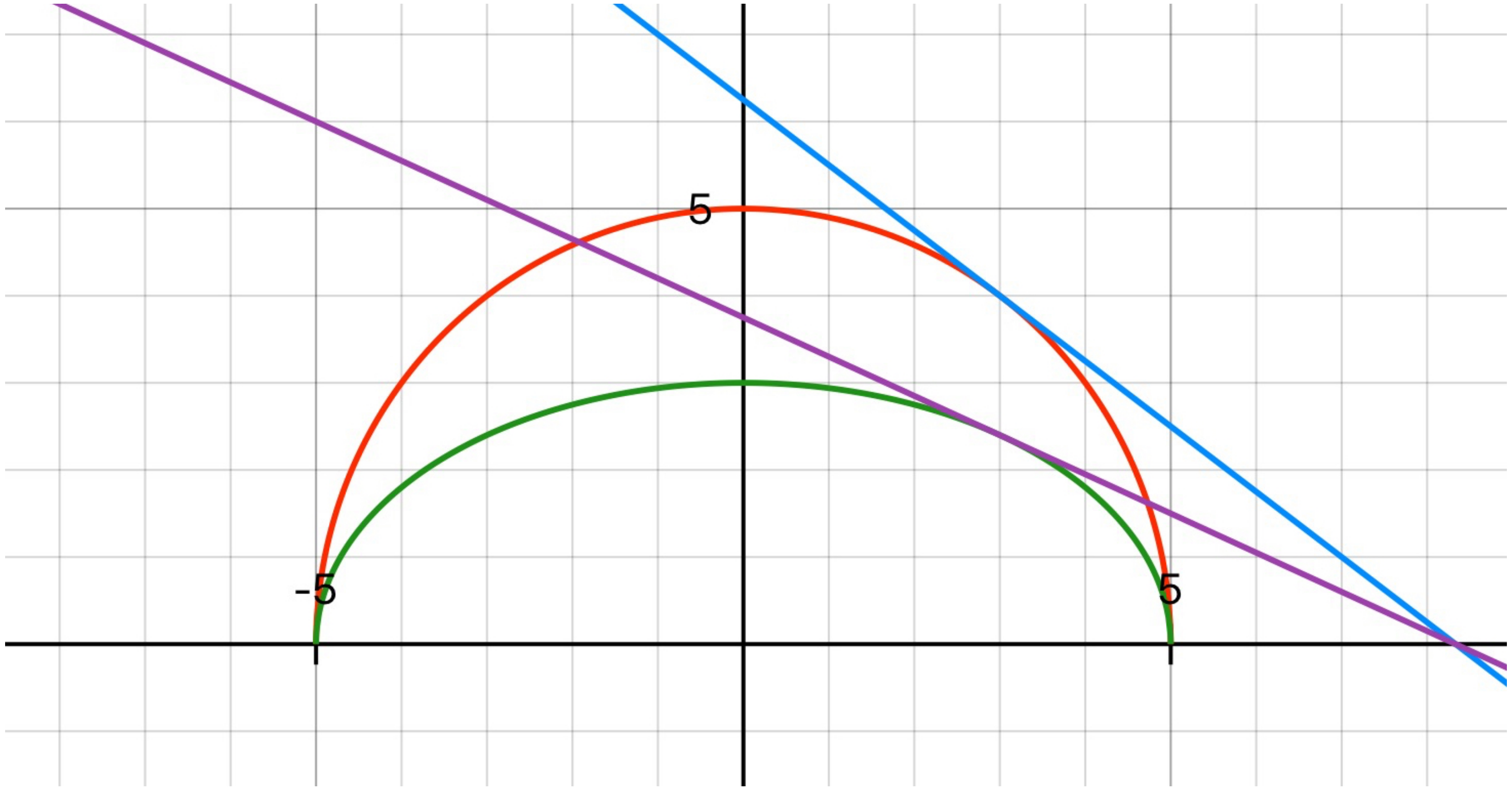
Finally, tan $\theta$ equals the slope of $\overleftrightarrow{CP}$, the angle bisector, and -1/tan $\theta$ = the slope of the tangent line.

But there is a much simpler way. See the next page.



# Ellipses (Method #2)

An ellipse is actually just a stretched or shrunken circle. For example, the green semi ellipse can be formed by shrinking the red semicircle in the y-direction.




Equation of semicircle:

$$y = \sqrt{25 - x^2}$$

Equation of semi ellipse:

$$y = \sqrt{9 - \frac{9x^2}{25}}$$

For a given x value, the y-coordinate for a point on the semi ellipse equals 3/5 times the y coordinate on the circle:

$$\frac{3}{5}\sqrt{25 - x^2} = \sqrt{\frac{9}{25} \cdot \left(25 - x^2\right)} = \sqrt{9 - \frac{9x^2}{25}}$$

The tangent to the semicircle at the point (3,4) becomes a tangent to the semi ellipse at $\left(3, \frac{3}{5} \cdot 4\right) = \left(3, \frac{12}{5}\right)$.

The slope of the tangent to the semi ellipse equals $\frac{3}{5}$ times the slope of the tangent to the semicircle: $\frac{3}{5} \cdot \frac{-3}{4} = \frac{-9}{20}$

The y-intercept of the semi ellipse's tangent equals $\frac{3}{5}$ times the y-intercept of the semicircle's tangent: $\frac{3}{5} \cdot \frac{25}{4} = \frac{15}{4}$

# Parabolas

The definition of a parabola is the set of all points equidistant from a given point (the focus) and a given line (the directrix). You can construct tangents to a parabola by paper folding:

1. Make a dot somewhere near to the bottom of a sheet of paper. This will be the focus of the parabola. The bottom of the sheet of paper will be the directrix.

2. Fold up one bottom corner of the paper so that the bottom edge goes right through the focus as shown in figure 1.

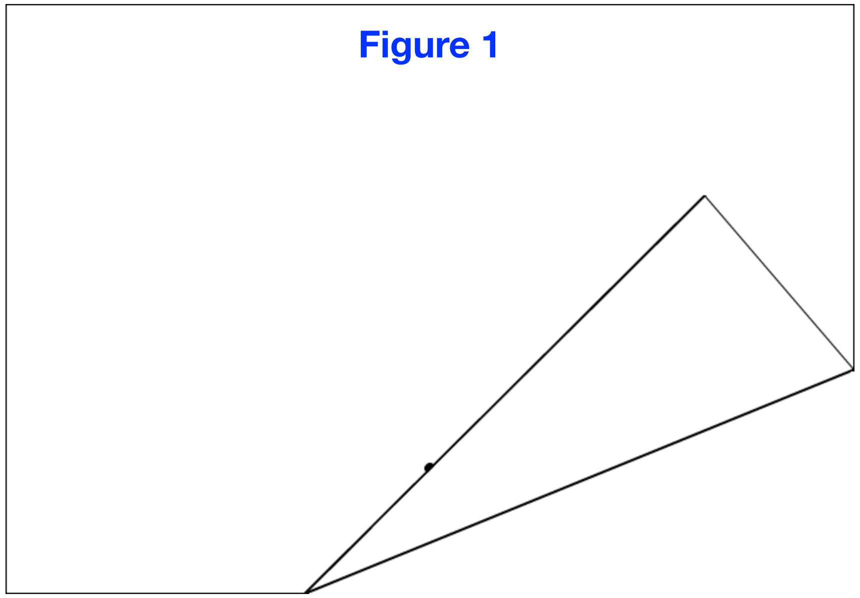


Figure 1

3. Unfold the paper. The crease, shown in figure 2, is your first tangent.

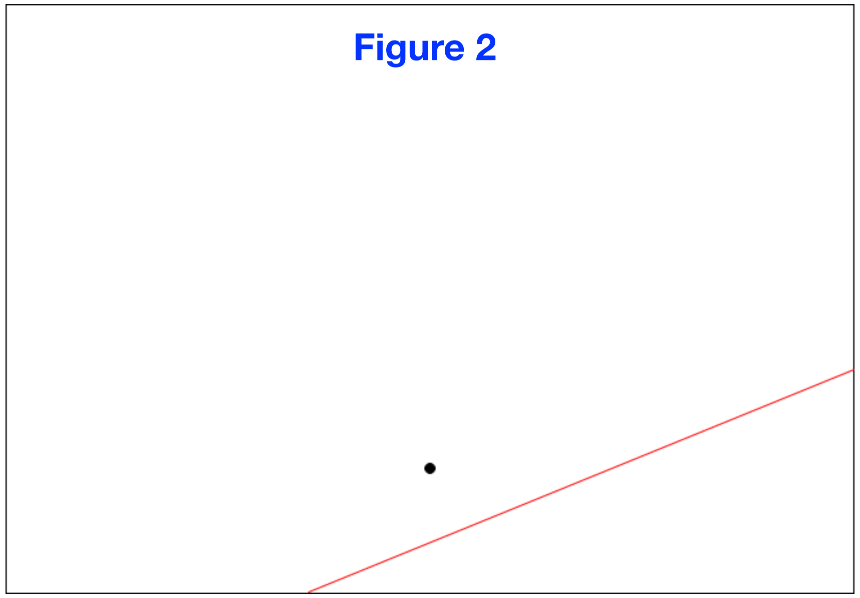


Figure 2

4. Continue this process, varying the amount of paper you fold over. You may also fold up the other bottom corner of the paper. You will end up with something that looks like figure 3. See the parabola?

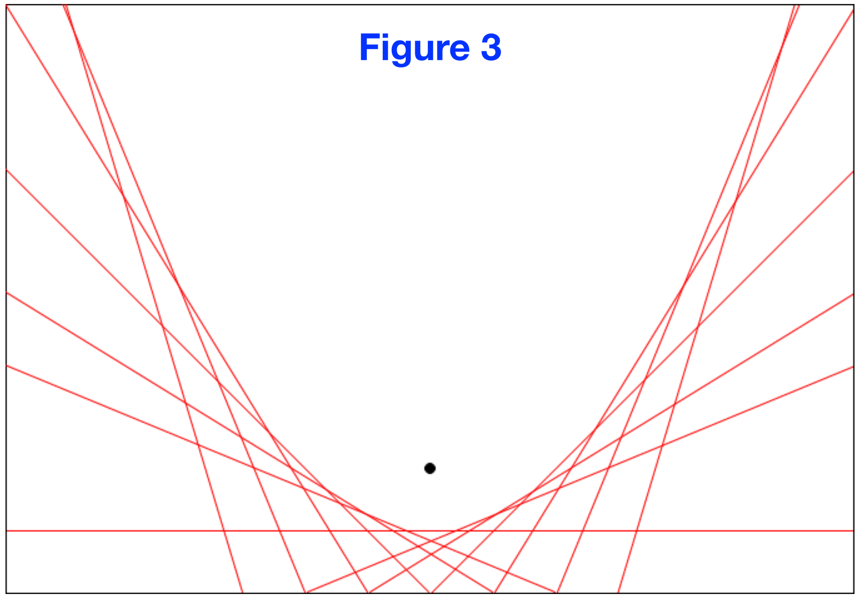
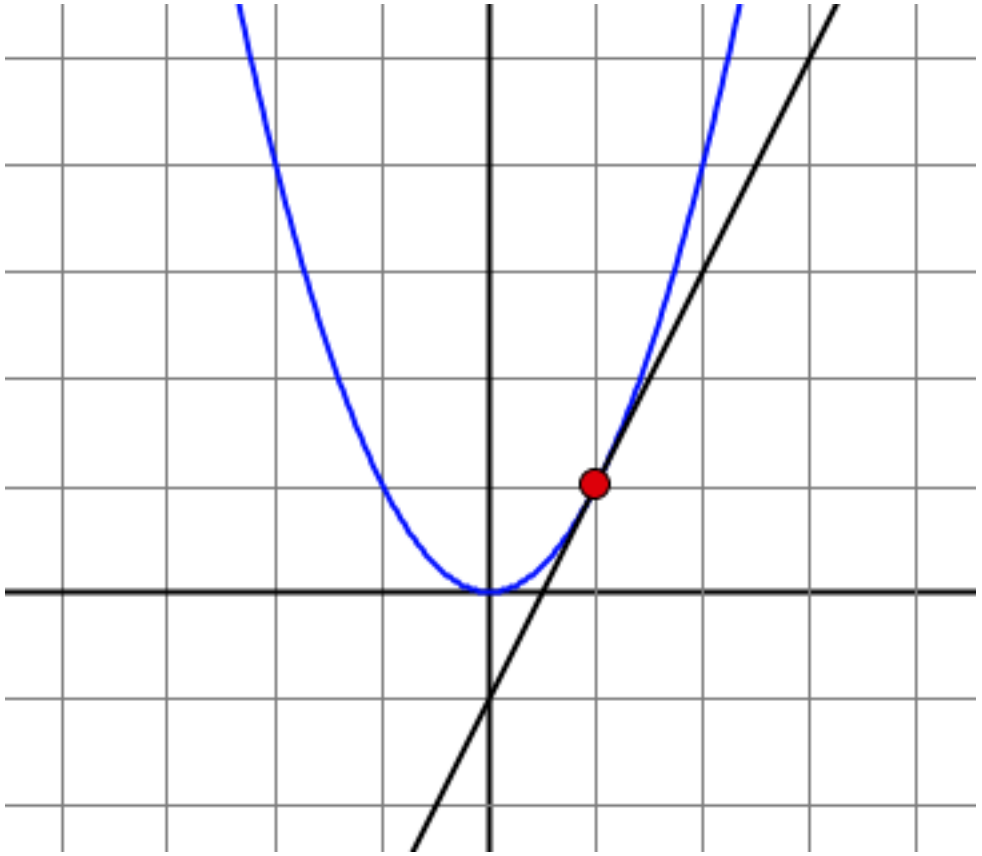


Figure 3

# 7.3 Method of Equal Roots

The tangent to a curve sometimes intersects that curve at only one point. Consider, for example, the parabola defined by y = $x^2$ and the tangent to that parabola at the point (1,1).



The equation of that tangent line is $y - 1 = m(x - 1)$ , but how can we figure out the correct value for m?

To find the point(s) where the tangent line intersects the parabola, substitute $x^2$ for y in the equation of the line:

$$x^2 - 1 = m(x - 1) \;\Rightarrow\; x^2 - mx + m - 1 = 0$$

You can use the quadratic formula to solve that quadratic equation for x:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{where } a = 1,\ b = -m,\ c = m - 1 .$$

This would give us two solutions for x, but clearly the tangent intersects the parabola at only one point.

Therefore, the discriminant must equal 0.

$$b^2 - 4ac = 0$$

Substitute for a, b, and c: $m^2 - 4(1)(m - 1) = 0$

Simplify: $m^2 - 4m + 4 = 0$

Factor: $(m - 2)(m - 2) = 0$

The slope, m, of the tangent equals 2, and the equation of the tangent line is $y = 2x - 1$ .

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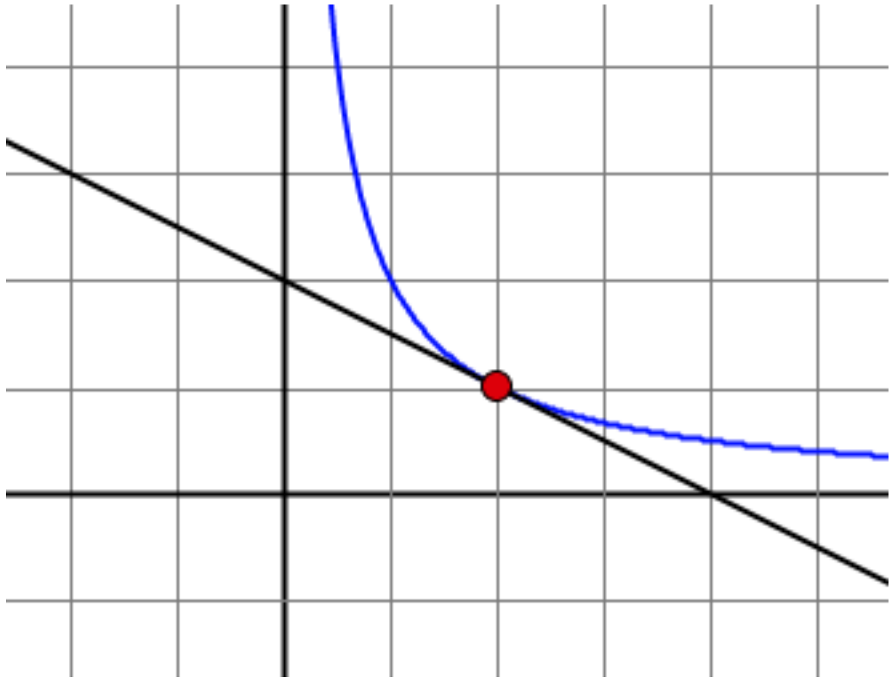
The same process can be applied to other curves. The next page shows two more examples.

**Inverse Function**

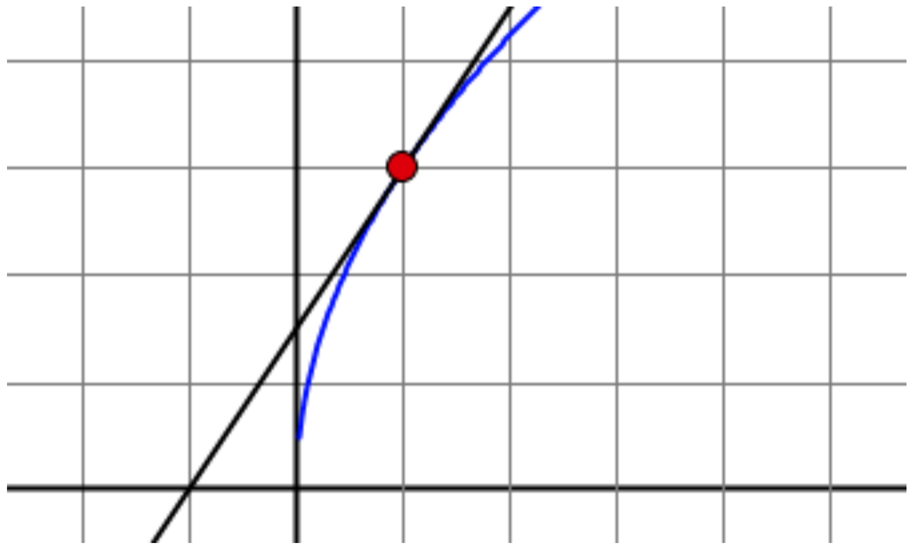
Suppose $y = \dfrac{2}{x}$. Find the slope of the tangent at (2,1).



1. $y - 1 = \dfrac{2}{x} - 1 = m(x - 2)$
2. $2 - x = mx^2 - 2mx$
3. $mx^2 - 2mx + x - 2 = mx^2 + (1 - 2m)x - 2 = 0$
4. *Discriminant:* $(1 - 2m)^2 - 4(m)(-2) = 0$
5. $4m^2 - 4m + 1 + 8m = 4m^2 + 4m + 1 = 0$
6. $(2m + 1)(2m + 1) = 0 \;\Rightarrow\; m = -1/2$

**Square Root Function**

Suppose $y = 3\sqrt{x}$. Find the slope of the tangent at (1,3).



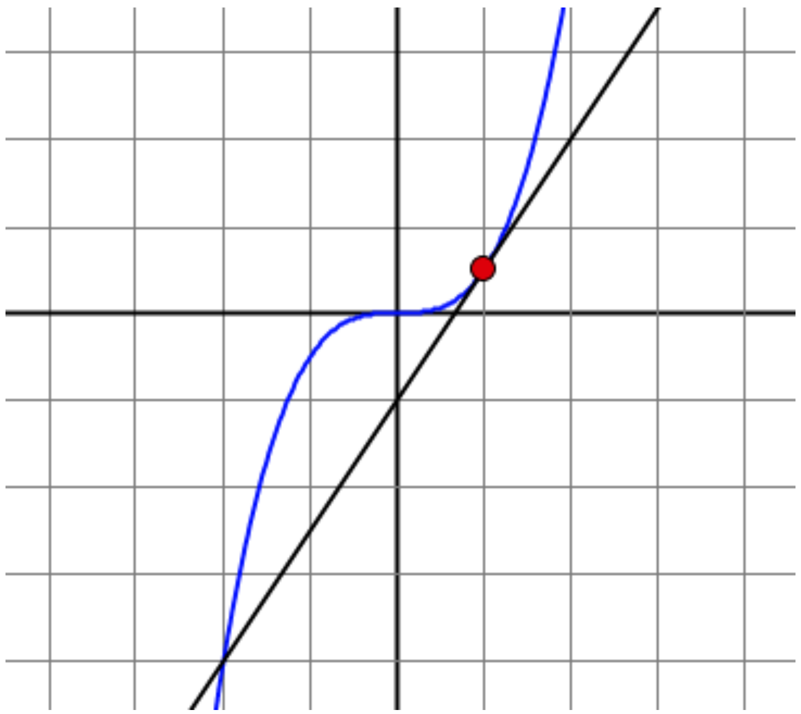
1. $y - 3 = 3\sqrt{x} - 3 = m(x - 1)$
2. *Let* $w = \sqrt{x}$. $3w - 3 = m(w^2 - 1)$
3. $mw^2 - 3w + 3 - m = 0$
4. *Discriminant:* $9 - 4(m)(3 - m) = 0$
5. $4m^2 - 12m + 9 = 0$
6. $(2m - 3)(2m - 3) = 0 \;\Rightarrow\; m = 3/2$

Once again, equal roots.

**Cubic Function**

Tangents to curves defined by cubic equations almost always intersect the curve in two places. Can the method of equal roots still be used to find their slopes?

Consider the tangent to the curve defined by $y = \frac{x^3}{2}$ at the point (1, $\frac{1}{2}$):



1. *Equation of tangent line:* $y - \frac{1}{2} = m(x-1)$
2. $\frac{x^3}{2} - \frac{1}{2} = m(x-1)$
3. $x^3 - 1 = 2mx - 2m$
4. $x^3 - 2mx + 2m - 1 = 0$
5. *Since we know that* $x = 1$ *is one of the solutions,* $x^3 - 2mx + 2m - 1$ *must be evenly divisible by* $x - 1$.
6. $\frac{x^3 - 2mx + 2m - 1}{x - 1} = x^2 + x - 2m + 1$
7. *Since there can be only two solutions for x, either 1 is a double root, or some other value for x must be a double root. If 1 is a double root, then* $x^2 + x - 2m + 1$ *must be evenly divisible by* $x - 1$.
8. $\frac{x^2 + x - 2m + 1}{x - 1} = x + 2$ *with a remainder of* $-2m + 3$, *so* $-2m + 3 = 0$ *and* $m = 3/2$. *As you can see by the graph, this is the correct slope for the tangent.*
9. *If some other value is the double root, then* $x^2 + x - 2m + 1$ *has to be a perfect square and* $x^2 + x - 2m + 1 = 0$ *can have only one solution.*
10. *Discriminant:* $1 - 4(1)(-2m + 1) = 1 + 8m - 4 = 0$
11. $m = 3/8$
12. $0 = x^2 + x - 2m + 1 = x^2 + x + \frac{1}{4} = \left(x + \frac{1}{2}\right)^2$
13. $x = \frac{-1}{2}$ *and* $y = \frac{x^3}{2} = \frac{-1}{16}$.

The green line is the tangent to the curve that you get when you assume that 1 is a double root of $x^3 - 2mx + 2m - 1 = 0$.

Slope: $\frac{3}{2}$

Equation: $y = \frac{3}{2}x - 1$

Point of tangency: $\left(1, \frac{1}{2}\right)$

-----------------------------------------------------------------------

The blue line is the tangent to the curve that you get when you assume that 1 is only a single root and some other number is the double root of $x^3 - 2mx + 2m - 1 = 0$.

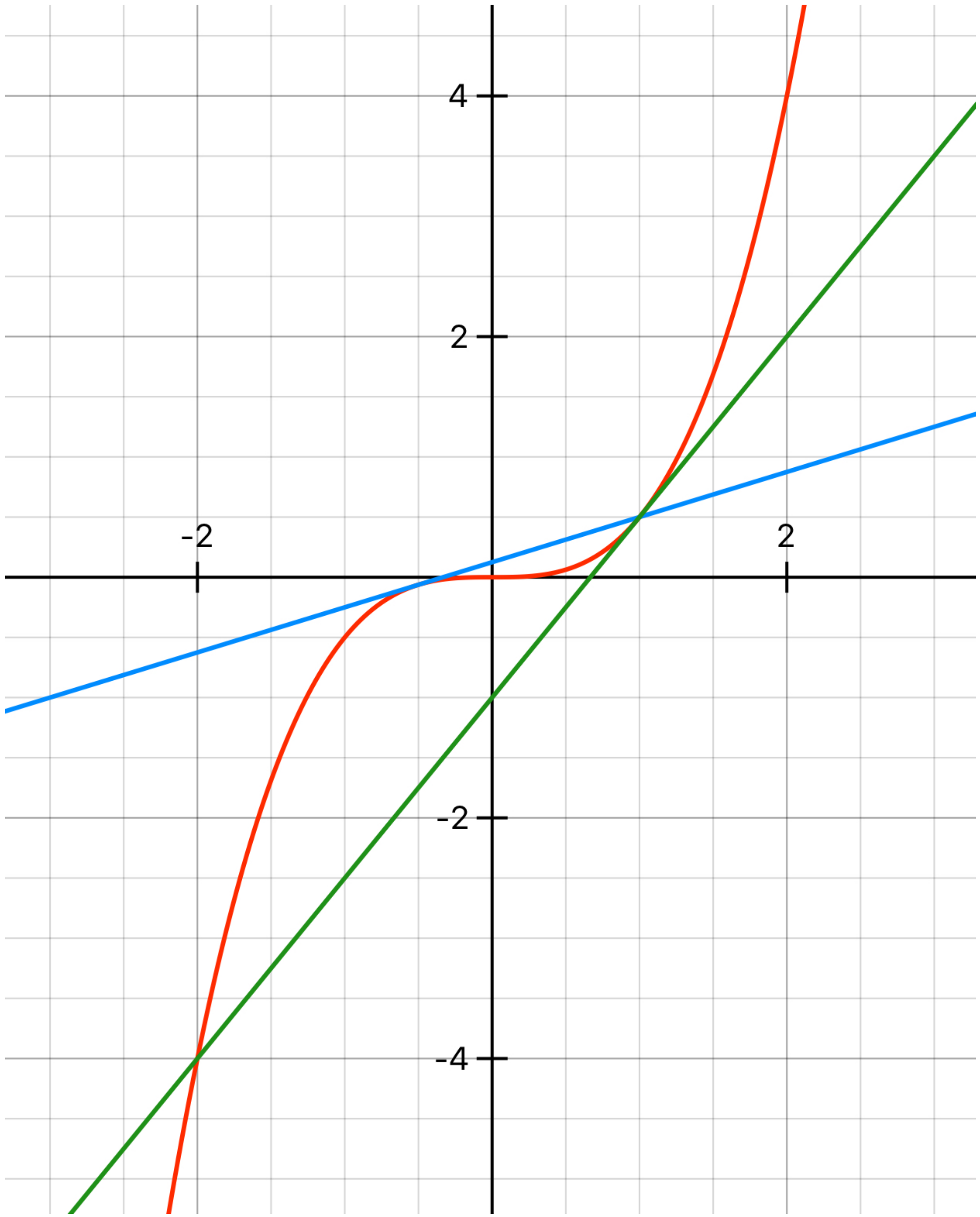
Slope: $\frac{3}{8}$

Equation: $y = \frac{3}{8}x + \frac{1}{8}$

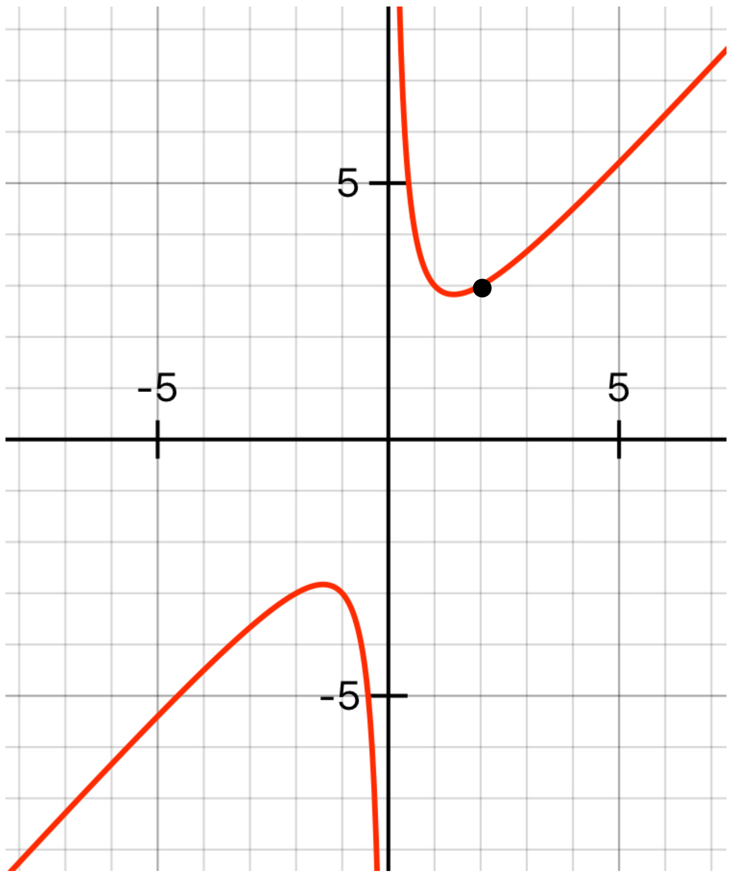
Point of tangency: $\left(\frac{-1}{2}, \frac{-1}{16}\right)$

-----------------------------------------------------------------------

The blue line does pass through the point at $\left(1, \frac{1}{2}\right)$, but it is not tangent there.

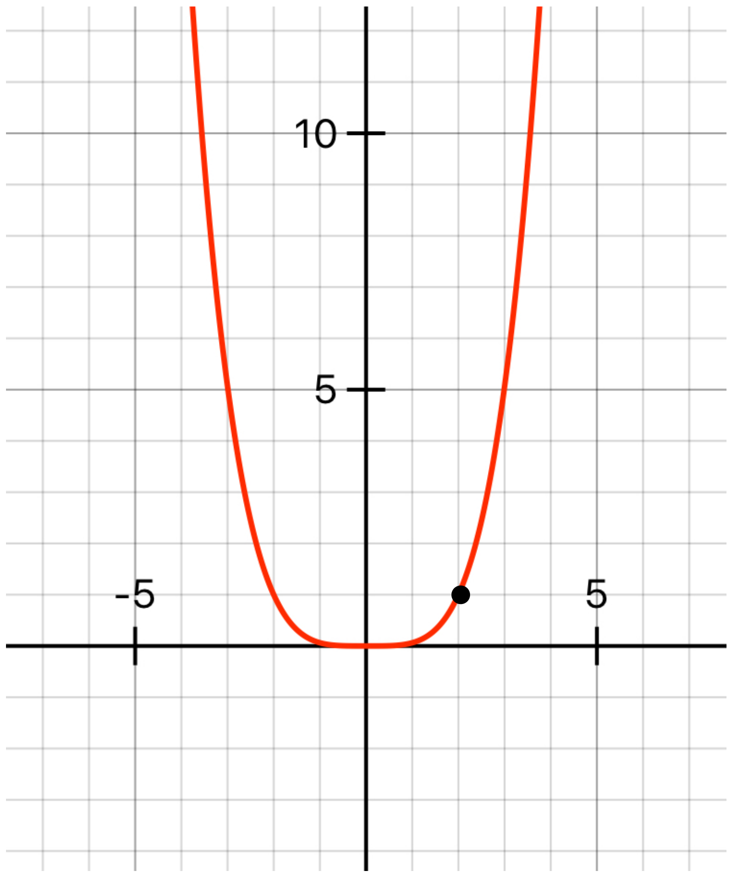


## 3.4 Problems




Find the slope and equation of the tangent to the curve $y = x + \dfrac{2}{x}$ at the point $(2, 3)$.

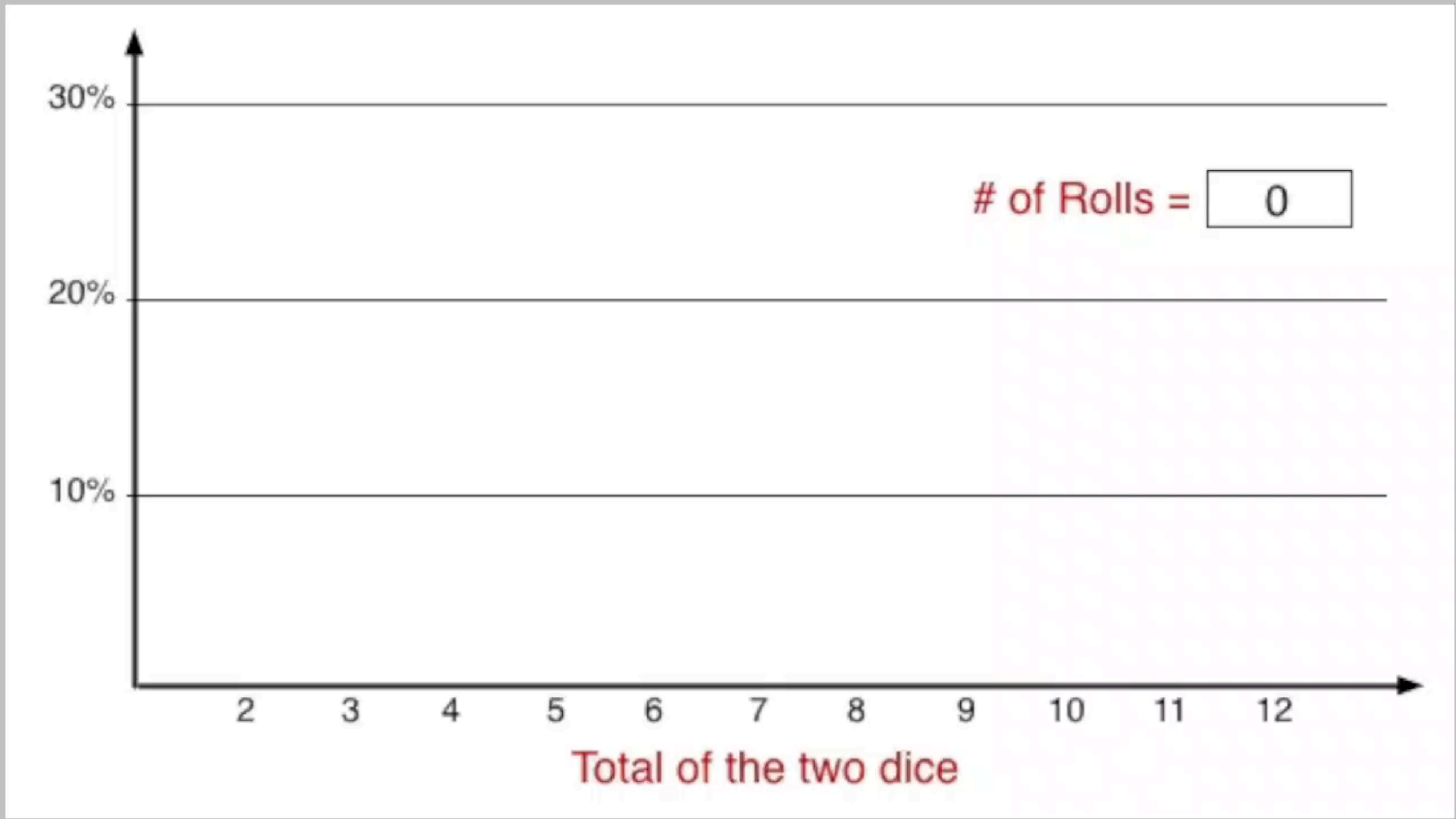
The solution is shown in Chapter 10.




Find the slope and equation of the tangent to the curve $y = \dfrac{x^4}{16}$ at the point $(2, 1)$ .

The solution is shown in Chapter 10.

# 4. Probability

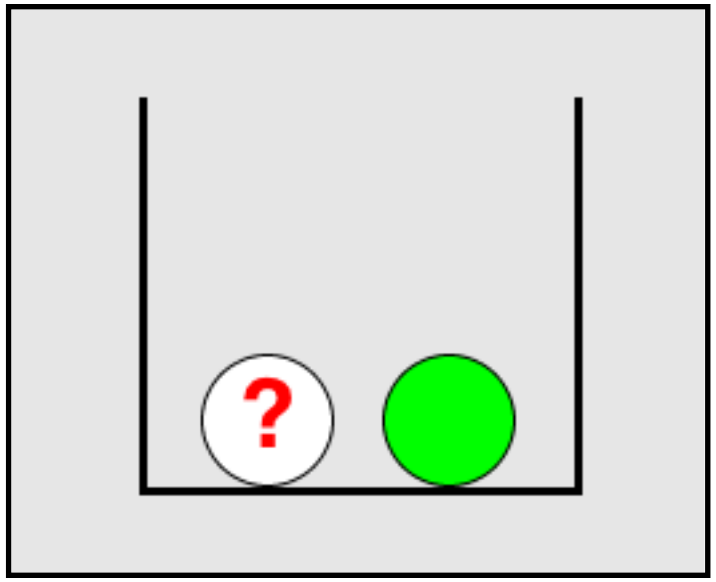


## 4.1 Introduction

Probability problems can be very tricky! It's often easy to understand the question, but difficult to come up with the right answer. It's just a matter of counting possible outcomes, but the "catch" is that those outcomes must all be "equally likely." For example, a person might mistakenly think that the probability of rolling 7 with a pair of dice is 1/11 because there are 11 different totals possible (2 through 12), but of course those 11 possible outcomes are not all equally likely. There are six ways to roll 7, but only one way to roll 2.

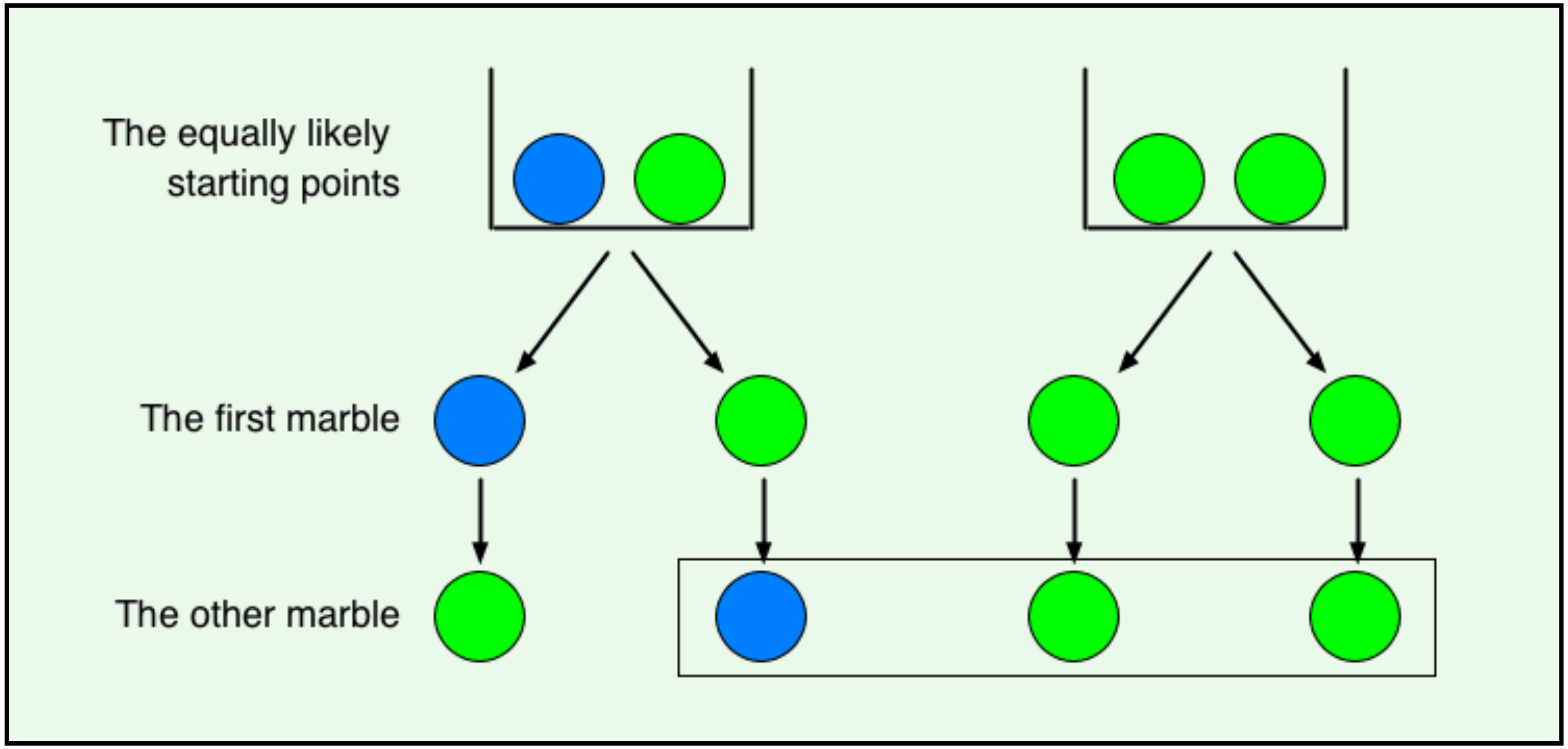
This chapter presents some well known but often misunderstood probability problems. See if you can solve them before looking at the proposed solutions. Even math teachers have in some cases been fooled by them! My **ProbSim** program can be used to explore these problems further. Simulating the problems may help convince you that the "solution" is correct. The program is available for free from my **Syzygy Shareware** site at http://tcbretl.weebly.com.

## 4.2 The Urn Problem



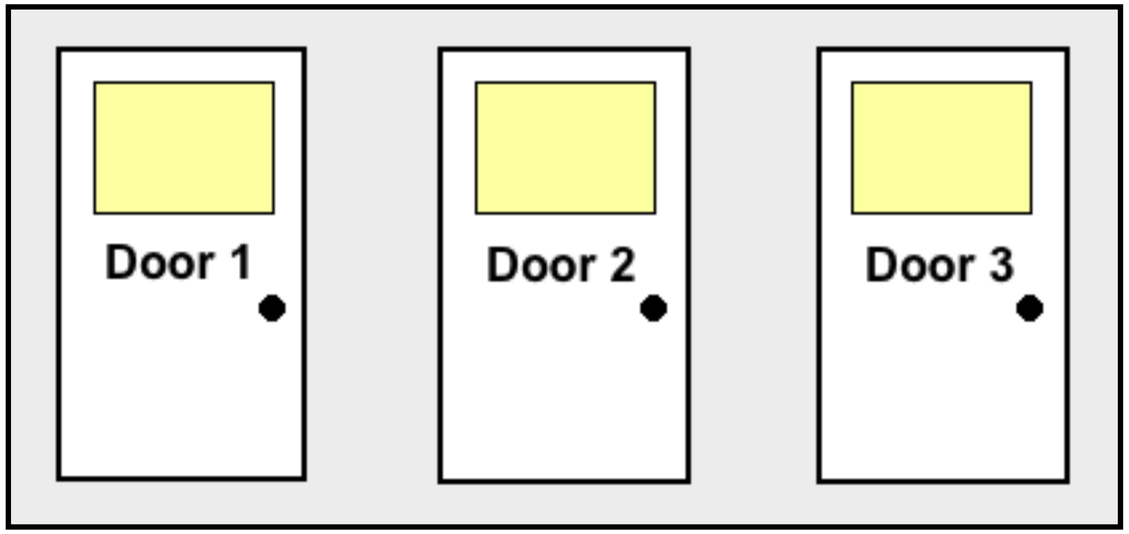

An urn contains a marble which is either blue or green. (There is a 50-50 chance of each.) You drop a second marble, which is definitely green, into the urn. You then reach in and randomly choose one of the marbles. Suppose it is green. What is the probability that the other marble is also green?

In other words, you are being asked to find the probability that the second marble is green given that the first marble was green. Do you think the answer is 1/2? Perhaps the diagram shown below will change your mind.




As you can see, when the first marble is green, the second marble is green 2/3 of the time.

## 4.3 The Game Show Problem

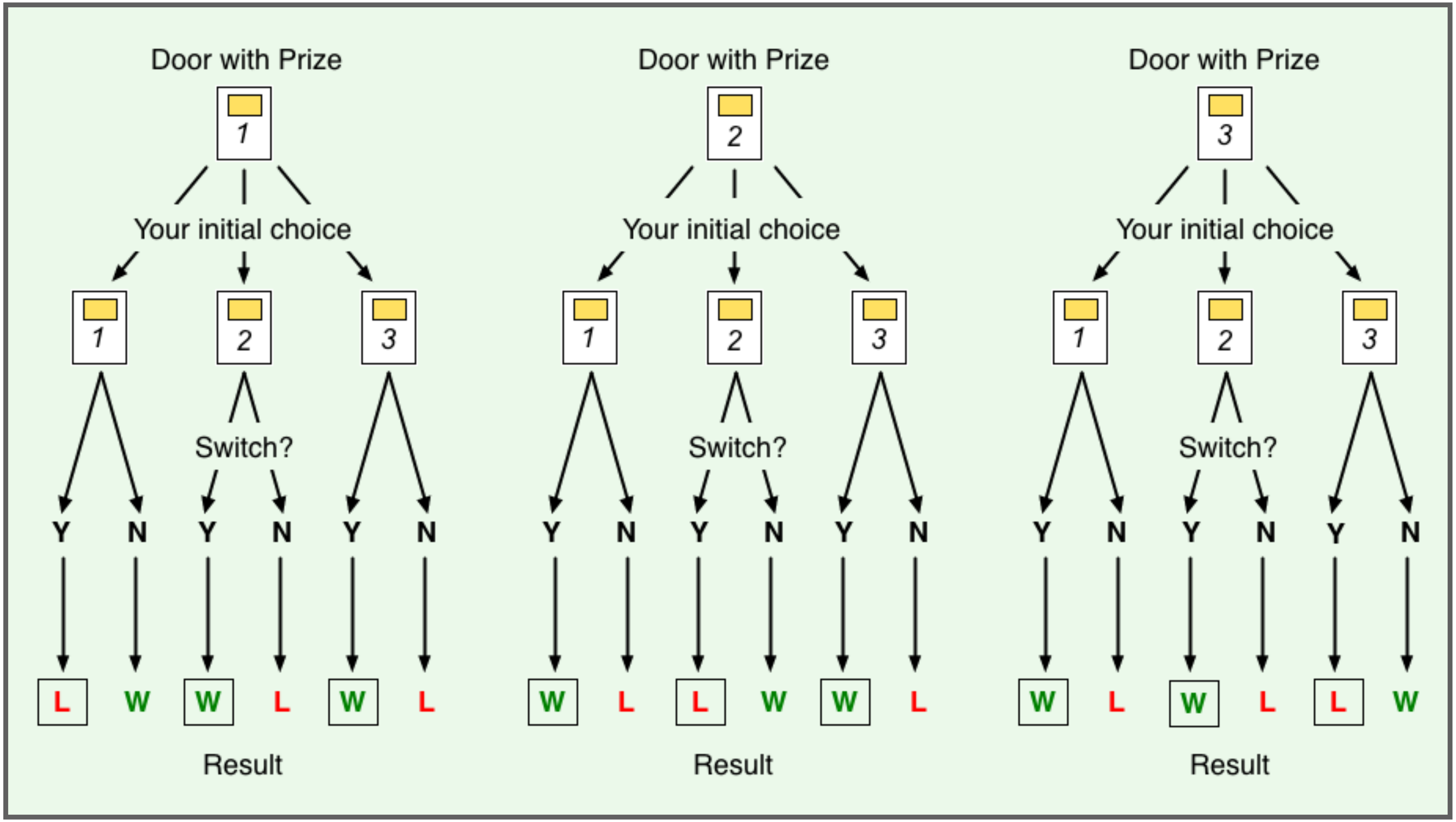



Suppose you are a contestant on a game show. You are presented with three doors, and are asked to choose between them. You will win whatever is behind the door you choose. Behind one of the doors is a terrific prize; behind the other two doors there is nothing of value. You choose door #1, but the show's emcee (who knows what is behind all three doors) then reveals that there is nothing of value behind door #2, and gives you the option of changing your mind. Should you choose door #3 instead of door #1?

This well known problem stirred up a lot of controversy back in 1990 when Marilyn vos Savant wrote that switching was the best strategy. She said that switching would give you a 2/3 chance of winning the prize. **Lots** of people disagreed. I, myself, was not convinced until I simulated the problem.

The diagram on the next page illustrates why switching is the best strategy. It shows all of the different possibilities in a tree diagram format. There are three possibilities for the location of the prize, three possibilities for your initial choice, and two possibilities for whether or not you decide to switch. That's 3 x 3 x 2 =18 different ways in which the game could proceed. Nine include switching and the other nine do not. If you switch, you win 6 out of 9 times. If you don't switch, you win only 3 out of 9 times.

# Game Show Possibilities



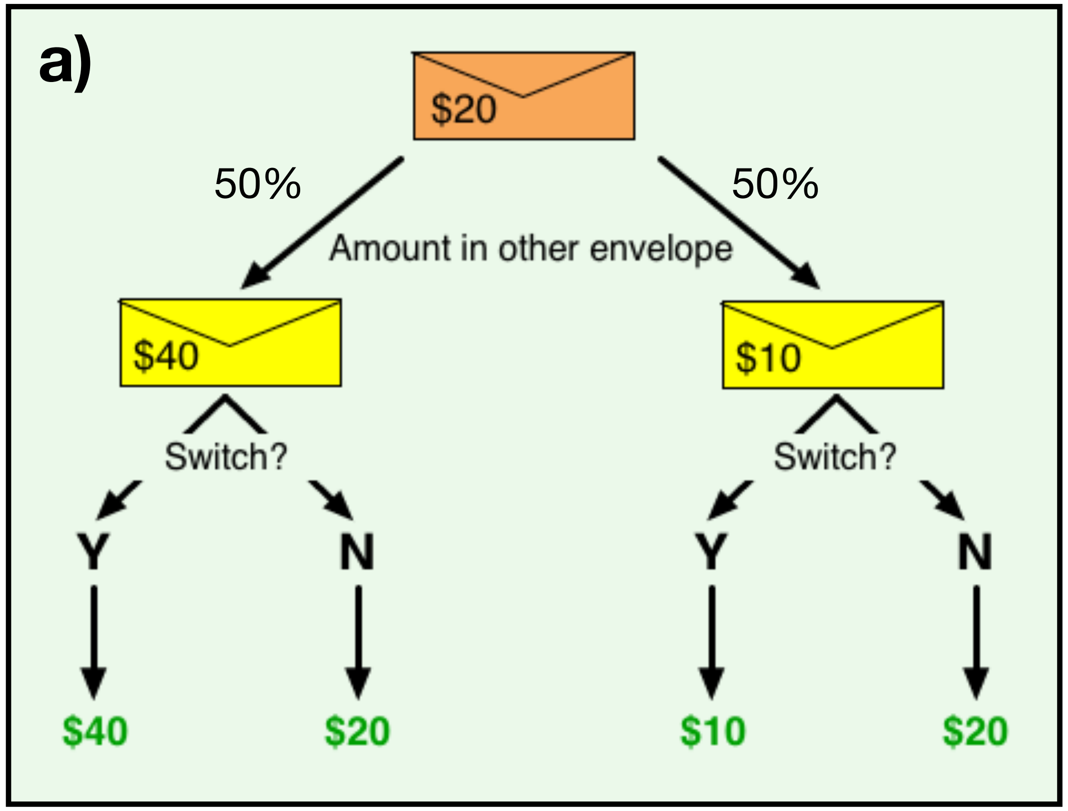
## 4.4 The Envelope Problem

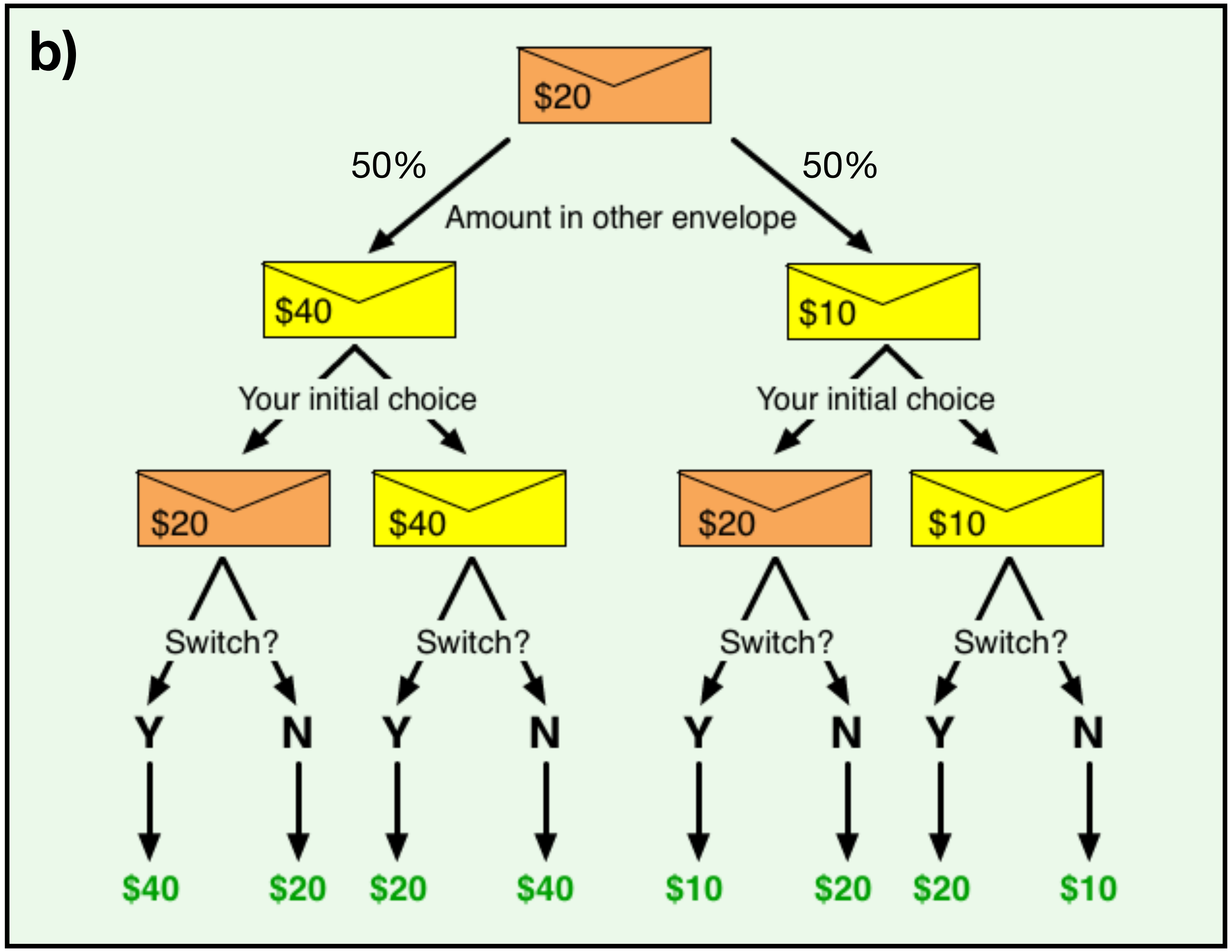
a) You are presented with two envelopes and are told that both of them contain some money. After you open one of the envelopes, you are told that the other envelope contains either half as much or twice as much money, a 50% chance of each. You can either keep the money from the first envelope or switch and take the money that is in the other, as yet unopened, envelope. Should you switch or should you keep what you've got?

b) You are presented with two envelopes. You are told that one of the envelopes contains twice as much money as the other one. You choose one of the envelopes and find out how much money it contains. You can either keep that amount or switch and take the money that is in the other, as yet unopened, envelope. Should you switch or should you keep what you've got?

> There is a subtle but significant difference between the two situations described in this problem.
>
> If, after having already selected and opened an envelope, you are told that the other envelope contains either half as much or twice as much money, switching envelopes is definitely the best strategy.
>
> For example, suppose the envelope you pick first contains $20, and you are told that the other envelope contains either $10 or $40 (50% chance of each). If you stick with your original choice, you obviously get $20, but if you switch to the other envelope, your expected amount is 1/2•(10 + 40) = $25.
>
> If, on the other hand, you are told, before making your selection, that one of the envelopes contains twice as much as the other, then changing your mind and switching envelopes makes no difference. The diagrams on the next page illustrate the difference between the two solutions.

These two diagrams illustrate the two different situations. To make things easier to follow, the envelopes are given colors. Unknown to you, the brown envelope contains $20, and the yellow one contains either $10 or $40.



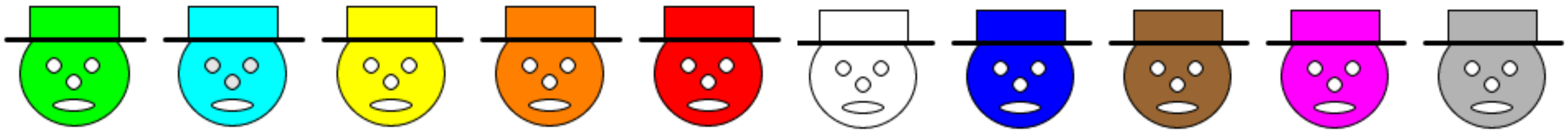

**a)** Here you choose and open the brown envelope, find the $20, and are then told that the other envelope contains either twice as much or half as much money. If you switch, you get an average of $25. If you don't switch, you get an average of $20. The key here is the fact that twice as much money means $20 more, but half as much money means only $10 less.

**b)** Here you are told that the yellow envelope contains either twice or half as much money as the brown one <u>before</u> you make your selection. You select one of the envelopes, and are then asked whether you would like to change your mind. If you switch, the average amount of money you get is (40+20+10+20)/4 = $22.50. If you don't switch, the average amount of money you get is (20+40+20+10)/4 = $22.50. There is no difference.

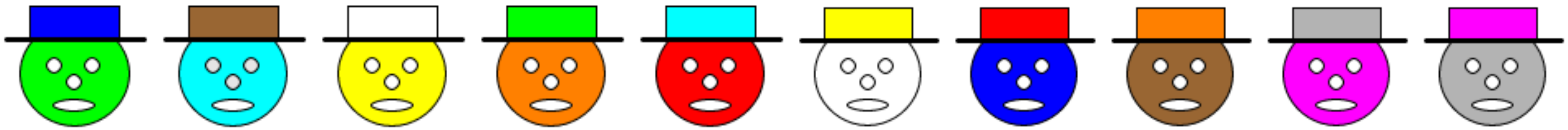
## 4.5 The Hatcheck Problem

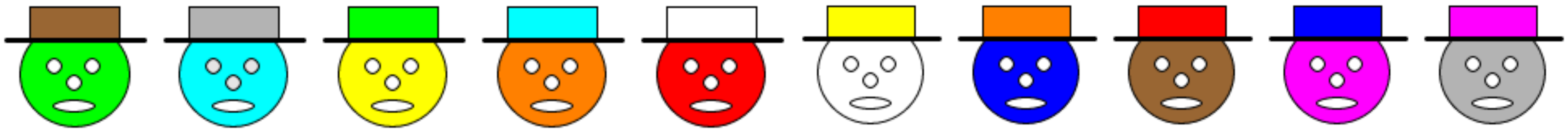
A group of N guests check their hats at a restaurant. Later, the hatcheck person returns their hats totally at random. What is the probability that nobody gets the right hat?

Here, for example, are 10 guests, all wearing their nice color-coordinated hats.



When the hatcheck person randomly returns the hats, the first guest could end up with any one of the 10 hats, the second guest could end up with any one of the 9 remaining hats, the third guest could end up with any one of the remaining 8 hats, etc. All together there are 10 x 9 x 8 x 7 x 6 x 5 x 4 x 3 x 2 = 3,628,800 different ways in which the hats could be returned! In some cases, everyone will get the wrong hat. These are called **derangements**. Here are two of them.





How many different derangements there are? What fraction of all the possible outcomes are derangements?

If N is a fairly small number, it is easy to answer the questions.

If N=2, the probability = 0.5 because there are only two possible equally likely outcomes:

1. First guest gets the correct hat, so the second guest must also get the correct hat.
2. First guest gets the wrong hat, so the second guest must also get the wrong hat.

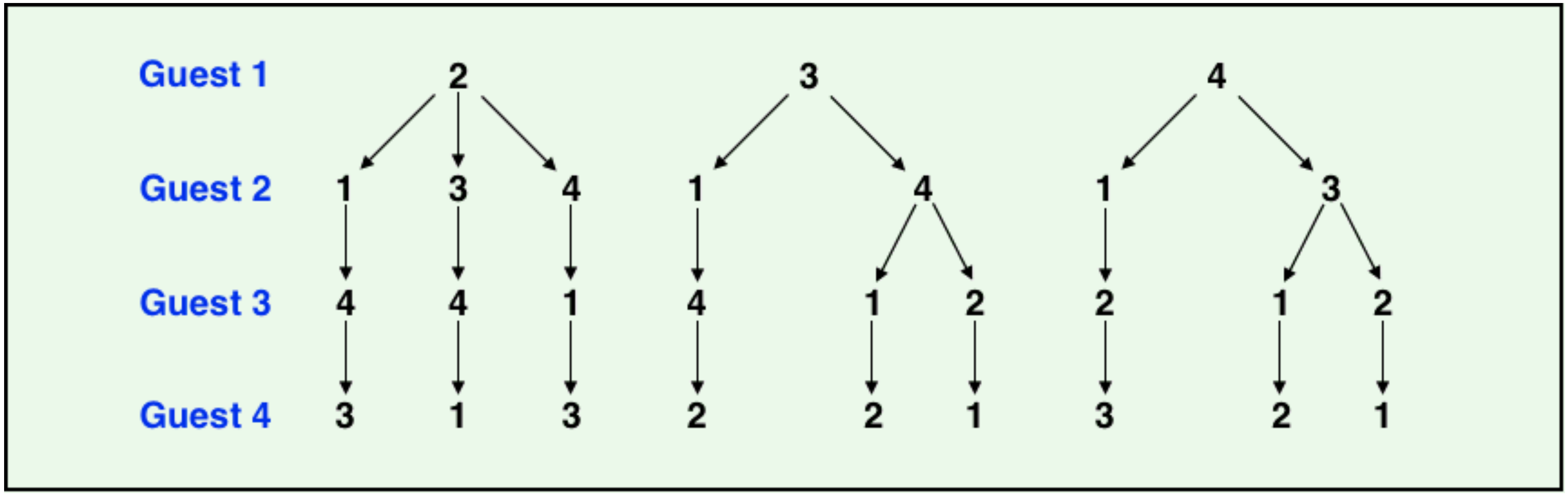
One is a derangement and the other is not.

If N=3, let's assume that hat1 belongs to guest1, hat2 belongs to guest2, and hat3 belongs to guest3. There are 6 ways to return the hats, and only two of them are derangements:

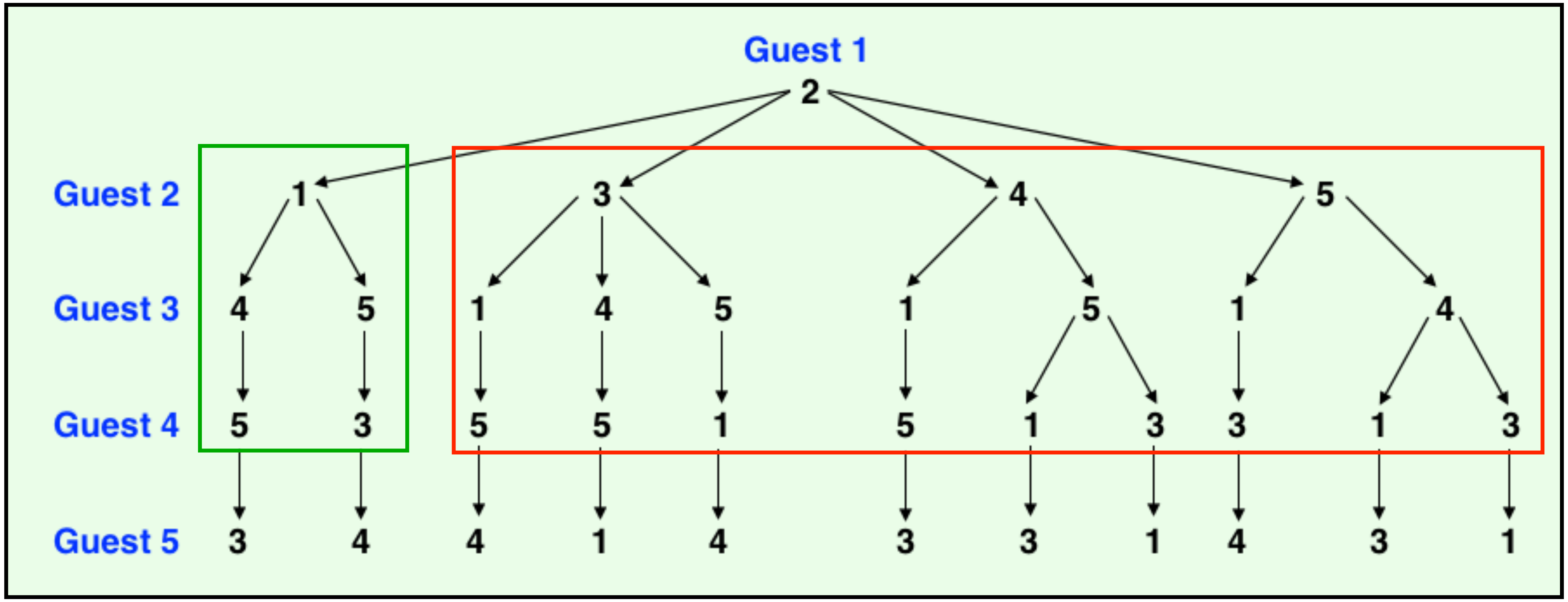
Guest1 gets hat2, guest2 gets hat3, guest3 gets hat1.
Guest1 gets hat3, guest2 gets hat1, guest3 gets hat2.

The probability of a derangement is 2/6 =1/3.

If there are four guests, a tree diagram makes it easier to count up all of the possible derangements. There are nine of them, so the probability of getting a derangement is 9 /(4 x 3 x 2) = 9/24 = 3/8.



As the number of guests increases, the possible derangements get more difficult to count. Some kind of formula would certainly help. Consider the situation when N = 5 and assume that guest1 gets back hat2.




As shown, there are 11 derangements if guest1 gets hat2 back. There are another 11 derangements if guest1 gets hat3 back,11 more if guest1 gets hat4 back, and 11 more if guest1 gets hat5 back. So altogether there are 44 possible derangements.

The part of the diagram boxed in red looks very much like the diagram on the previous page for N = 4, and the part boxed in green, with only two outcomes, is the same as for when N = 3. So if we let $\mathrm{D}(n)$ stand for the number of derangements for *n* guests, $\mathrm{D}(5) = 4\big(\mathrm{D}(4) + \mathrm{D}(3)\big)$.

In general, this suggests that $\mathrm{D}(n) = (n-1)\big(\mathrm{D}(n-1) + \mathrm{D}(n-2)\big)$.

| N | D | p(D) |
|---|---|---|
| 1 | 0 | 0 |
| 2 | 1 | 0.50000 |
| 3 | 2 | 0.33333 |
| 4 | 9 | 0.37500 |
| 5 | 44 | 0.36666 |
| 6 | 265 | 0.36806 |
| 7 | 1854 | 0.36786 |

The chart shown to the right, based on the recursive formula from the previous page, gives the number of derangements (D) as a function of the number of guests (N). It also calculates the probability of getting a derangement by dividing D by the total number of ways that the hats could be returned (N factorial).

As the number of guests increases, the number of derangements increases rapidly! Also, p(D) alternates between increasing and decreasing and appears to be approaching a limit.

The values of p(D) also follow a pattern:

$$\mathrm{p}(1) = 1 - \frac{1}{1!} = 0$$

$$\mathrm{p}(2) = 1 - \frac{1}{1!} + \frac{1}{2!} = \frac{1}{2}$$

$$\mathrm{p}(3) = 1 - \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} = \frac{2}{6}$$

$$\mathrm{p}(4) = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} = \frac{9}{24}$$

$$\mathrm{p}(5) = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} = \frac{44}{120}$$

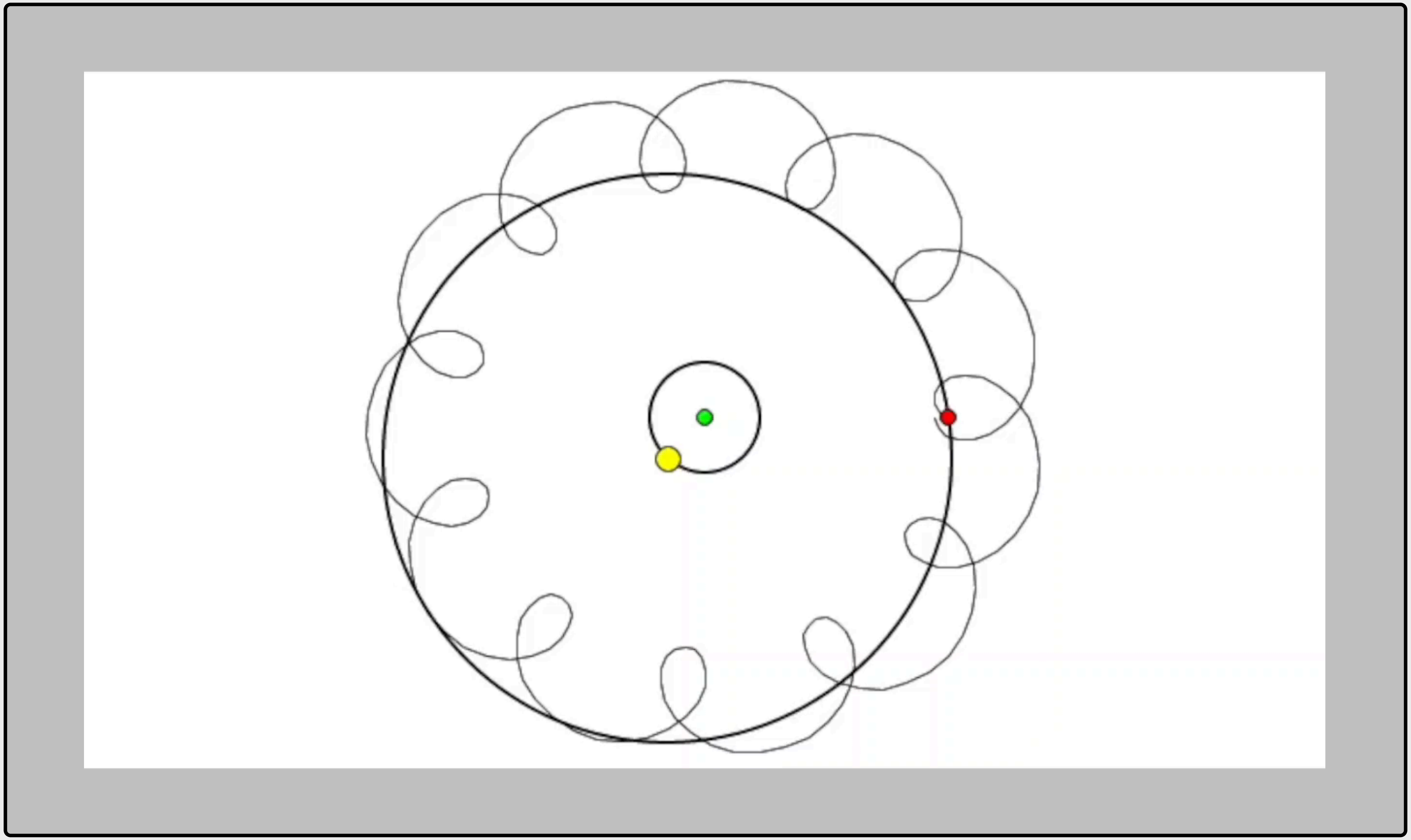
$$\mathrm{p}(6) = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} = \frac{265}{720}$$

Although the proof is beyond the level of this book, in general,

$$\mathrm{p}(n) = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} - \frac{1}{9!} + \cdots + \frac{(-1)^n}{n!}$$ and as $n \to \infty$, $\mathrm{p}(n) \to 1/e$.

Yes, the probability of derangements involves the transcendental constant e.

# 5. Celestial Dances



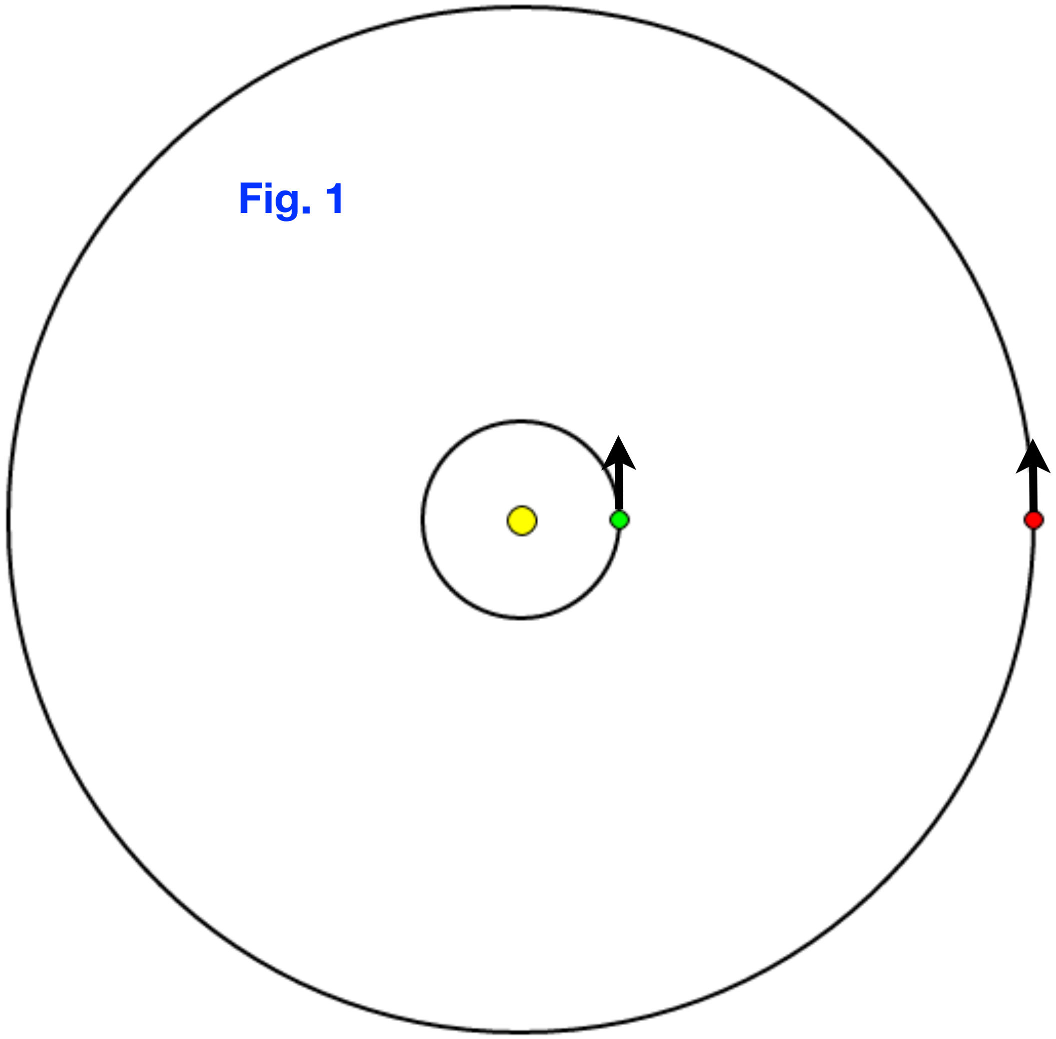
## 5.1 Introduction

The orbits of the planets in our solar system are pretty much concentric circles with the sun at the center. When seen from the viewpoint of the sun, that doesn't seem like a recipe for creating anything very surprising or interesting. But things are different when you observe the motion of one planet from the viewpoint of one of the other planets. To us, for example, Earth seems to be in the center of the solar system. The other planets revolve around the sun, but the sun revolves around us. This is the Tychonic model for the solar system, proposed by 16th century Danish astronomer Tycho Brahe as a compromise between the Earth centered Ptolemaic system and the sun centered Copernican system.

This chapter examines some of the amazingly beautiful geometric relationships that arise when you look at the motion of one planet or moon in our solar system from the point of view of another planet or moon. My **Celestial Dances** program, which simulates planetary motion, can be used to discover many more. I was inspired to write the program after reading A Little Book of Coincidence by John Martineau, a wonderful little book that reveals all sorts of unexplained "coincidences." The program is available for free from my **Syzygy Shareware** site at http://tcbretl.weebly.com.

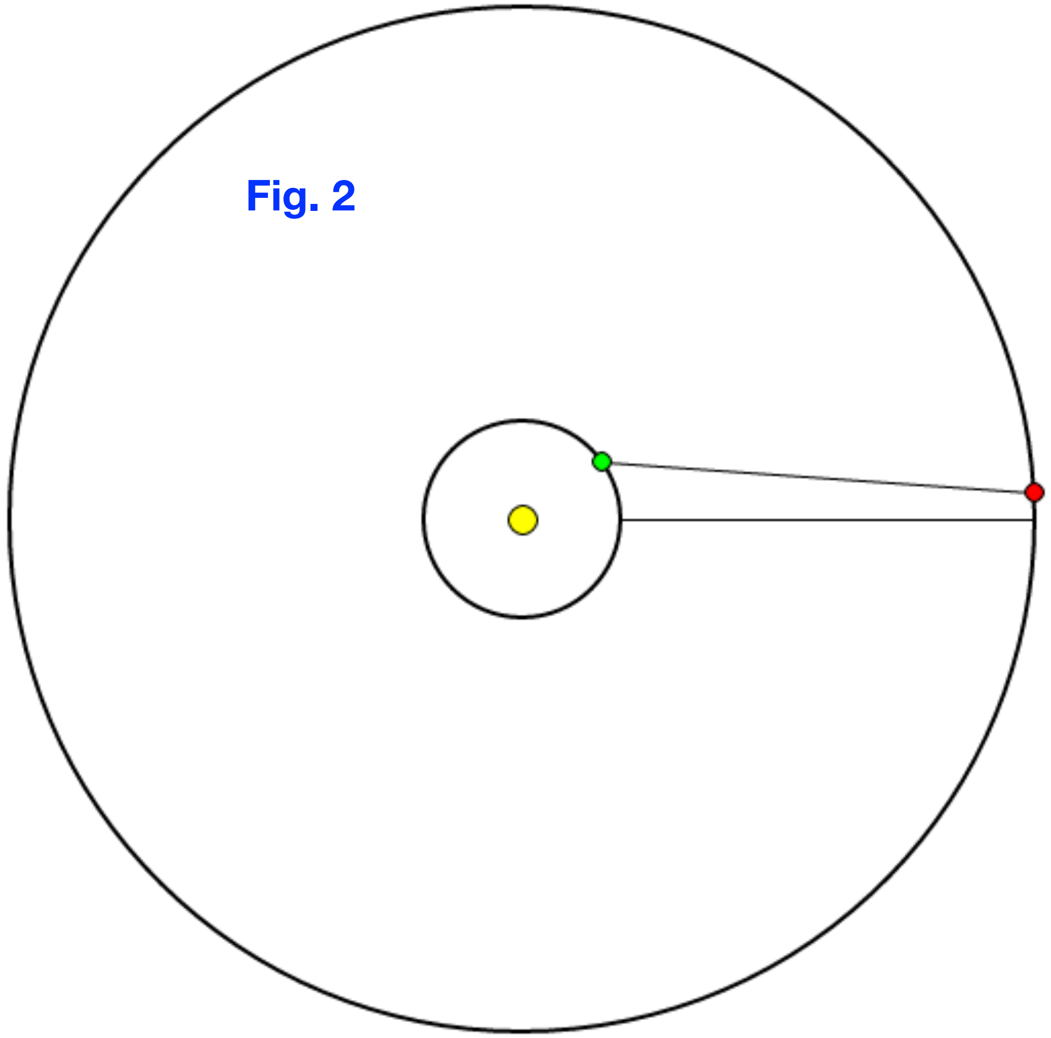
## 5.2 Earth and Jupiter

Figure 1 shows the orbits of Earth and Jupiter in a sun centered solar system as seen looking down from above the Earth's north pole. Earth is green, Jupiter is red, and the sun is yellow. Jupiter is approximately 5.2 times as far away from the sun as Earth, and takes almost 12 years to revolve once around the sun.




Both planets move counter-clockwise around their orbits. Figure 1 also shows Jupiter at **opposition**, a time when it would appear highest in the night sky at about midnight.

Earth moves faster in its orbit than Jupiter does. As we pass Jupiter, it will appear to go backwards (like when you pass another car on the highway). It's called **retrograde** motion. Figure 2 shows the planets' positions 1/10 of a year after opposition, with their positions connected.




Jupiter has moved towards the top of the page, but to us on Earth it seems to have moved towards the bottom.

Fig. 3

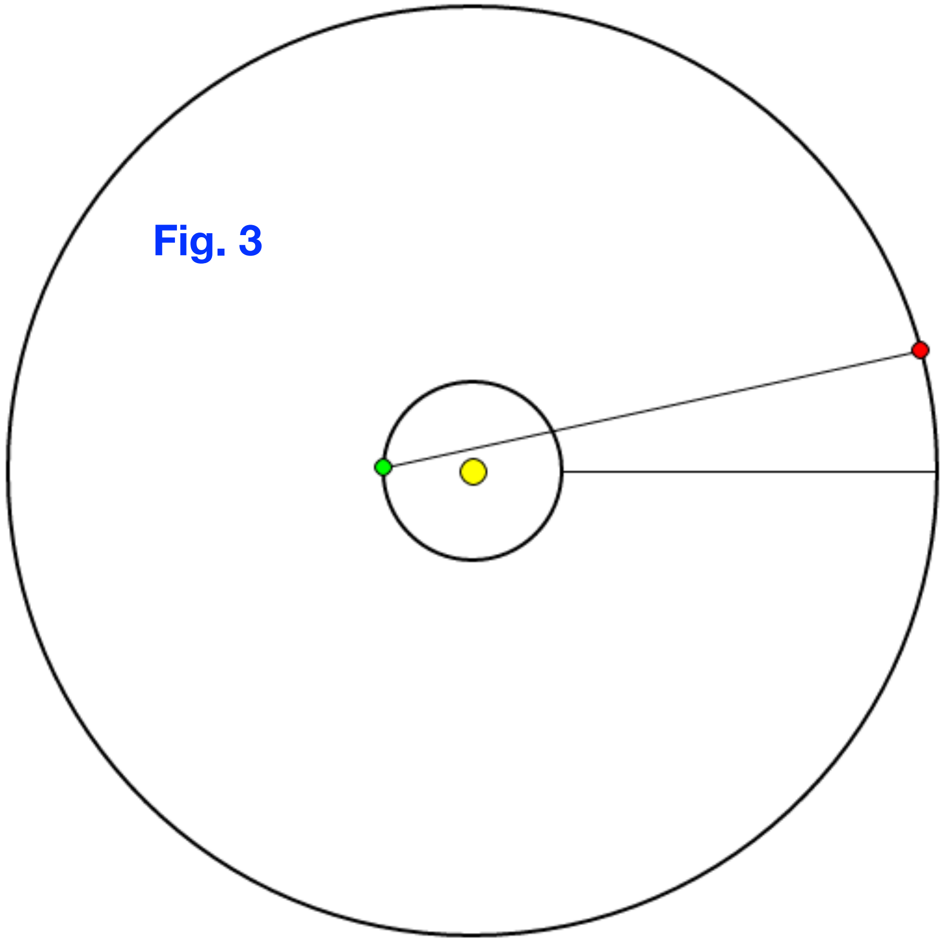
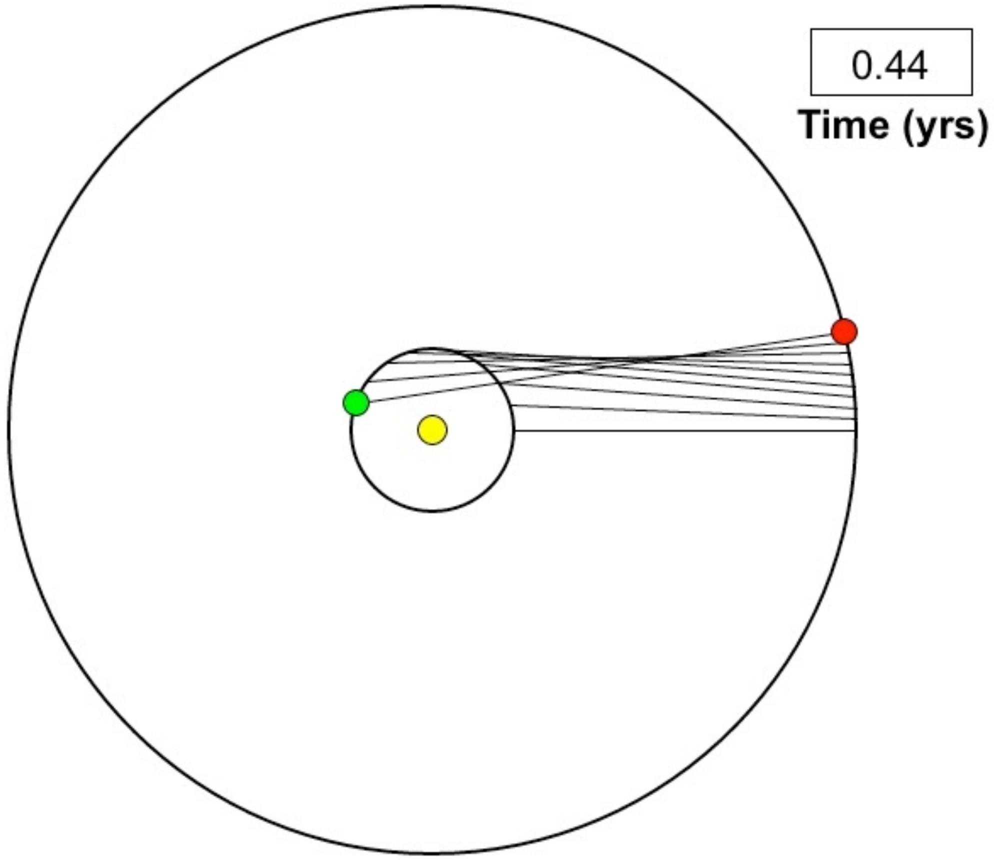
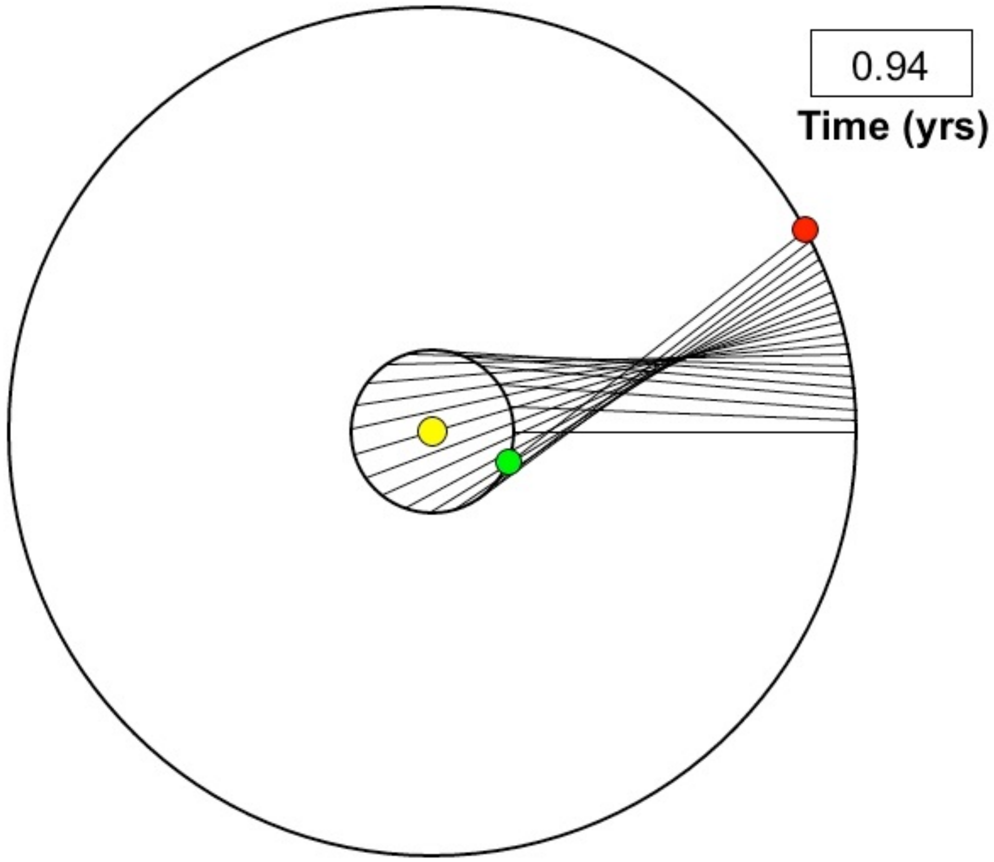
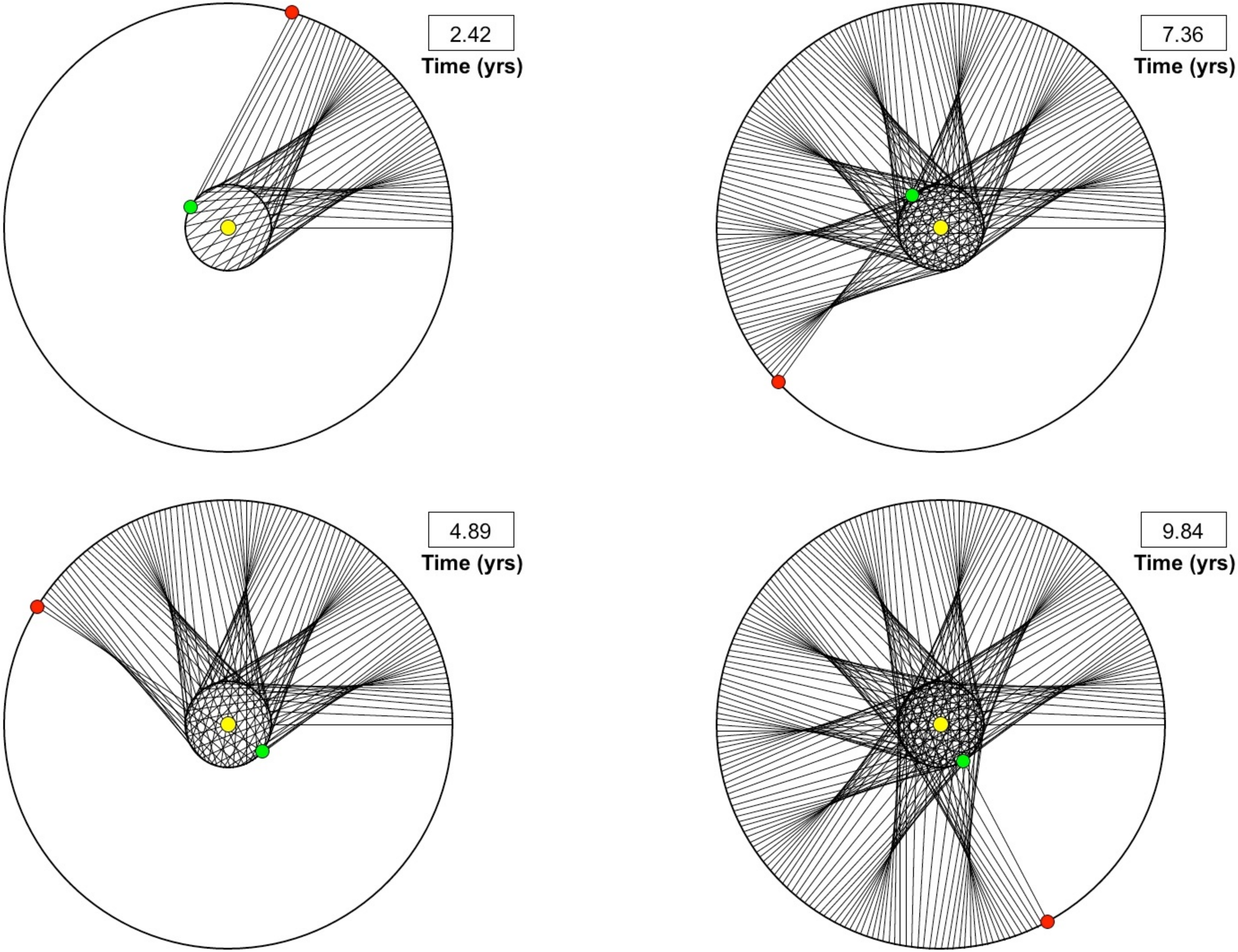


Figure 3 shows the positions of the two planets 1/2 year after opposition. It no longer looks to us like Jupiter is moving backwards. That will continue until Earth starts to catch up and pass Jupiter again. Every 13 months, approximately, there is another opposition and another period of retrograde motion. Finally, after approximately 12 years, the two planets return again to almost the same positions as they were in figure 1.

If you plot and connect the positions of Earth and Jupiter approximately 20 times per year over a period of 12 years, here is what happens.

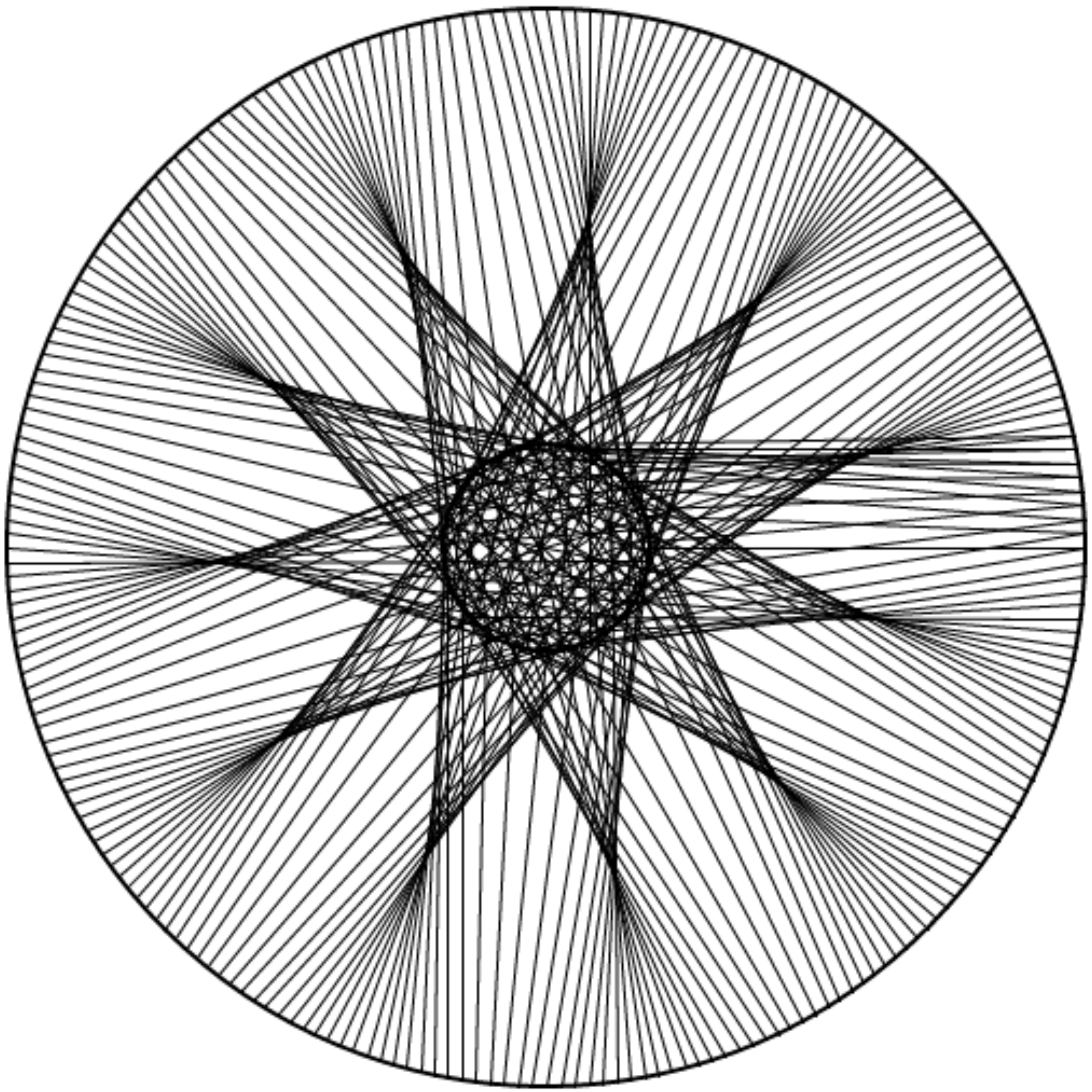



2.42
Time (yrs)
7.36
Time (yrs)
4.89
Time (yrs)
9.84
Time (yrs)

Finally, after 12 years, you get this string-art pattern.

The oppositions occur halfway between the points of the star.



# Putting Earth at the Center

Now consider how things look in the Tychonic solar system. Earth doesn't move, the sun revolves around Earth, and Jupiter revolves around the sun as shown in figure 4.

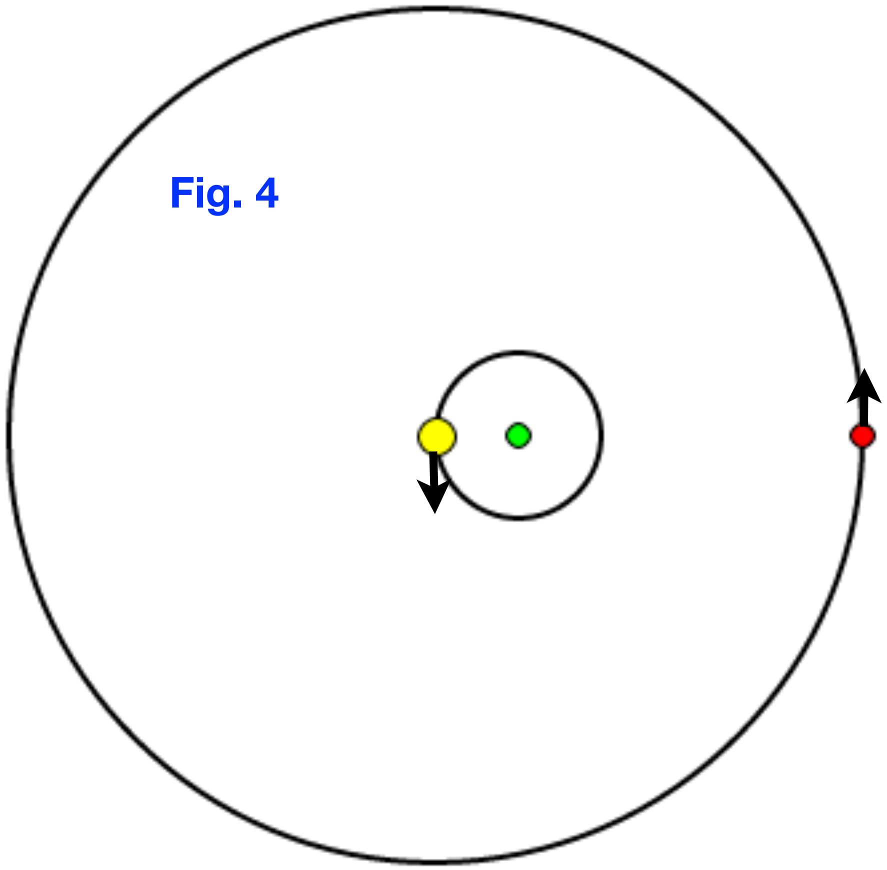


Fig. 4

During the next three years, Jupiter moves 1/4 of the way around the sun in a nice circular orbit, but because of the sun's motion around Earth, Jupiter appears (as seen by us on Earth) to follow the looping path shown in figure 5.

Each loop marks a period of retrograde motion, and the part of the loop closest to Earth marks the time of opposition. Geometrically, this model is exactly equivalent to the

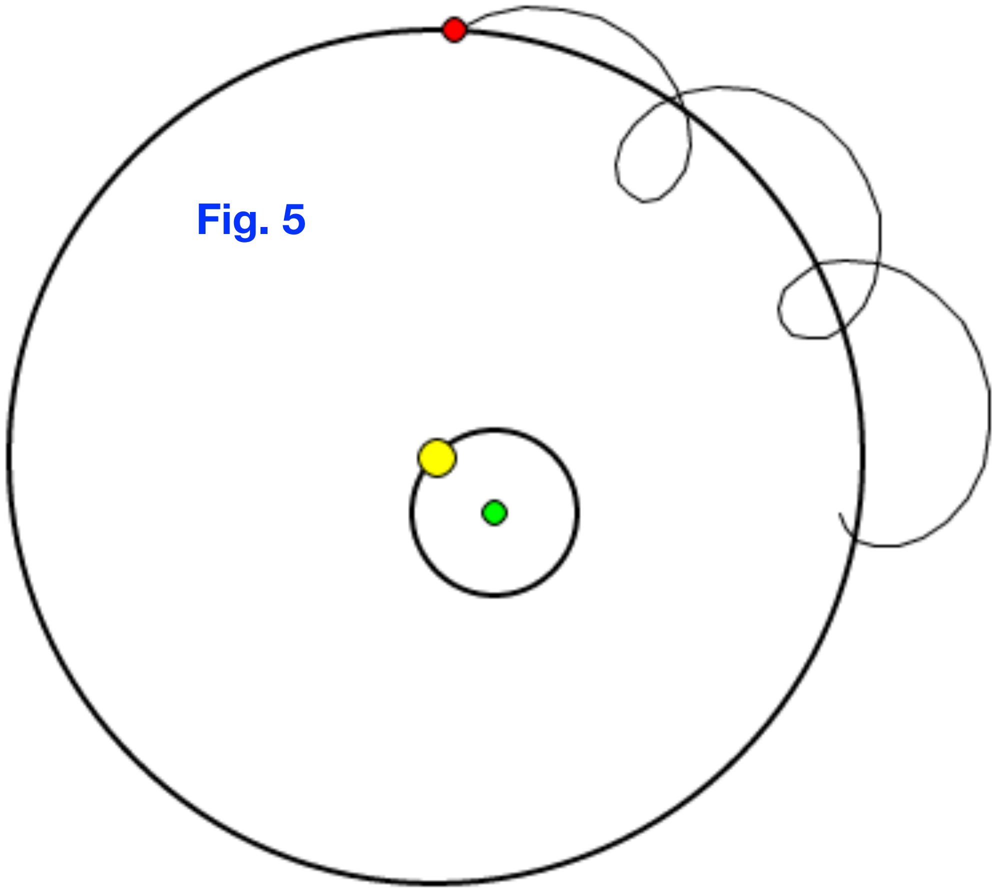
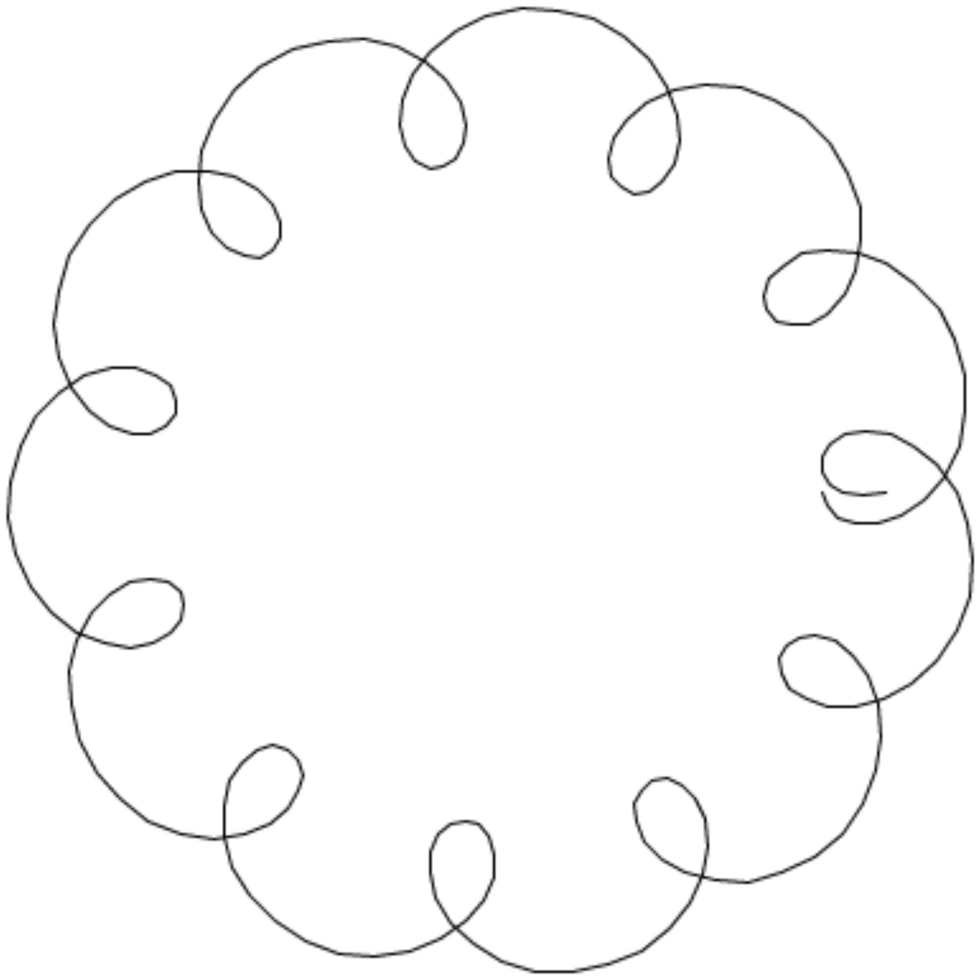


Fig. 5

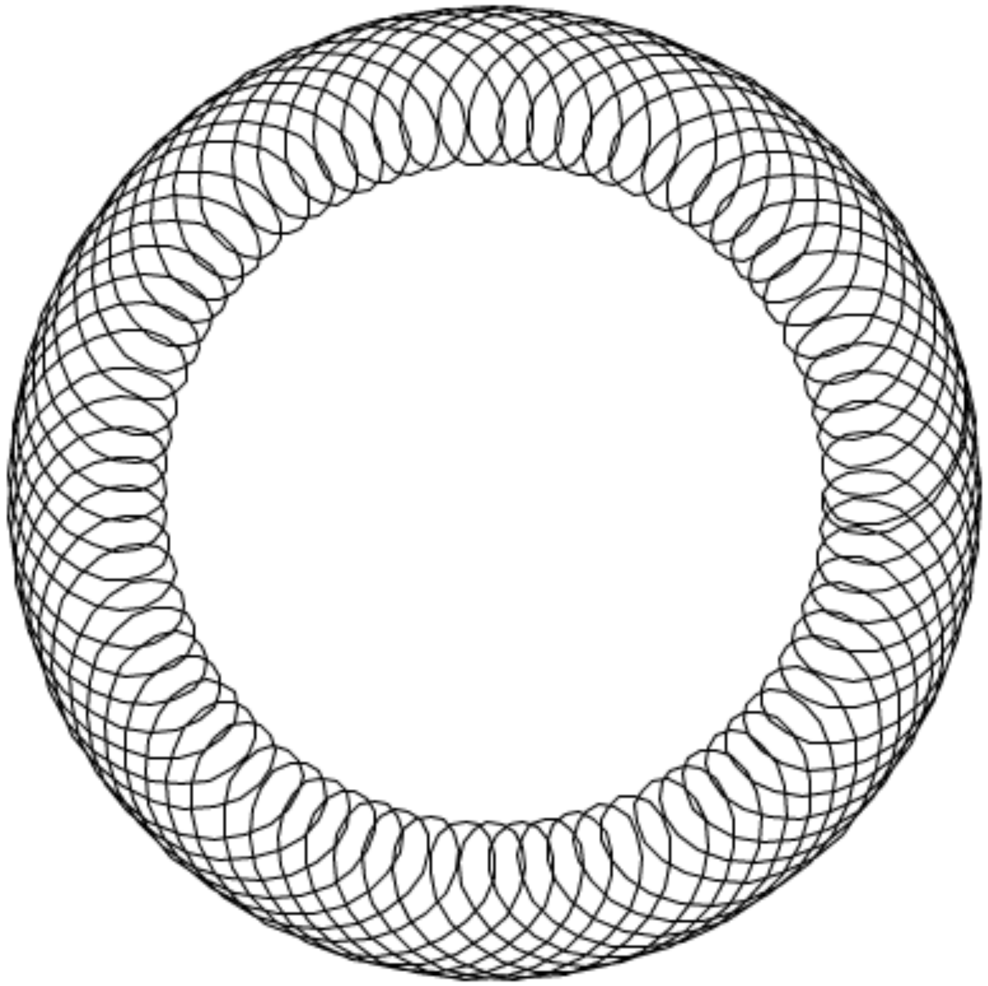
Copernican one. What has changed? Only that we are now observing Jupiter's motion relative to Earth instead of relative to the sun. To fully understand and appreciate how this works, obtain the **Celestial Dances** program and use it to simulate the motion.

If you run the simulation for 12 years, Jupiter's motion will look like this:



Things don't connect up perfectly in the end because Jupiter's period is not exactly 12 years.

If you run the simulation for about 80 years, Jupiter's motion will look like this:

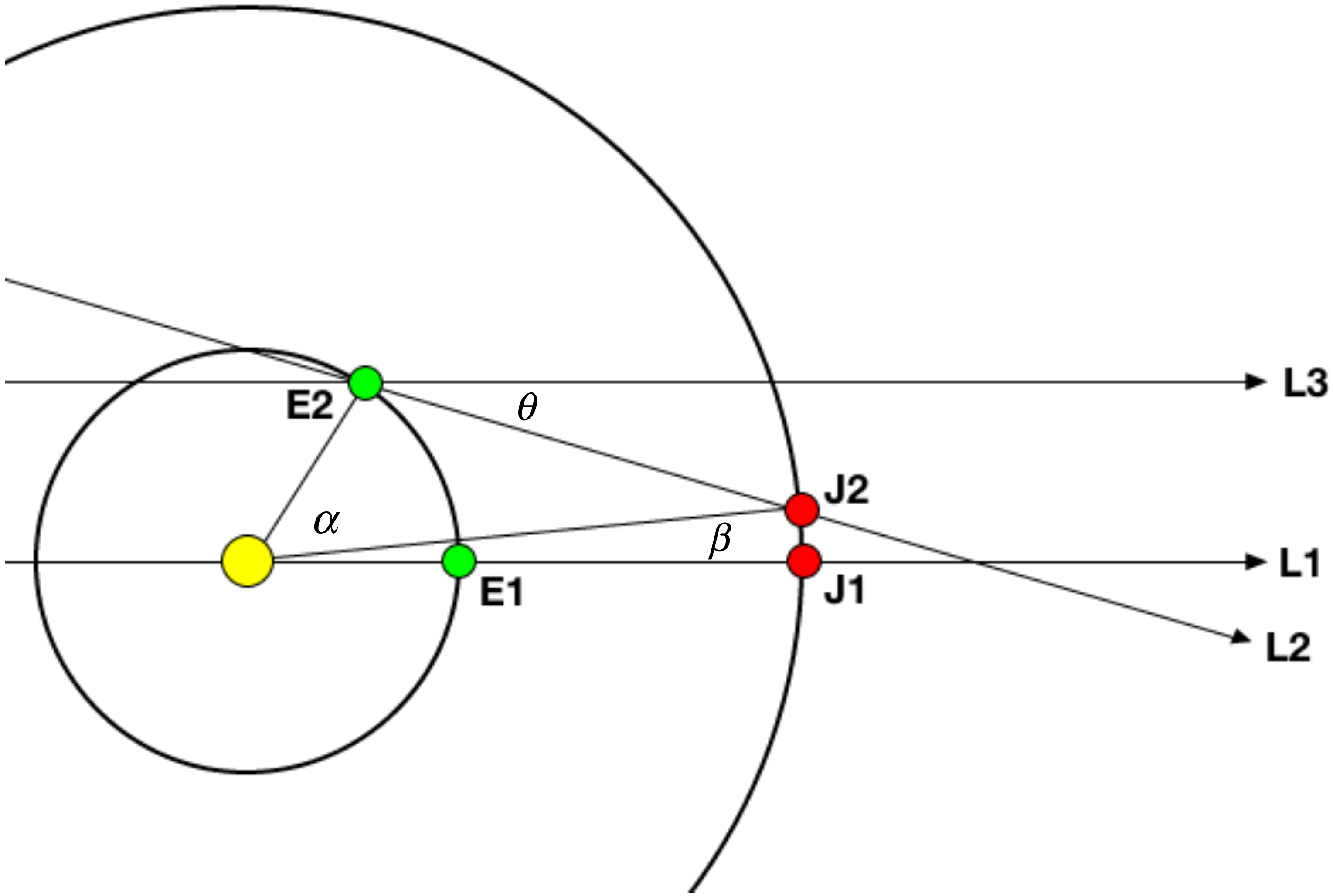


Things still don't connect up perfectly, but it's close!

## 5.3 Retrograde Motion

Suppose Earth and Jupiter start out lined up with the sun along line L1. Earth then revolves through an angle $\alpha$ to point E2 in the same time it takes for Jupiter to revolve through an angle $\beta$ to point J2. To an observer on Earth, the direction to Jupiter shifts from L1 to L2, and since L3 is parallel to L1, $\theta$ is the angular shift in Jupiter's position.

Let the sun be the origin, and L1 the x-axis, of a coordinate system. Also let r be the radius of Earth's orbit, and R be the radius of Jupiter's orbit. The coordinates of E2 are $\left(r\text{Cos}\alpha, r\text{Sin}\alpha\right)$. The coordinates of J2 are $\left(R\text{Cos}\beta, R\text{Sin}\beta\right)$.



Measured in **astronomical units**, r = $1$ and R = $5.2$, so

$$\tan(\theta) = \text{m (the slope of line L2)} = \frac{(5.2)\sin\beta - \sin\alpha}{(5.2)\cos\beta - \cos\alpha}$$

Because $\tan(\theta) = \dfrac{(5.2)\sin\beta - \sin\alpha}{(5.2)\cos\beta - \cos\alpha}$,

$$\theta = \arctan\left(\frac{(5.2)\sin\beta - \sin\alpha}{(5.2)\cos\beta - \cos\alpha}\right)$$

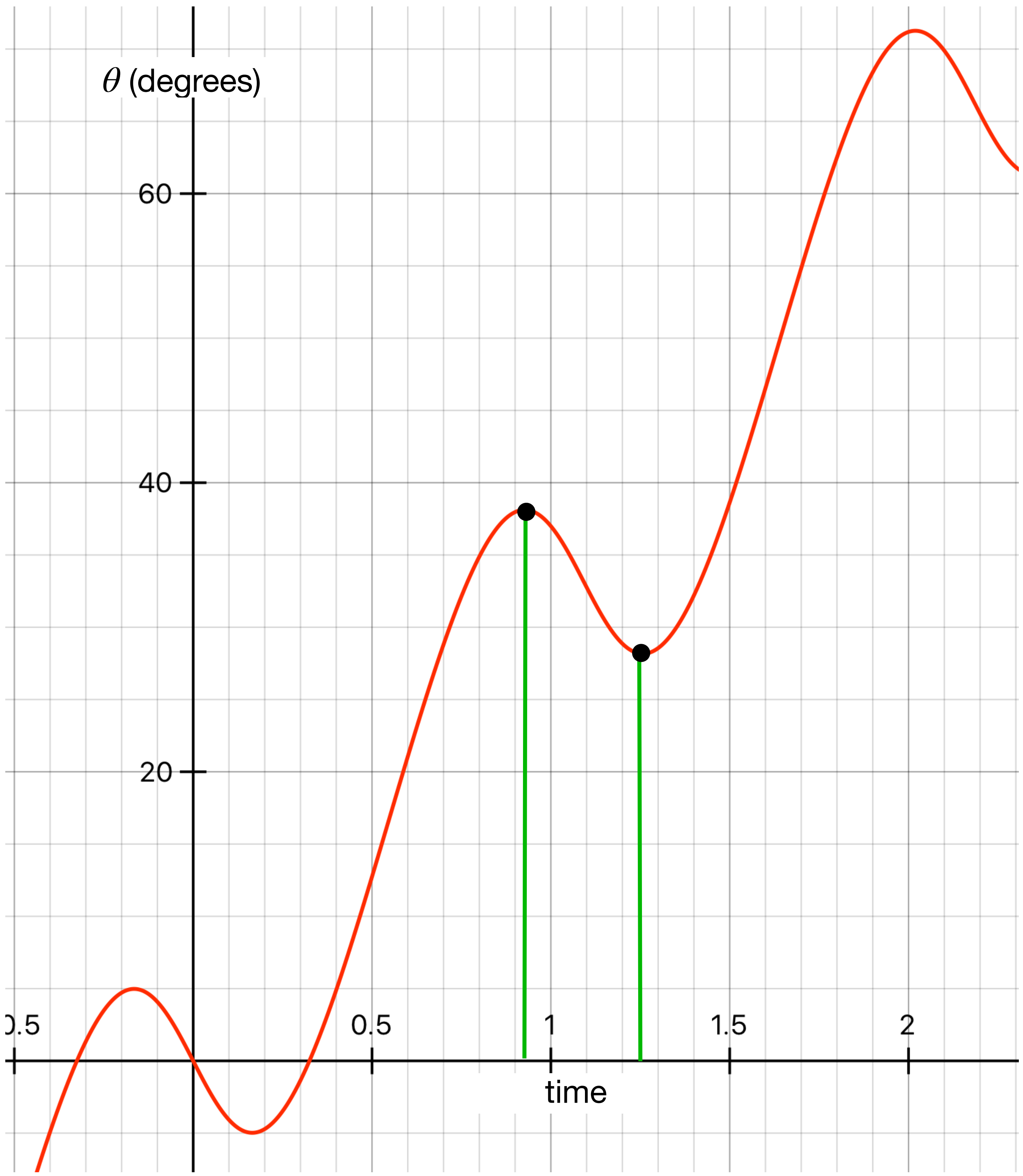
If t is the time, measured in years since Earth and Jupiter were at positions E1 and J1, then

$\alpha$ = 360t because earth revolves 360° per year and

$\beta$ = 30.35t because Jupiter revolves only 30.35° per year.

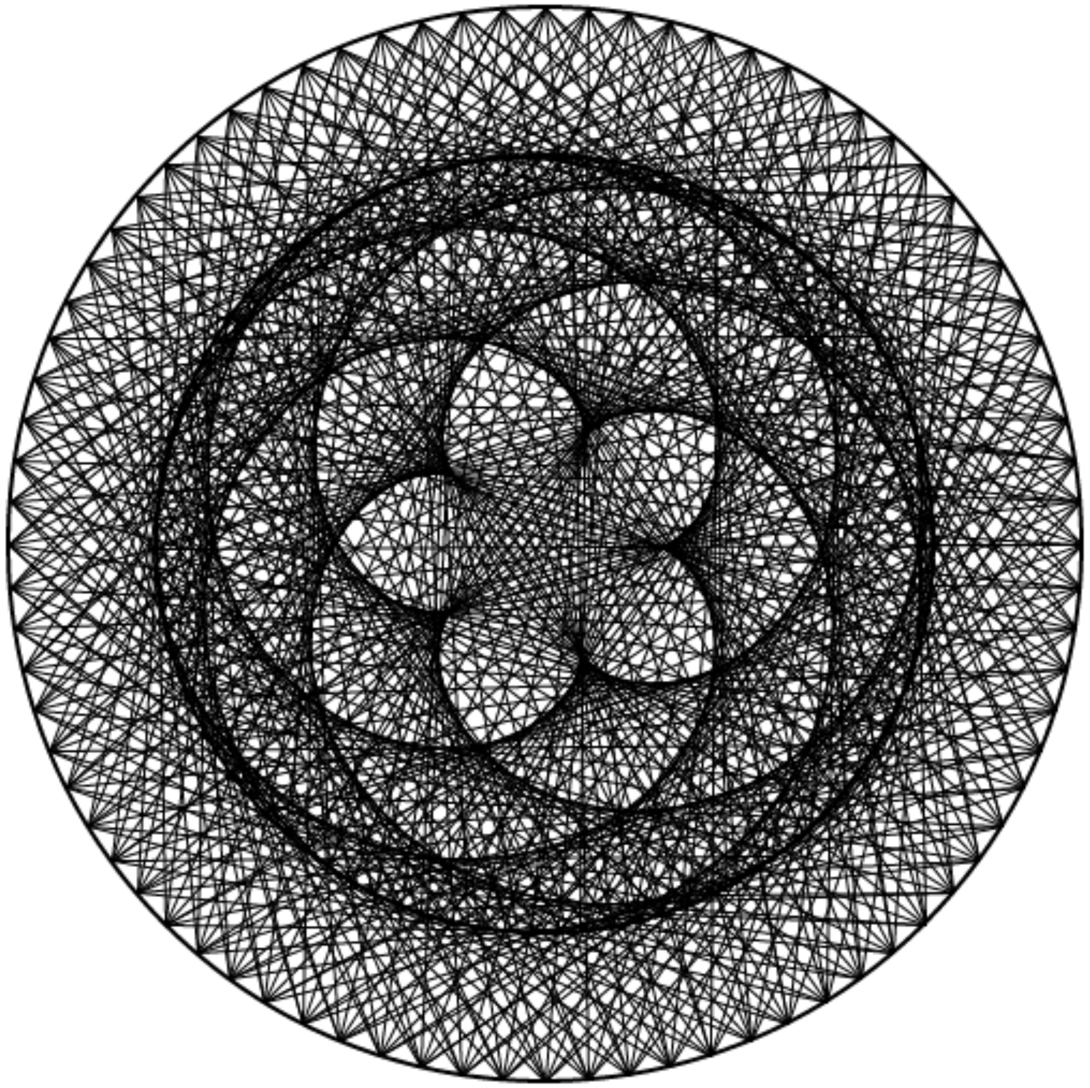
$$\text{So } \theta = \arctan\left(\frac{(5.2)\sin(30.35t) - \sin(360t)}{(5.2)\cos(30.35t) - \cos(360t)}\right)$$

Here is a graph of that function. Retrograde motion is occurring when $\theta$ is decreasing. One place this occurs is between t = 0.93 and t = 1.26 years. So the retrograde motion lasts for 0.33 years or about 4 months.



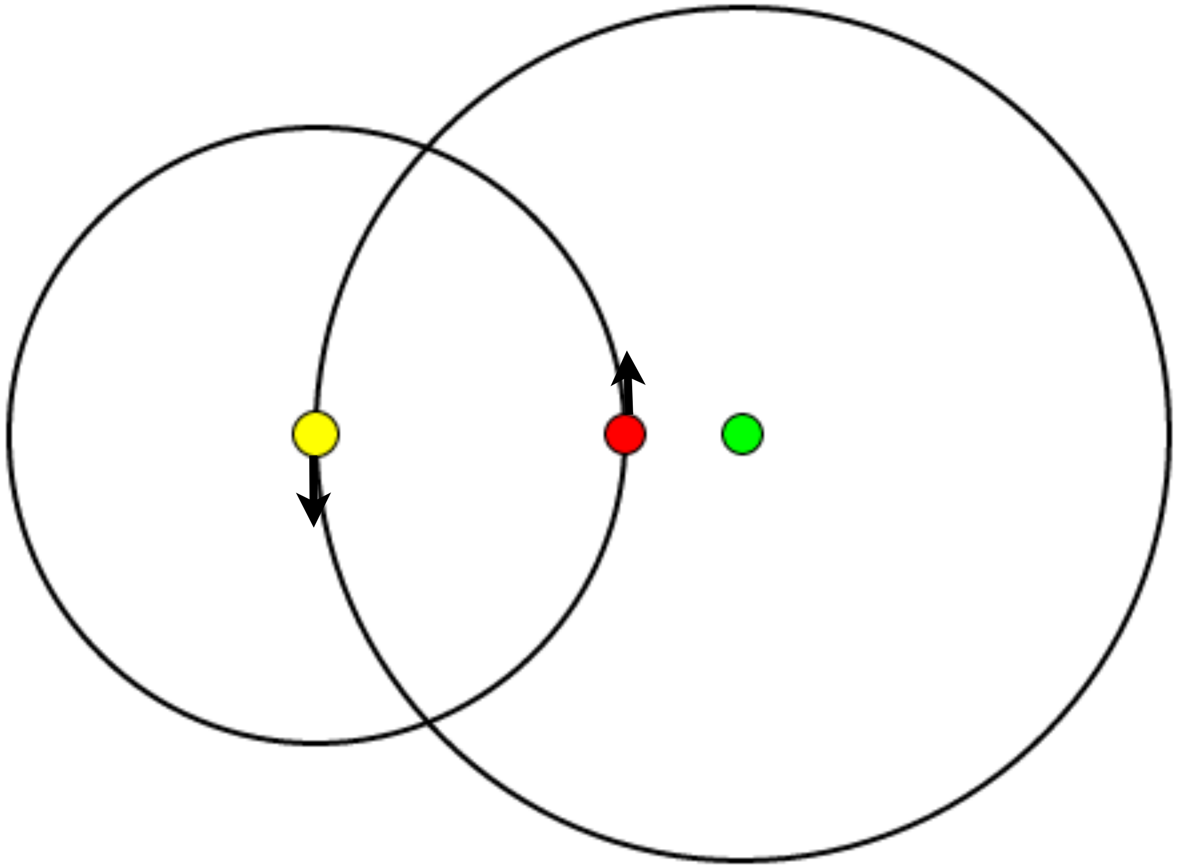
## 5.4 Earth and Venus

The distance from Venus to the sun is a little less than three-fourths the distance from Earth to the sun. Venus completes one revolution around the sun in 224.7 days, so 13 Venus years almost exactly equals 8 Earth years. If you connect the positions of Earth and Venus as they revolve around the sun, this is what the pattern looks like after 8 years.

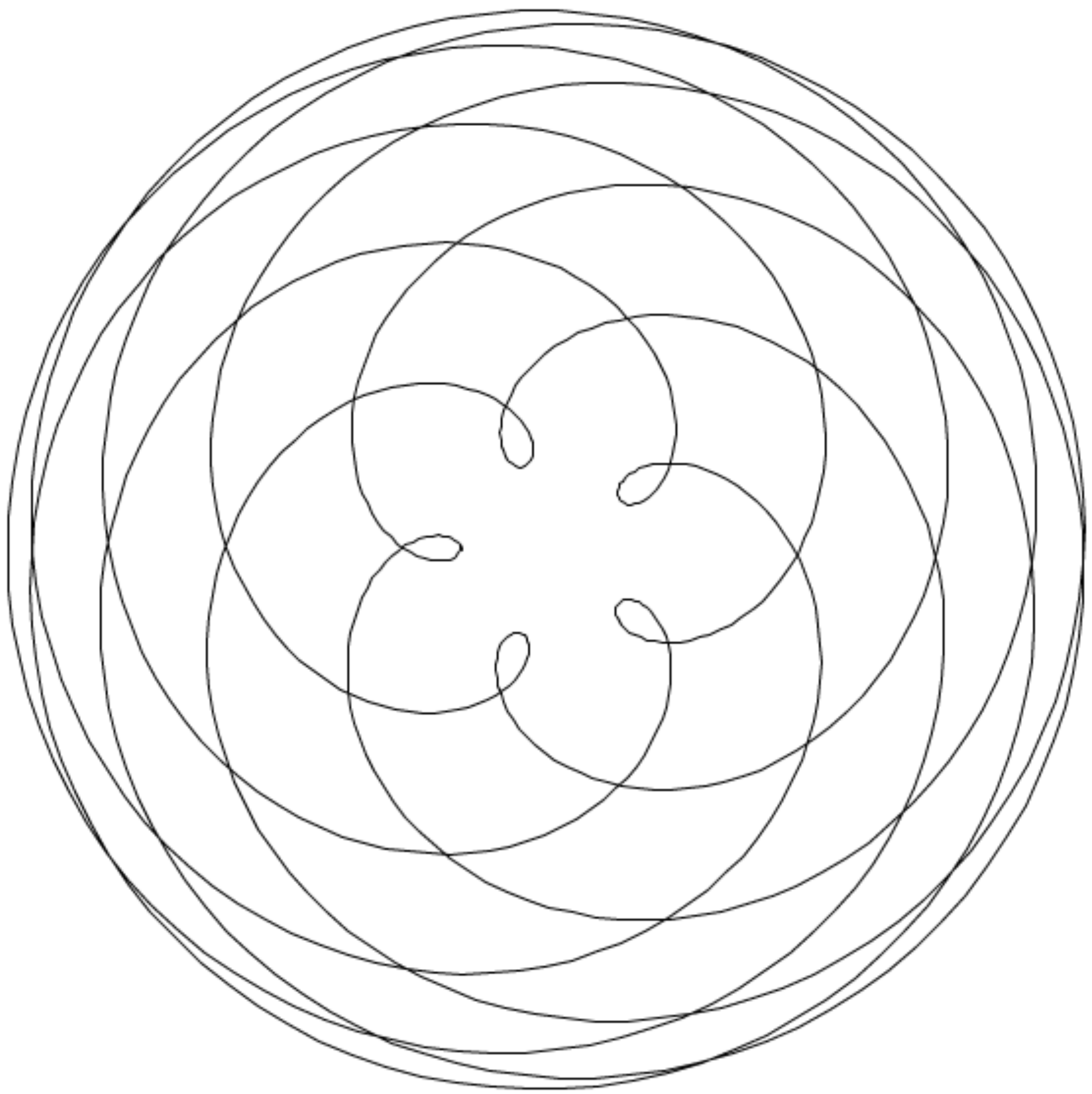


The flower in the middle has 5 petals because Venus has to catch up and pass Earth 5 times during the 8 years. If you continue beyond 8 years, the pattern repeats almost exactly.

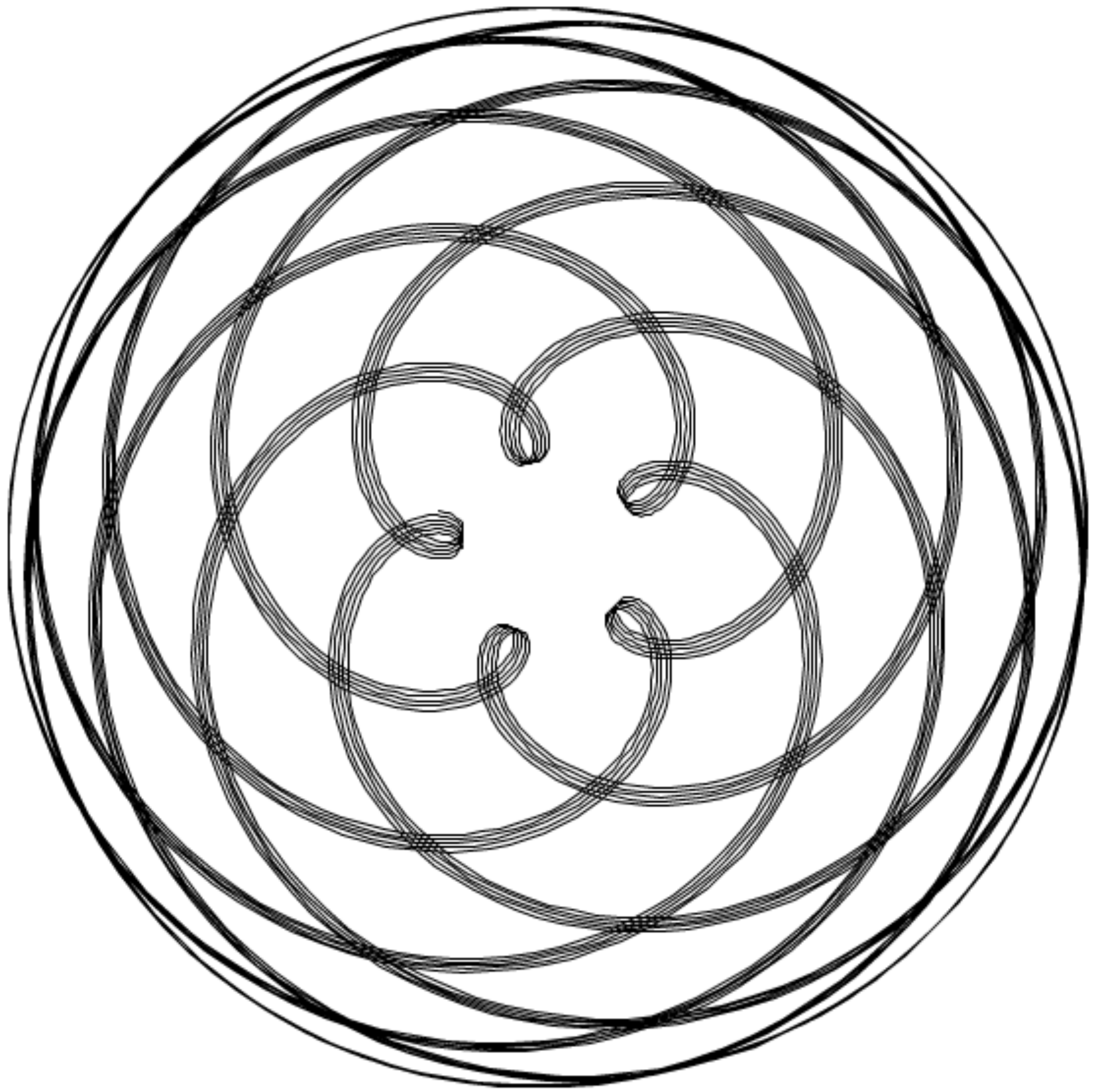
Again consider how things would look in a Tychonic solar system. Earth is stationary, and Venus revolves around the sun as the sun revolves around Earth. Earth is green, Venus is red, and the sun is yellow.



Assuming that the planets start in the positions shown on the previous page, Venus spends 8 years tracing out the loopy path shown below. Note that Venus's starting position is between Earth and the sun. It is in **conjunction** with the sun.



During ensuing 8 year periods, Venus almost retraces the same path. Approximately 40 years of motion is shown below. The loops are very slowly rotating in a clockwise direction. It will take 243 years for them to rotate all the way back to their original positions.



The slow change in Venus's path relative to Earth is the reason a **transit** of Venus is so rare. A transit occurs when Venus passes directly between Earth and the sun. If the plane of Venus's orbit were the same as Earth's, a transit would occur five times every 8 years. But because the plane of Venus's orbit is inclined by 3.4°, most of the time Venus passes either north or south of the sun.

There are two times during one Venus year when Venus crosses the plane of Earth's orbit. The places where this happens are called **nodes**. They are on opposite sides of Venus's orbit. A transit occurs only when Venus is in conjunction with the sun at the same time it is at, or very near, one of the nodes. In the diagrams on this page, assume that the nodes are at points directly to left or right of the sun.

Figure 1 traces Venus's motion during the first 8 years. Venus is both in conjunction and at a node. A transit is taking place. Approximately 8 years later, a conjunction will take place a little before Venus reaches that same node, so a transit may not occur.

Figure 2 traces Venus's motion between years 64 and 72. Now Venus reaches conjunction well before it reaches a node. It is neither directly to the left nor directly to the right of the sun. There will definitely be no transit.

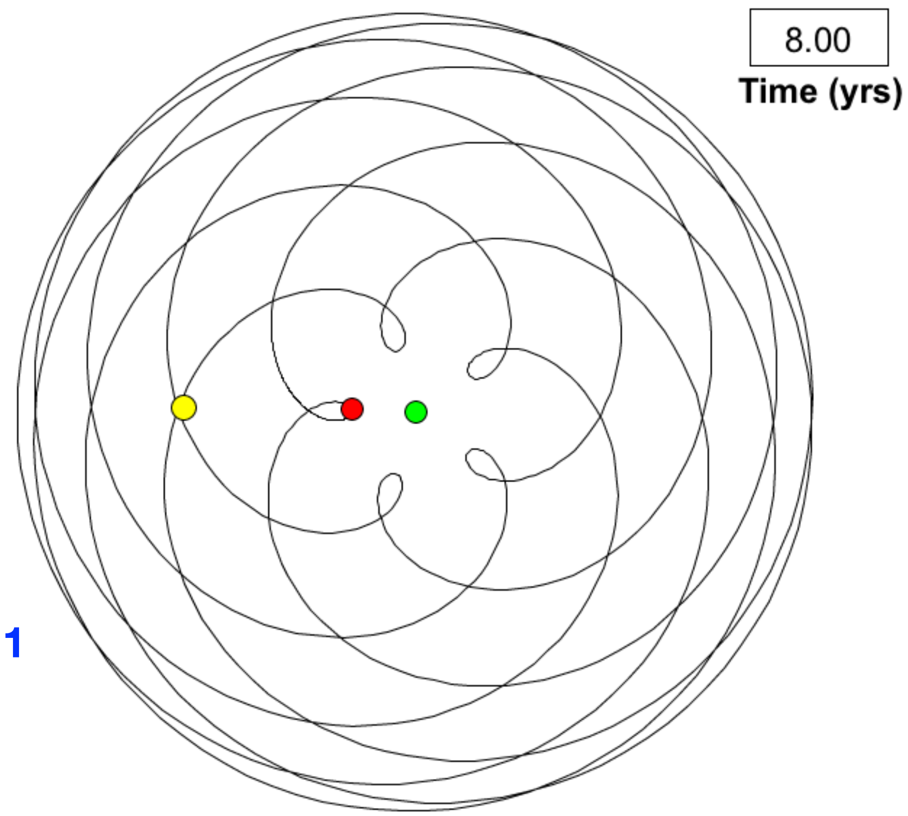


**Fig. 1**

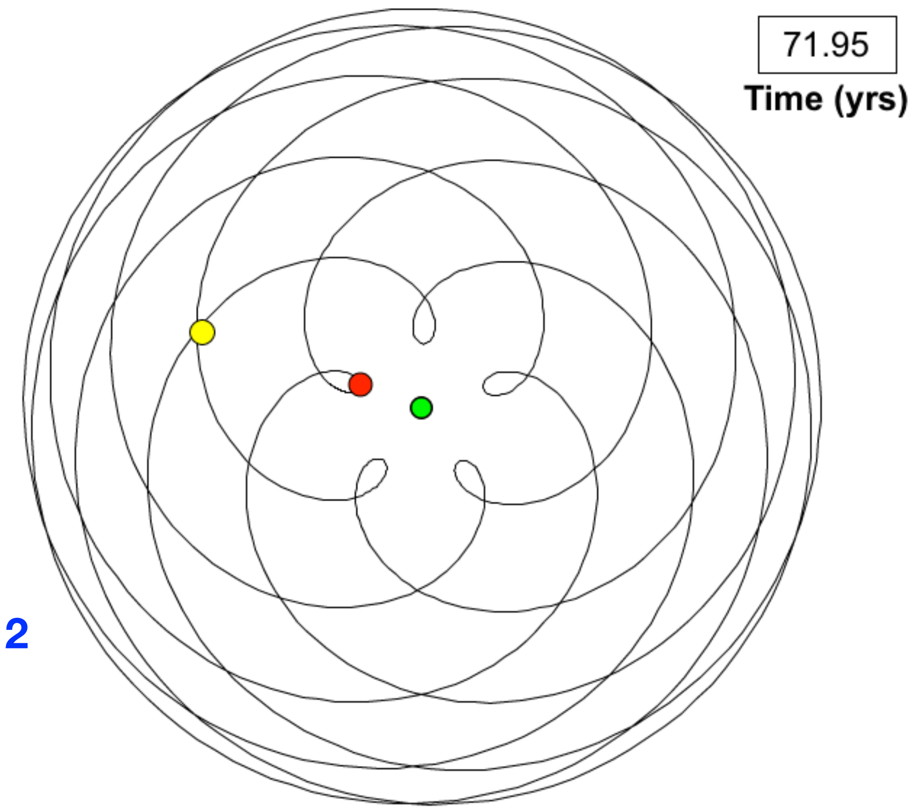
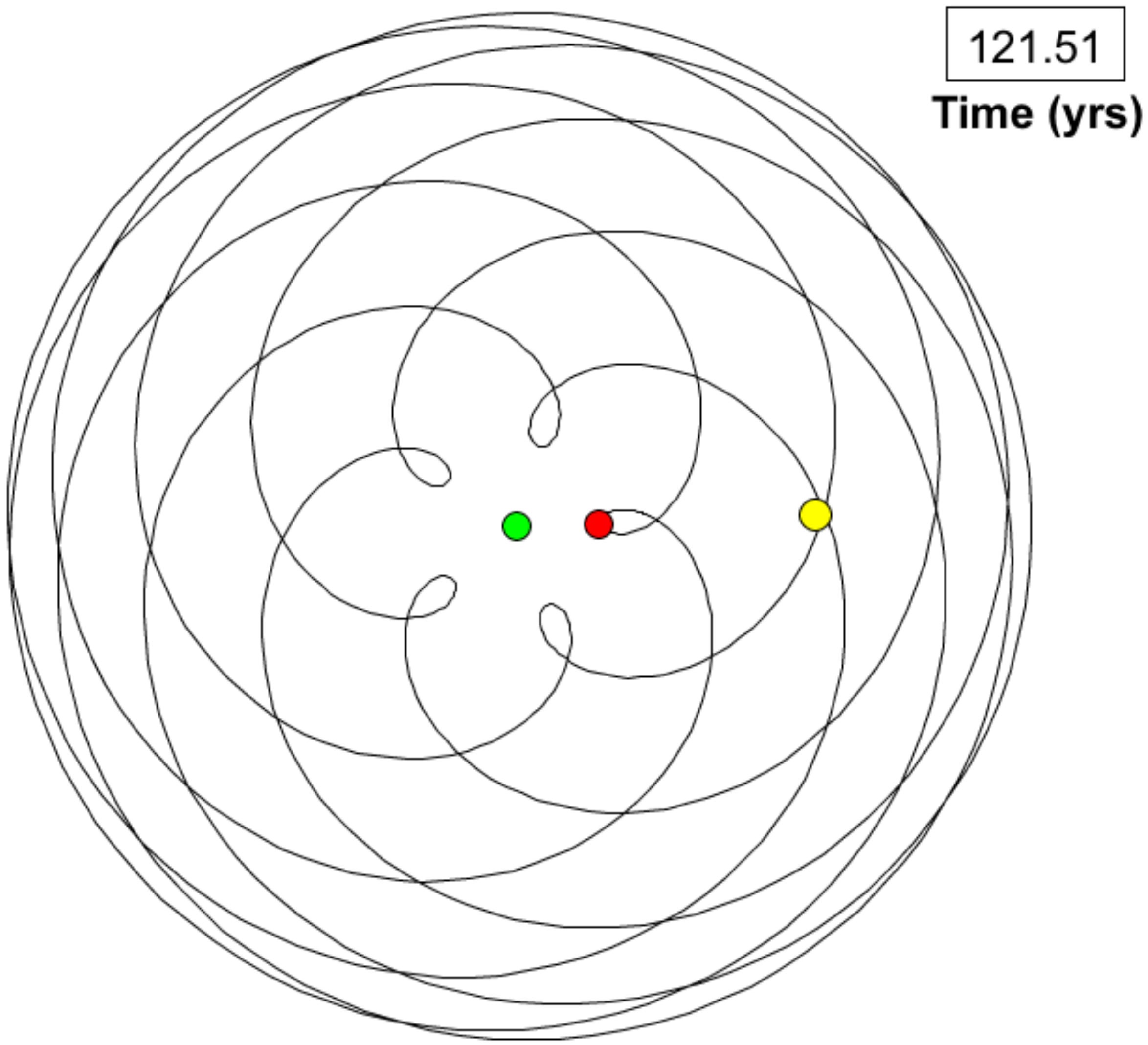


**Fig. 2**

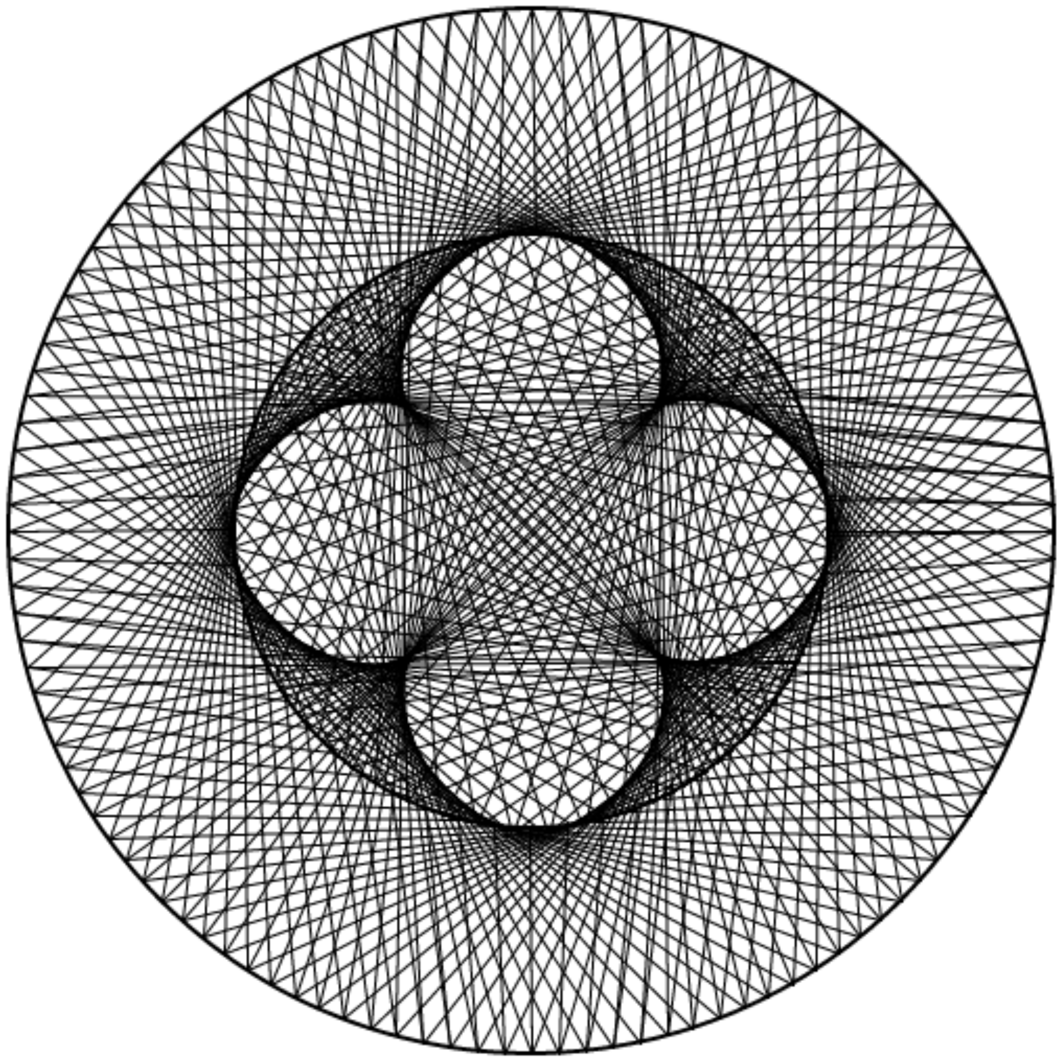
Finally, here's a trace of Venus's motion between years 113 and 122. Venus is shown both in conjunction and at a node. But it is not the same node. This time Venus is directly to the left of the sun.

During its conjunction 8 years later, Venus may still be near enough to this node for a transit to occur. After that, there will be another gap of over 100 years!

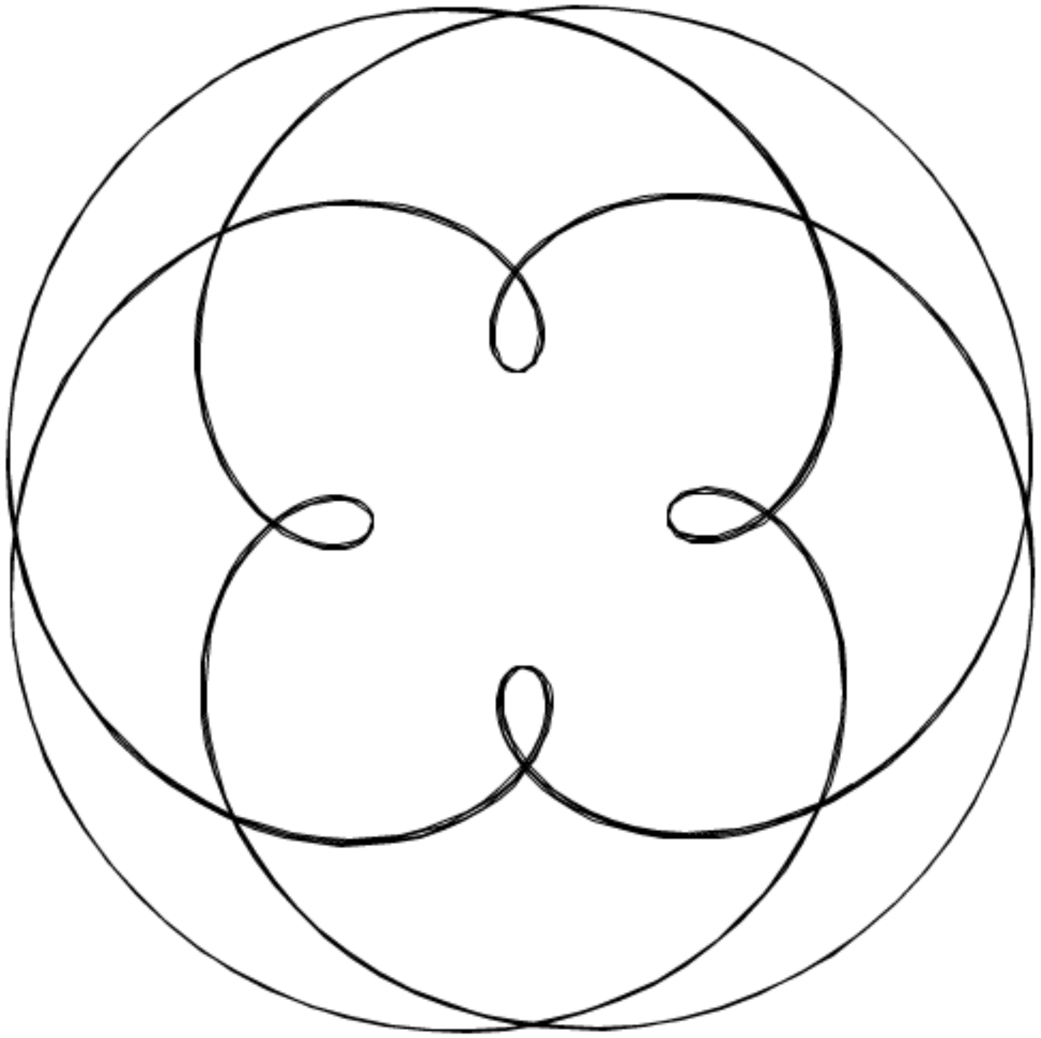


## 5.5 Moons of Jupiter

Similar string-art patterns are created by the relative motions of satellites. Here, for example, is the relationship between the motions of Ganymede and Callisto, two moons of Jupiter. This four-lobed pattern repeats almost exactly.



**Jupiter in the Center**



**Ganymede in the Center**

# 6. Magic Squares

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | 9 |

## 6.1 Introduction

The Chinese knew about magic squares more than 2000 years ago, and much has been written about them since that time. There are many different types of magic squares, and many different strategies for creating them.This chapter touches on just a few things that perhaps will catch your interest.

My **SquArray** program makes it easy for you to “play around” with various types of magic squares. See if you can re-create some known ones, or, perhaps, discover some new ones. You can explore the Lo Shu Square, prime squares, multiplicative squares, doubly even squares, the Sagrada Familia square, rotatable squares, and squares of squares.

As with all the other programs discussed in this book, **SquArray** is available for free from my **Syzygy Shareware** site at http://tcbretl.weebly.com. SquArray solutions can be found at http://reckonsupport.blogspot.com/p/squarray-solutions.html.

## 6.2 Magic Squares (3x3)

The Lo Shu Square is a basic 3x3 magic square that dates back to a very old Chinese legend about a turtle with symbols representing the numbers 1 though 9 on its back.

To make it magic, you must rearrange the numbers so that all three rows, all three columns, and both diagonals add up to the same thing.

One “common sense” strategy is to leave the 5 in the middle (since it is the average value), spread out the three smallest numbers so that no two of them fall in the same row or column, and put the 2 in a corner spot (why?).

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | 9 |

| | | |
|---|---|---|
| 8 | 4 | 2 |
| 3 | 5 | 6 |
| 7 | 1 | 9 |

Since the sum of all nine numbers is 45, each row and each column must total 15 (the **Magic Constant**). That makes it easy to rearrange the rest of the numbers into their proper positions. The top of the middle column, for example, must be 9 because 9+5+1 = 15, and that means that the top left number must be 4 because 4+9+2 = 15. Try to finish the magic square on your own. There is really only one solution. Other solutions are just rotations or reflections of it. The solution is shown on the next page.

| | | |
|---|---|---|
| 4 | 9 | 2 |
| 3 | 5 | 7 |
| 8 | 1 | 6 |

**Lo Shu Square**

Here is another way to figure out how the numbers must be arranged to make the Lo Shu Square:

1. List the eight different ways there are to get a sum of 15 using three different numbers between 1 and 9.

   1 + 5 + 9     2 + 6 + 7

   1 + 6 + 8     3 + 4 + 8

   2 + 4 + 9     3 + 5 + 7

   2 + 5 + 8     4 + 5 + 6

2. Count up the total number of times each number appears somewhere in those eight sums.

   1 appears twice     2 appears three times

   3 appears twice     4 appears three times

   7 appears twice     6 appears three times

   9 appears twice     8 appears three times

   5 appears four times

3. The number 5 must go in the middle because that is the only spot where it will contribute to four sums (a row, a column, and both diagonals).

4. The numbers 2, 4, 6, and 8 must go in the corner spots where each will contribute to exactly three sums (a row, a column, and one diagonal). To make the diagonals add up to 15, the 2 must be in the opposite corner from the 8; the 4 in the opposite corner from the 6.

5. The numbers 1, 3, 7, and 9 must go in the middle spots of the top row, bottom row, left column, and right column where each will contribute to only two sums (one row and one column). To make the rows and columns add up to 15, the 9 must be opposite the 1; the 7 opposite the 3.

## Prime Square

All the numbers in the array below are prime numbers. It can be turned into a magic square by using exactly the same arrangement as in the Lo Shu Square. Simply rank the numbers from smallest to largest, and then use those ranks to place the numbers. 5 is smallest number, for example, so it should go where the 1 went in the Lo Shu Square.

| | | |
|---|---|---|
| 5 | 17 | 29 |
| 47 | 59 | 71 |
| 89 | 101 | 113 |

But how did someone know that it would be possible to rearrange this particular set of numbers to make a magic square?

Examine the numbers closely:

| | |
|---|---|
| 5 = 17 - 12 | 29 = 17 +12 |
| 47 = 59 - 12 | 71 = 59 +12 |
| 89 = 101 - 12 | 113 = 101 + 12 |

Things work out nicely if you put 17, 59, and 101 on one diagonal, and place the other numbers in such a way that the minus 12's cancel out with the plus 12's.

| | | |
|---|---|---|
| **59-12** | **101+12** | **17** |
| **17+12** | **59** | **101-12** |
| **101** | **17-12** | **59+12** |

This same technique can be used to create other magic squares. The hard part is to use only prime numbers. See if you can do it! Another example is given in Chapter 10.

## Multiplicative Magic Squares

In a multiplicative magic square, the product of the numbers in each row, each column, and each diagonal are the same. Consider, for example, this 3x3 array of numbers:

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 6 | 9 |
| 12 | 18 | 36 |

The product of all nine numbers is 10,077,696, so the numbers must be rearranged in a way that makes the product of each row, column, and diagonal equal to 216 (the cube root of 10,077,696).

If you factor the nine numbers, however, you find that they all involve only powers of 2 and 3.

$1 = 2^0 \cdot 3^0$

$2 = 2^1 \cdot 3^0$

$3 = 2^0 \cdot 3^1$

$4 = 2^2 \cdot 3^0$

$6 = 2^1 \cdot 3^1$

$9 = 2^0 \cdot 3^2$

$12 = 2^2 \cdot 3^1$

$18 = 2^1 \cdot 3^2$

$36 = 2^2 \cdot 3^2$

The product of all nine numbers is $2^9 \cdot 3^9$, so the product of each row, each column, and each diagonal = $2^3 \cdot 3^3$.

It's all a matter of adding exponents. In each row, column, and diagonal, the powers of 2 and the powers of 3 must both add up to 3. Therefore, in each row, column, and diagonal, there must be:

1. One number with $2^0$ as a factor, one number with $2^1$ as a factor, and one number with $2^2$ as a factor.

2. One number with $3^0$ as a factor, one number with $3^1$ as a factor, and one number with $3^2$ as a factor.

Leave 6, the median number, in the middle, and try to rearrange the others so that powers of both the 2's and the 3's always add up to 3.

Give up? See Chapter 10.

## 6.3 Magic Squares (4x4)

How can you rearrange the numbers 1 through 16 in a 4x4 array to create a magic square? The sum of the numbers is 136, so the magic constant is 34.

For the 3x3 magic square there were a total of 8 rows, columns, and diagonals to fill, and only 8 ways to get a total of 15. Now there are a total of 10 rows, columns, and diagonals to fill, but 86 ways to get a total of 34. Clearly, most of the possible number combinations will not be used in a solution! There are actually 880 distinctly different ways to make this square magic.

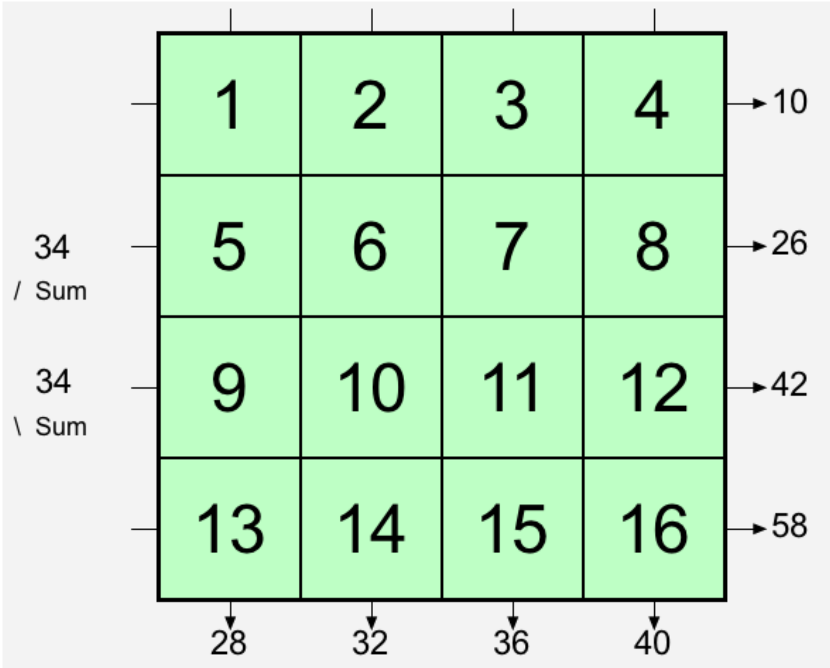
| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 5 | 6 | 7 | 8 |
| 9 | 10 | 11 | 12 |
| 13 | 14 | 15 | 16 |

| | | | |
|---|---|---|---|
| 16 | 15 | 2 | 1 |
| 16 | 14 | 3 | 1 |
| 16 | 13 | 4 | 1 |
| 16 | 13 | 3 | 2 |
| 16 | 12 | 5 | 1 |
| 16 | 12 | 4 | 2 |
| 16 | 11 | 6 | 1 |
| 16 | 11 | 5 | 2 |
| 16 | 11 | 4 | 3 |
| 16 | 10 | 7 | 1 |
| 16 | 10 | 6 | 2 |
| 16 | 10 | 5 | 3 |
| 16 | 9 | 8 | 1 |
| 16 | 9 | 7 | 2 |
| 16 | 9 | 6 | 3 |
| 16 | 9 | 5 | 4 |
| 16 | 8 | 7 | 3 |
| 16 | 8 | 6 | 4 |
| 16 | 7 | 6 | 5 |
| 15 | 14 | 4 | 1 |
| 15 | 14 | 3 | 2 |
| 15 | 13 | 5 | 1 |
| 15 | 13 | 4 | 2 |
| 15 | 12 | 6 | 1 |
| 15 | 12 | 5 | 2 |
| 15 | 12 | 4 | 3 |
| 15 | 11 | 7 | 1 |
| 15 | 11 | 6 | 2 |
| 15 | 11 | 5 | 3 |

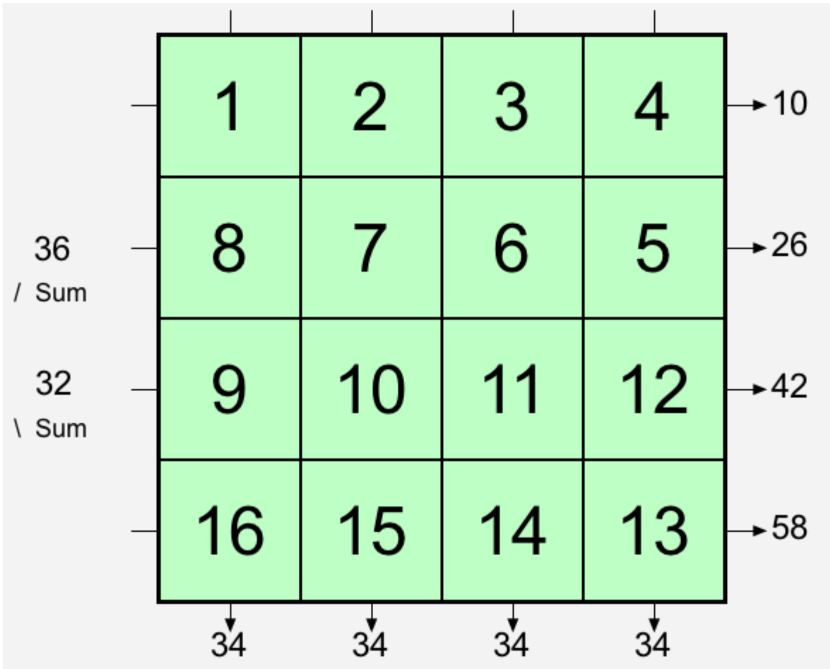
| | | | |
|---|---|---|---|
| 15 | 10 | 8 | 1 |
| 15 | 10 | 7 | 2 |
| 15 | 10 | 6 | 3 |
| 15 | 10 | 5 | 4 |
| 15 | 9 | 8 | 2 |
| 15 | 9 | 7 | 3 |
| 15 | 9 | 6 | 4 |
| 15 | 8 | 7 | 4 |
| 15 | 8 | 6 | 5 |
| 14 | 13 | 6 | 1 |
| 14 | 13 | 5 | 2 |
| 14 | 13 | 4 | 3 |
| 14 | 12 | 7 | 1 |
| 14 | 12 | 6 | 2 |
| 14 | 12 | 5 | 3 |
| 14 | 11 | 8 | 1 |
| 14 | 11 | 7 | 2 |
| 14 | 11 | 6 | 3 |
| 14 | 11 | 5 | 4 |
| 14 | 10 | 9 | 1 |
| 14 | 10 | 8 | 2 |
| 14 | 10 | 7 | 3 |
| 14 | 10 | 6 | 4 |
| 14 | 9 | 8 | 3 |
| 14 | 9 | 7 | 4 |
| 14 | 9 | 6 | 5 |
| 14 | 8 | 7 | 5 |
| 13 | 12 | 8 | 1 |
| 13 | 12 | 7 | 2 |

| | | | |
|---|---|---|---|
| 13 | 12 | 6 | 3 |
| 13 | 12 | 5 | 4 |
| 13 | 11 | 9 | 1 |
| 13 | 11 | 8 | 2 |
| 13 | 11 | 7 | 3 |
| 13 | 11 | 6 | 4 |
| 13 | 10 | 9 | 2 |
| 13 | 10 | 8 | 3 |
| 13 | 10 | 7 | 4 |
| 13 | 10 | 6 | 5 |
| 13 | 9 | 8 | 4 |
| 13 | 9 | 7 | 5 |
| 13 | 8 | 7 | 6 |
| 12 | 11 | 10 | 1 |
| 12 | 11 | 9 | 2 |
| 12 | 11 | 8 | 3 |
| 12 | 11 | 7 | 4 |
| 12 | 11 | 6 | 5 |
| 12 | 10 | 9 | 3 |
| 12 | 10 | 8 | 4 |
| 12 | 10 | 7 | 5 |
| 12 | 9 | 8 | 5 |
| 12 | 9 | 7 | 6 |
| 11 | 10 | 9 | 4 |
| 11 | 10 | 8 | 5 |
| 11 | 10 | 7 | 6 |
| 11 | 9 | 8 | 6 |
| 10 | 9 | 8 | 7 |

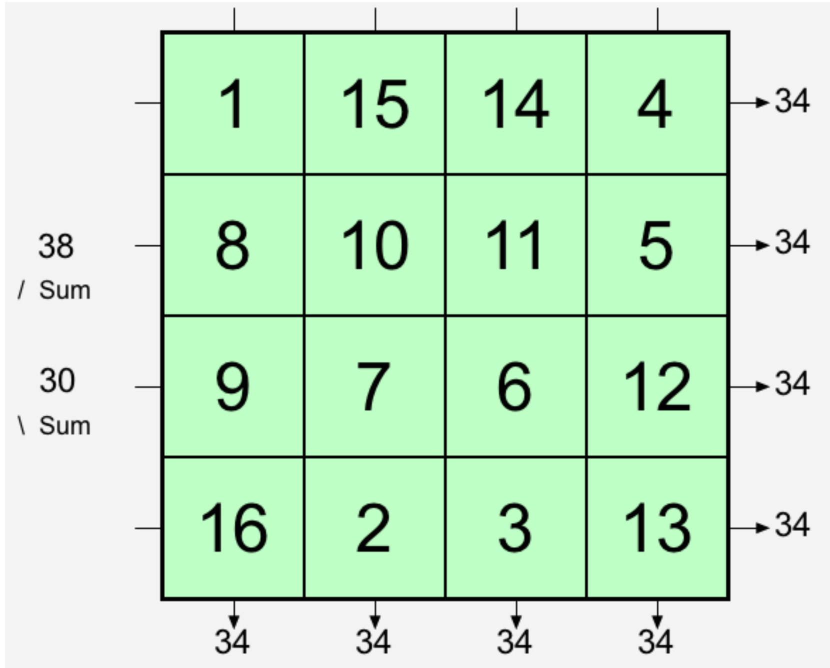
# Creating a 4x4 magic square:




1. Reverse the order of rows 2 and 4. This makes all four columns total 34.

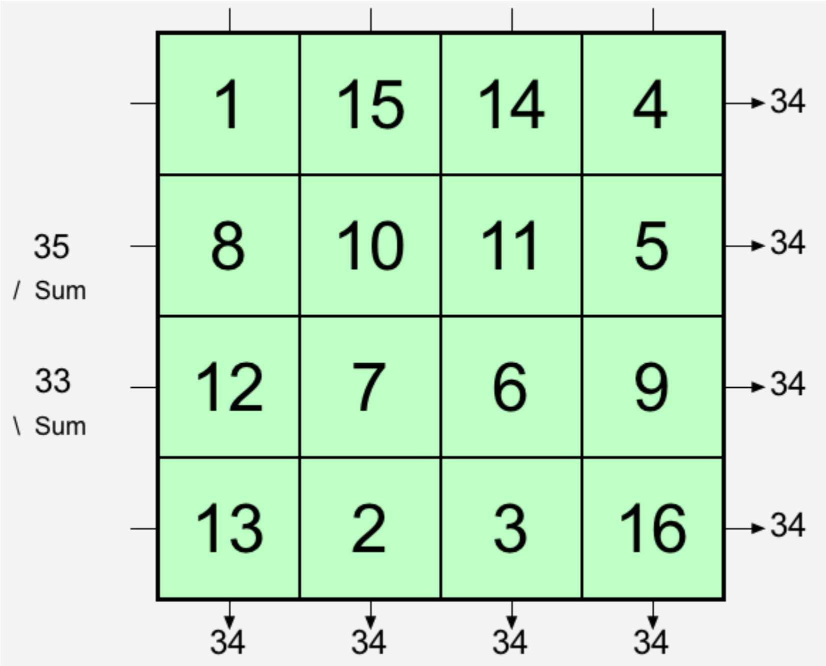



2. Reverse the order of the two middle columns. This makes all 4 rows total 34. Only the diagonals still need to be adjusted.

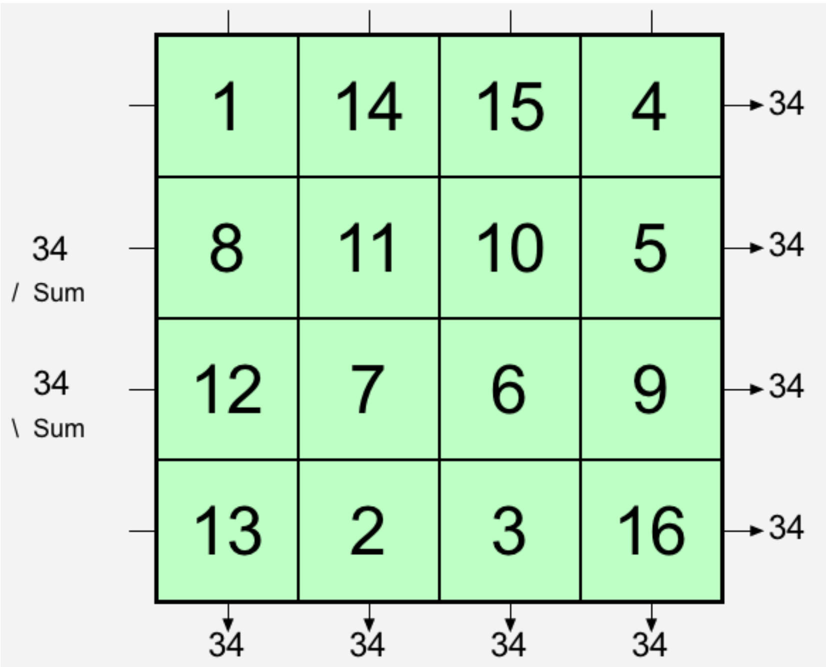


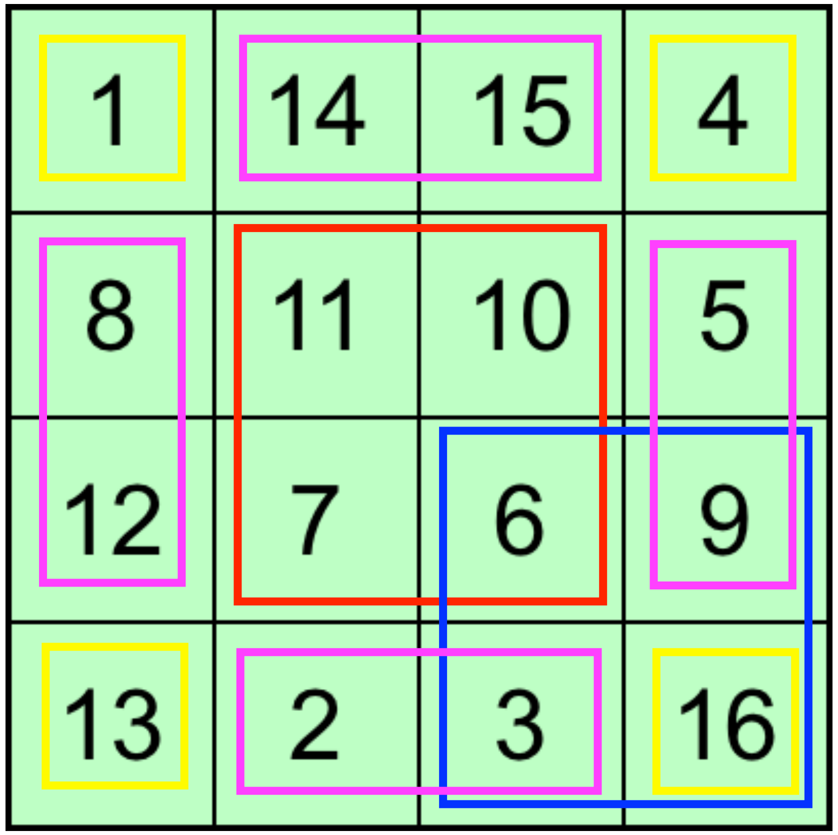

3. Let D1 be the diagonal that currently totals 38, and D2 be the diagonal that currently totals 30. Notice that you can switch the 9 with the 12 and the 16 with the 13 without messing up the column totals, but D1 would decrease by 3 and D2 would increase by 3. Also notice that you can switch the 15 with the 14 and the 10 with 11 without messing up the column totals, but D1 would decrease by 1 and D2 would increase by 1. Total decrease for D1 is 4; total increase for D2 is 4. Perfect!

After switching 9 and 16 with 12 and 13




After switching 15 and 10 with 14 and 11




**This square has lots of special properties:**

- The sum of the four numbers in the **middle** is 34.
- The sum of the four numbers in the **corners** is 34.
- The sum of the numbers in each of the four **quadrants** is 34.
- The sum of the **middle two numbers of row 1** and the middle two numbers of row 4 is 34.
- The sum of the **middle two numbers of column 1** and the middle two numbers of column 4 is 34.

Can you arrange the numbers differently and still make a magic square? See Chapter 10 for another solution.

# Multiplicative 4x4 Magic Square

The product of the numbers 1 through 16 is huge! Its 4th root is 6720. Try to rearrange the numbers below so that the product of the entries in every row, column, and diagonal is 6720. It helps to put the numbers into four groups.

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 5 | 6 | 7 | 8 |
| 10 | 12 | 14 | 20 |
| 24 | 28 | 40 | 56 |

Group 1: odd numbers (1, 3, 5, 7)

Group 2: numbers $2^1$ times larger (2, 6, 10, 14)

Group 3: numbers $2^2$ times larger (4, 12, 20, 28)

Group 4: numbers $2^3$ times larger (8, 24, 40, 56)

In order for the products of the rows, columns, and diagonals to be the same, only one member of each group can appear in each row, column, and diagonal. Color coding helps keep this requirement easier to remember.

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 5 | 6 | 7 | 8 |
| 10 | 12 | 14 | 20 |
| 24 | 28 | 40 | 56 |

Each row, column, and diagonal needs to include one square filled with each color.

But there are four more groups to consider:

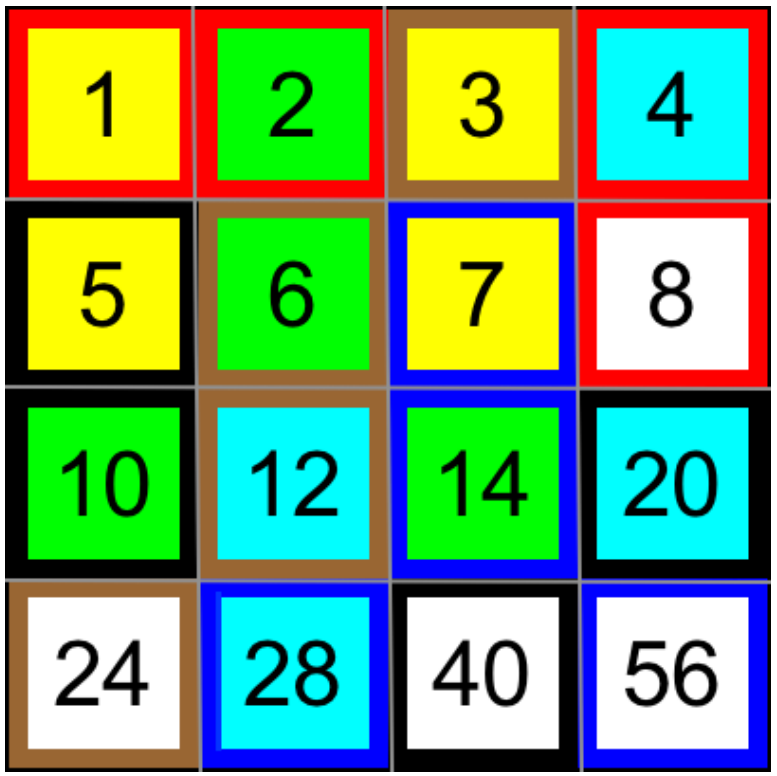
Group 5: powers of 2 (1, 2, 4, 8)

Group 6; 3 times powers of 2 (3, 6, 12, 24)

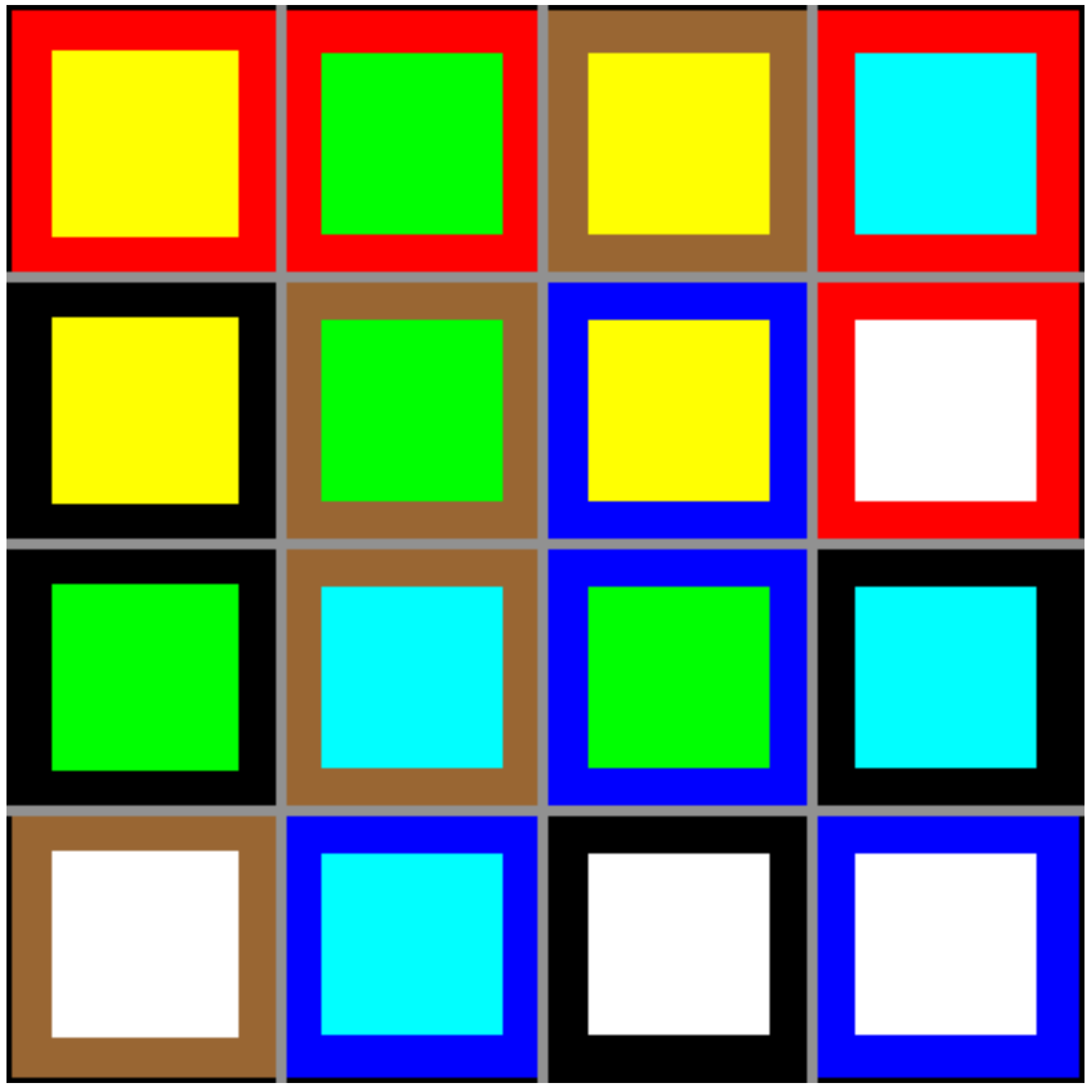
Group 7: 5 times powers of 2 (5, 10, 20, 40)

Group 8: 7 times powers of 2 (7, 14, 28, 56)

Again, in order for the products of the rows, columns, and diagonals to be the same, only one member of each group can appear in each row, column, and diagonal. Color coding the borders helps make this easier to remember. Each row, column, and diagonal must include one square bordered by each color.

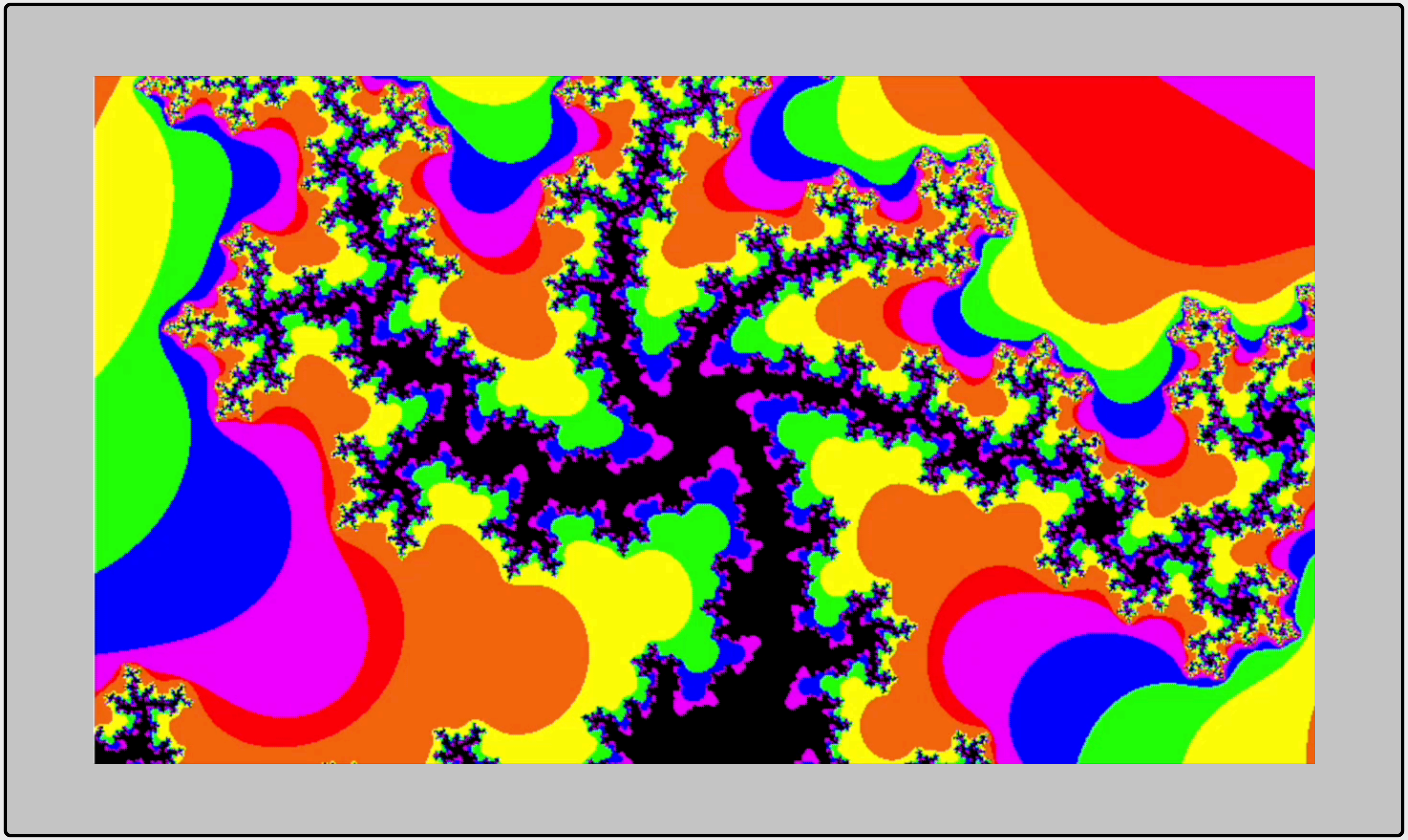



Each of the 16 numbers has a unique color combination, so there is really no need to include the numbers at all. Think of this as a geometric puzzle - rearrange the colored blocks so that no color (in the center or on the border) appears twice in the same row, column, or diagonal. If you satisfy the color requirements, the numbers will be okay and all of the products will be equal.



Give up? See Chapter 10.

# 7. Recursion



## 7.1 Introduction

A recursive formula defines a sequence of numbers by supplying both a starting point and a rule for getting from one element of the sequence to the next. The first element determines the second, the second determines the third, and so on. It's just like a row of closely spaced dominoes - once the first falls, the others follow. My appreciation of recursion formulas grew enormously in 1980 when a friend introduced me to the VisiCalc spreadsheet "replicate" command. You could use it to copy a formula from one cell to another using relative references. When you entered 1 in cell A1, =A1+1 in cell A2, and then replicated that formula (using the "relative" option) in cells A3 through A10, you ended up with the numbers 1 through 10 appearing in those cells. Each cell added one to the preceding cell.

Many sequences can be defined either recursively or explicitly. Consider, for example, the infinite sequence 5, 8, 11,14, 17, ...

**Explicit formula**: $x(n) = 2 + 3n$ where n is any positive integer and x(n) is the nth element of the sequence.

**Recursive formula**: $x_1 = 5$ and $x_n = x_{n-1} + 3$. You get $x_n$, the nth element of the sequence, by adding 3 to $x_{n-1}$, the previous element of the sequence. Subscripts are usually used to designate the different elements of the sequence.

Some mathematical relationships are easier to understand by thinking recursively; others can only be understood in that way. This chapter presents some applications of recursion that I find particularly interesting.

My **Stepwising** program recursively plots the motion of stars, planets, and satellites; my **Life** program, based on Conway's Game of Life, recursively determines the fate of a group of living cells; and my **Mandelbrot** program zooms into the Mandelbrot set by determining whether a recursively defined sequence of complex numbers is bounded.  In all three cases, everything depends on a small set of initial conditions. Both programs are available for free from my **Syzygy Shareware** site at [http://tcbretl.weebly.com](http://tcbretl.weebly.com).

## 7.2 Stepwising

I wrote my first computer program as a teacher in 1975. I borrowed a very expensive desktop calculator from the school secretary, and used it to create a stepwise approximation of an orbit for my physics class. It was based on what I had read in **The Feynman Lectures on Physics.** Here is the basic idea.

1. Start with an imaginary sun which has one planet. Assume that the sun is much more massive than the planet. Choose an initial distance between the sun and the planet, and an initial velocity for the planet (relative to the sun).

2. Use Newton's Law of Gravitation to calculate the planet's acceleration. Based on that acceleration, find the planet's velocity one second later.

3. Use that velocity to calculate the approximate the distance travelled by the planet during the first 2 seconds, and determine its new position.

4. Use the planet's new position to find the planet's distance from the sun. Then re-calculate its acceleration.

5. Add the acceleration from step 4 to the previous velocity to approximate the new velocity.

6. Use the new velocity from step 5 to approximate the planet's new position.

7. Repeat steps 4 through 6. Velocities after an odd number of seconds determine positions after an even numbers of seconds (but the units don't really need to be seconds).

Here are the details in a recursive format. The location of the planet is (x,y), its acceleration in the x-direction is ax, its acceleration in the y-direction is ay, its velocity in the x-direction is vx, its velocity in the y-direction is vy, and its distance from the sun is r. The time interval between calculations is $\Delta$t, and the mass of the sun is M.

To get things started (steps 1 through 3):

$$ax = -\frac{GMx}{r^3} \qquad ay = -\frac{GMy}{r^3}$$

$$vx_1 = vx_0 + \frac{1}{2}ax \cdot \Delta t \qquad vy_1 = vy_0 + \frac{1}{2}ay \cdot \Delta t$$

$$x_1 = x_0 + vx_1 \cdot \Delta t \qquad y_1 = y_0 + vy_1 \cdot \Delta t$$

For steps 4 through 6:

$$r = \sqrt{x^2 + y^2}$$

$$ax = -\frac{GMx}{r^3} \qquad ay = -\frac{GMy}{r^3}$$

$$vx_n = vx_{n-1} + ax \cdot \Delta t \qquad vy_n = vy_{n-1} + ay \cdot \Delta t$$

$$x_n = x_{n-1} + vx_n \cdot \Delta t \qquad y_n = y_{n-1} + vy_n \cdot \Delta t$$

Here are the initial values. The first 5 are set by the program; the last 2 can be changed by the user.

Gravitational constant • sun's mass = GM = 1.

Location of the sun is (0,0).

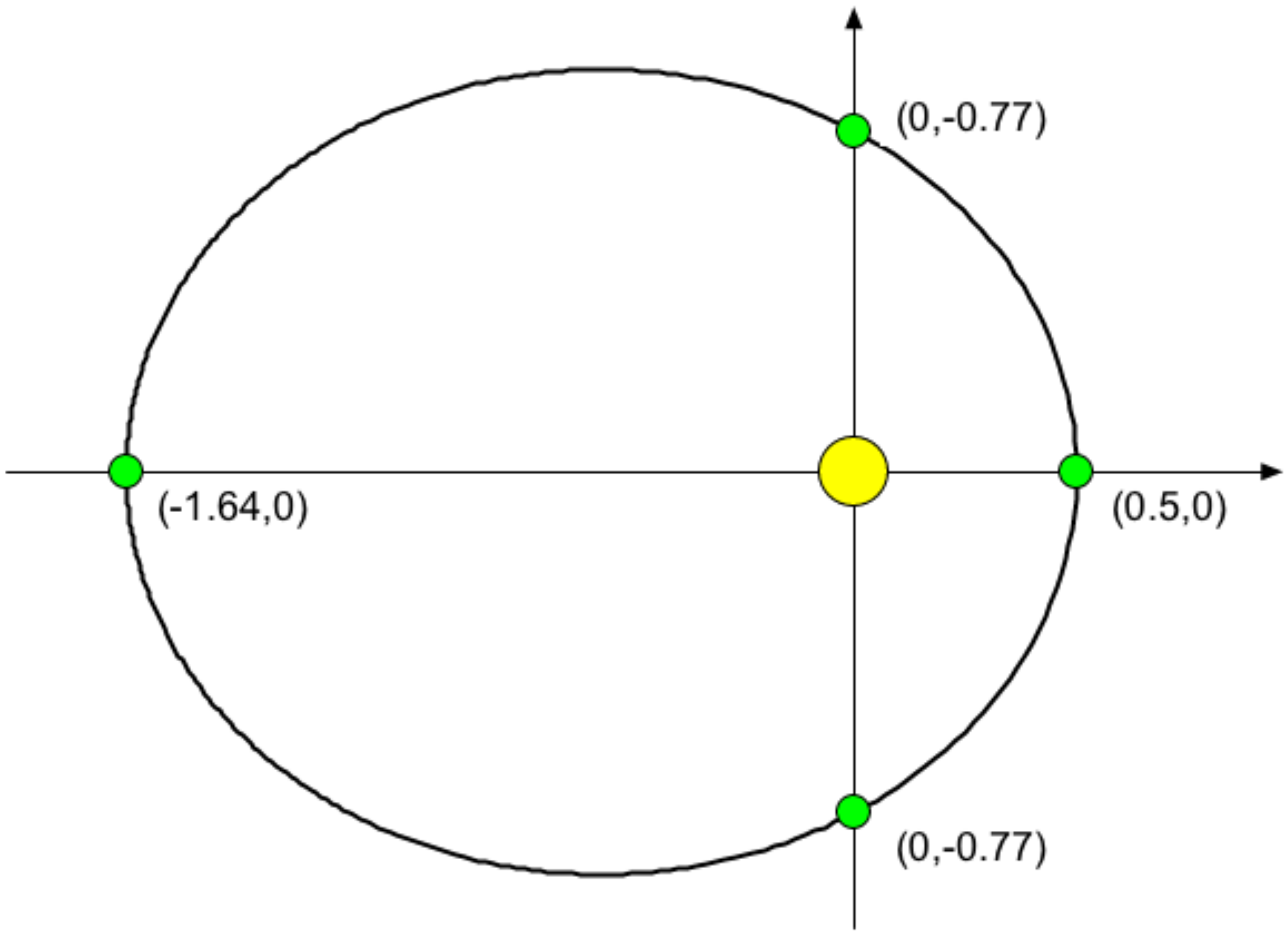
Planet's y-coordinate is 0.

Planet's velocity in the x-direction (vx) = 0.

$\Delta$t, the time interval between calculations = .02.

Planet's x-coordinate is 1/2, so r = 1/2.

Planet's velocity in the y-direction (vy) = 1.75.




Shown to the right is a printout of the resulting orbit. It is only an approximation because the force on the planet, and thus the planet's velocity, changes constantly.

Determining the planet's path recursively assumes that its velocity remains constant for a short period of time, and that is true only if the planet's distance from the sun does not change. As long as the time interval between calculations is very small, however, the distance will not change by much, and the approximation will be very good.

Could you come up with an explicit formula for this motion? Yes, but it would involve a lot more work. You would have to solve some differential equations.

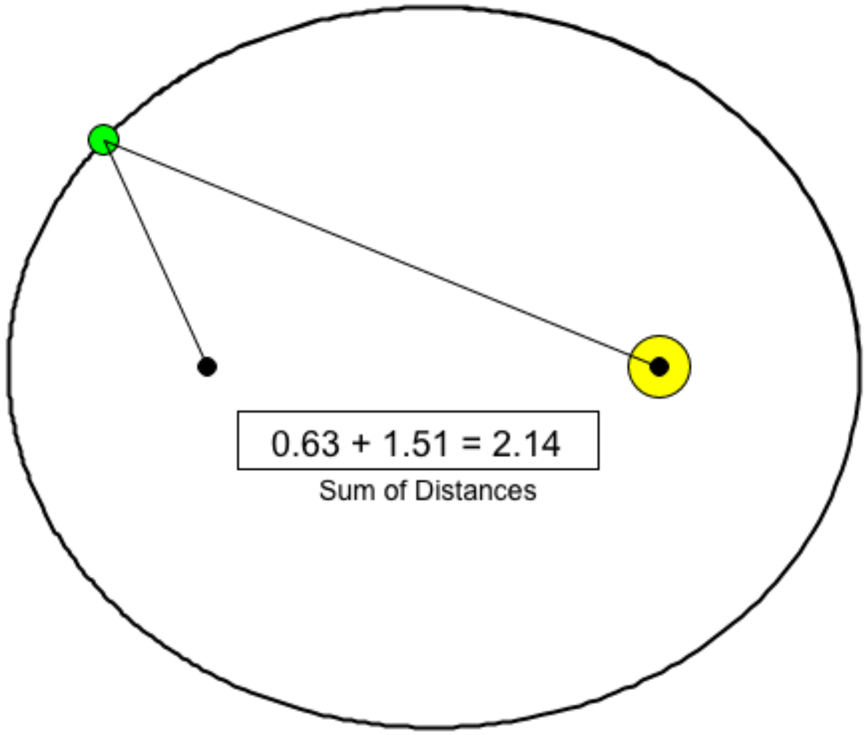
**Kepler's three laws of planetary motion:**

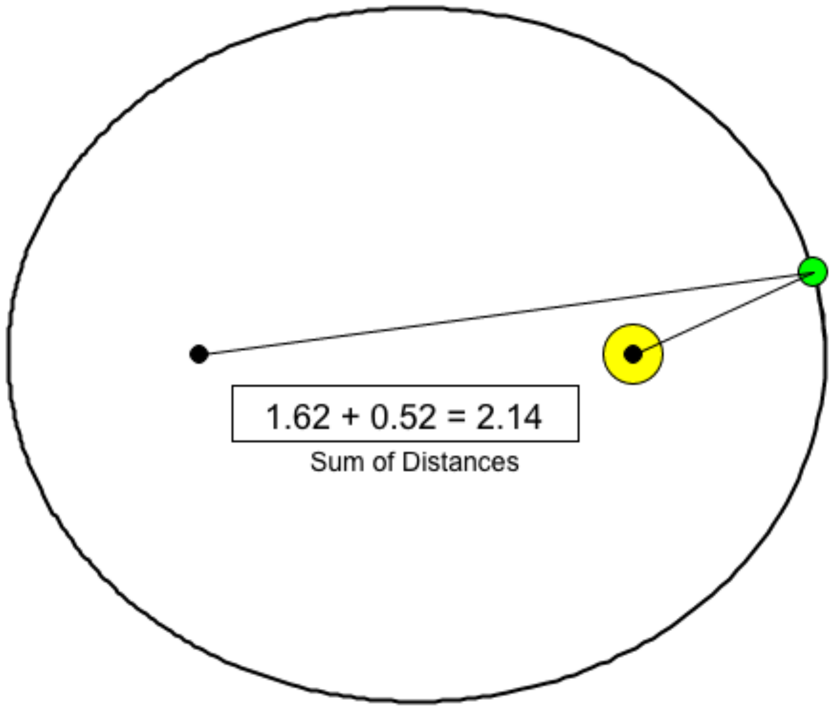
1. Planetary orbits are ellipses, with the sun at one focus.
2. A line segment connecting a planet to the sun sweeps out equal areas in equal times.
3. For different planets revolving around the same sun, $R^3/T^2$ is a constant (where R = the semi-major axis, and T = the period of revolution).

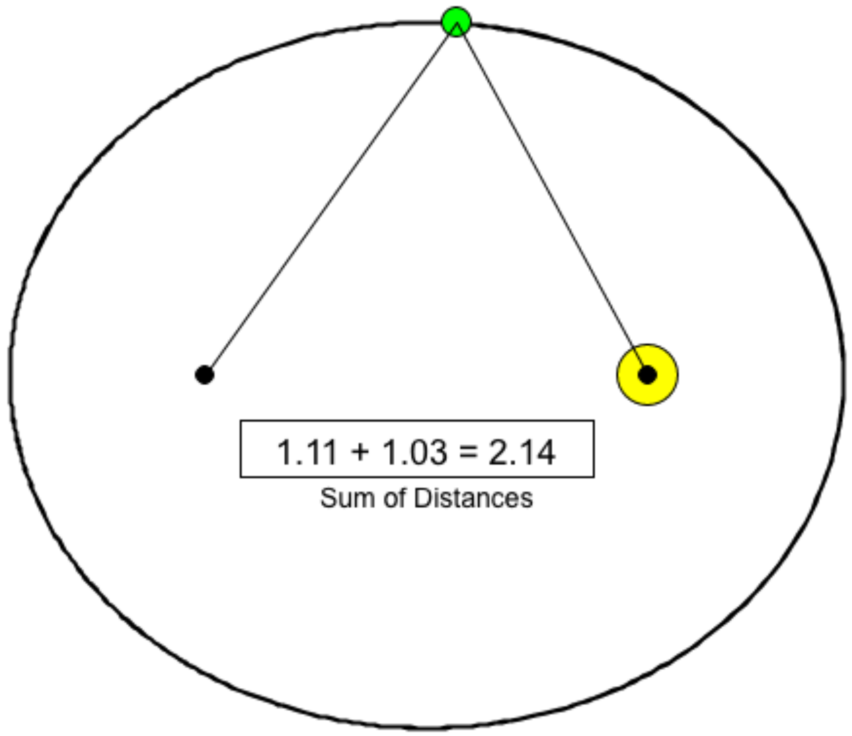
Let's see if the stepwise approximation supports these laws.

**Definition of an ellipse**: the path followed by a point moving in a plane in such a way that the sum of its distances from two fixed points (the foci) is a constant.

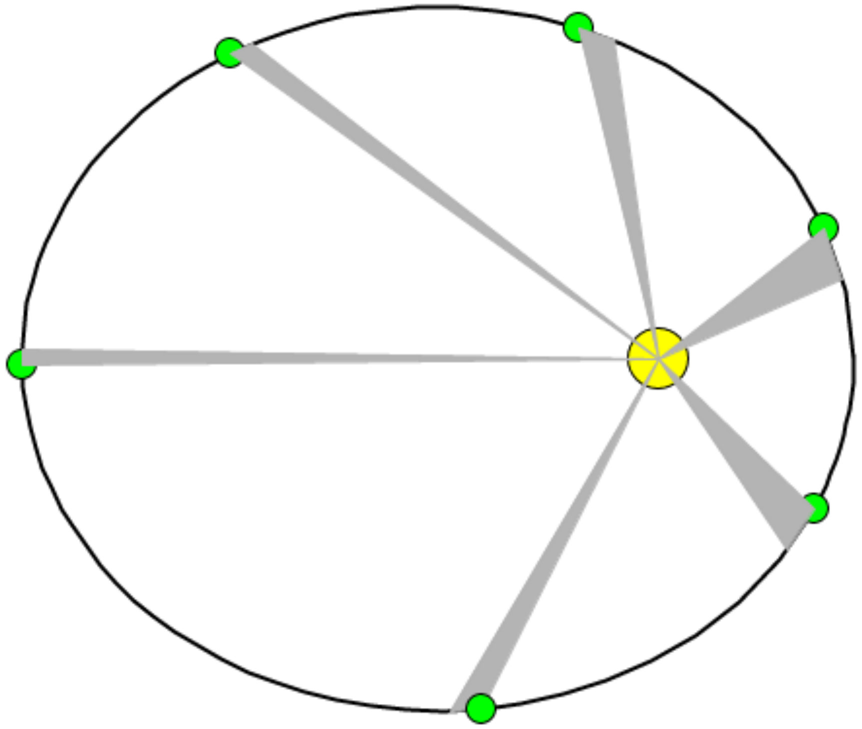
The program printouts on this page show the planet in three different positions. In each case the sum of the distances equals 2.14. The same is true for any other point that lies on the path.



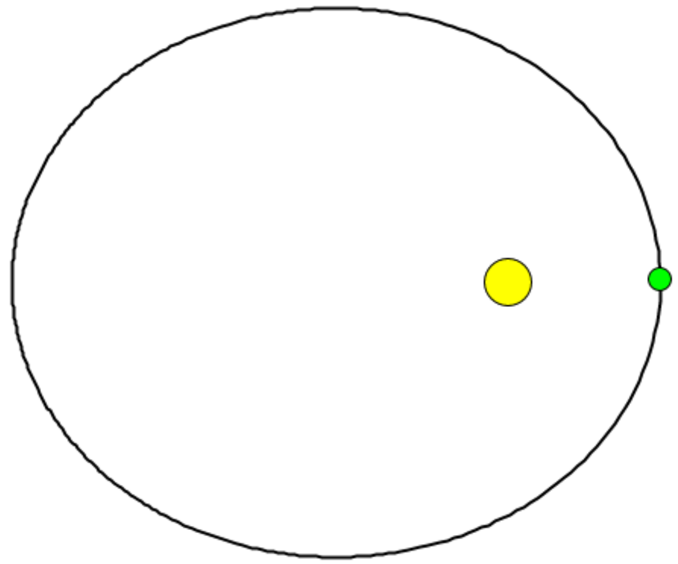
What about Kepler's 2nd Law? Using the same initial conditions, the shaded regions shown in the computer printout below were all swept out in the same amount of time. Their approximate areas were calculated by using Heron's formula for triangular regions. All of the areas equaled 0.044 when rounded to two significant digits.



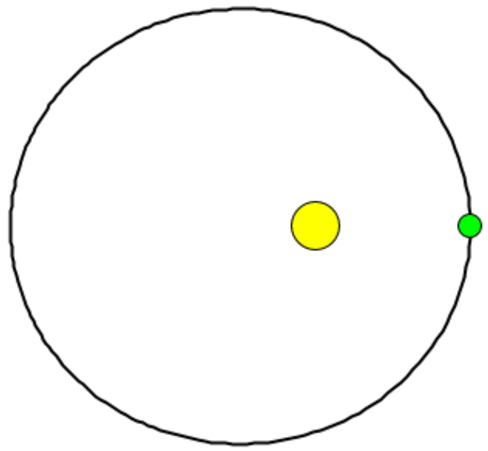
Finally, to check Kepler's 3rd law, we need to look at several orbits resulting from different initial conditions.

For the orbits shown to the right, the initial conditions are listed together with the resulting values for R, T, and $R^3/T^2$. As you can see, the values for $R^3/T^2$ are equal when rounded to two three significant digits.

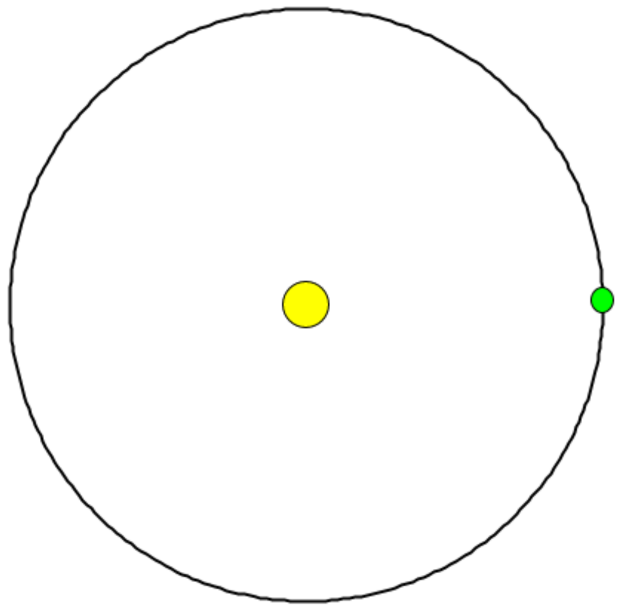
Initial r = 1/2
Initial vy = 1.75
R = 1.067
T = 1.101
$R^3/T^2$ = 1.002



Initial r = 1/2
Initial vy = 1.63
R = 0.745
T = .0.643
$R^3/T^2$ = 1.000

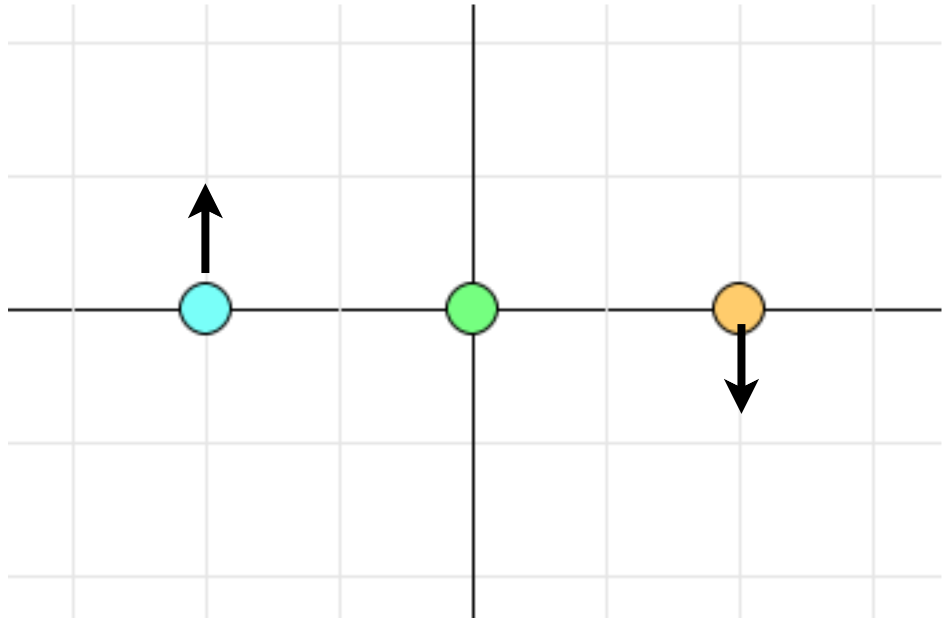


Initial r = 1
Initial vy = 1.0
R = 1.0
T = 1.0
$R^3/T^2$ = 1.000



# The 3-Body Problem

What about the motion of three objects moving through space? Unlike the 2-body problem discussed in the previous seection, there is no general solution to this problem. Depending upon the initial conditions (the positions, masses, and velocities of the three objects), there may or may not be a predictable and stable result.



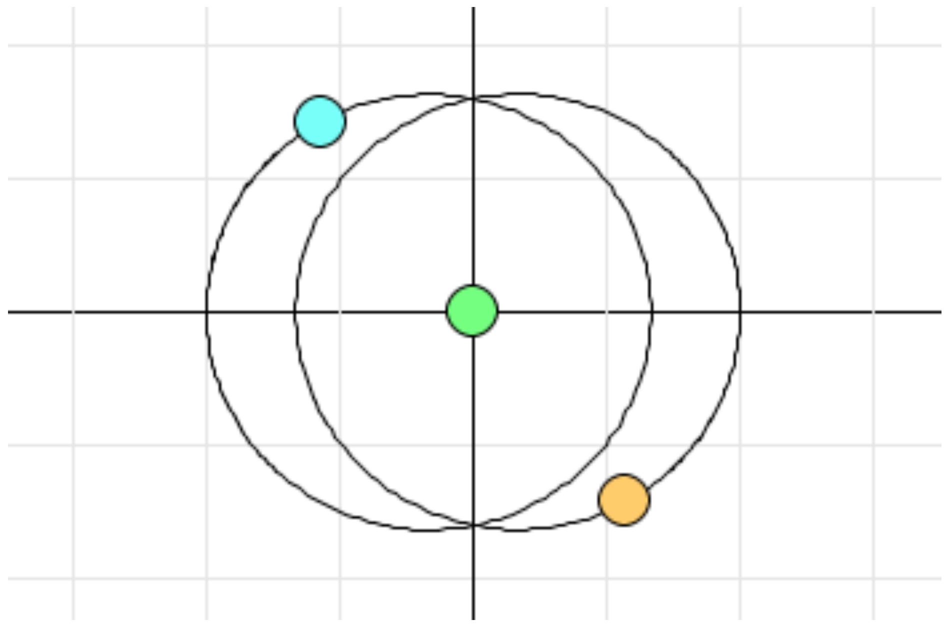
The **Stepwising** program can again be used to simulate the objects' motion to a high degree of accuracy. There are just more calculations to make.

Assume that the three objects all move in the same plane. Here is a set of possible initial conditions:5

The blue object is initially moving towards the top of the screen, the orange object towards the bottom of the screen. The green object has no initial velocity.

| Object | Coordinate | Mass | Velocity in x direction | Velocity in y direction |
|---|---|---|---|---|
| 1 (blue) | (-100,0) | 5 | 0 | 1 |
| 2 (Orange) | (100,0) | 5 | 0 | -1 |
| 3 (Green) | (0,0) | 5 | 0 | 0 |

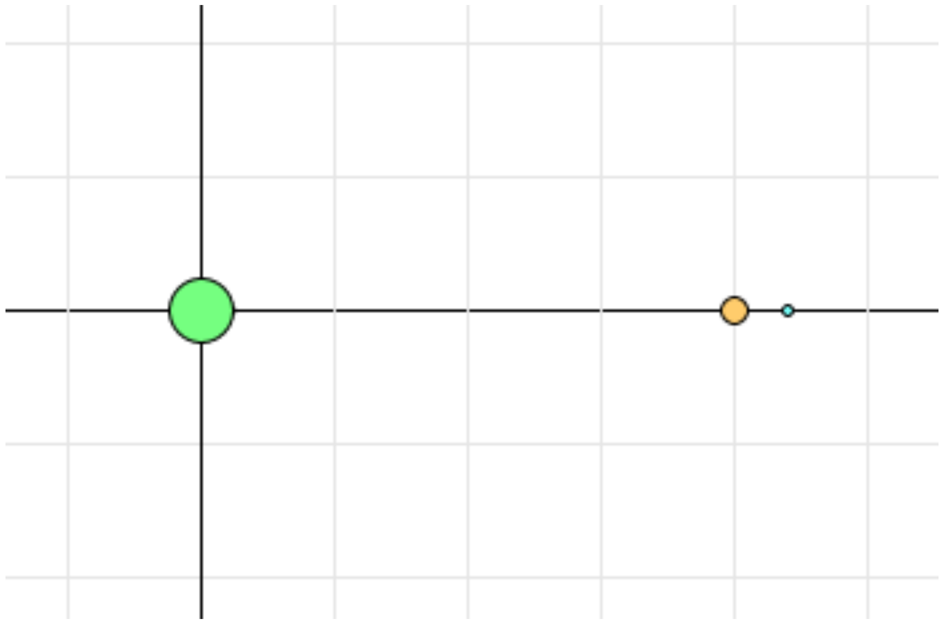
Here are the paths that the objects will follow. The green object remains stationary - equally attracted to the other two.

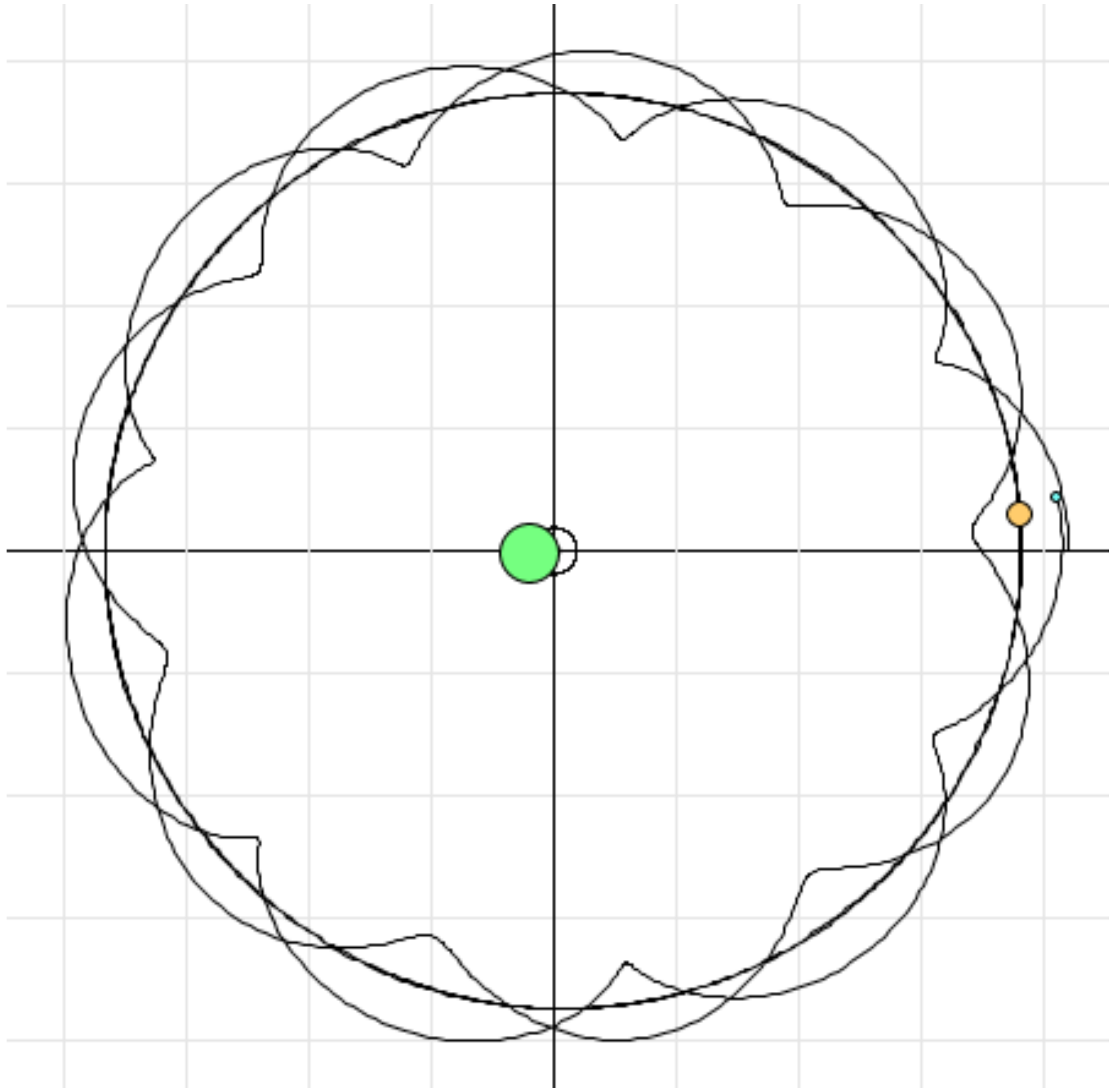


**Example 2:**

| Object | Coordinate | Mass | Velocity in x direction | Velocity in y direction |
|---|---|---|---|---|
| 1 (blue) | (220,0) | 0.01 | 0 | 1.7 |
| 2 (Orange) | (200,0) | 0.5 | 0 | 1 |
| 3 (Green) | (0,0) | 10 | 0 | 0 |

The tiny blue satellite orbits around the more massive orange planet as the planet orbits around the still more massive green sun.



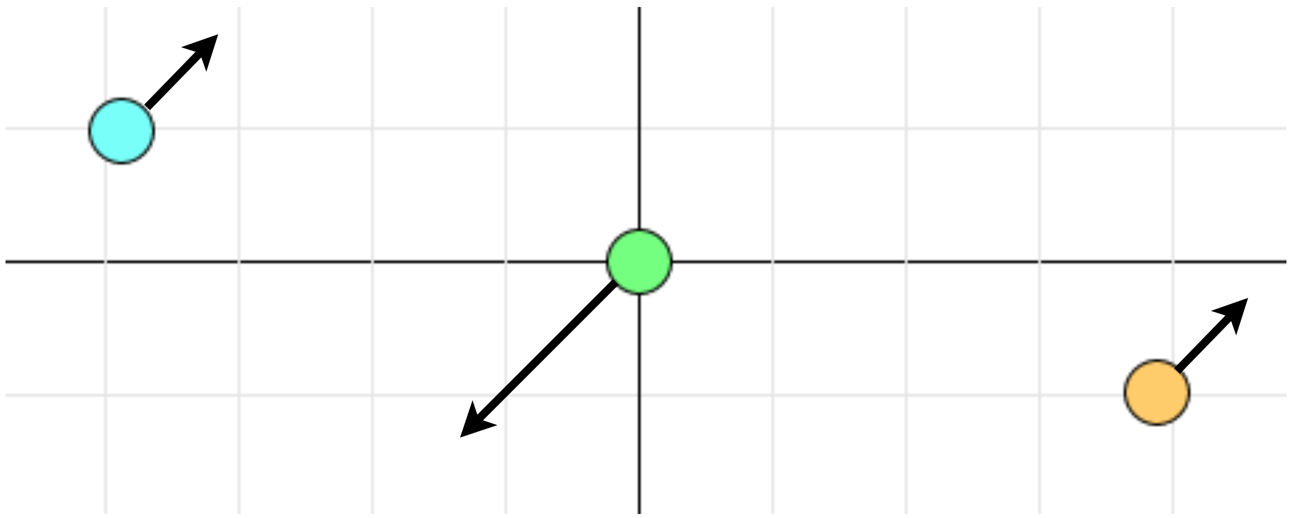


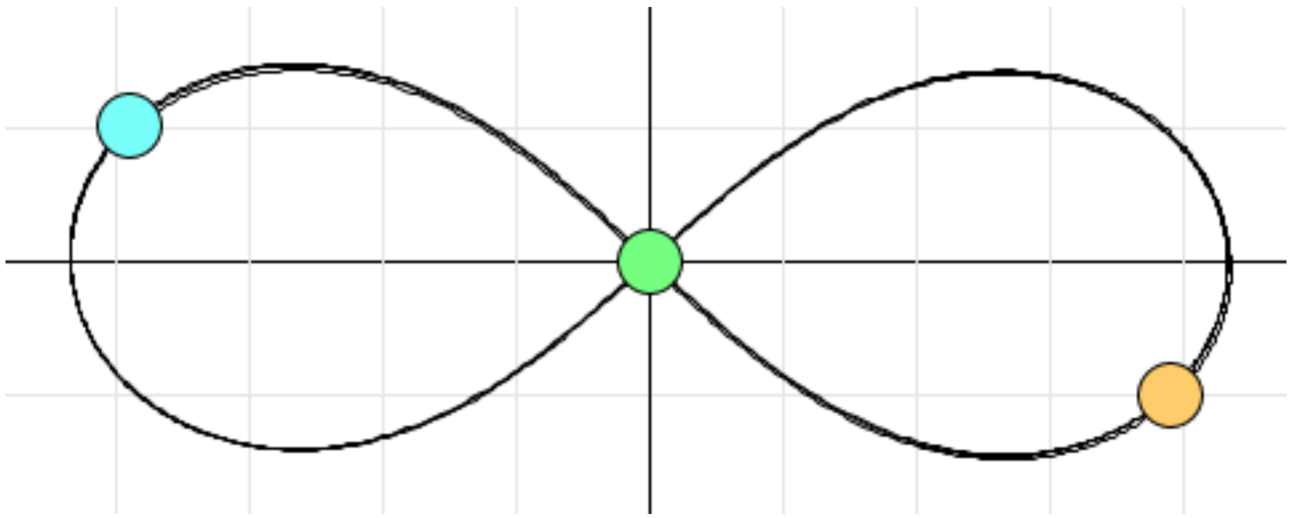
**Example 3:**

| Object | Coordinate | Mass | Velocity in x direction | Velocity in y direction |
|---|---|---|---|---|
| 1 (blue) | (-194,49) | 10 | 0.47 | 0.43 |
| 2 (Orange) | (194,-49) | 10 | 0.47 | 0.43 |
| 3 (Green) | (0,0) | 10 | -0.94 | -0.86 |

All three objects have the same mass, and they follow each other along a figure 8 pattern.

This is not a very stable configuration. If the initial conditions are just a little bit off, the pattarn does not last for long.





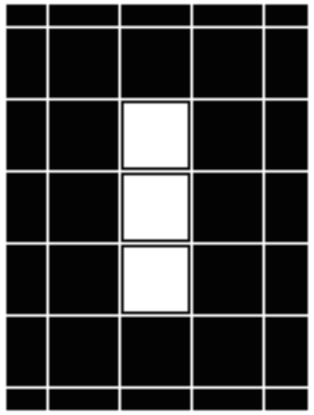
# 7.3 The Game of Life

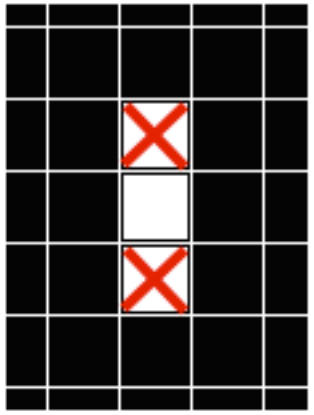
Conway's Game of Life was created in 1970 by John Horton Conway. It really isn't a game, but it is fascinating to watch. Once the initial conditions are set, everything happens automatically. The "game board" consists of an infinite grid of cells. You start by designating which of those cells are "living." Then two very simple rules determine what happens next.

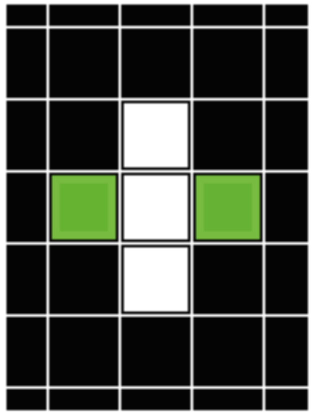
1. A living cell survives if and only if it has 2 or 3 living neighbors (living cells that share a side or vertex with it).

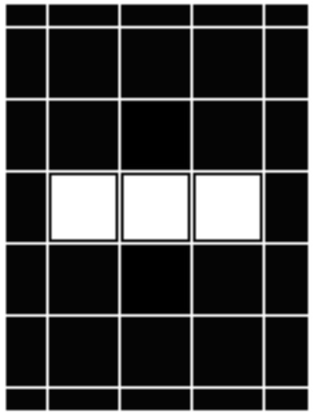
2. A "birth" occurs in any empty cell that has exactly 3 living neighbors. The births and deaths happen simultaneously (i.e. both are determined by what previously existed, before any changes have been made).

Here's what happens to a very simple pattern of three living cells. The vertical row becomes a horizontal one. Next, of course, the horizontal pattern will switch back to the original vertical one.


Original pattern

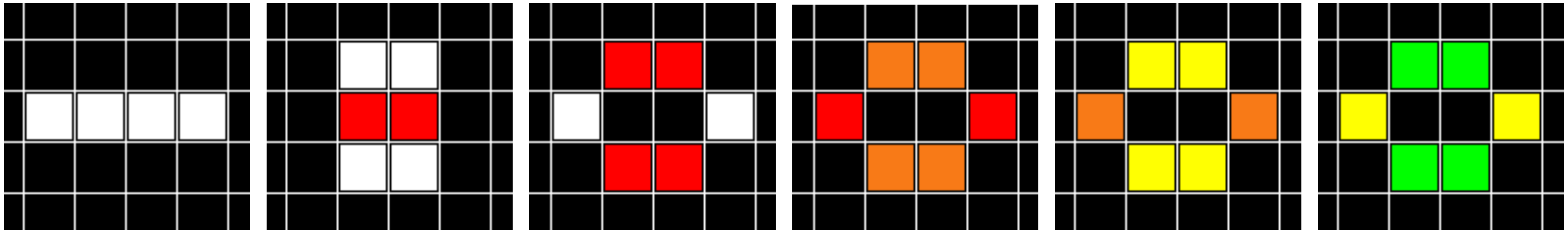

X cells die


Two new cells are born

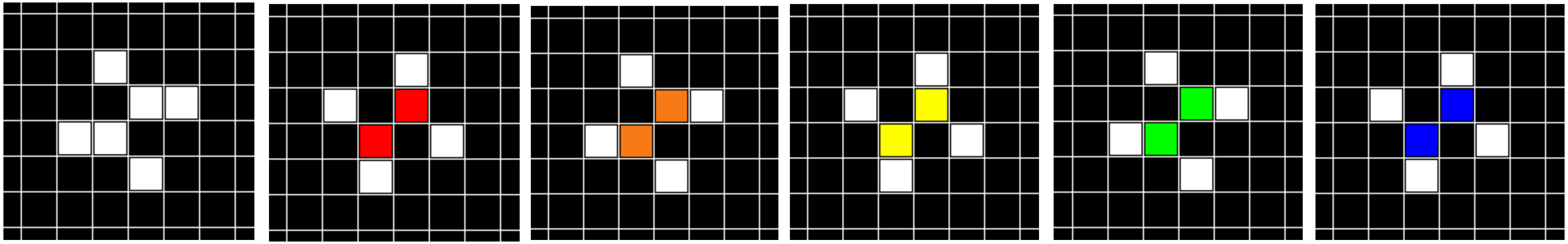

The new pattern

Some initial living cell groups quickly die out completely, some eventually form a pattern that doesn't change at all, some repeat a never ending cycle of patterns, and some expand and grow forever.

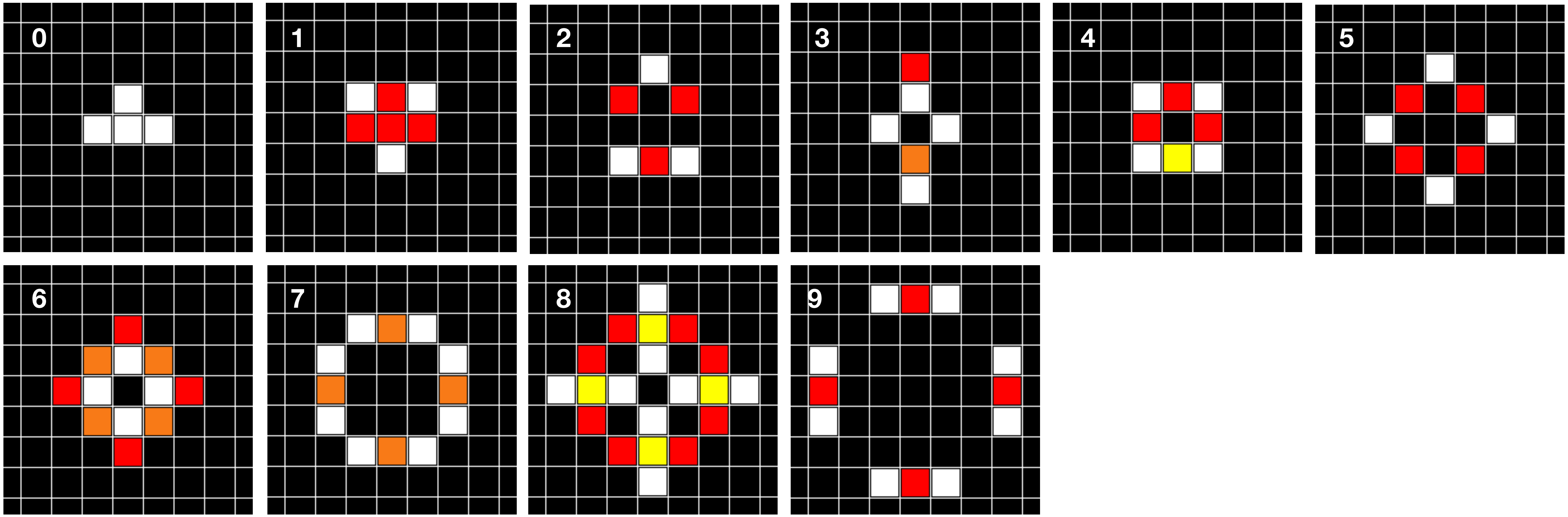
The examples on the next page are color coded to indicate the "age" of each living cell. The age is the number of recursive steps that have been calculated since the cell was born.



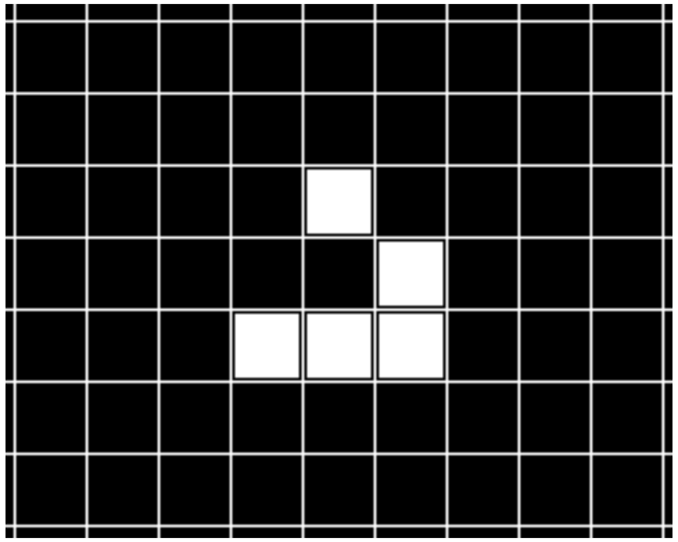
0 1 2 3 4 5

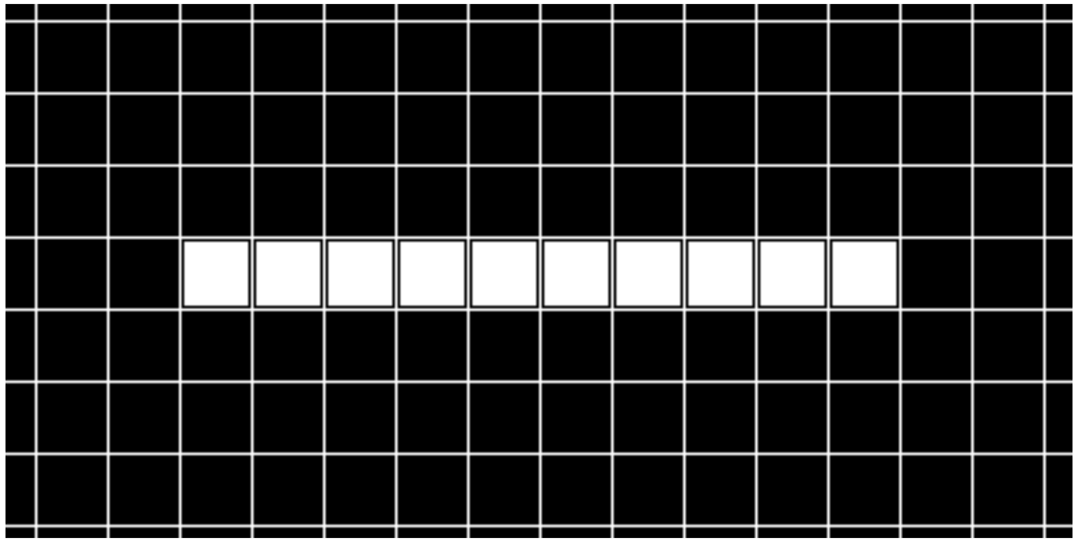


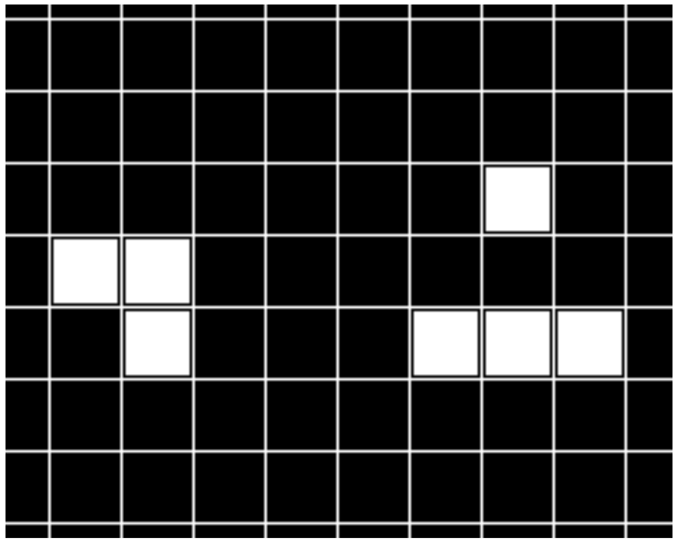
0 1 2 3 4 5

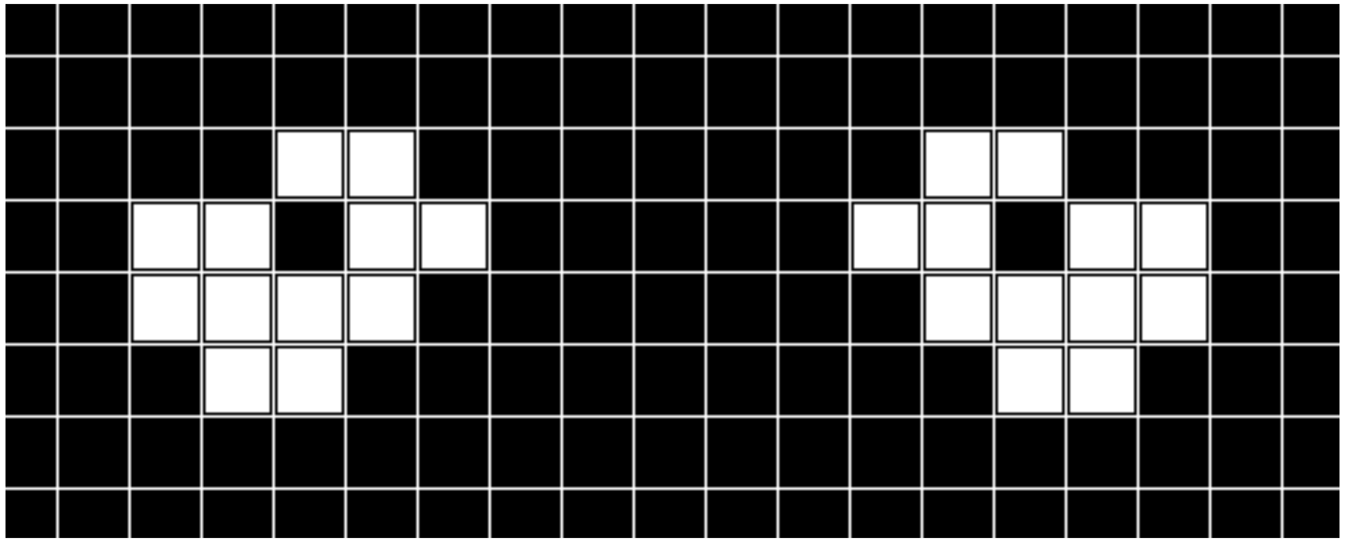


# Some other interesting initial cell patterns to try:



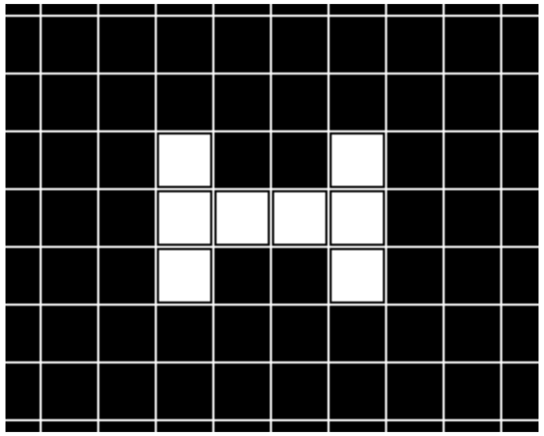




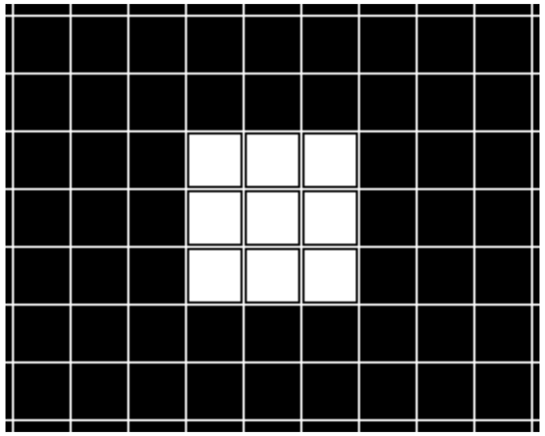


## 7.4 Life Problems

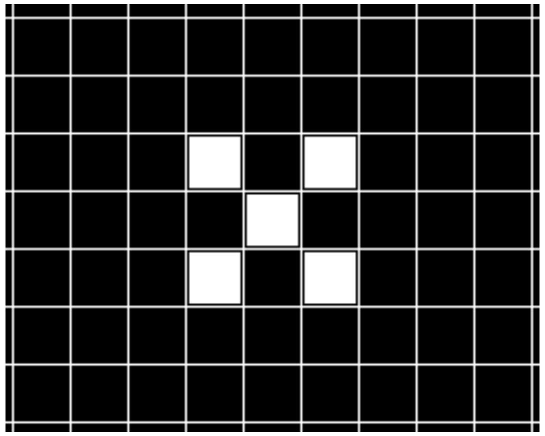
Try to predict what happens to these 8 initial groups of living cells after 1 “year.” Then try to predict their ultimate fate.



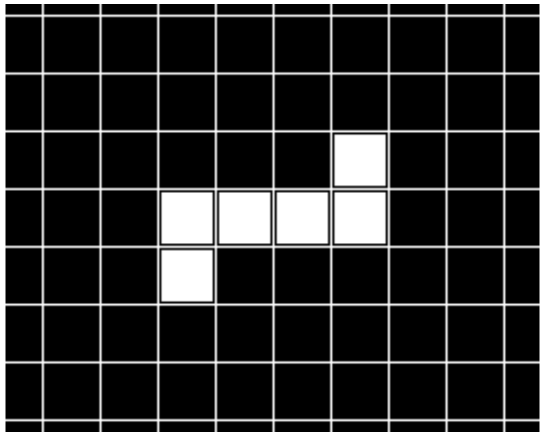
1



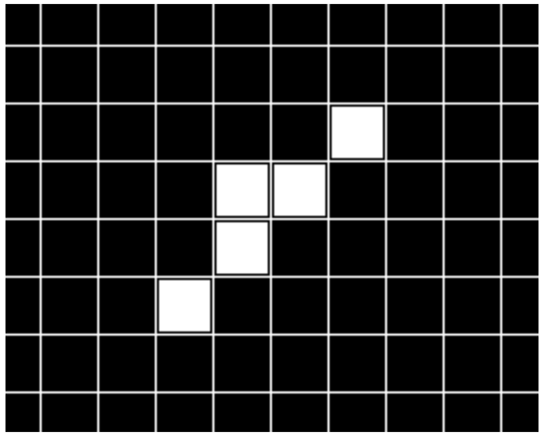
2



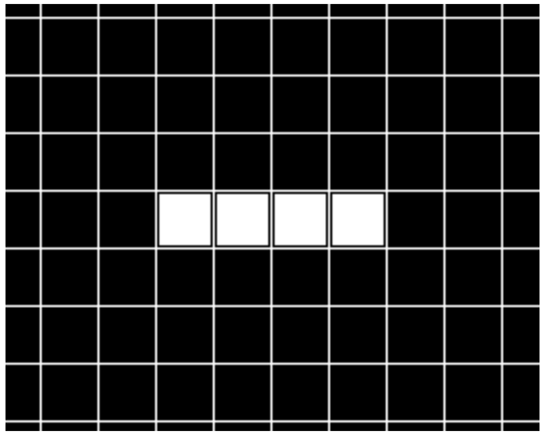
3



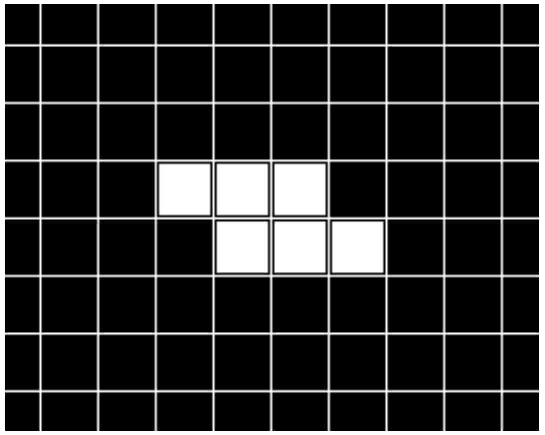
4



5



6



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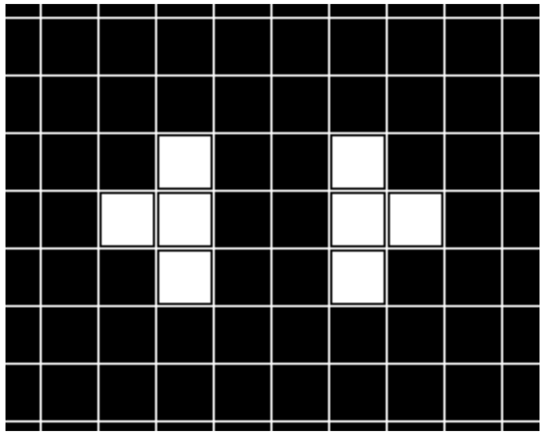
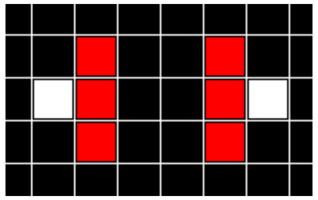
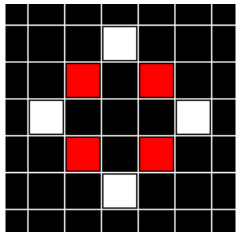
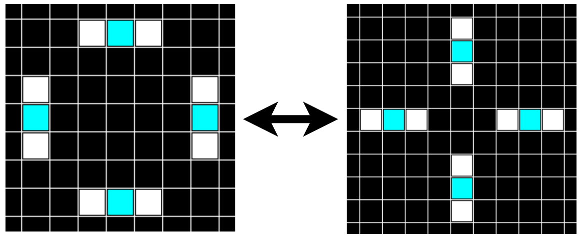
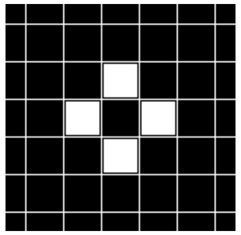
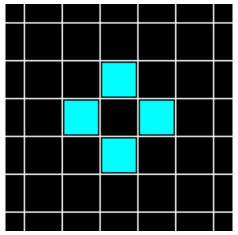
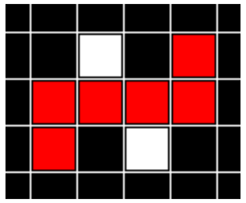
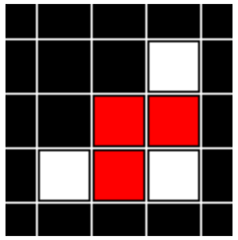
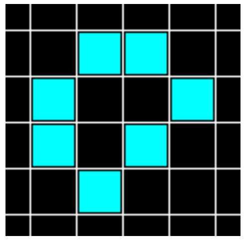
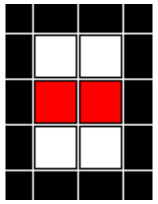
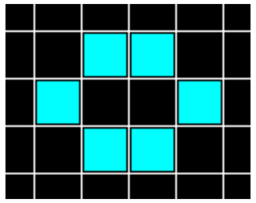
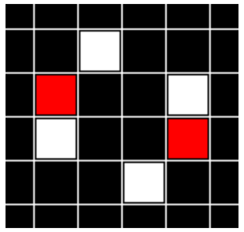
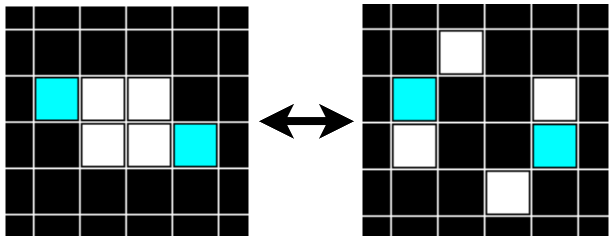
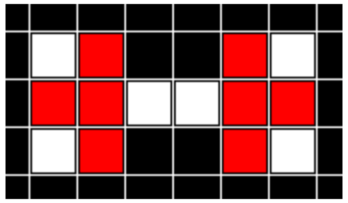


8

Check by running the program, or see the next page.

# Solutions

| | After 1 year | Ultimate fate |
|---|---|---|
| 1. |  | Extinction |
| 2. |  |  |
| 3. |  |  |
| 4. |  | Extinction |
| 5. |  |  |
| 6. |  |  |
| 7. |  |  |
| 8. |  | Extinction |

## 7.5 The Mandelbrot Set

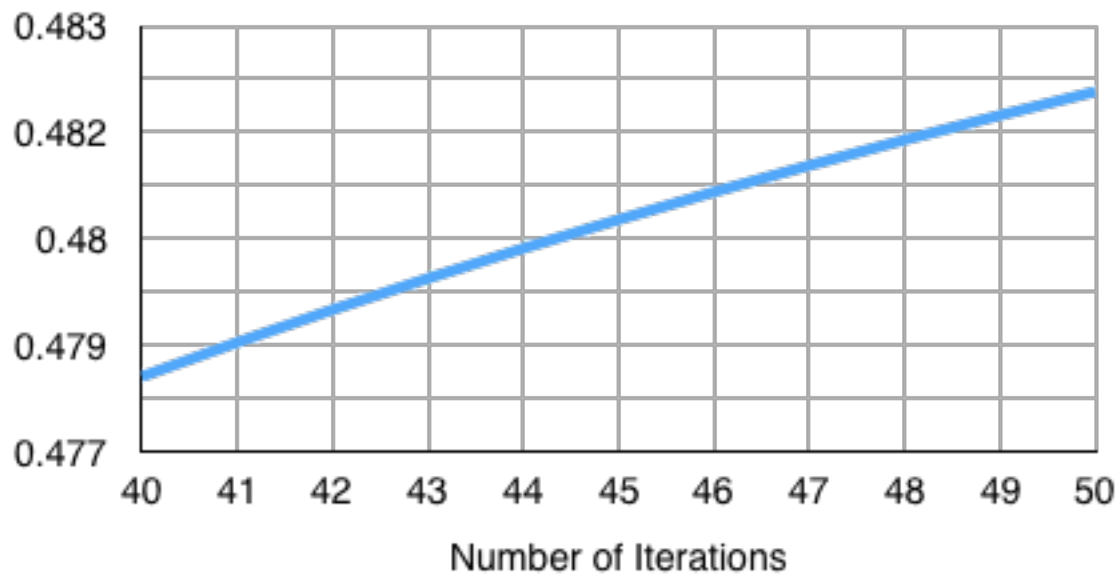
The definition of the Mandelbrot Set is perhaps the quintessential example of an amazing recursive formula. The formula is simple:

$z_n = (z_{n-1})^2 + c$, where $z_0 = 0$, and c is a complex number.

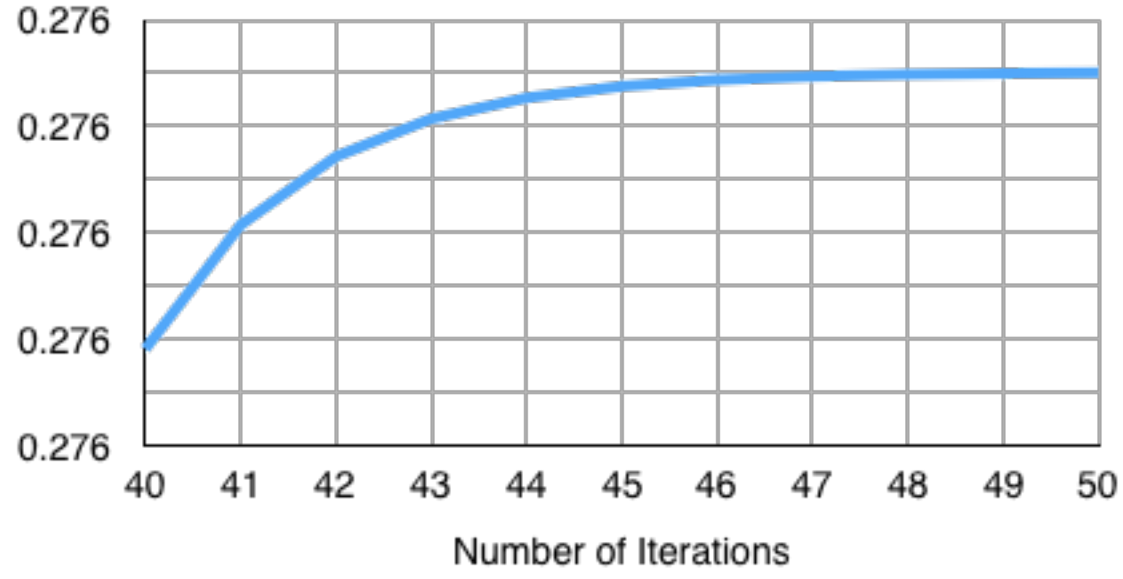
If the resulting sequence is bounded, then c is a member of the Mandelbrot set.

Even when c is a real number, this recursive formula can produce a variety of different sequences. For some values of c, the value of z quickly gets extremely large, but for other values of c, the sequence is bounded. The following graphs show a few of the possibilities.

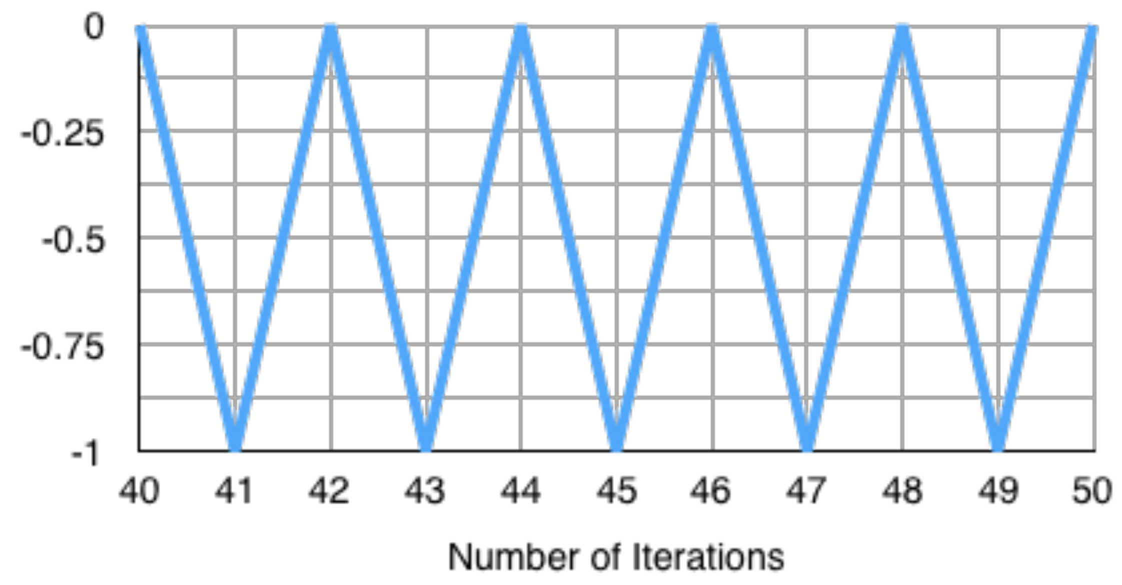
If c = 0.25, z steadily increases.



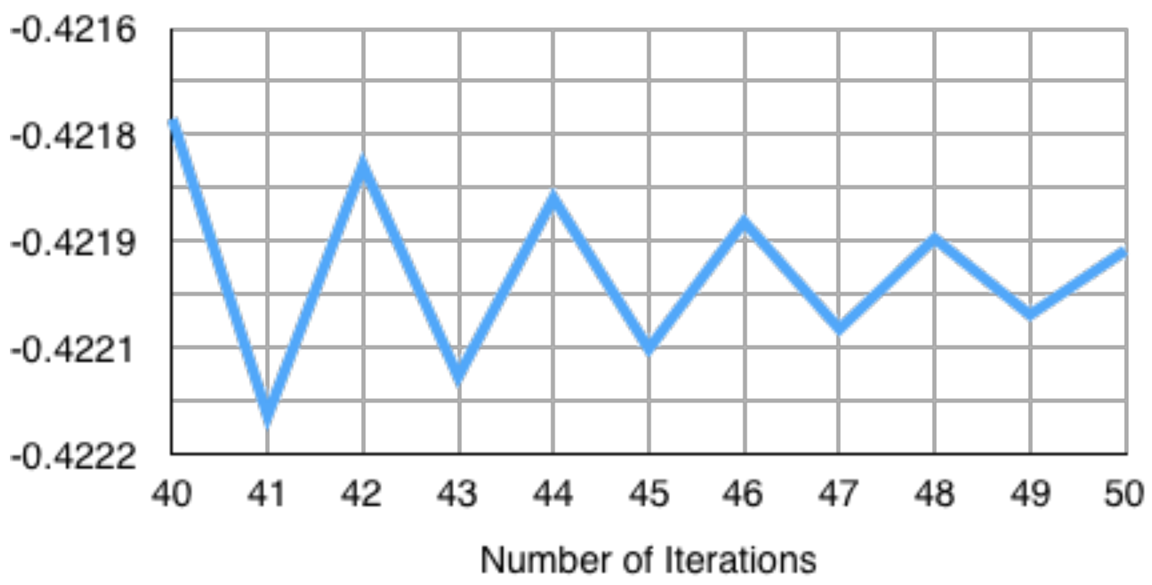

If c = 0.20, z appears to be leveling off.



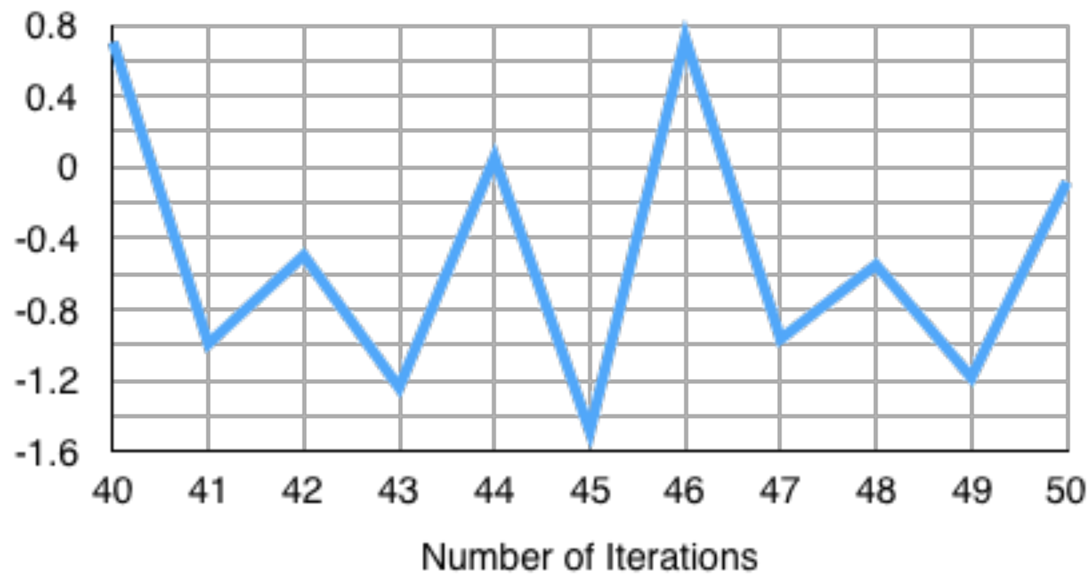

If c = -1, z simply jumps between 0 and -1.



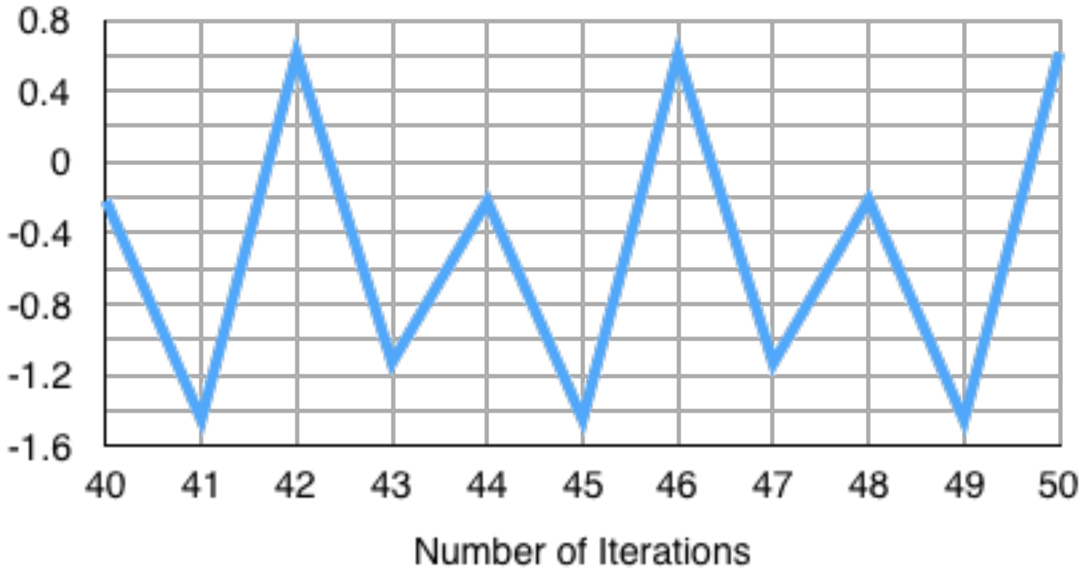
If c = -0.60, z jumps up and down but seems to be leveling off.



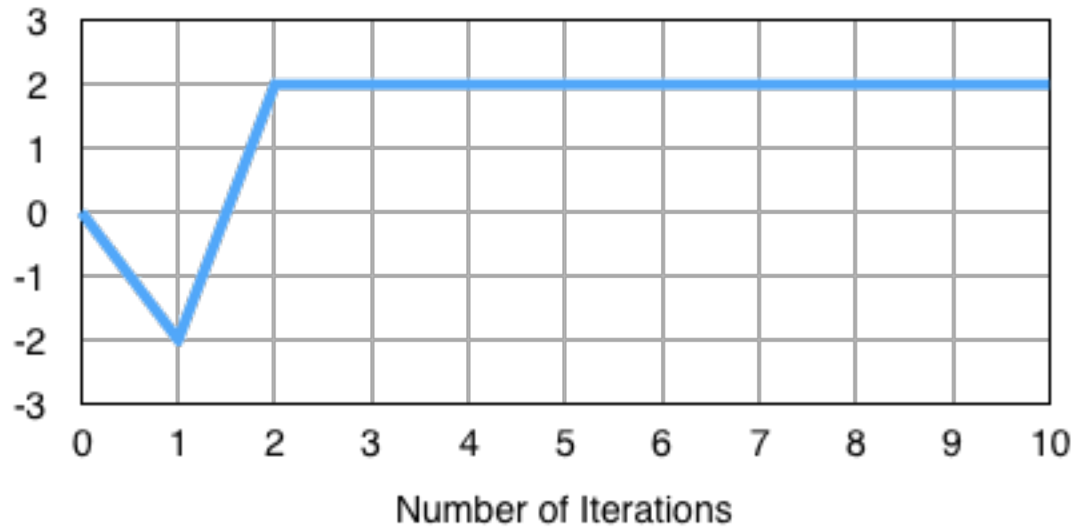

But if c = -1.49, z jumps between 6 different values. Just a small change in c makes a significant difference!




If c = -1.50, z jumps between 4 different values.




If c = -2, $z_n$ equals 2 for all values of greater than 1.



Even stranger things happen when you include all of the complex numbers. The values of c are no longer restricted to the real axis; instead they cover the entire complex plane.

The Mandelbrot set is shown to the right. Points in the black areas represent values of c for which the recursive formula produces a sequence which is bounded. Points in the other colored areas represent values of c for which an element of the sequence has an absolute value greater than 2, an indication that the sequence is going to be divergent.

The specific color indicates how quickly the sequence diverges; sequences with values of c in the blue area diverge quickly; sequences with values of c in the red area diverge much more slowly.

As with the real numbers, some complex c values produce a sequence which quickly converges to a limit:

- 0.1 + 0.1i → - 0.1 + 0.08i → - 0.096 + 0.084i →

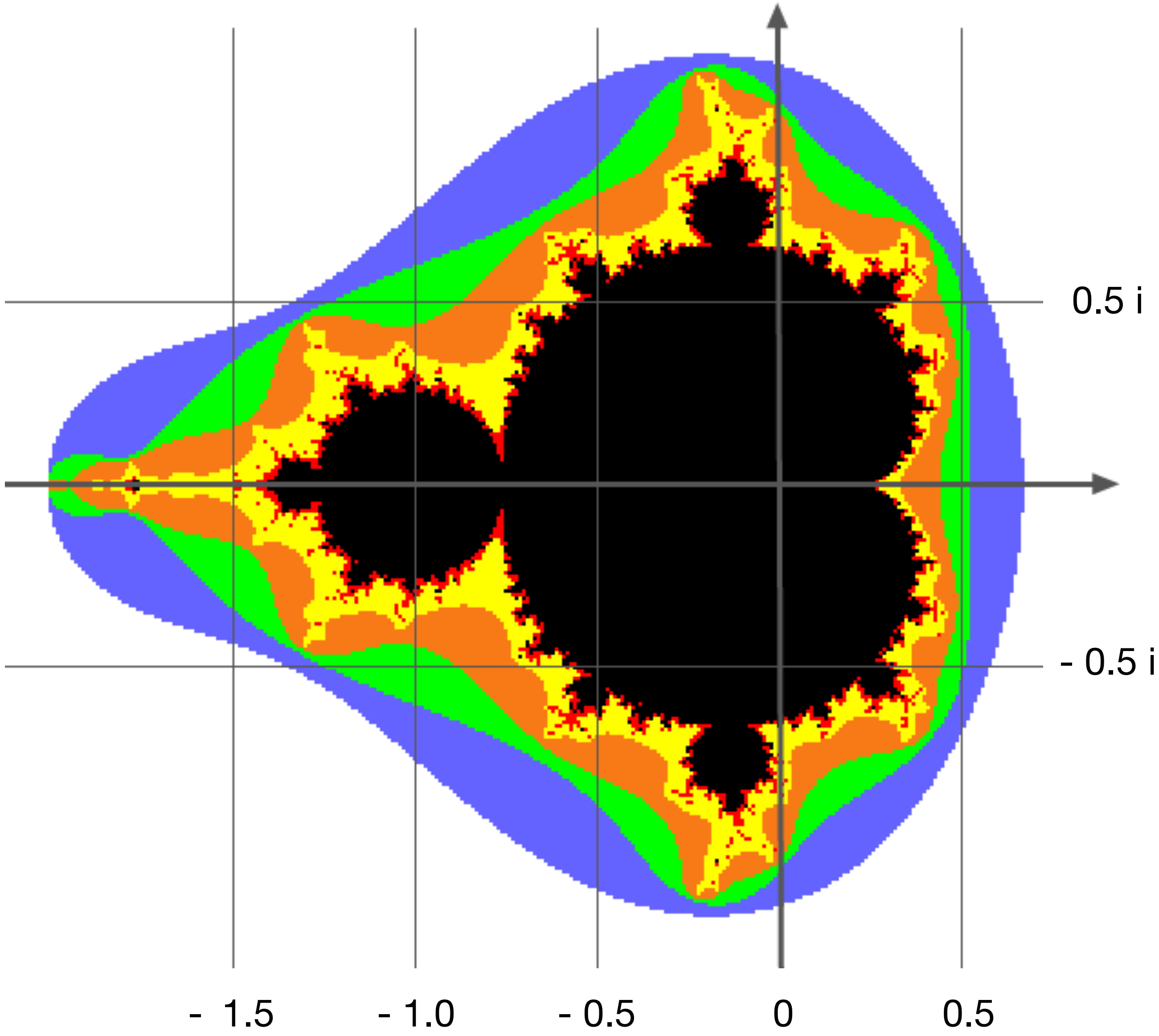
- 0.098 + 0.084i → - 0.097 + 0.084i → 0.097 + 0.084i

Other c values produce a sequence which is bounded but alternates between several different points:

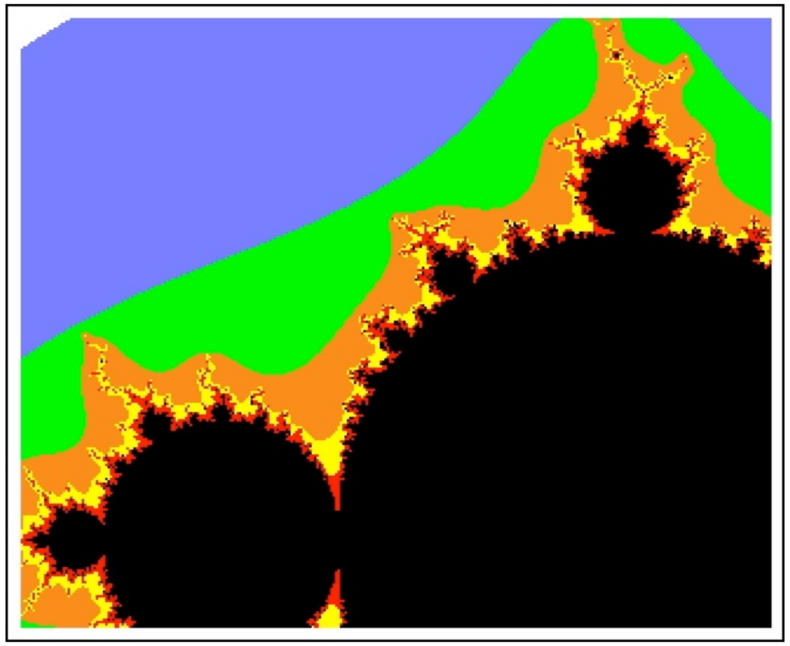
-1.135 + 0.235i → 0.098 - 0.298i → -1.214 + 0.177i →

0.307 - 0.194i → -1.078 + 0.116i → 0.014 - 0.015i →

-1.135 + 0.235i → 0.098 - 0.298i → -1.214 + 0.177i . . .




The following three pages show what happens when you zoom in on one particular region near the edge of the Mandelbrot Set. The complex number at the center of each printout is -0.650911+ 0.440255i, and each printout has 1/2 the length and width of the previous one. N = the number of iterations.



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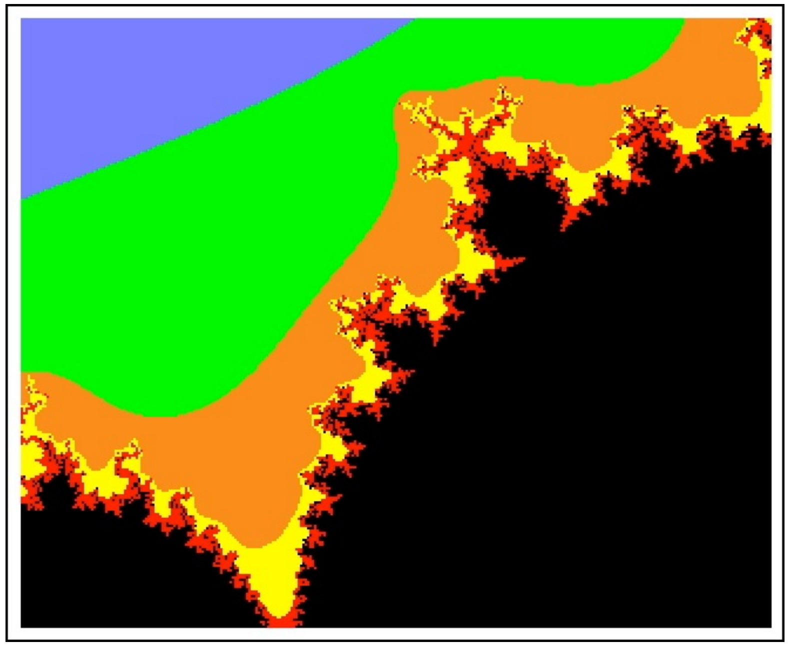
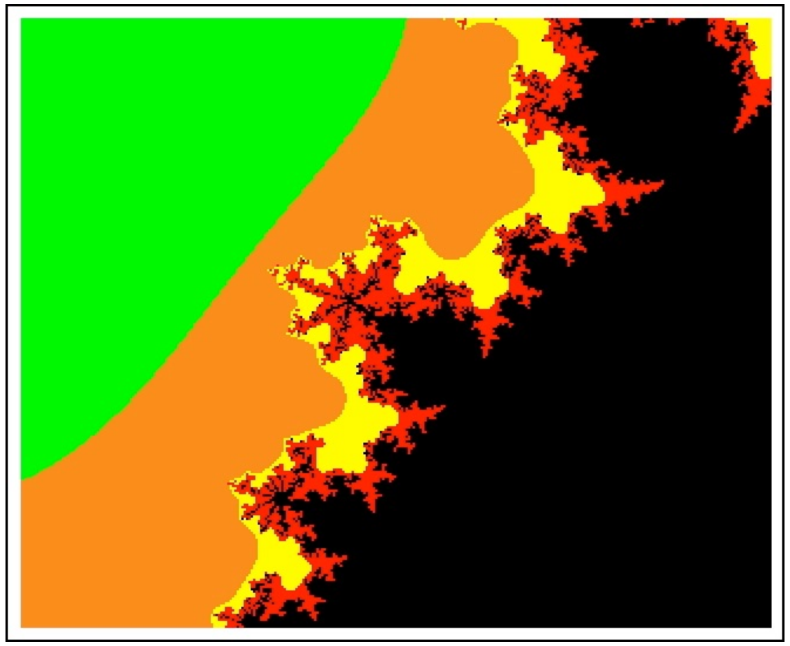
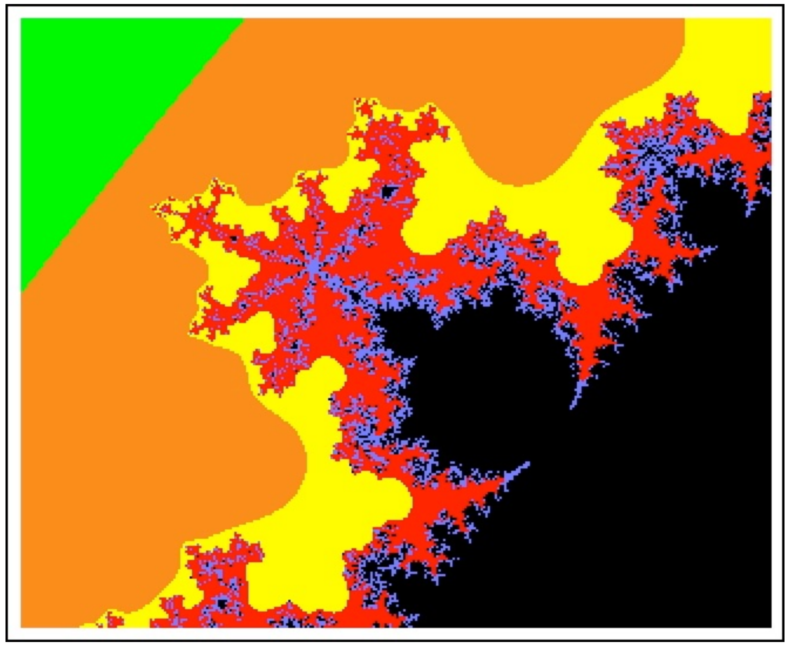


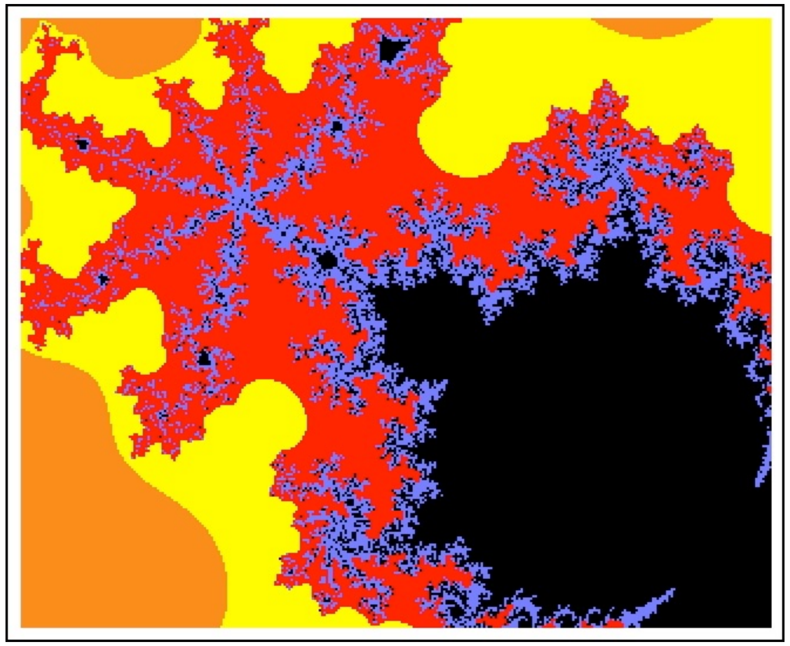
2) N = 50



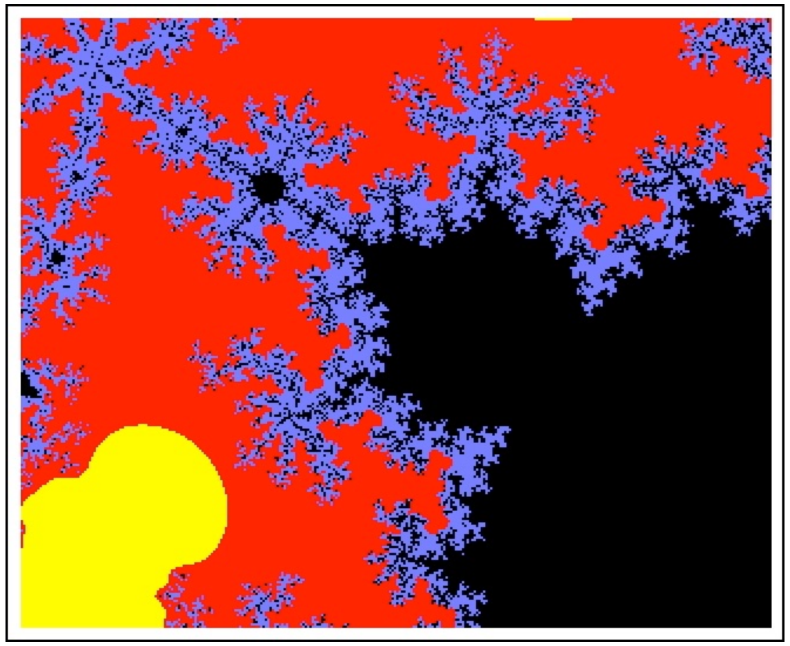
3) N = 50



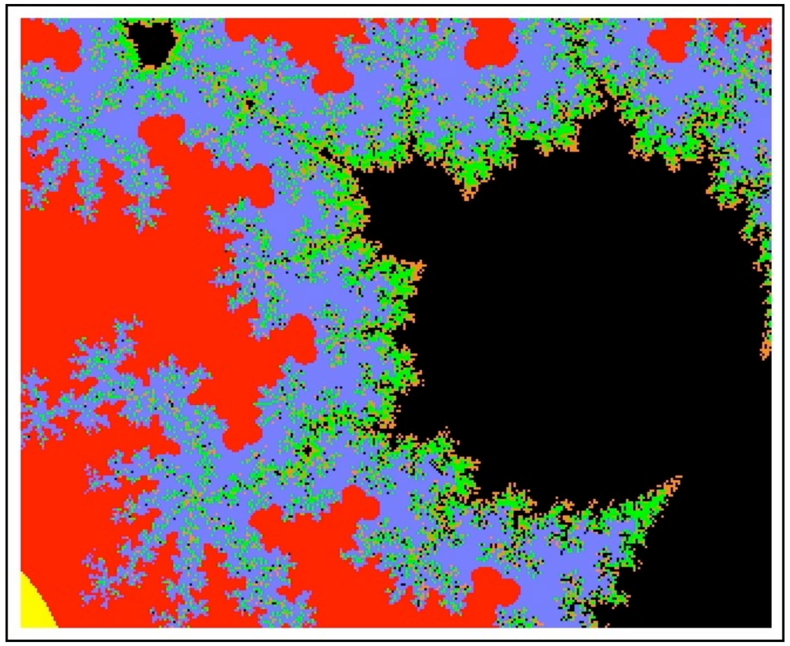
4) N = 100



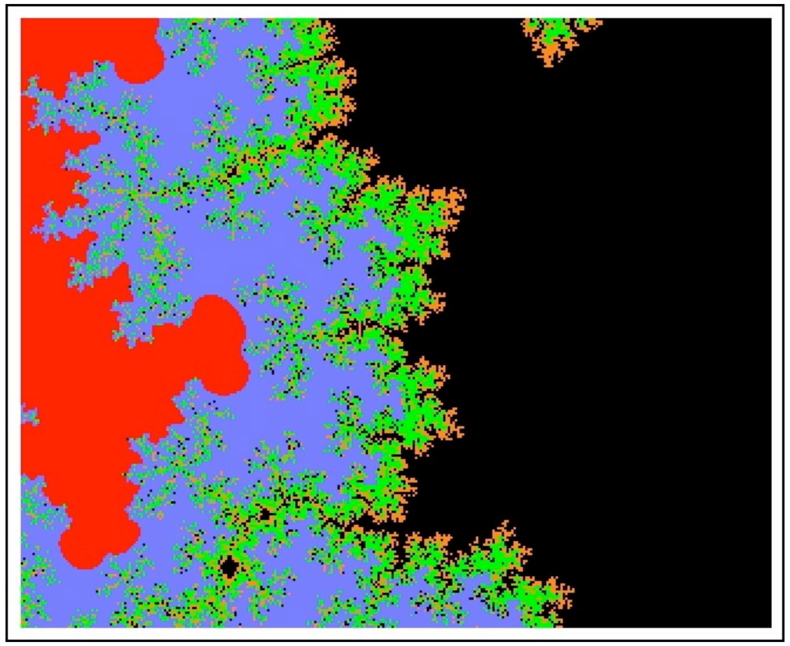
5) N = 100



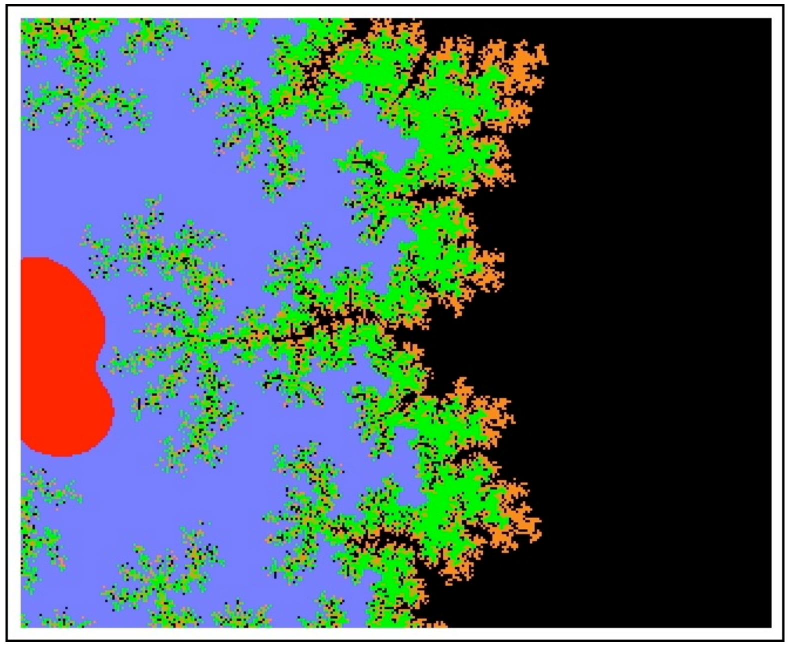
6) N = 100



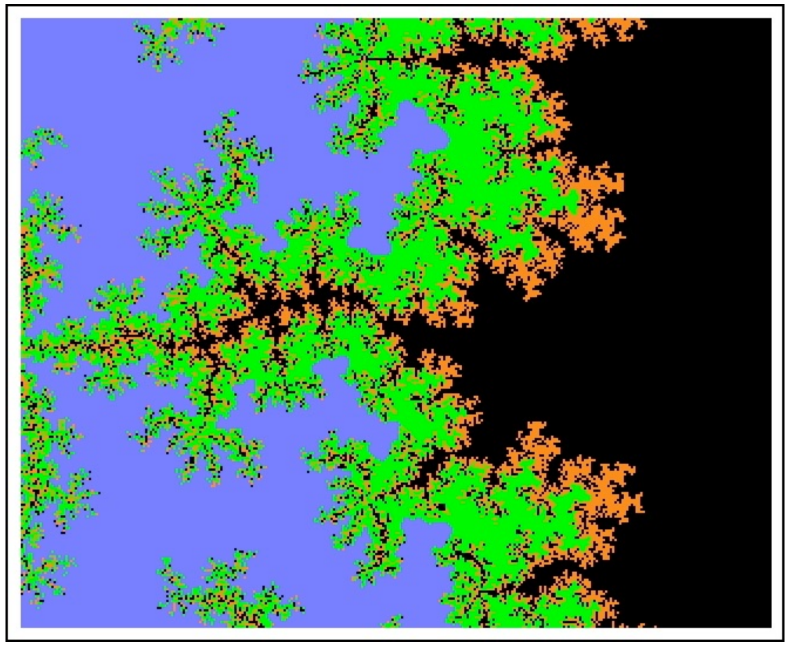
7) N = 200



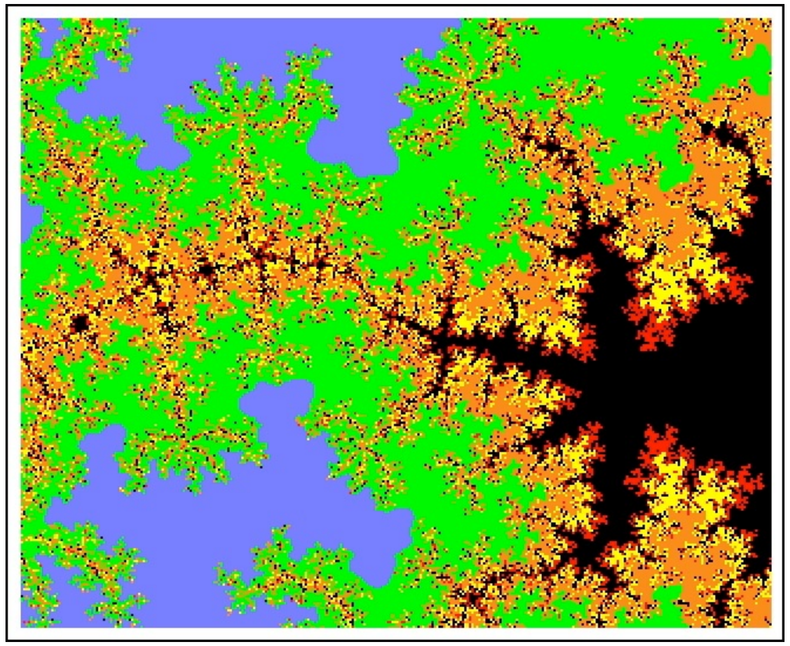
8) N = 200



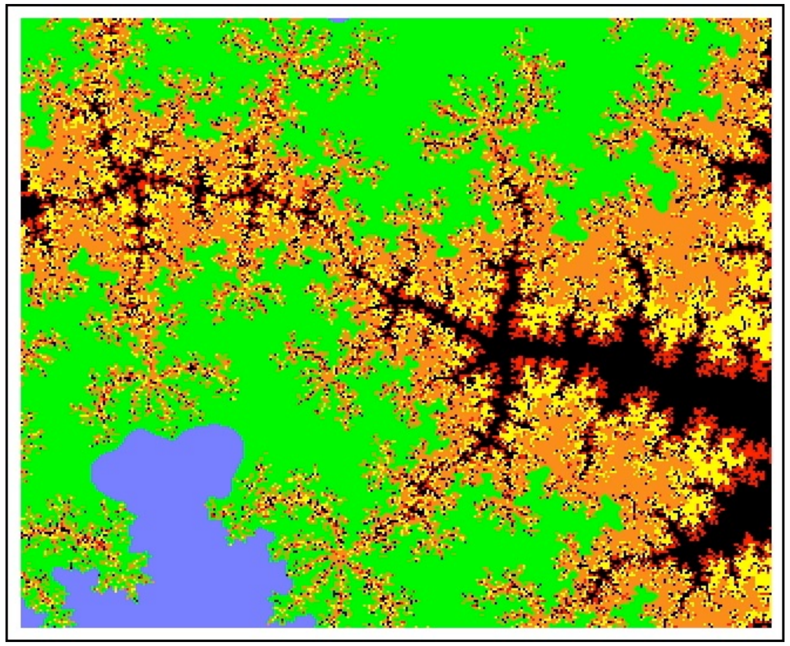
9) N = 200



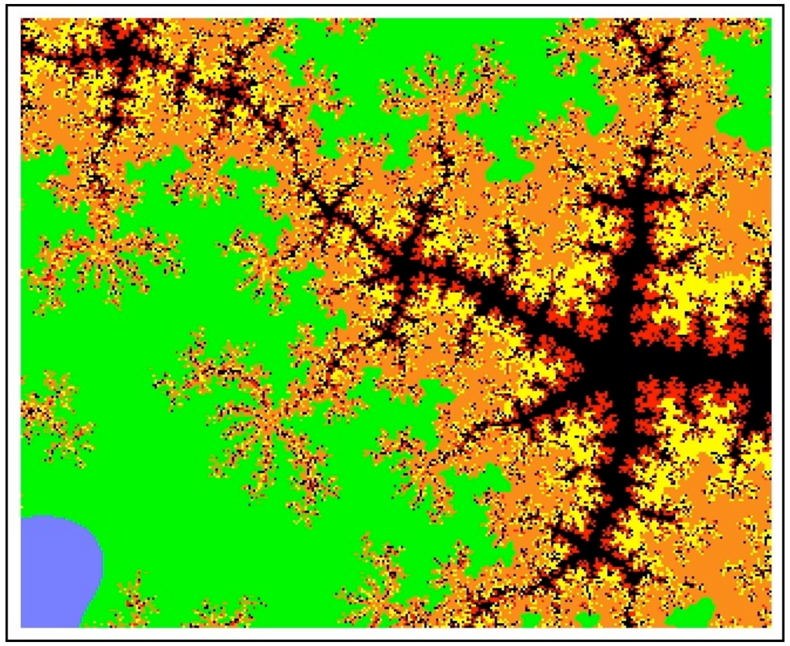
10) N = 200



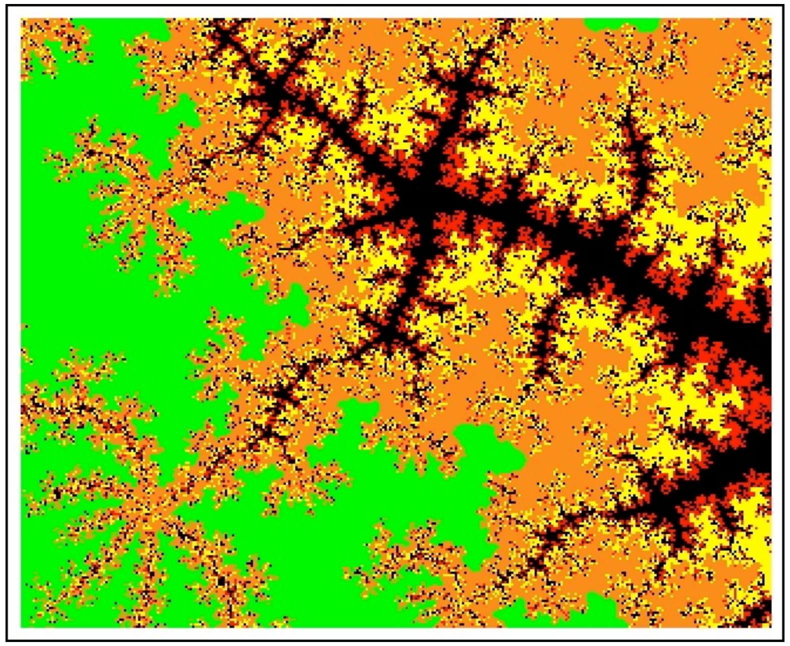
11) N = 300



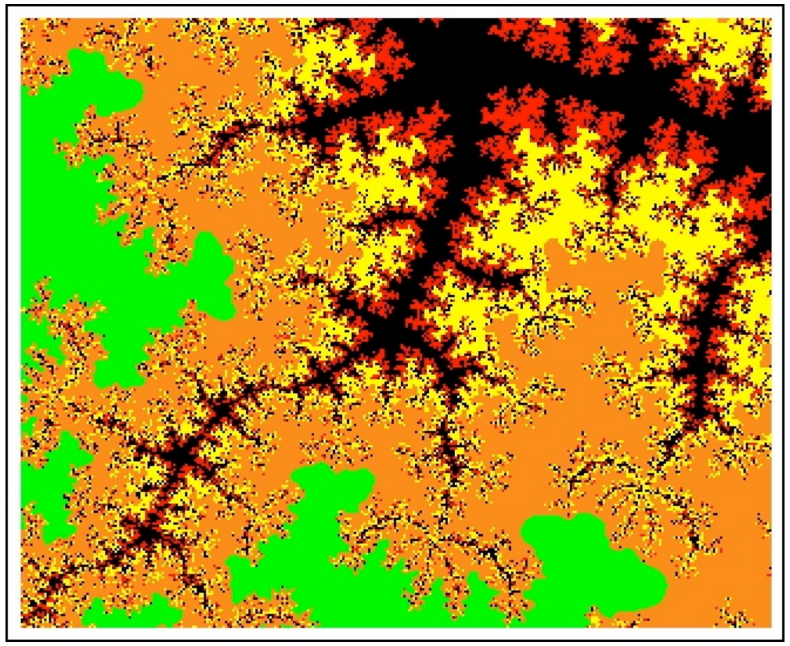
12) N = 300



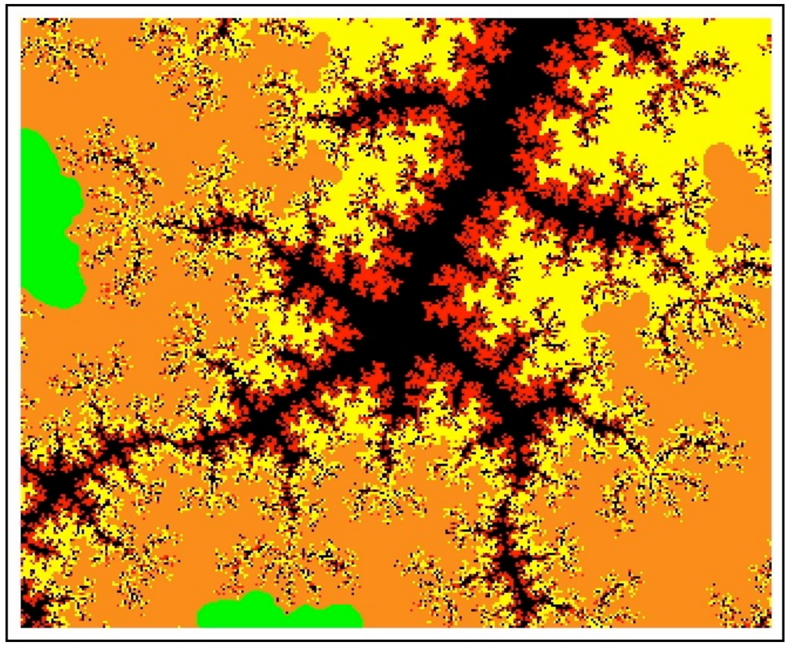
13) N = 300



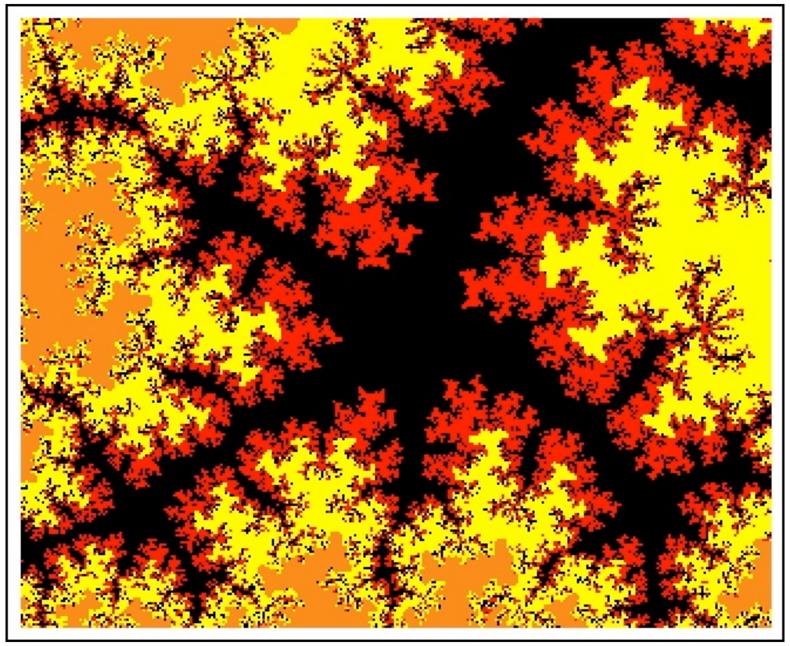
14) N = 300



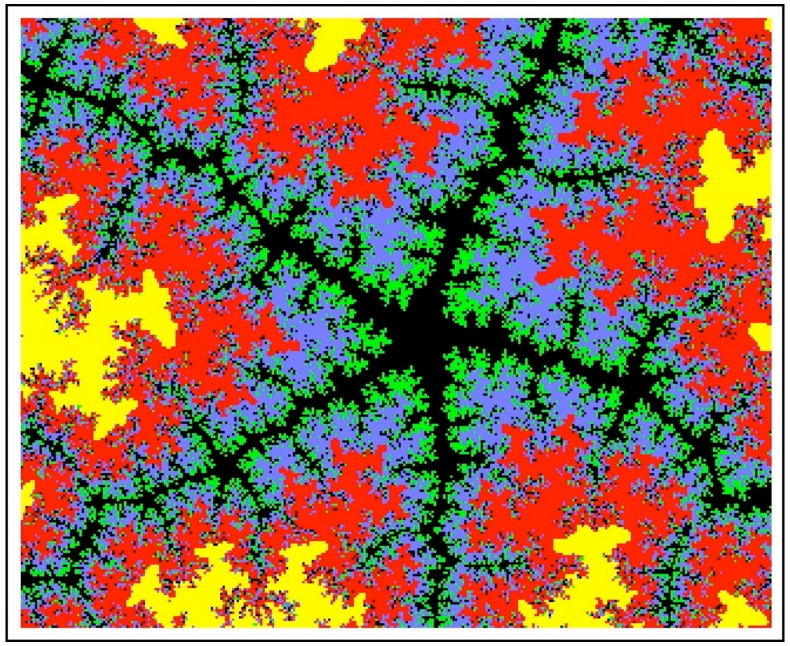
15) N = 300



16) N = 300



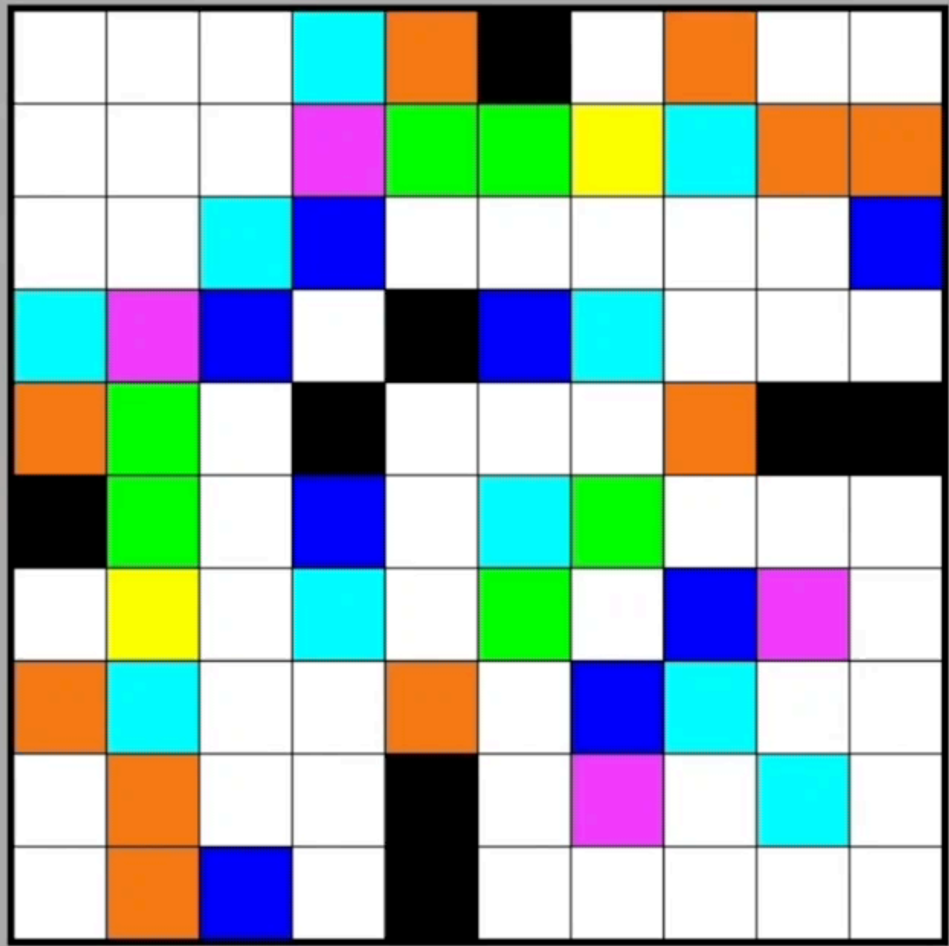
17) N = 300



18) N = 400

# 8. Connections

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 58 | 91 | 67 | 134 | 88 | 16 | 95 | 88 | 91 | 33 |
| 91 | 67 | 56 | 43 | 32 | 32 | 31 | 134 | 88 | 88 |
| 67 | 56 | 134 | 19 | 112 | 23 | 12 | 8 | 91 | 19 |
| 134 | 43 | 19 | 95 | 16 | 19 | 134 | 12 | 40 | 7 |
| 88 | 32 | 112 | 16 | 35 | 91 | 11 | 88 | 16 | 16 |
| 16 | 32 | 23 | 19 | 91 | 134 | 32 | 67 | 8 | 70 |
| 95 | 31 | 12 | 134 | 11 | 32 | 33 | 19 | 43 | 55 |
| 88 | 134 | 8 | 12 | 88 | 67 | 19 | 134 | 24 | 22 |
| 91 | 88 | 91 | 40 | 16 | 8 | 43 | 24 | 134 | 46 |
| 33 | 88 | 19 | 7 | 16 | 70 | 55 | 22 | 46 | 91 |



## 8.1 Introduction

This chapter revisits some of the topics from the previous chapters and explores connections between them. The first section deals with connections between groups, permutations, and matrices; the second deals with with connections between two digit numbers, the Mandelbrot Set, recursive functions and Quilt patterns; and the final section deals with connections between Pascal's Triangle, the Binomial Theorem, binomial probabilities, Sierpinski's Triangle, and the Game of Life.

The **Arithmetic Quilts** is revisited and a new program, called **Dot-dots**, is introduced. As with all the other programs discussed in this book, they are available for free from my **Syzygy Shareware** site at http://tcbretl.weebly.com.

## 8.2 Permutation Groups

All of the groups discussed in Chapter 1 were based on arithmetic, and all of them were abelian groups. Permutation groups are quite different.

Consider all of the ways you can rearrange three objects, labelled here as A, B, and C. Each rearrangement results in a different permutation of the three objects.

| | | |
|---|---|---|
| 0. | Don’t rearrange them at all - | A B C |
| 1. | Switch the positions of the last two - | A C B |
| 2. | Switch the positions of the first two - | B A C |
| 3. | Move the first one to the end - | B C A |
| 4. | Move the last one to the front - | C A B |
| 5. | Switch the positions of first and last - | C B A |

What happens when one of these rearrangements follows another? If rearrangement #2 is followed by rearrangement #3 for example, do you see that A B C becomes A C B, the same as rearrangement #1.?

If we let the + operation mean “followed by”, then 2 + 3 = 1. (Rearrangement 2 followed by rearrangement 3 is the same as rearrangement 1.)

Note, however, that 3 + 2 does not equal 1. Instead, A B C becomes C B A which is the same as #5.

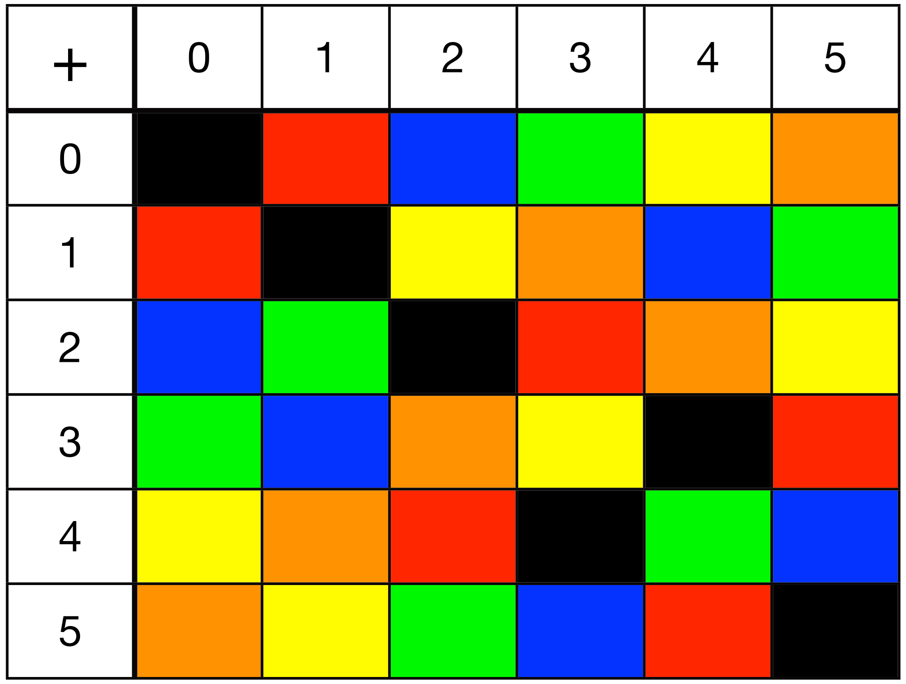
Here’s an “addition” table for all six of the rearrangements. The order matters. The first rearrangement is the row number and the second rearrangement is the column number.

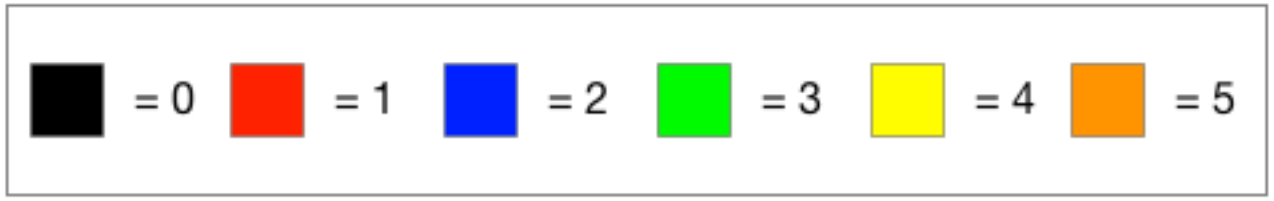
2nd

| 1st | + | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| | 0 | 0 | 1 | 2 | 3 | 4 | 5 |
| | 1 | 1 | 0 | 4 | 5 | 2 | 3 |
| | 2 | 2 | 3 | 0 | 1 | 5 | 4 |
| | 3 | 3 | 2 | 5 | 4 | 0 | 1 |
| | 4 | 4 | 5 | 1 | 0 | 3 | 2 |
| | 5 | 5 | 4 | 3 | 2 | 1 | 0 |

The six possible rearrangements form a non-abelian group. 0 is the identity element, and every rearrangement has an inverse. It is left to the reader to show that the associative property holds. There are two subgroups: {0,1} and {0,3,4}

When color coded, it looks like this:




There’s another way to look at this permutation group. Represent each rearrangement as a 3x3 matrix and the ordered objects as elements of a 3x1 matrix. Matrix multiplication changes the order of the objects:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}A\\B\\C\end{bmatrix}=\begin{bmatrix}1{\bullet}A+0{\bullet}B+0{\bullet}C\\0{\bullet}A+1{\bullet}B+0{\bullet}C\\0{\bullet}A+0{\bullet}B+1{\bullet}C\end{bmatrix}=\begin{bmatrix}A\\B\\C\end{bmatrix}$$

$$\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}A\\B\\C\end{bmatrix}=\begin{bmatrix}0{\bullet}A+1{\bullet}B+0{\bullet}C\\0{\bullet}A+0{\bullet}B+1{\bullet}C\\1{\bullet}A+0{\bullet}B+0{\bullet}C\end{bmatrix}=\begin{bmatrix}B\\C\\A\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}A\\B\\C\end{bmatrix}=\begin{bmatrix}1{\bullet}A+0{\bullet}B+0{\bullet}C\\0{\bullet}A+0{\bullet}B+1{\bullet}C\\0{\bullet}A+1{\bullet}B+0{\bullet}C\end{bmatrix}=\begin{bmatrix}A\\C\\B\end{bmatrix}$$

$$\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}A\\B\\C\end{bmatrix}=\begin{bmatrix}0{\bullet}A+0{\bullet}B+1{\bullet}C\\1{\bullet}A+0{\bullet}B+0{\bullet}C\\0{\bullet}A+1{\bullet}B+0{\bullet}C\end{bmatrix}=\begin{bmatrix}C\\A\\B\end{bmatrix}$$

$$\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}A\\B\\C\end{bmatrix}=\begin{bmatrix}0{\bullet}A+1{\bullet}B+0{\bullet}C\\1{\bullet}A+0{\bullet}B+0{\bullet}C\\0{\bullet}A+0{\bullet}B+1{\bullet}C\end{bmatrix}=\begin{bmatrix}B\\A\\C\end{bmatrix}$$

$$\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}A\\B\\C\end{bmatrix}=\begin{bmatrix}0{\bullet}A+0{\bullet}B+1{\bullet}C\\0{\bullet}A+1{\bullet}B+0{\bullet}C\\1{\bullet}A+0{\bullet}B+0{\bullet}C\end{bmatrix}=\begin{bmatrix}C\\B\\A\end{bmatrix}$$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is rearrangement 0.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is rearrangement 1.

$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is rearrangement 2.

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ is rearrangement 3.

$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is rearrangement 4.

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is rearrangement 5.

The "followed by" operation becomes matrix multiplication (with the "followed by" matrix on the left):

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0\cdot0+1\cdot1+0\cdot0 & 0\cdot1+1\cdot0+0\cdot0 & 0\cdot0+1\cdot0+0\cdot1 \\ 0\cdot0+0\cdot1+1\cdot0 & 0\cdot1+0\cdot0+1\cdot0 & 0\cdot0+0\cdot0+1\cdot1 \\ 1\cdot0+0\cdot1+0\cdot0 & 1\cdot1+0\cdot0+0\cdot0 & 1\cdot0+0\cdot0+0\cdot1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Rearrangement 2 followed by rearrangement 3 = rearrangement 1.

Order makes a difference:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0\cdot0+1\cdot0+0\cdot1 & 0\cdot1+1\cdot0+0\cdot0 & 0\cdot0+1\cdot1+0\cdot0 \\ 1\cdot0+0\cdot0+0\cdot1 & 1\cdot1+0\cdot0+0\cdot0 & 1\cdot0+0\cdot1+0\cdot0 \\ 0\cdot0+0\cdot0+1\cdot1 & 0\cdot1+0\cdot0+1\cdot0 & 0\cdot0+0\cdot1+1\cdot0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Rearrangement 3 followed by rearrangement 2 = rearrangement 5.

# 8.3 Recursive Quilts

The Mandelbrot Set program is really a Quilts program with tiny squares, but the color of each square depends on a recursive formula instead of simple arithmetic. Other recursive formulas can also be used to create interesting quilts.

Start with any one or two digit number. Square the ones digit and square the tens digit. Add the results. Repeat the process. If the sum includes a hundreds digit, then square and add all three digits. What happens in the long run?

**Example:**

Start with 24.

2 squared + 4 squared = 20

→ 2 squared + 0 squared = 4

0 squared + 4 squared = 16

1 squared + 6 squared = 37

3 squared + 7 squared = 58

5 squared + 8 squared = 89

8 squared + 9 squared = 145

1 squared + 4 squared + 5 squared = 42

4 squared + 2 squared = 20

The sequence cycles through just 8 different values.

Create a number grid containing integers between 0 and 99. Iterate all of them simultaneously.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 00 | 01 | 02 | 03 | 04 | 05 | 06 | 07 | 08 | 09 |
| 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 |
| 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 |
| 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 |
| 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 |
| 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 |
| 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 |
| 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 |

After only 9 iterations, every number on the grid equals 4, 16, 20, 37, 58, 89, 145, or 42, except for 0 and those that end up stuck on the number 1.

**Example**:

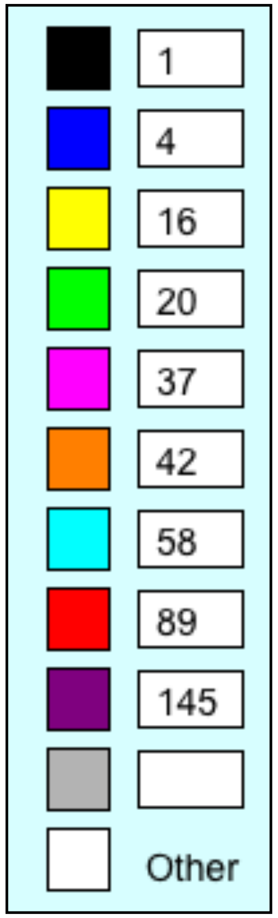
7 → 49 → 97 → 130 →

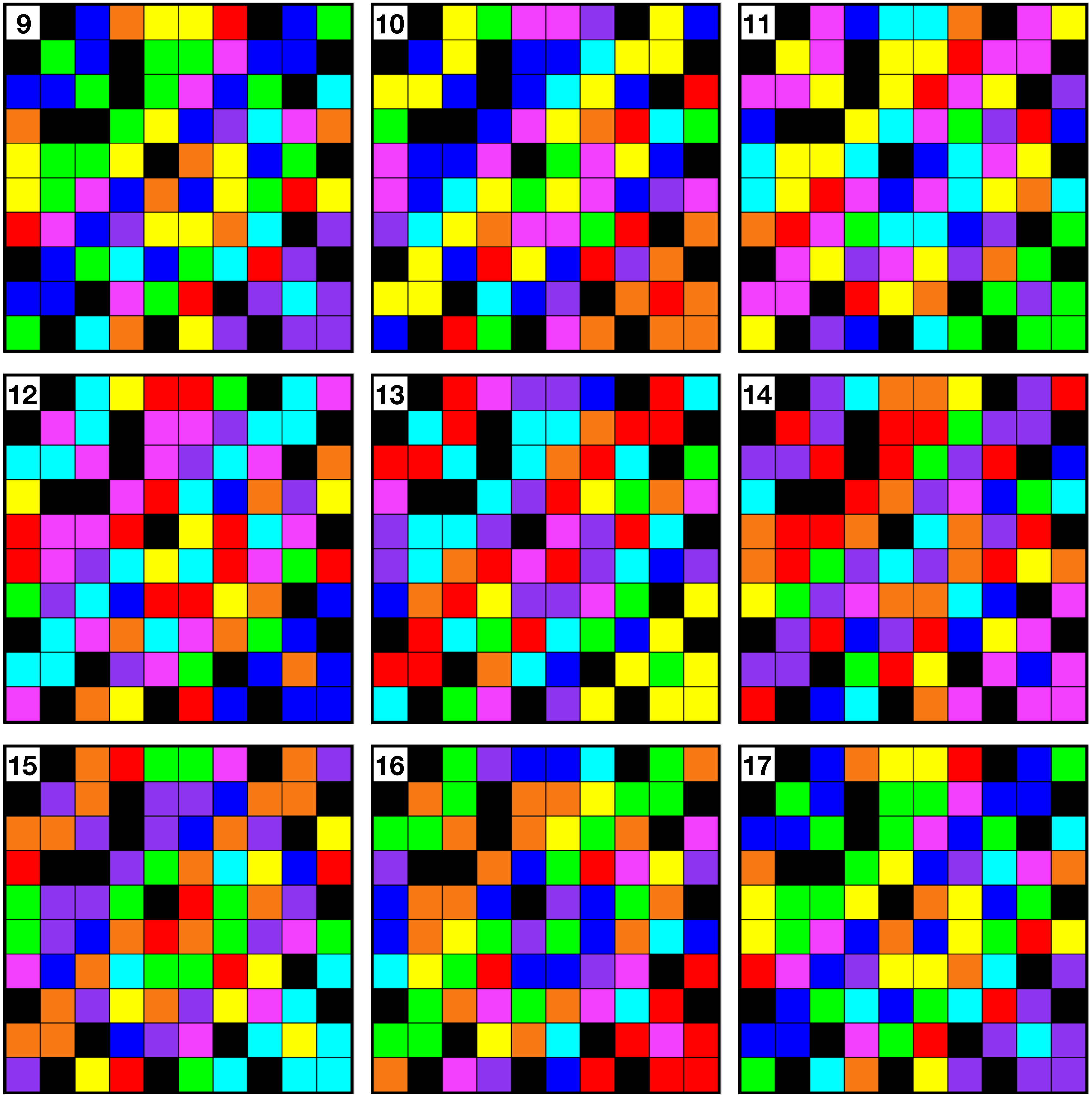
10 → 1 → 1 → 1 ...

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 4 | 42 | 16 | 16 | 89 | 1 | 4 | 20 |
| 1 | 20 | 4 | 1 | 20 | 20 | 37 | 4 | 4 | 1 |
| 4 | 4 | 20 | 1 | 20 | 37 | 4 | 20 | 1 | 58 |
| 42 | 1 | 1 | 20 | 16 | 4 | 145 | 58 | 37 | 42 |
| 16 | 20 | 20 | 16 | 1 | 42 | 16 | 4 | 20 | 1 |
| 16 | 20 | 37 | 4 | 42 | 4 | 16 | 20 | 89 | 16 |
| 89 | 37 | 4 | 145 | 16 | 16 | 42 | 58 | 1 | 145 |
| 1 | 4 | 20 | 58 | 4 | 20 | 58 | 89 | 145 | 1 |
| 4 | 4 | 1 | 37 | 20 | 89 | 1 | 145 | 58 | 145 |
| 20 | 1 | 58 | 42 | 1 | 16 | 145 | 1 | 145 | 145 |

Here is a color coded sequence of iterations 9 through 17. Notice that iteration 17 is the same as 9.

Then it all starts over again.



Numbers that a recursively defined sequence gets "stuck on" are sometimes called "black holes."

### An Example with more Black Holes

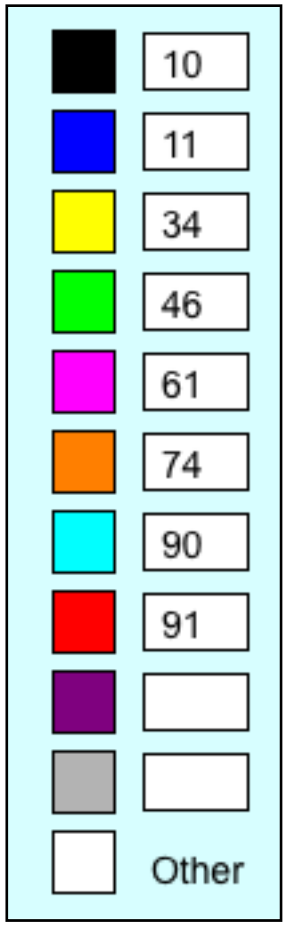
Square and add the ones and tens digits as before, but this time also add on 9. After several iterations, the number grid will alternate back and forth between just two patterns. 10, 11, 34, 74, 90, and 91 are black holes, while 46 becomes 61 and 61 becomes 46.

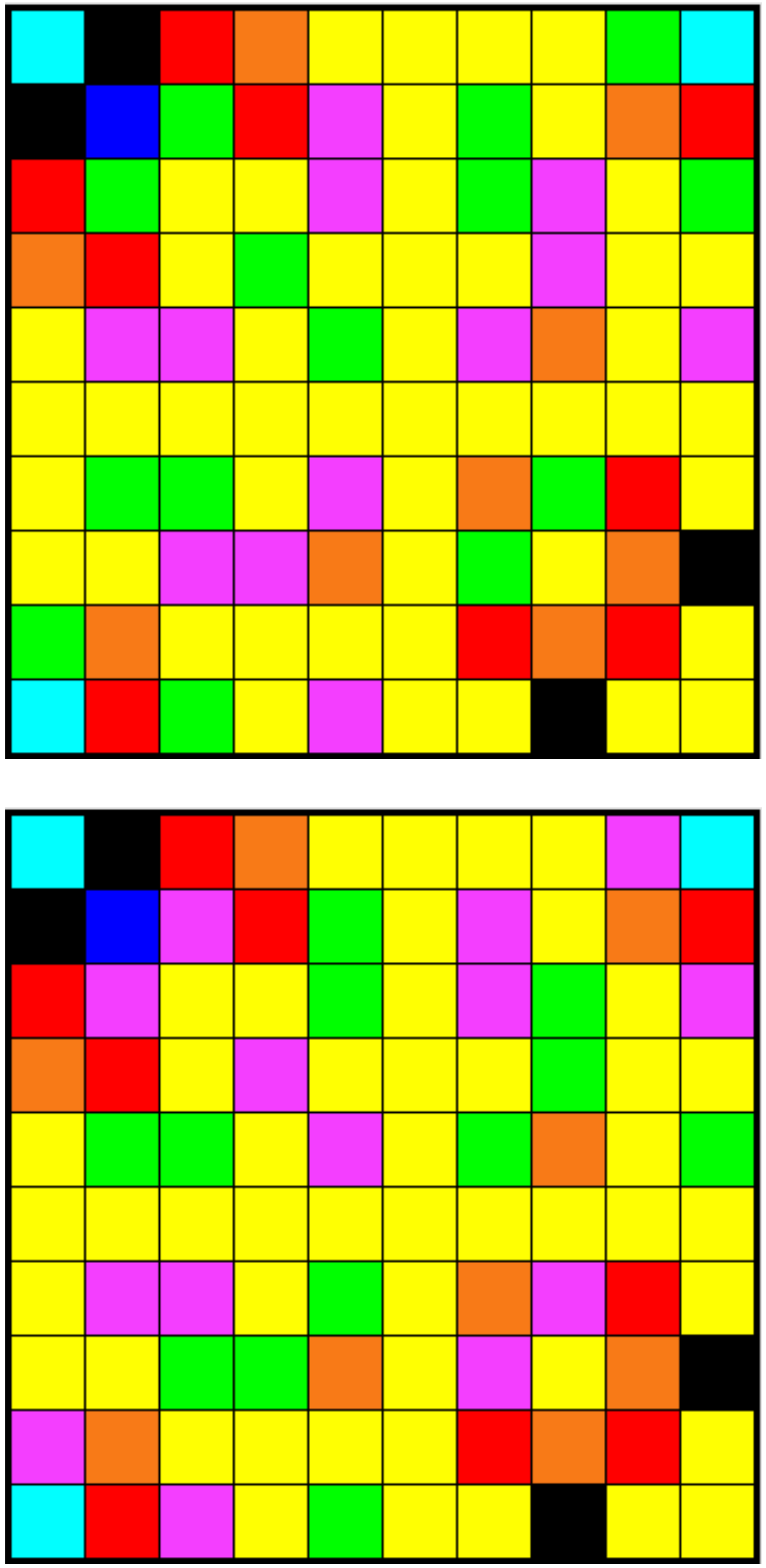
A little algebra reveals why this sequence has so many black holes. Let x be the tens digit and y be the ones digit, so the number = 10x + y. Assume it is a black hole:

$$x^2 + y^2 + 9 = 10x + y$$

$$x^2 - 10x + 9 = y - y^2$$

$$(x-1)(x-9) = y(1-y)$$



**Four easy solutions:**

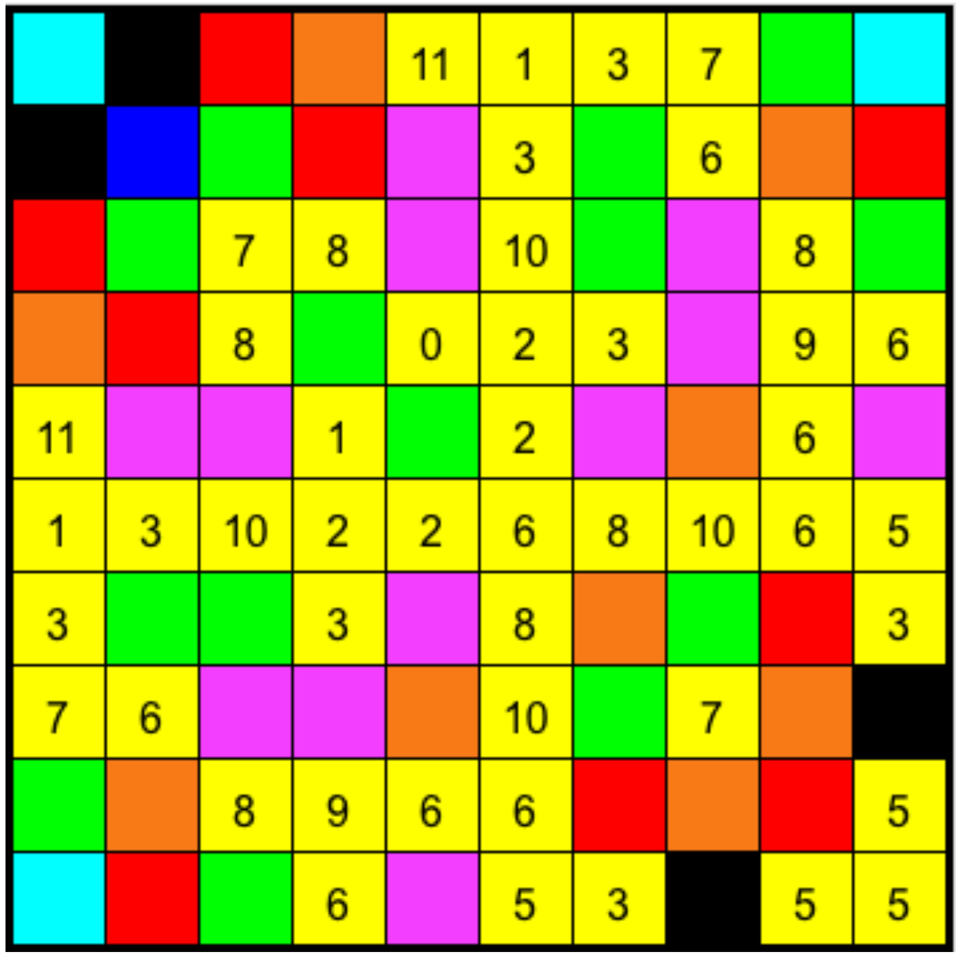
x = 1 and y = 0, x = 1 and y = 1, x = 9 and y = 0, x = 9 and y = 1

Adding 9 really makes a difference. It is the only single digit number you can add to $x^2 - 10x$ and end up with an expression that can be factored using only integers.

**Two more solutions:** x = 3 and y = 4, x = 7 and y = 4. In both cases, $x^2 + y^2 + 9 = 10x + y = 34$.

Some starting numbers get recursively sucked into a black hole more quickly than others.

The numbers overlaid on the grid shown here tell how many iterations were needed for 48 “doomed” sequences to reach 34, turning the color of their cells permanently yellow.



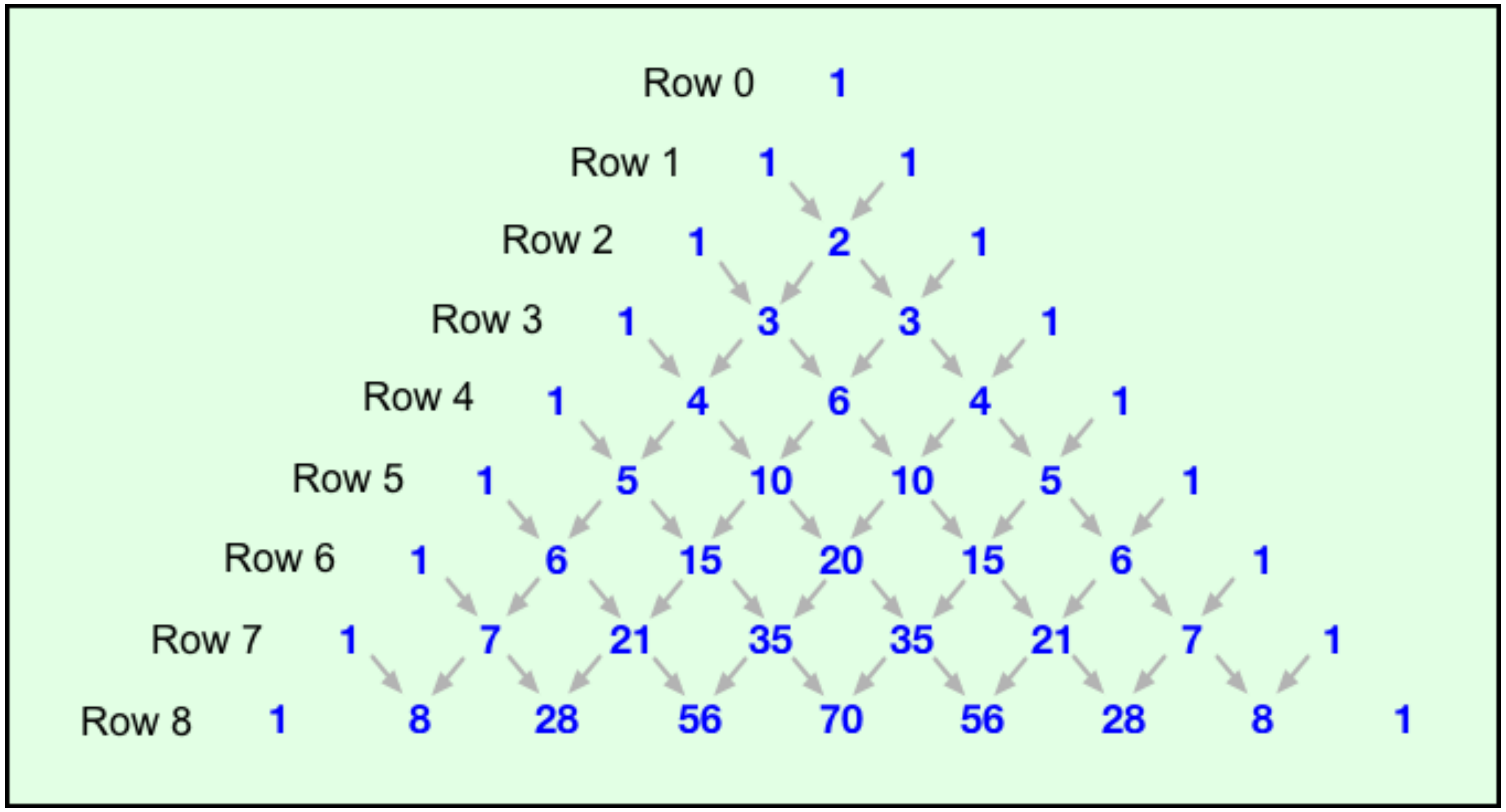

Try replacing 9 in the recursive formula with some other integer (either positive or negative). To avoid negative results, always take the absolute value of each iteration before proceeding: next term = $\left|x^2 + y^2 + n\right|$, where $x$ and $y$ are the tens and ones digits, respectively, from the previous term of the sequence, and n is a constant.

Can you find a value for n that makes all of the sequences end in a black hole? The same black hole?

Can you find a value for n that makes all of the sequences switch back and forth between just two values?

# 8.4 Pascal's Triangle

As shown below, each number in Pascal's triangle (except for the 1's along its sides) equals the sum of the two numbers directly above it.




The patterns found within this triangle have an extraordinary number of "connections" to a variety mathematical concepts. Here are just a few of them:

### 1. The Binomial Theorem

The nth row of Pascal's triangle tells you the coefficients in the expansion of $(x+y)^n$

Example: $(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

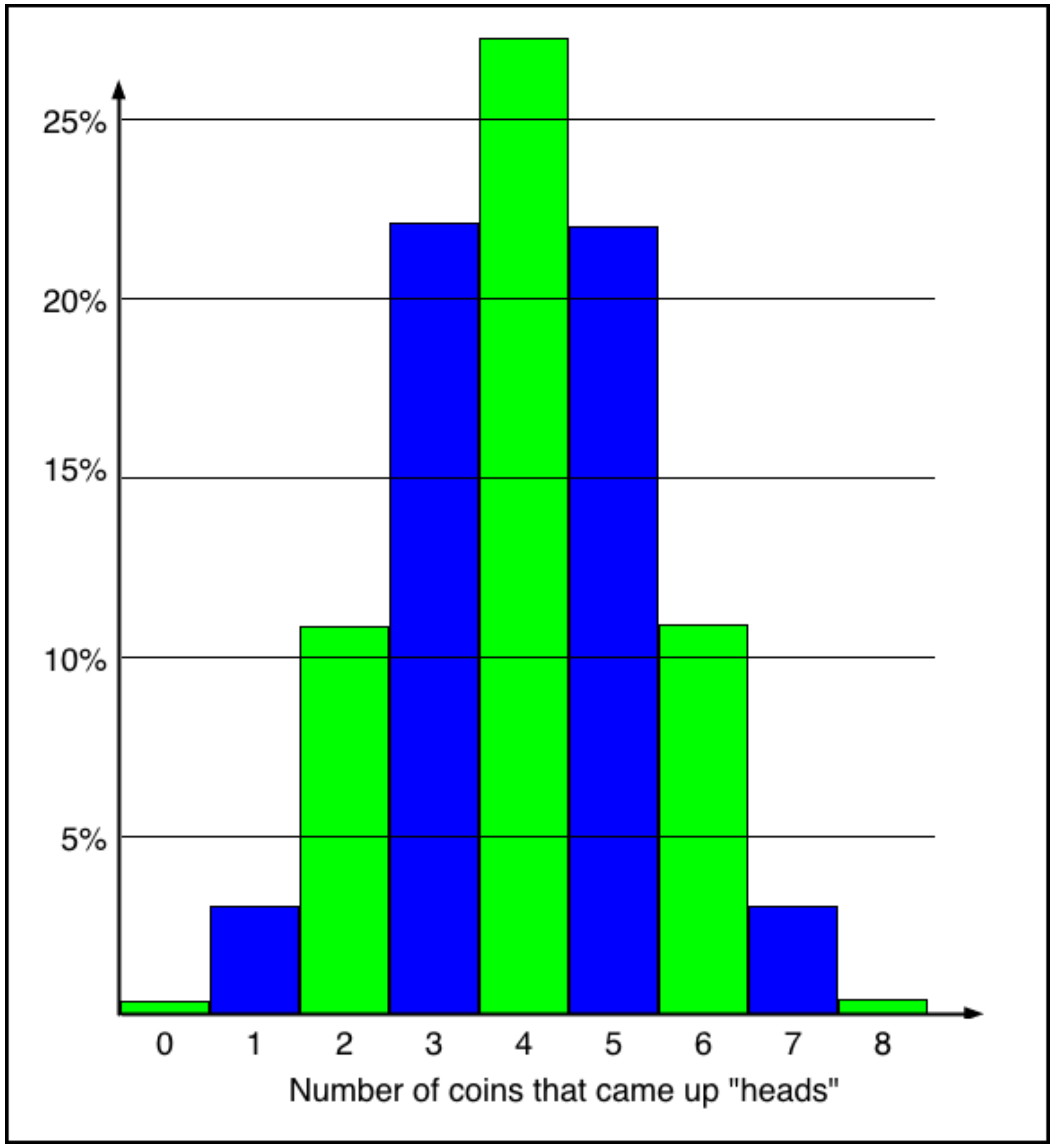
## 2. Binomial Probabilities

The sum of the numbers in a row of Pascal's triangle is always a power of 2. The sum of the nth row is $2^n$. This also equals the number of possible outcomes when you flip a coin n times (or flip n coins one time).

The rth number in the nth row equals nCr, the number of combinations of n things taken r at a time. This also equals the number of times you would expect (on the average) to get r *heads* when you flip a coin n times, so p(r), the probability of getting r *heads*, is nCr divided by $2^n$.

| r | ${}_8C_r$ | Probability |
|---|---|---|
| 0 | 1 | 1/256 ≈ 0.4 % |
| 1 | 8 | 8/256 ≈ 3.1 % |
| 2 | 28 | 28/256 ≈ 10.9 % |
| 3 | 56 | 56/256 ≈ 21.9 % |
| 4 | 70 | 70/256 ≈ 27.3 % |
| 5 | 56 | 56/256 ≈ 21.9 % |
| 6 | 28 | 28/256 ≈ 10.9 % |
| 7 | 8 | 8/256 ≈ 3.1 % |
| 8 | 1 | 1/256 ≈ 0.4 % |

If you flip a coin 8 times, there are 256 possible outcomes. The table below lists the probabilities, and the bar chart to its right shows the results obtained when the **Probability Simulator** program was used to simulate 8 coin flips 50,000 times. Note the close agreement!



**Another example:** Suppose you roll a pair of dice, and your goal is to roll a total of either 8 or 9. If you roll 6 times, how many times should you expect, on the average, to succeed?

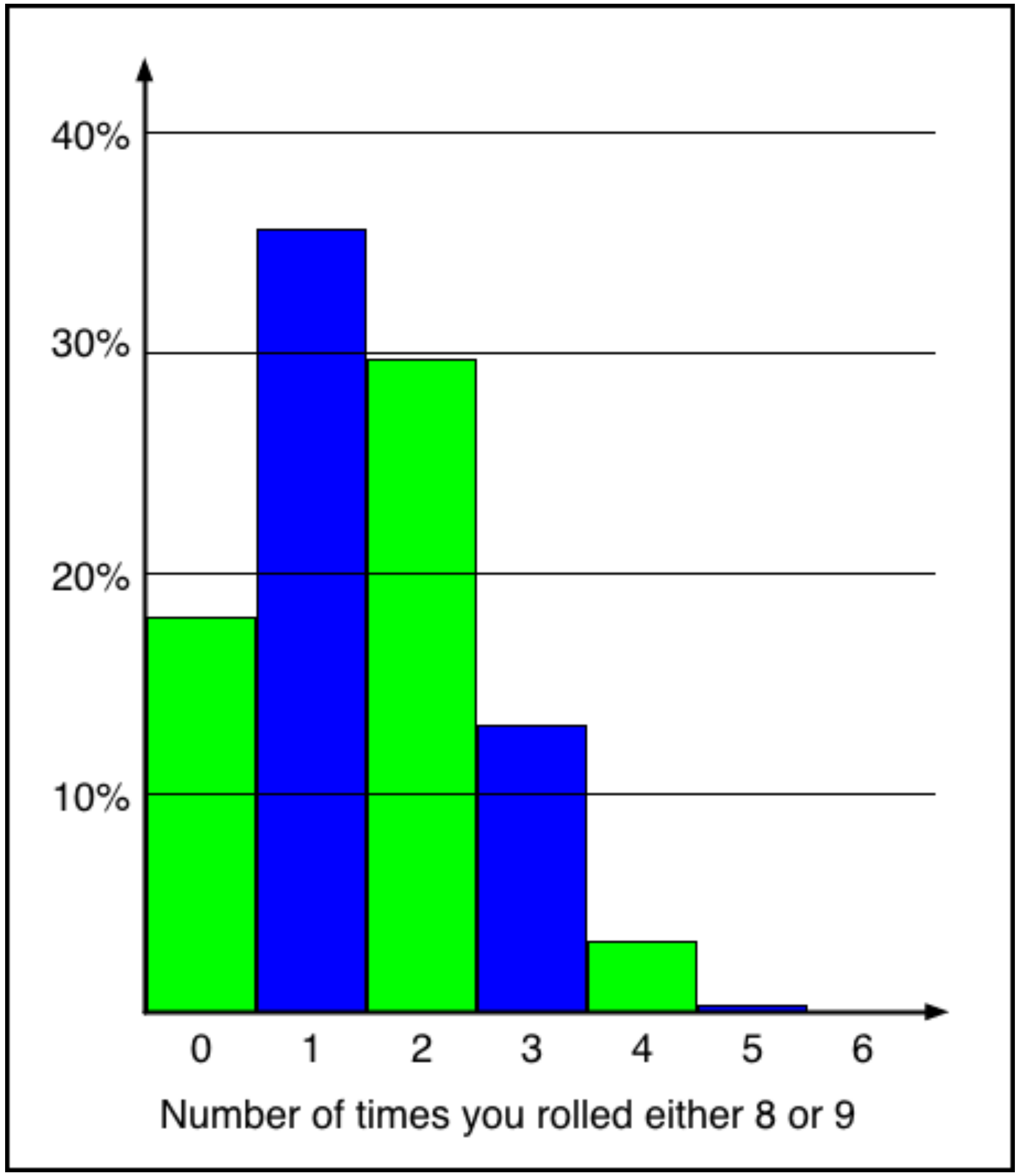
Each time you roll, there are 6x6 = 36 different equally likely outcomes. Nine of these outcomes successfully give you a total of 8 or 9: 5+3, 3+5, 6+2, 2+6, 4+4, 5+4, 4+5, 6+3, and 3+6. So on any individual roll, the probability of success = 9/36 = 1/4, and the probability of failure = 27/36 = 3/4.

Let S be the probability of success on a single roll and F be the probability of failure on a single roll.

$$(S+F)^6 = S^6 + 6S^5F + 15S^4F^2 + 20S^3F^3 + 15S^2F^4 + 6SF^5 + F^6$$

Substituting 1/4 for S and 3/4 for F reveals the probabilities for different numbers of successes and failures.The 1st term tells you the probability of succeeding all 6 times, the 2nd term tells you the probability of succeeding 5 out of 6 times, and so on. The table below shows of all the calculated probabilities (rounded to four decimal places). The histogram to its right shows the results obtained when the **Probability Simulator** program was used to simulate 6 rolls of the dice 50,000 times. Again, notice the close agreement!

| Successes | Failures | Term | Probability |
|---|---|---|---|
| 6 | 0 | $S^6$ | 0.0002 |
| 5 | 1 | $6S^5F$ | 0.0011 |
| 4 | 2 | $15S^4F^2$ | 0.0330 |
| 3 | 3 | $20S^3F^3$ | 0.1318 |
| 2 | 4 | $15S^2F^4$ | 0.2966 |
| 1 | 5 | $6SF^5$ | 0.3560 |
| 0 | 6 | $F^6$ | 0.1870 |

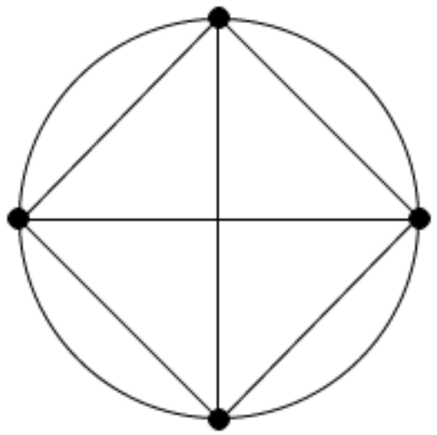


### 3. Fibonnacci's Sequence

Lots of interesting sequences are hidden within Pascal's triangle. Some are easy to find, like the diagonal containing 1, 3, 6, 10, 15, 21...

The nth term of that sequence equals the number of line segments that can be drawn connecting n+1 points on a circle.

Six line segments connecting 4 points

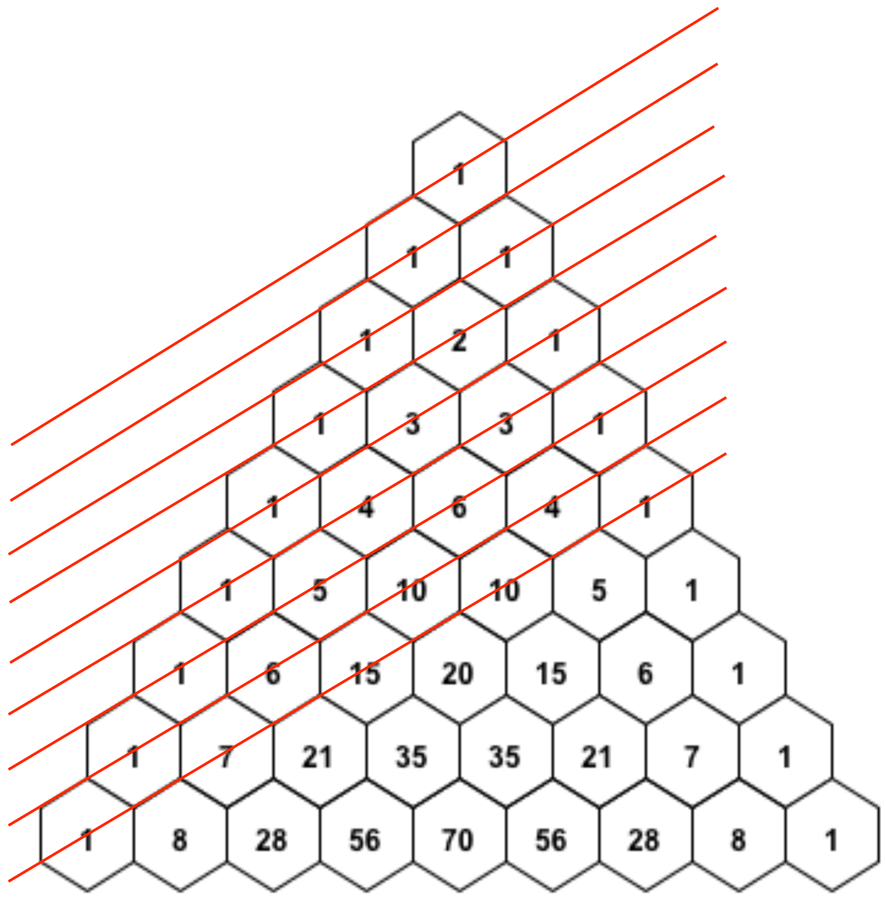


Harder to find is Fibonnacci's Sequence, the sequence defined by: $x_1 = 1$, $x_2 = 1$, and $x_n = x_{n-2} + x_{n-1}$ for n ≥ 3.

1, 1, 2, 3, 5, 8, 13, 21, 34...

To make things easier, first put the contents of Pascal's triangle into adjoining hexagons. Then draw diagonal lines that are collinear with the sides of those hexagons.

Add the numbers that each line passes directly over.




| Line | Sum |
|---|---|
| 1 | 1 |
| 2 | 1 |
| 3 | 1 + 1 = 2 |
| 4 | 1 + 2 = 3 |
| 5 | 1 + 3 + 1 = 5 |
| 6 | 1 + 4 + 3 = 8 |
| 7 | 1 + 5 + 6 + 1 =13 |
| 8 | 1 + 6 + 10 + 4 = 21 |
| 9 | 1 + 7 + 15 + 10 + 1 = 34 |

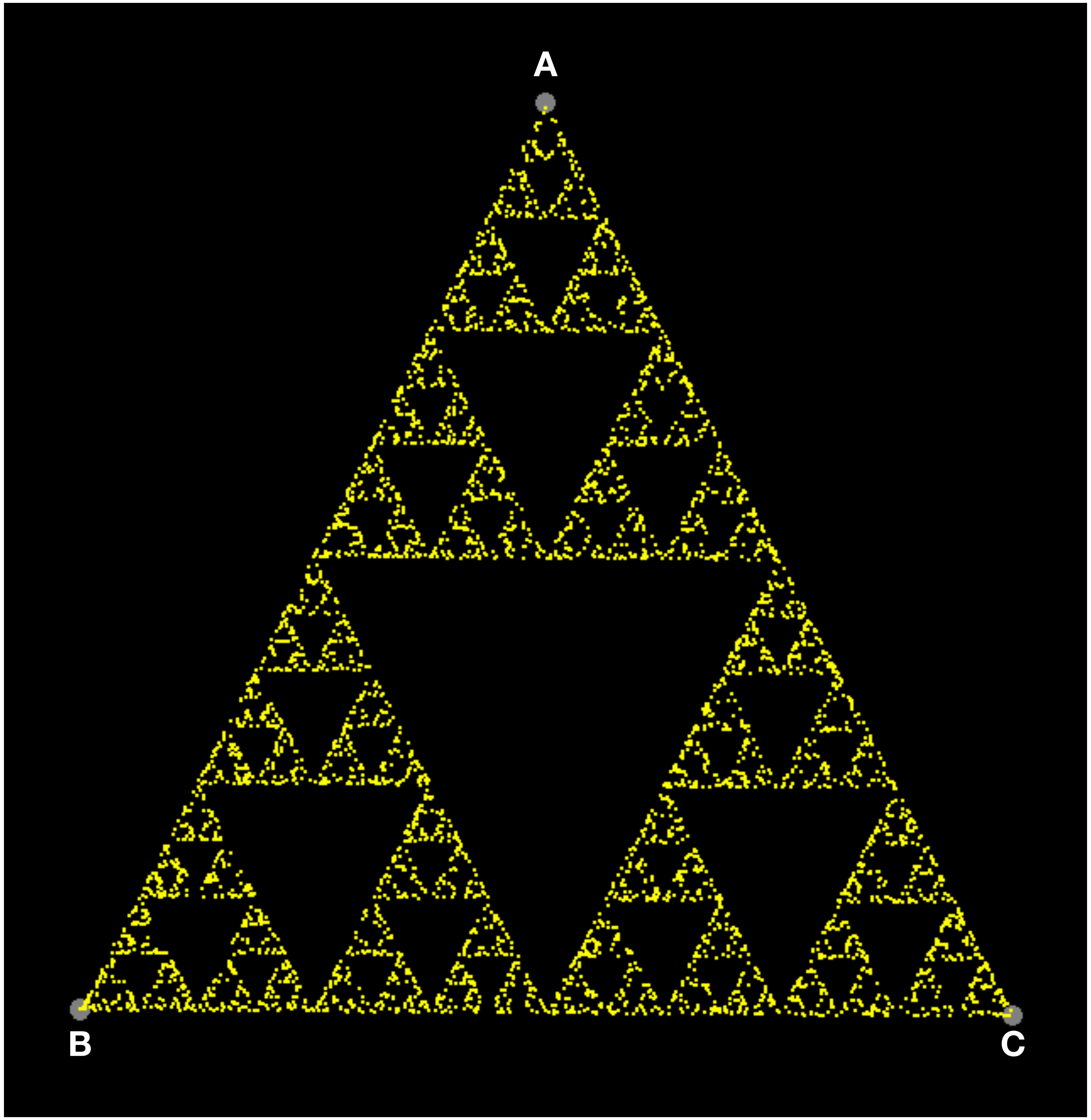
**4. Sierpinski's Triangle**

Sierpinski's triangle is a fractal that consists of an equilateral triangle that has been recursively subdivided into an unending pattern of smaller equilateral triangles.

One way to construct it is by following this procedure:

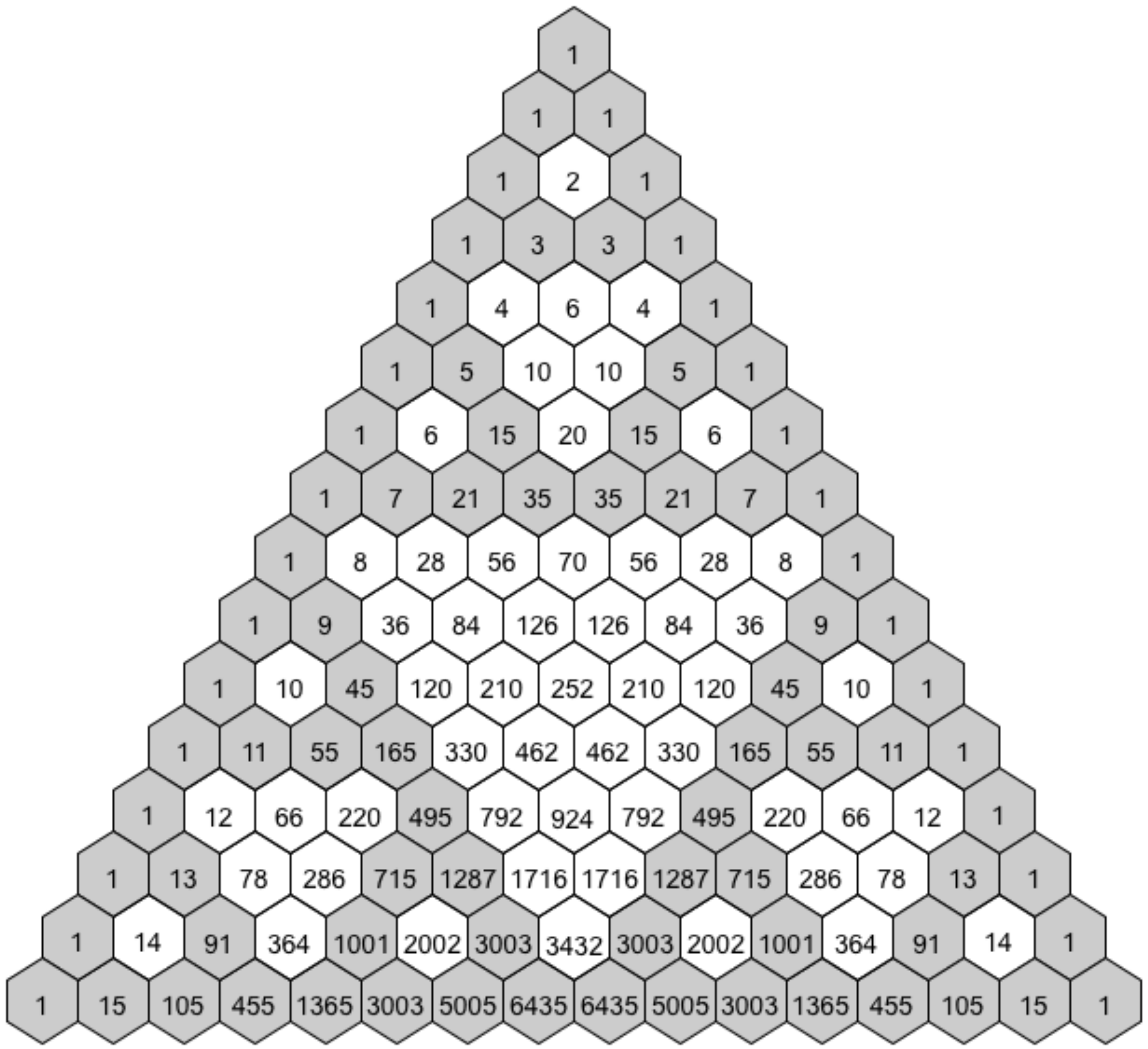
Choose three points (A, B, and C) as the vertices of an equilateral triangle in a coordinate plane. Randomly choose another point in the plane and call it point D. Spin a spinner that has three equal divisions labeled A, B, and C. If the spinner points at A, go half way from point D to point A and plot a new point; if the spinner points at B, go half way from point D to point B and plot a new point; if the spinner points at C, go half way from point D to point C and plot a new plot. Relabel this new point as D and recursively repeat the entire process.

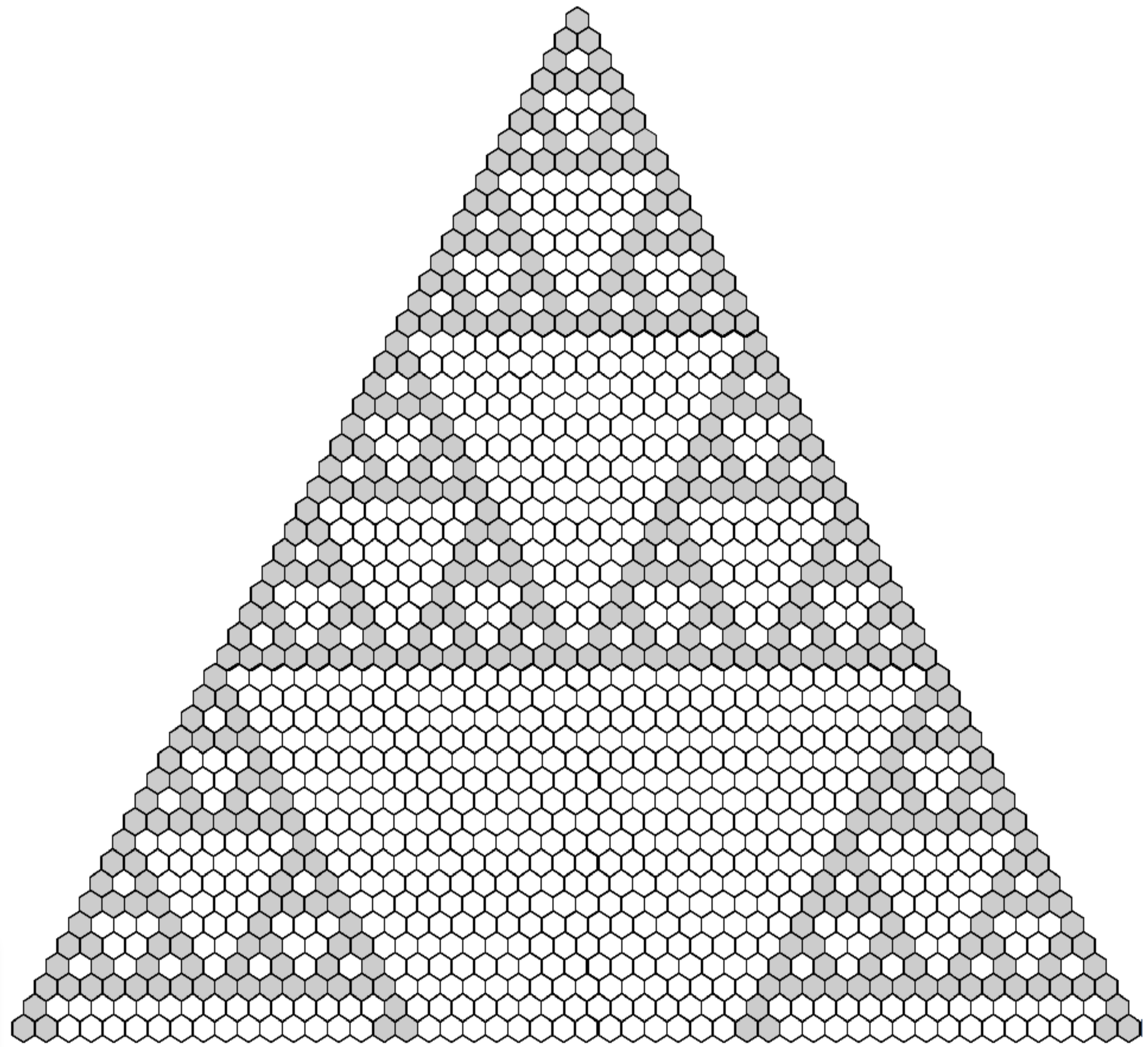
After 5000 iterations the **Dot-dots** program produced the image shown here. More iterations at a higher resolution would reveal increasingly small equilateral triangles.



Pascal's triangle can be used to construct an approximation of Sierpinski's triangle. Fill in all of the hexagons that contain an odd number.

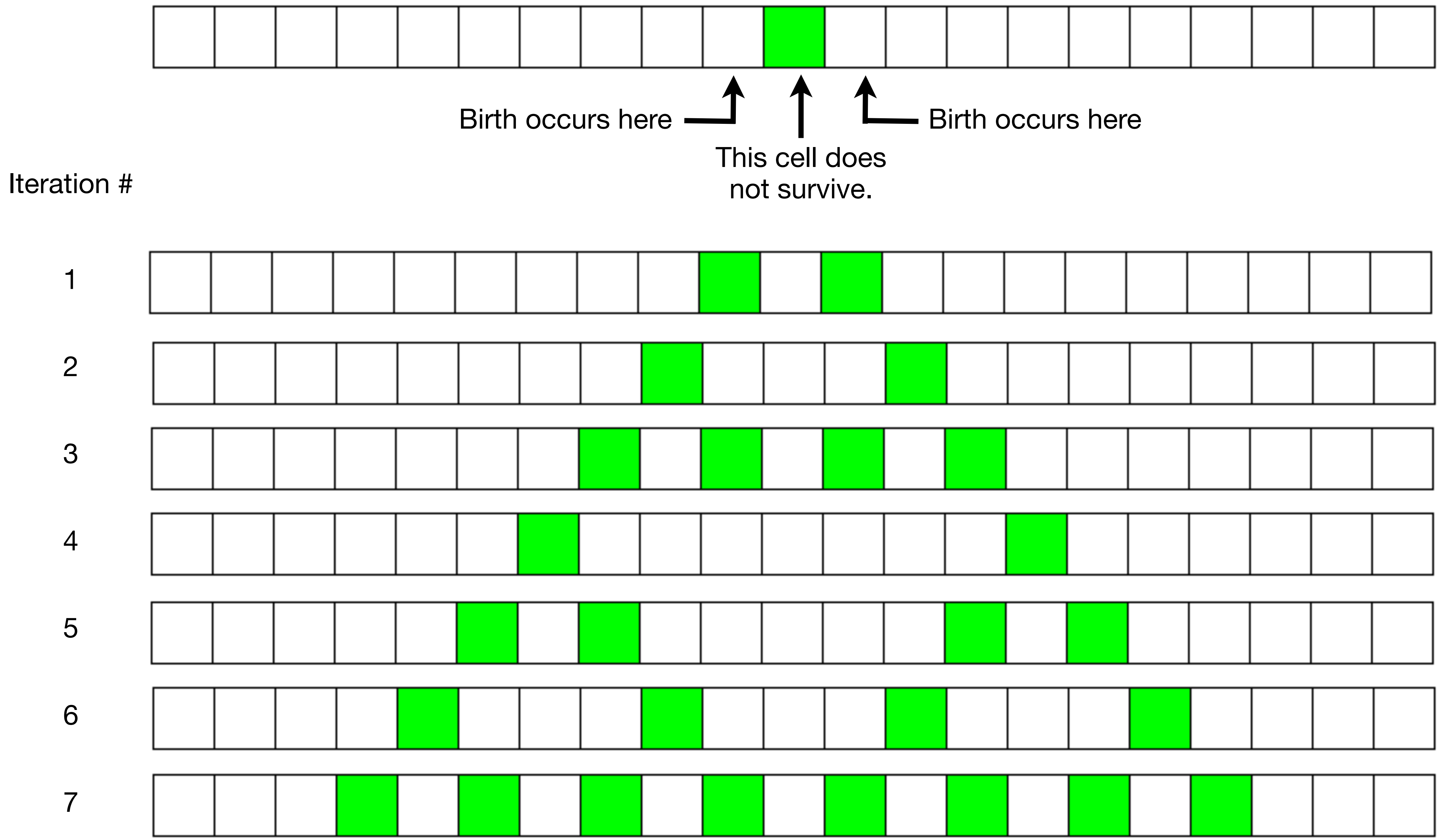
For a better approximation, add more rows and shrink the result to a reasonable size. The image on the right includes five copies of the image on the left.



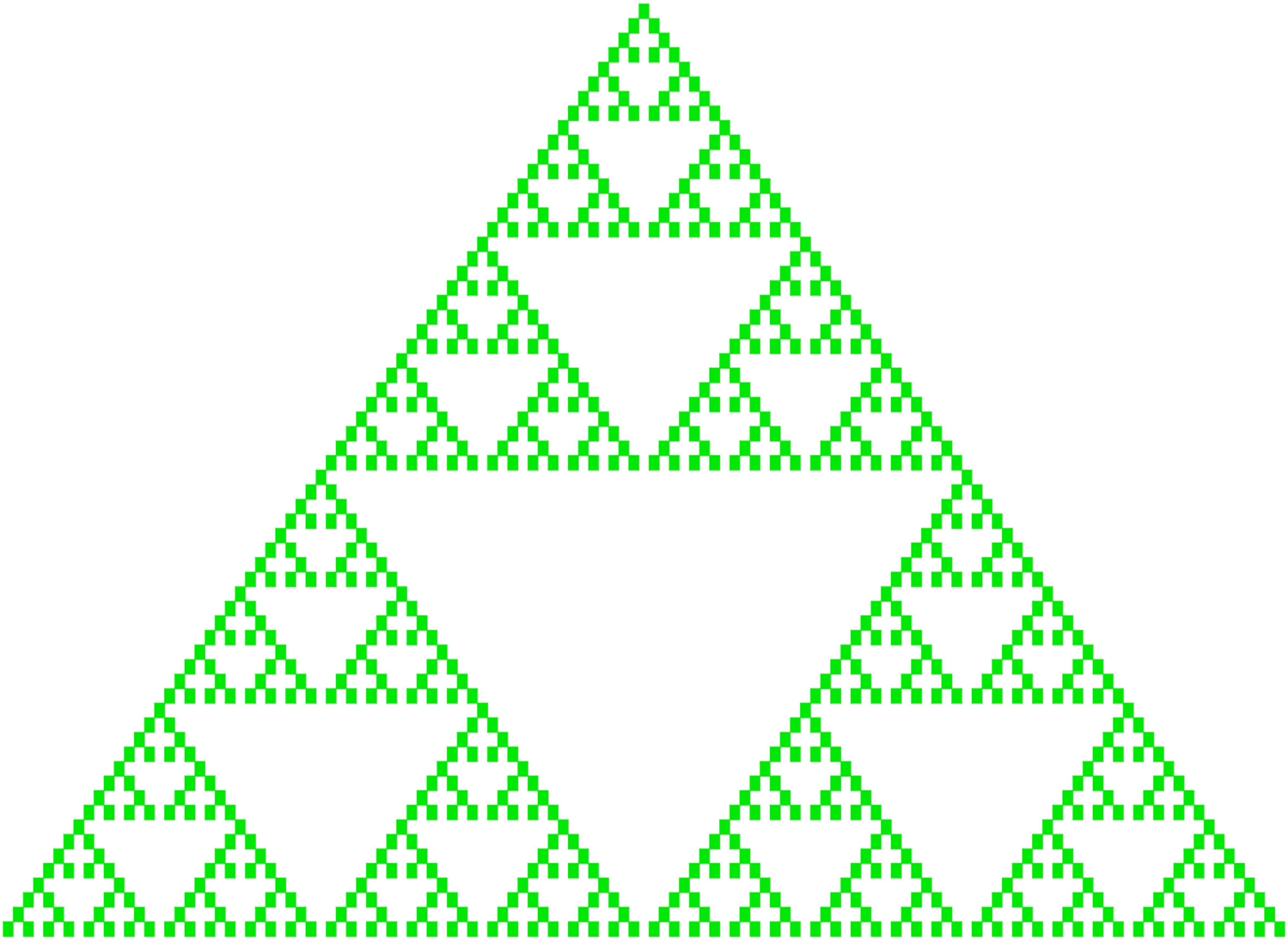



Because Sierpinski's triangle is a fractal, it isn't easy to determine exactly where the image on the left fits into the image on the right. The closer you look, the more you see the same thing!

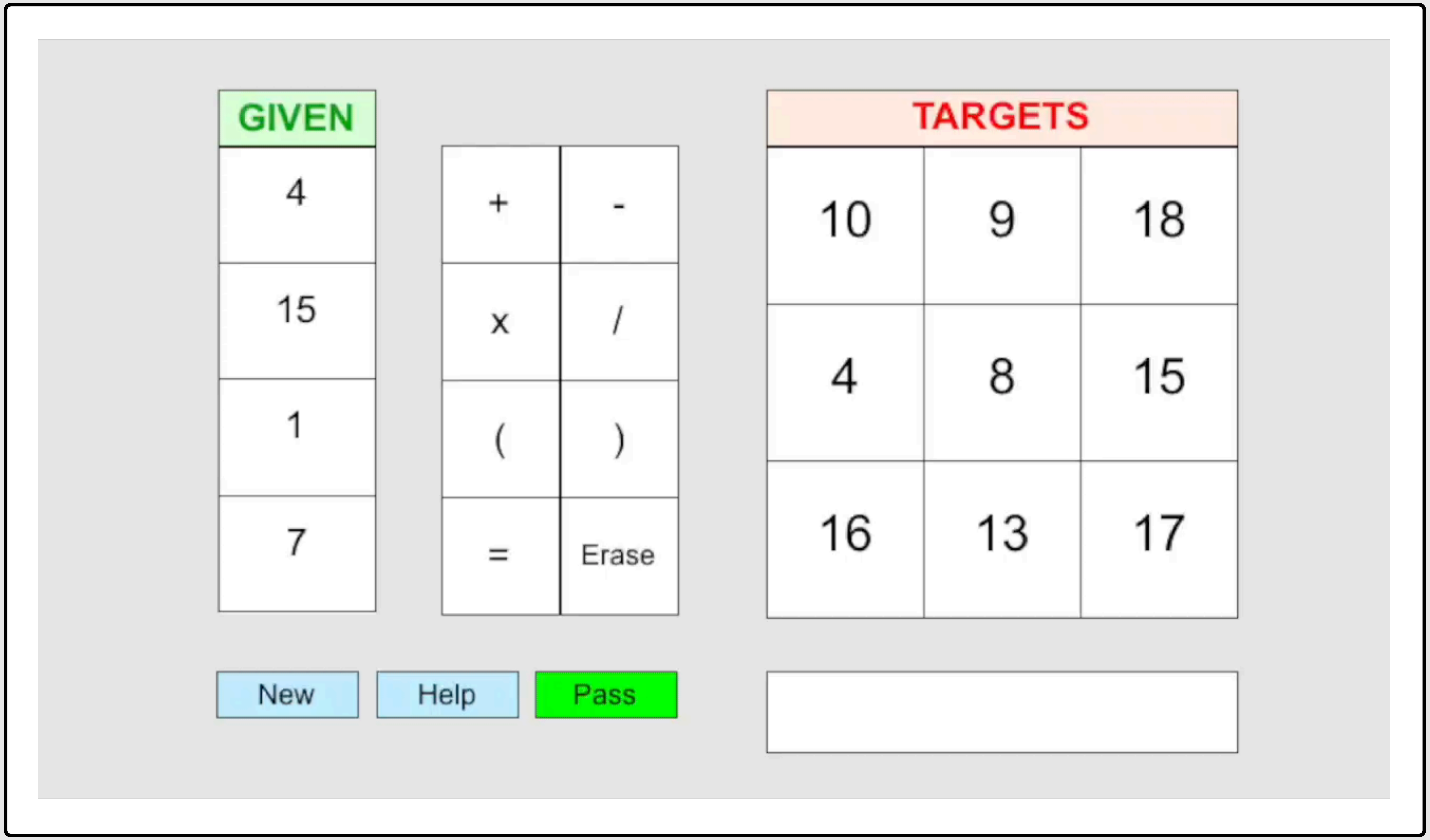
Another way to construct Sierpinski's triangle is by playing a one dimensional game of Life, starting with just one living cell.The rules are simple: A living cell never survives, but a birth occurs in an empty cell that previously had exactly one living neighbor.




If you do 64 iterations, combine them into one continuous image, and shrink that image to fit on one page, you get →



# 9. Games



## 9.1 Introduction

I have mixed feelings about “educational” math games. Too often they seem mostly “game” without much “math” - drill and practice exercises with little educational depth. I have tried to write and distribute programs that are instead designed to increase a student’s understanding of mathematical concepts and help develop his or her problem solving skills. I have also tried to make my programs flexible enough to present each user with just the right level of difficulty. “Too easy” makes a task boring and uninteresting; “too hard” makes a task frustrating and unhelpful. It’s important, I think, to try to keep a student on a fine line between those two extremes.

Four programs are introduced in this chapter: **Algernon** is about spatial reasoning and estimation skills, **ArithmeDarts** is about plotting fractions, decimals, and whole numbers on a number line, **Reckonings** is about arithmetic expressions and the order of operations, and **FactorMan** is about primes, factors, and the importance of thinking ahead. All three are available in various formats from my **Syzygy Shareware** site at http://tcbretl.weebly.com.

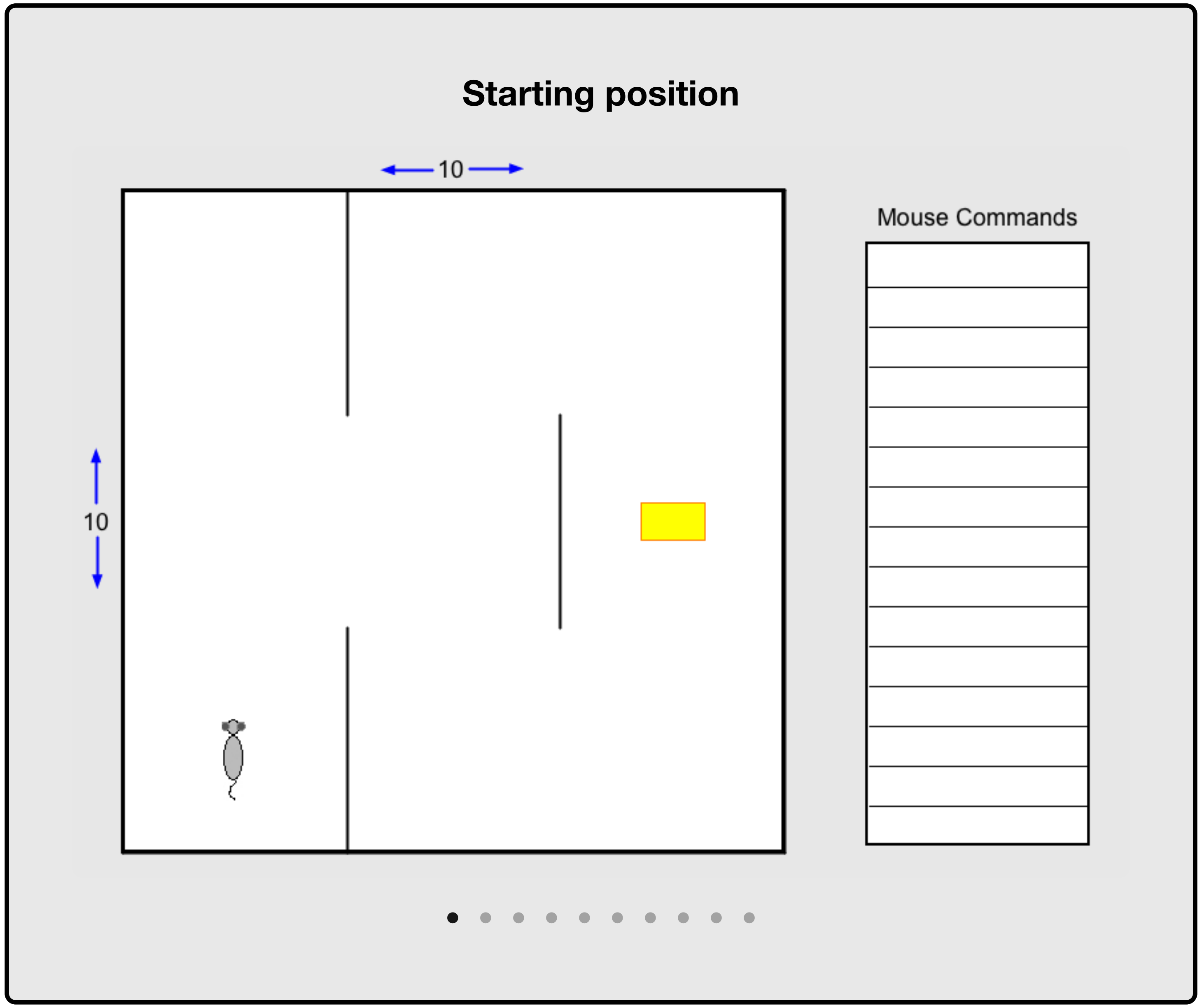
## 9.2 Algernon

I wrote Algernon after being introduced to the LOGO programming language and its turtle graphics. A mouse named Algernon replaces the turtle, and the object is to program him to travel safely through a maze and find the cheese. You do this by telling him to turn right, turn left, or move foward a specified number of “mouse steps.” You can make Algernon follow each instruction immediately, or you can have him wait until you have entered an entire sequence of instructions. If he crashes into a wall of the maze, you must debug your program and try again.

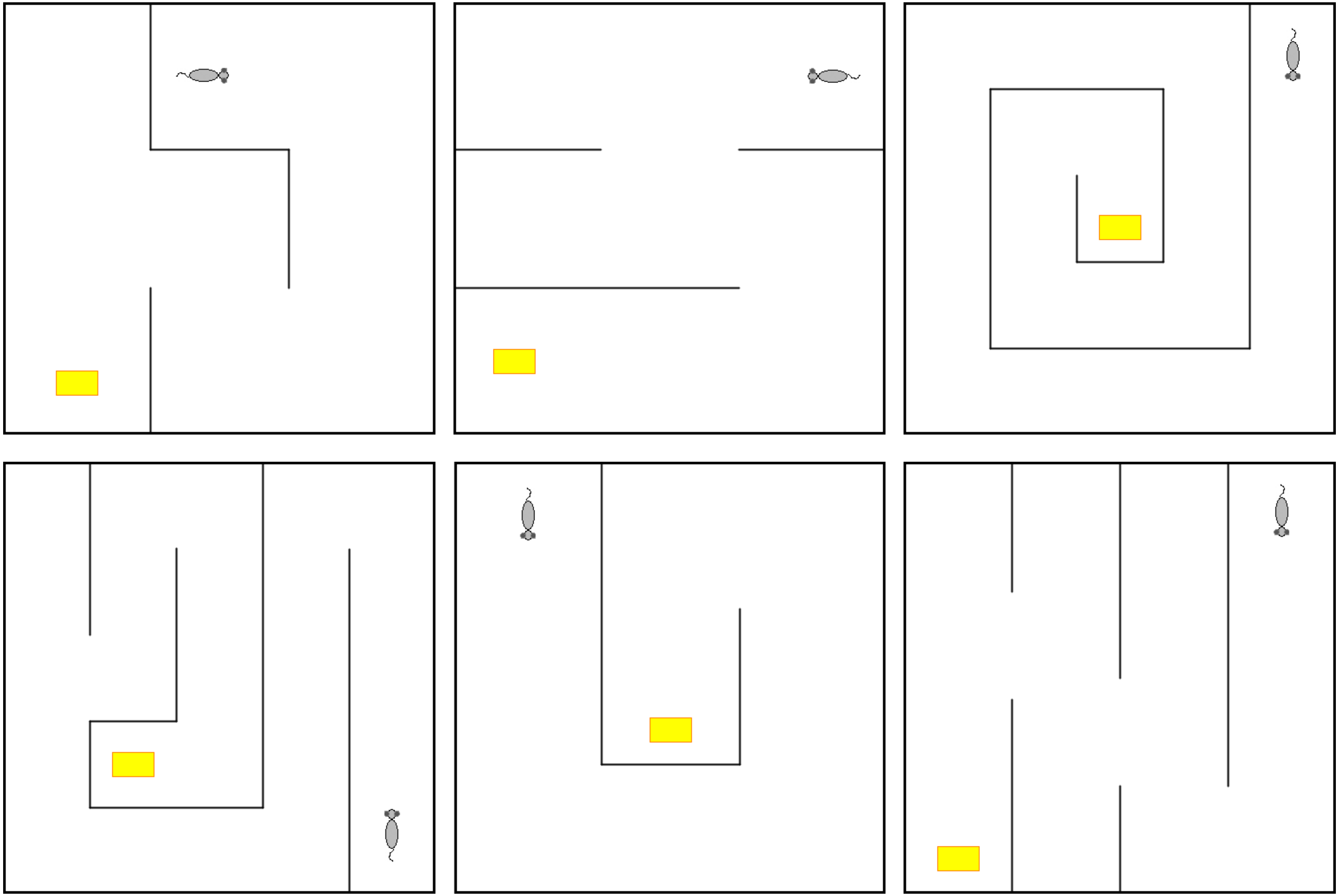
Young students find this game to be both fun and challenging. They have to get used to the fact that the meaning of “left” and “right” depends on the direction that the mouse is currently heading. The “right” command does not mean “turn towards the right side of the screen.”

Students also have to develop reasonably good estimation skills. Since a maximum number of 20 commands is allowed, an ultra-conservative strategy of moving forward only a few steps at a time usually leaves Algernon stranded in the middle of the maze.

Below is a series of screenshots showing one possible sequence of instructions.



## Some other Algernon mazes:



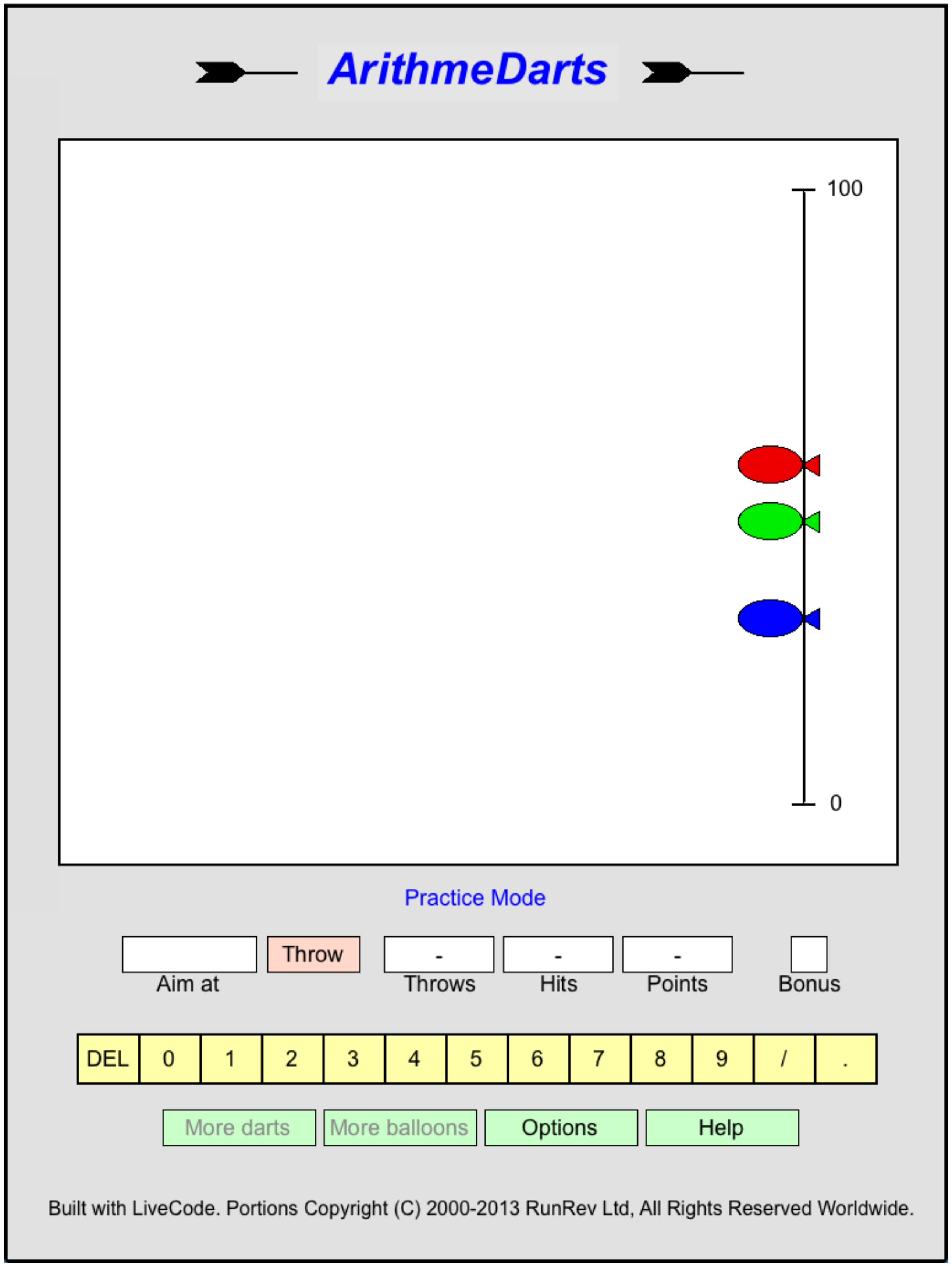
## 9.3 ArithmeDarts

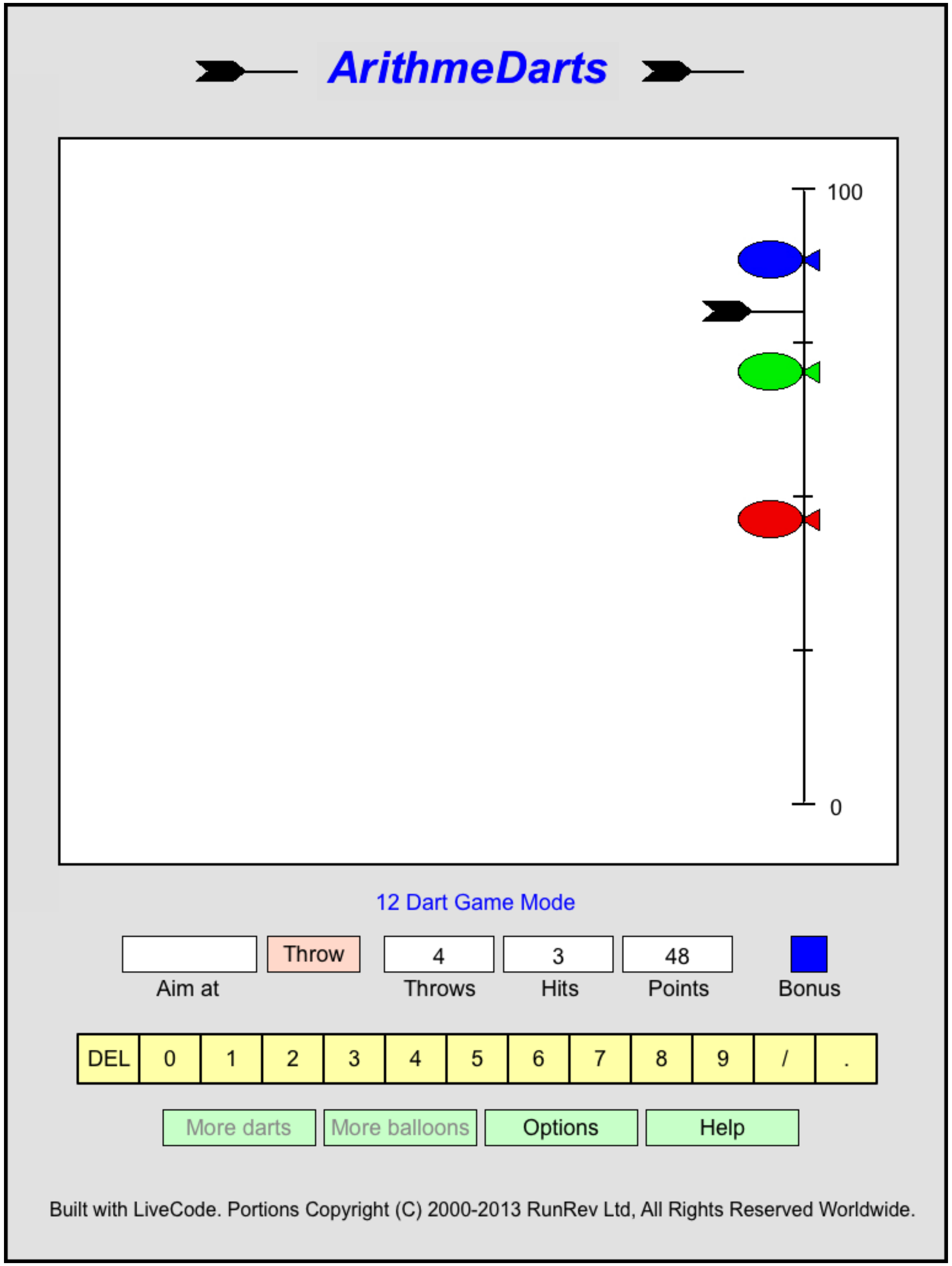
This game tests your mental arithmetic and estimation skills by challenging you to throw "arithmeDarts" at target balloons that are attached to a number line. You aim the darts by estimating the coordinates of the balloons. There is a practice mode, a game mode in which your goal is to accumulate as many points as possible using only 12 darts, and another game mode in which your goal is to accumulate 200 points using as few darts as possible.

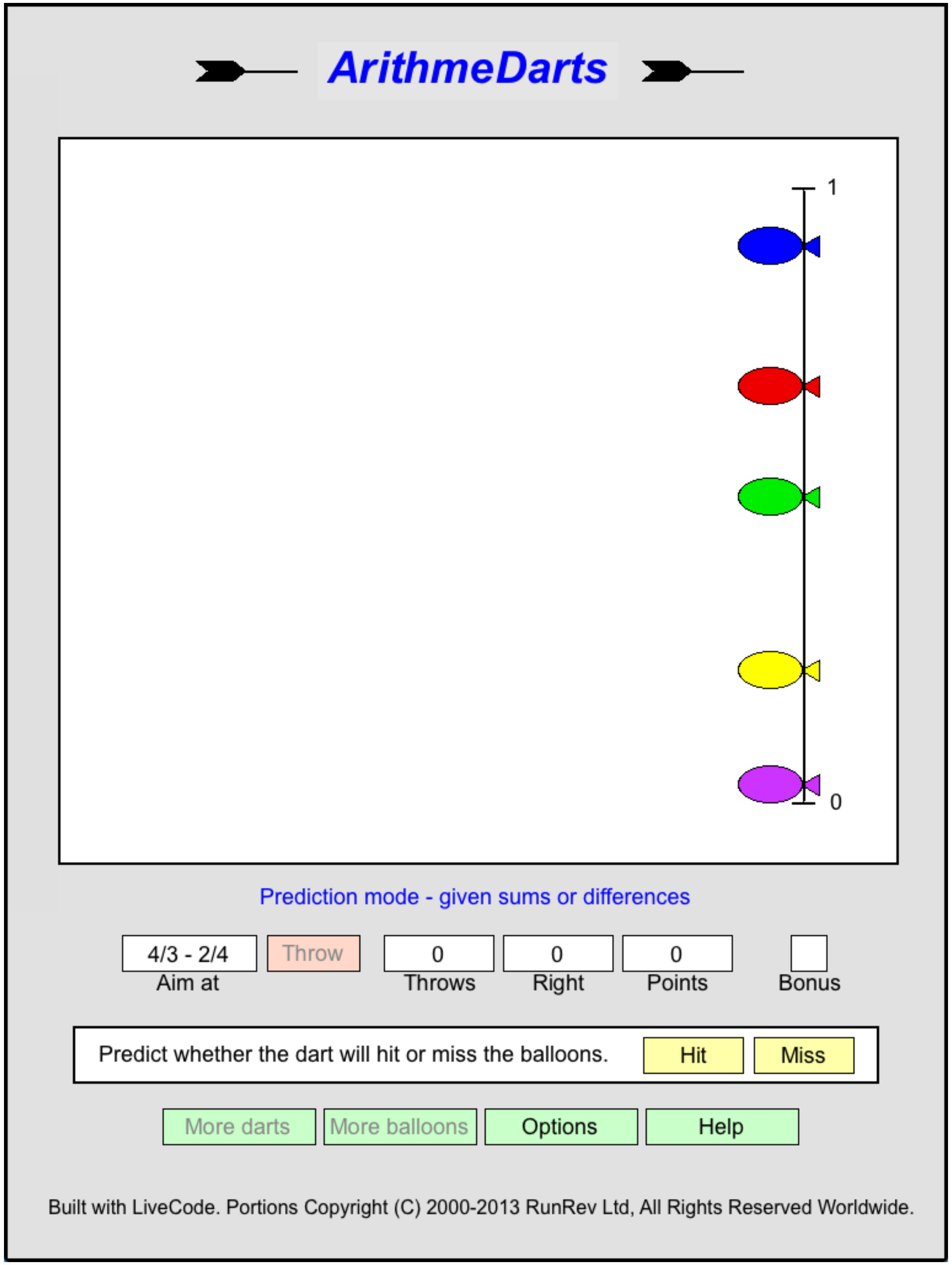
Instead of “aiming” the darts yourself, you can also try to predict whether a dart already aimed at a randomly selected point will hit a balloon. The The coordinate of the selected point might be expressed as a single number, or the sum, difference, product, or quotient of two numbers.

In all of these modes, you have several options: The numbers on the dart board can range from 0 to 100, 0 to 10, or 0 to 1, and to make estimations a little easier, 2, 3, 4, 5, or 10 equally spaced tick marks can be added along the number line.

More screen shots are shown on the next page.




ArithmeDarts
100
0
12 Dart Game Mode
Throw
4
3
48
Aim at
Throws
Hits
Points
Bonus
DEL
0
1
2
3
4
5
6
7
8
9
/
.
More darts
More balloons
Options
Help
Built with LiveCode. Portions Copyright (C) 2000-2013 RunRev Ltd, All Rights Reserved Worldwide.


ArithmeDarts
1
0
Prediction mode - given sums or differences
4/3 - 2/4
Throw
0
0
0
Aim at
Throws
Right
Points
Bonus
Predict whether the dart will hit or miss the balloons.
Hit
Miss
More darts
More balloons
Options
Help
Built with LiveCode. Portions Copyright (C) 2000-2013 RunRev Ltd, All Rights Reserved Worldwide.

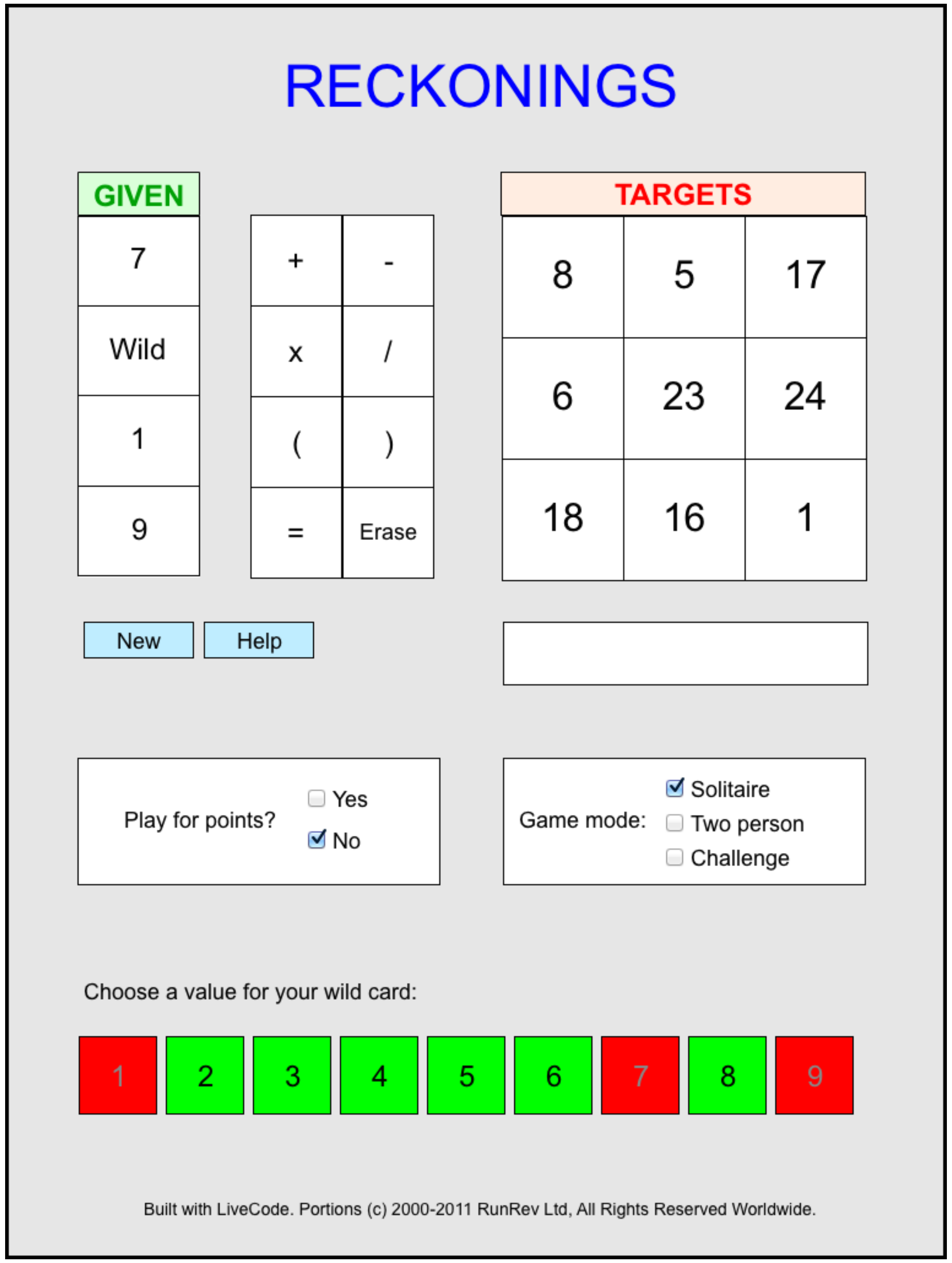
# 9.4 Reckonings

Reckonings is a game that challenges players to hit nine target numbers by using addition, subtraction, multiplication and division to combine four given numbers. You can play it alone or as a competitive two-person game. You can also challenge the Reckon Master, an extremely good cyber-opponent, to a two-person game. Points are awarded based on how many given numbers and operations are used to hit the target.

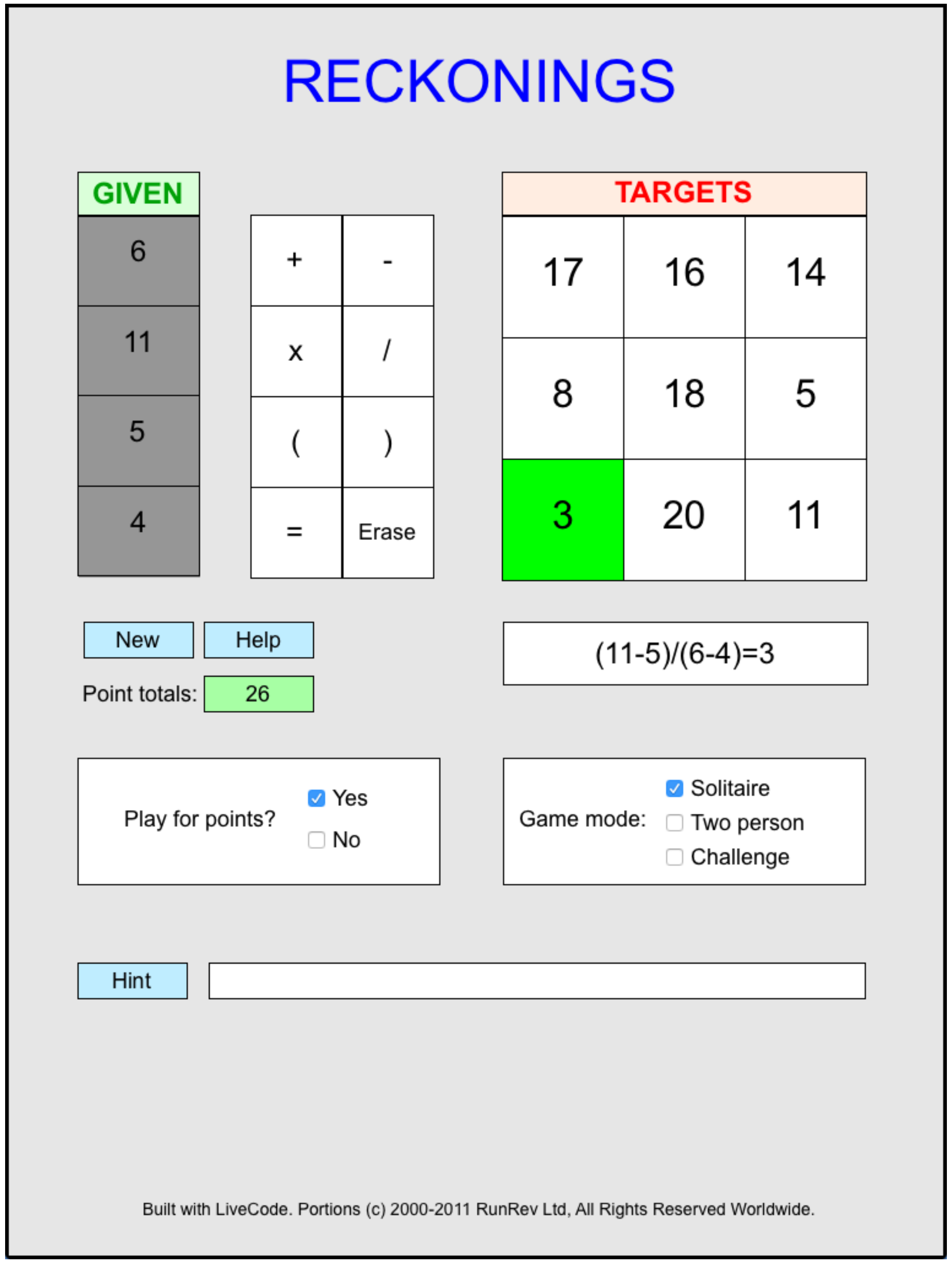
This program was used extensively by a group of second graders at a school where I worked. The students did not like to play against each other (especially when a “time limit” feature was included), but often chose to work together as a team. This led to the later addition of the “challenge” feature, allowing students to work together while competing against the Reckon Master.

We also found that it was good to have a teacher in the background, quietly observing the action, ready to give help and encouragement. If a student said the game was too easy, the teacher might say, “try to use all four given numbers to get the next target number.” If the student said the game was too hard, the teacher might say “I think there are two given numbers that can be combined to get the target number. See if you can find them.” A “hint” feature was added later, as an aid for when no teacher was present.

Sometimes a wild card shows up as a given number. Making a wise choice for it adds an extra challenge to the game.



The program rewards a good understanding of the order of operations. You get extra points when you use all four of the given numbers and/or use parentheses.




The given and target numbers are randomly chosen, so you may not always be able to hit all of the targets. In the game displayed below, for example, there is no way to get 22.

# 9.5 FactorMan

This game is played against the superhero FactorMan. You earn points by choosing numbers between 1 and 100. If you choose 5, you get 5 points; if you choose 18, you get 18 points. FactorMan gets the sum of all of the factors of the number you have chosen (except for the number itself). If you choose 12, for example, you get 12 points, but FactorMan gets 1+2+3+4+6=16 points!

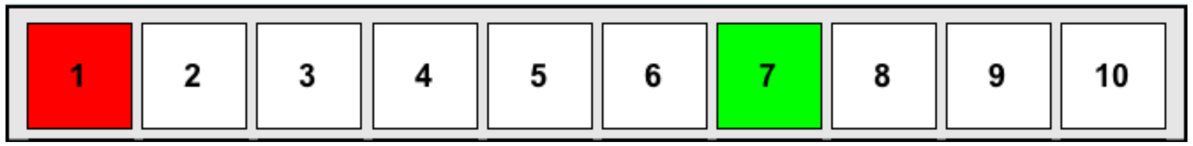
Any number you choose must have at least one factor left on the board. Once a number has been chosen, it and all of its factors are removed from the board. When you are unable to make another legal choice, FactorMan gets all of the remaining unchoosable numbers, so he often comes from behind to win!

To win this game you must have good arithmetic skills and develop a careful strategy. Beginning players may prefer to limit the maximum number to something less than 100, ask for hints, or highlight all of the numbers that can no longer be chosen.

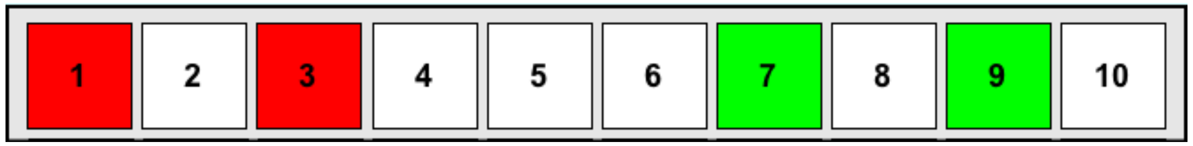
Shown to the right is a sequence of winning moves when 10 is the maximum number on the board. Your choices are highlighted in green; FactorMan's numbers are highlighted in red.

Final score: 40 for you; only 15 for FactorMan.

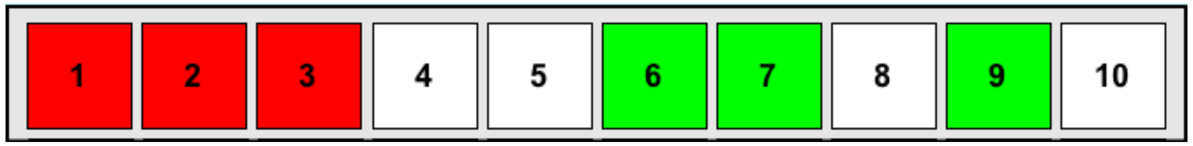
1) Choose 7. You get 7 points; FactorMan gets 1 point.



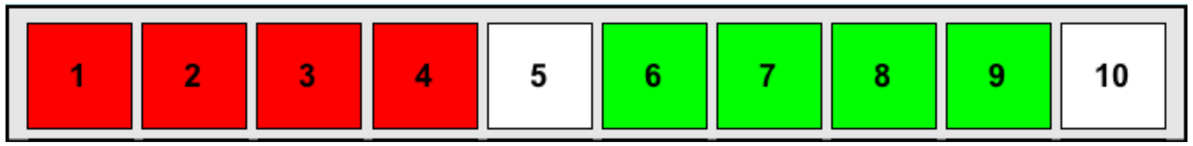

2) Choose 9. You get 9 points; FactorMan gets 3 points.



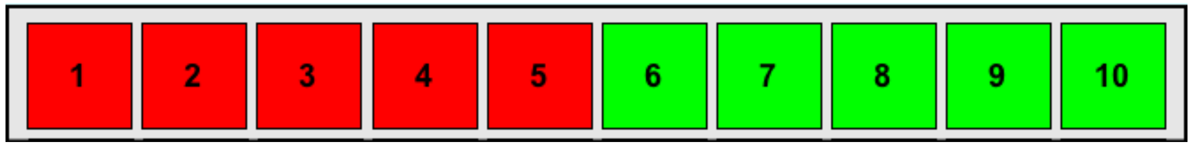

3) Choose 6. You get 6 points; FactorMan gets 2 points.




4) Choose 8. You get 8 points; FactorMan gets 4 points.

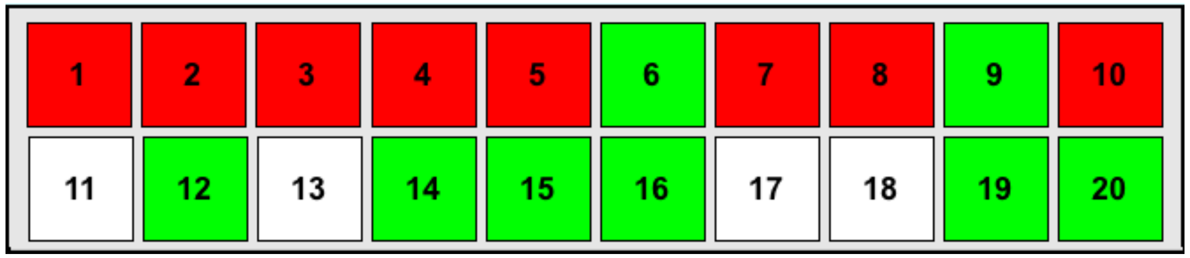



5) Choose 10. You get 10 points: FactorMan gets 5 points.



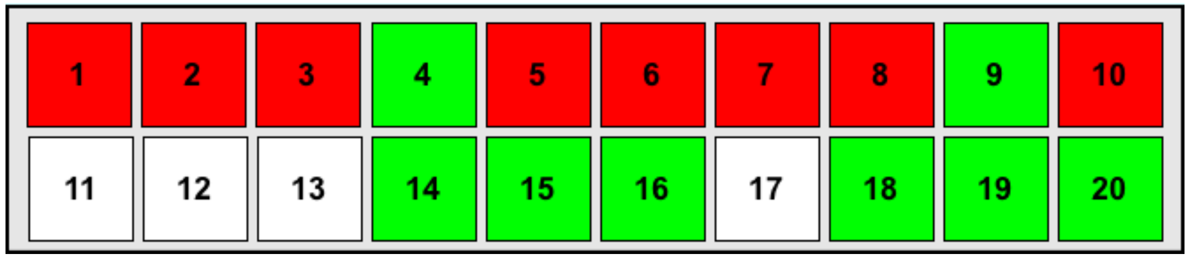

You get the five largest numbers; he gets the five smallest! In this case there were no unused numbers for him to claim at the end of the game.

Things get trickier when you increase the maximum number to 20. If your first choice is 19 (the largest prime number), then 11, 13, and 17 will definitely be left for FactorMan at the end of the game. If your choices, in order, are 19, 9, 6, 12, 14, 15, 16, and 20, then the game ends like this:




You get a total of 111 points. FactorMan gets 40 points from the factors and 59 points from the unused numbers for a total of 99 points. You win, but is it possible to do better?

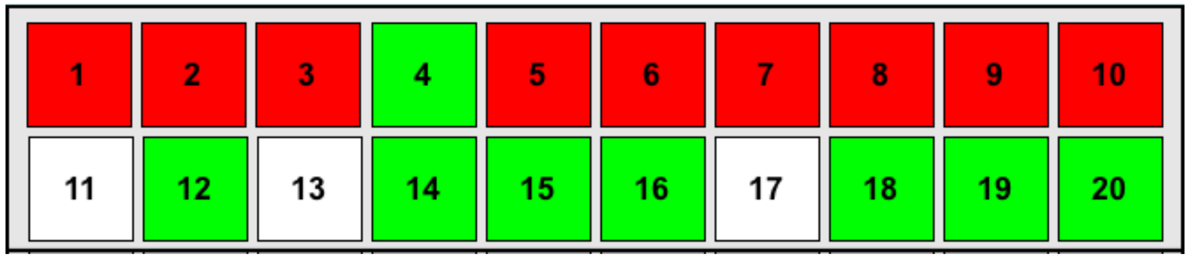
If your choices, in order, are 19, 9, 4, 14, 15, 16, 18 and 20, then the game ends like this:




You get a total of 115 points. FactorMan gets 42 points from the factors and 53 points from the unused numbers for a total of 95 points. You win by a greater amount this time, but is it possible to do even better?

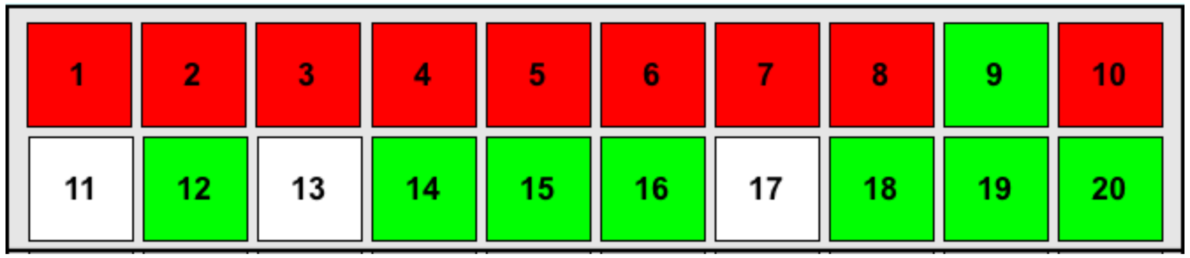
Perhaps there is a way to end up with 11, 13, and 17 as the only unused numbers. If so, will that give you more points, or will FactorMan gain too many factor points?

Try to figure out a way to end the game like this:




You get a total of 118 points. FactorMan gets 51 points from the factors and 41 points from the unused numbers for a total of 92 points. You can't do any better with the unused numbers, but can you give up fewer or smaller factors?

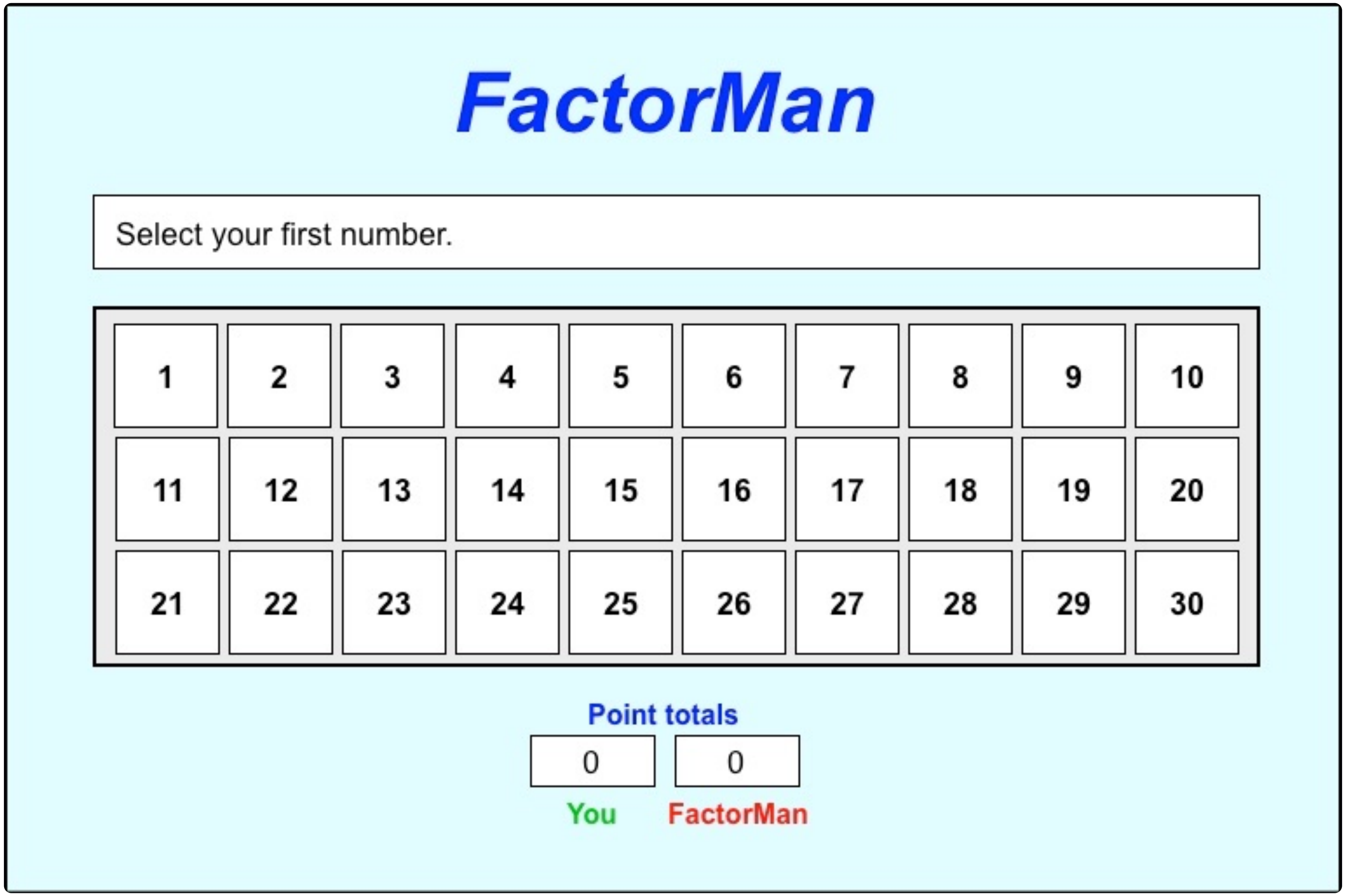
Try to figure out a way to end the game like this:




You get a total of 123 points. FactorMan gets 46 points from the factors and 41 points from the unused numbers for a total of 87 points. Is this the best possible outcome?

**Games with greater maximum numbers**

The greater the maximun number on the board, the trickier things become. The order in which you make your choices is very important. Here is a slide show of screenshots showing a possible FactorMan game with maximum number equal to 30.

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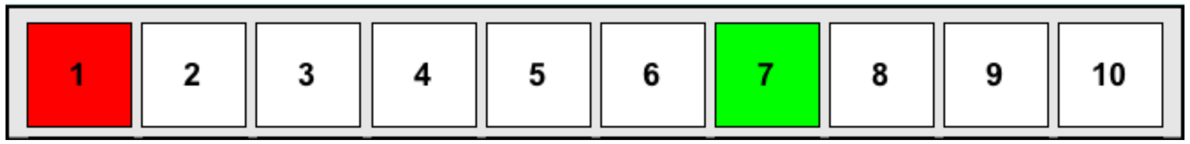
## Two Person Game

FactorMan also includes a two player option. The rules remain much the same. When Player #1 chooses a number, Player #2 gets all of the factors; when Player #2 chooses a number, Player #1 gets all of the factors. The game ends when no legal choices remain. Neither player gets points from the unused numbers.

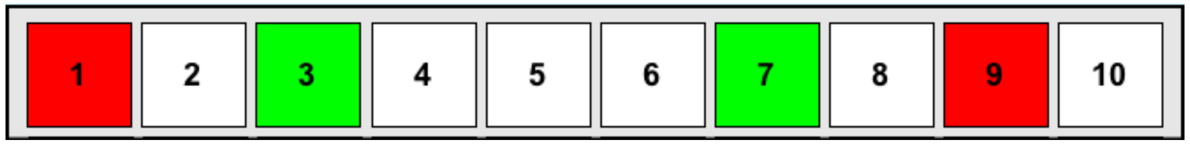
Because the unused numbers do not help either player, it may  sometimes be smart to give up more factor points to your opponent. Suppose, for example, that the only numbers remaining on the board were 11, 22, 33, and 44. If playing against the FactorMan, you would first choose 33 (giving him 11) and then choose 44 (giving him 22). This would give you a total of 77 and him a total of only 33. In the two player game, however, this strategy would give both you and your opponent the same number of points. First you would get 33, and he would get 11; then he would get 44, and you would get 22. It would be much better for you to first choose 44. That would give him 33 points (for the factors 11 and 22), but would leave the 33 as unusable.

Since going first appears to be an advantage, the program gives bonus points to Player #2. When the maximum number on the board is 10, for example, a bonus of three points is given. If both players make the logical choices shown to the right, this leads to a tie game.

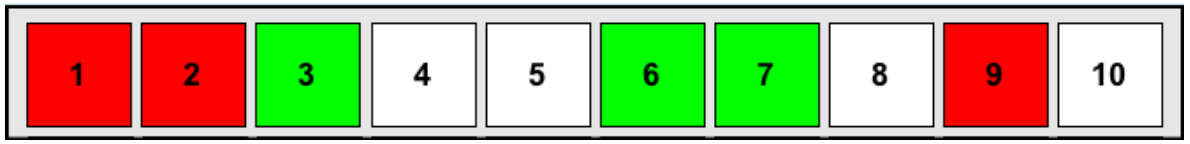
Player #1 chooses 7 Green: 7 Red: 1



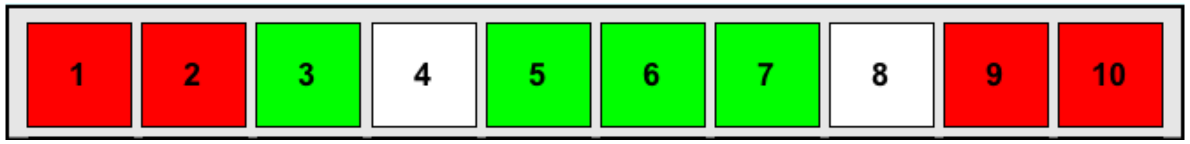

Player #2 chooses 9 Green: 10 Red: 10



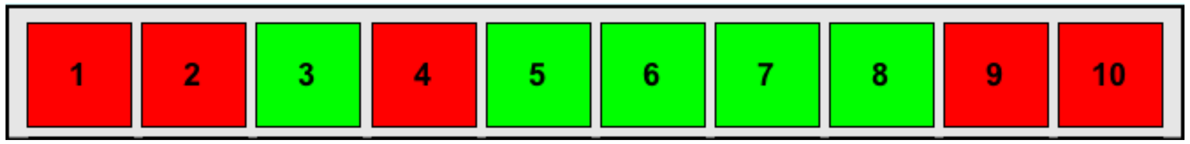

Player #1 chooses 6 Green: 16 Red: 12




Player #2 chooses 10 Green: 21 Red: 22



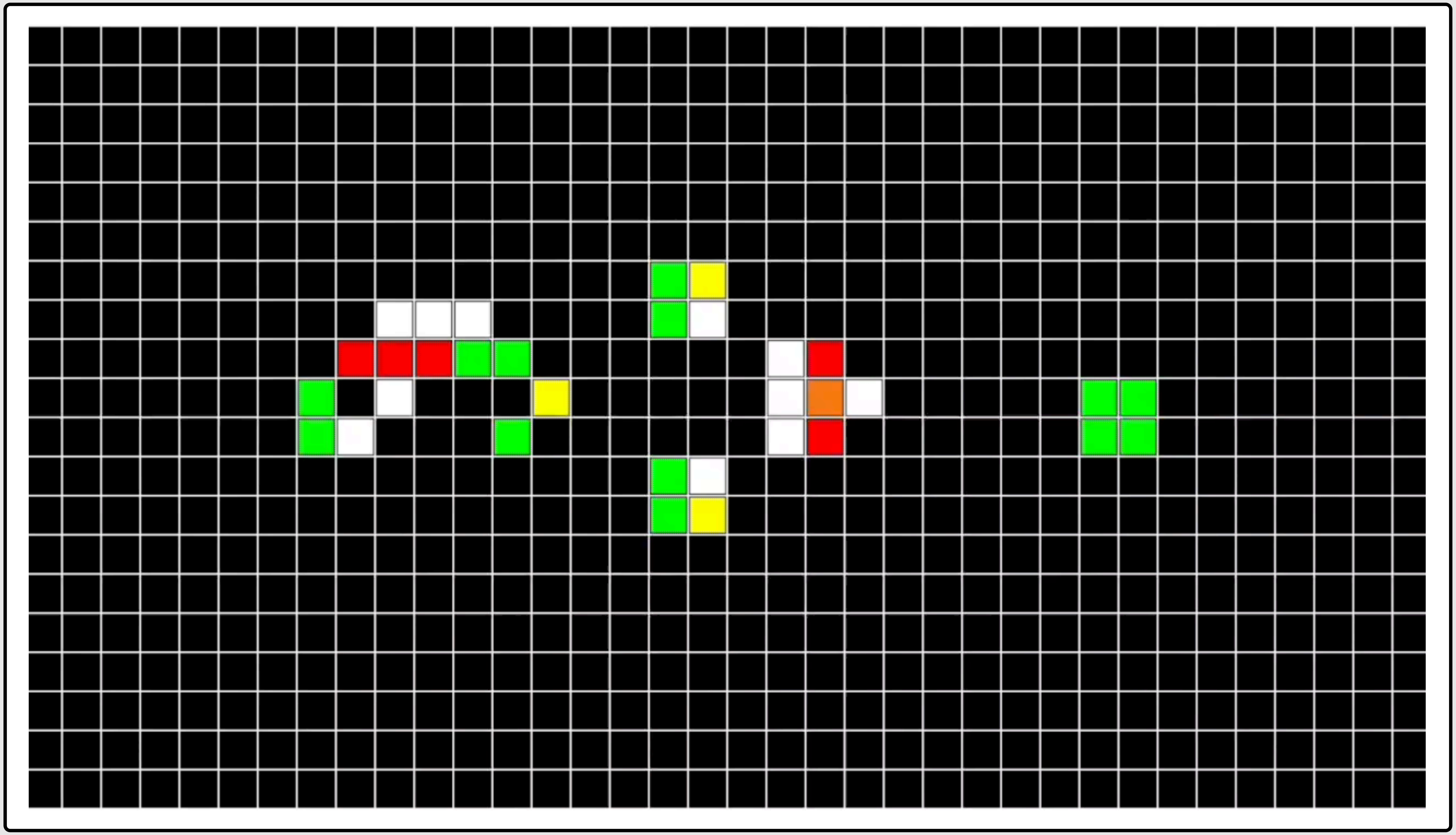

Player #1 chooses 8 Green: 29 Red: 26



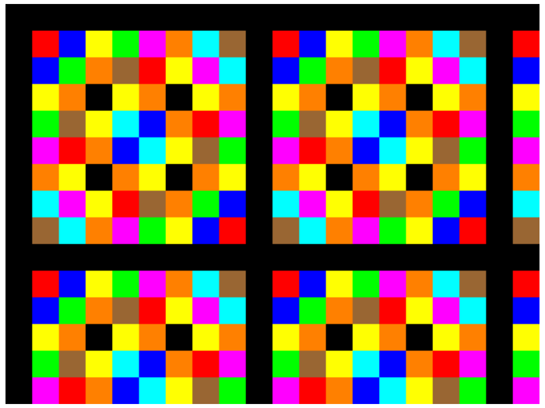

Player #2 (Red) gets a three point bonus, so both players end up with 29 points.

When the maximum number on the board is 20, twelve bonus points are given. When the maximum number on the board is greater than 20, then 20 bonus points are given.

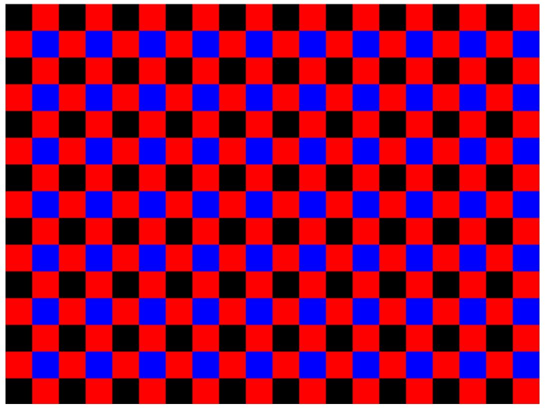
# 10. Problem Solutions



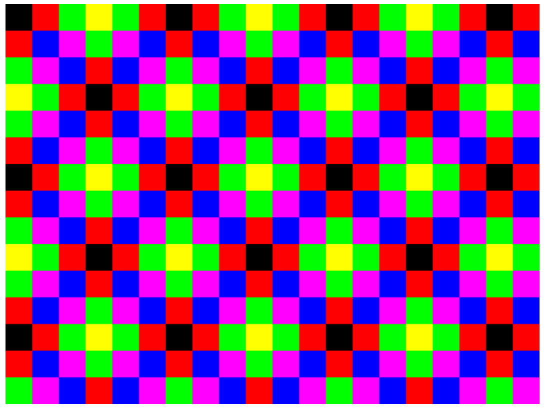
## 10.1 Chapter 2 Solutions



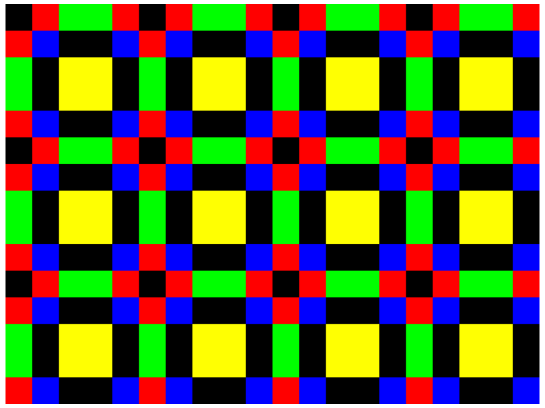
x • y = xy mod 9



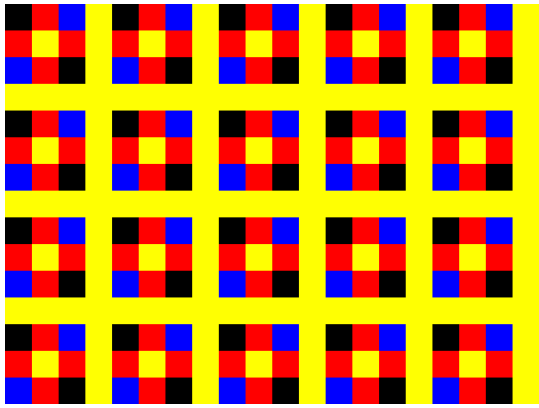
x • y = $x^2+y^2$ mod 4



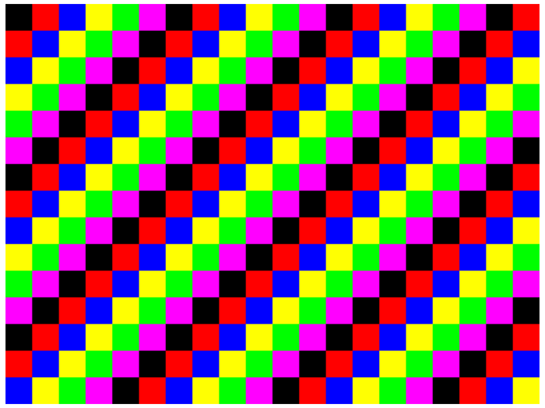
x • y = $x^2+y^2$ mod 6



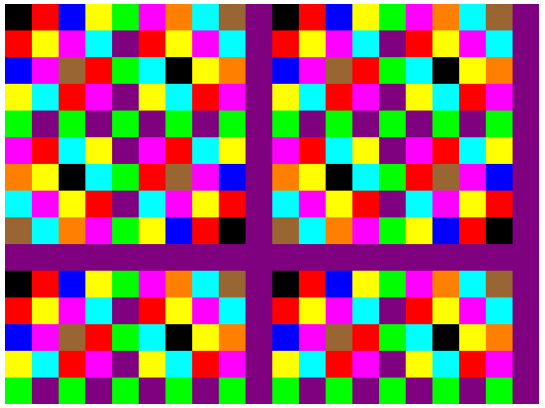
x • y = $x^2+y^2$ mod 5



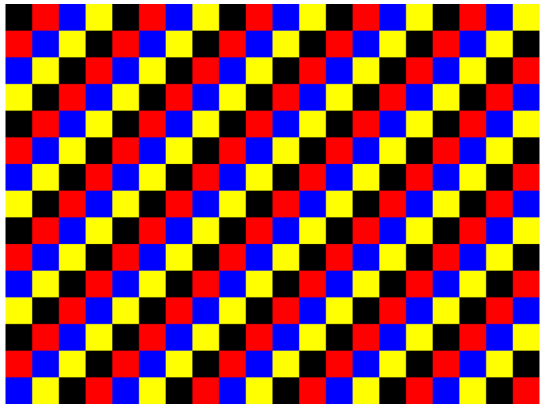
x • y = x + y + xy mod 4



x • y = x + y mod 6



x • y = x + y + xy mod 10



x • y = x + y mod 4

## 10.2 Chapter 3 Solutions

**Tangent to the curve $y = x + \frac{2}{x}$ at the point $(2, 3)$.**

$y - 3 = x + \frac{2}{x} - 3 = m(x - 2)$

$x^2 + 2 - 3x = mx^2 - 2mx$

$(1 - m)x^2 + (2m - 3)x + 2 = 0$

*Discriminant*:

$(2m - 3)^2 - 4(1 - m)(2) = 0$

$4m^2 - 12m + 9 - 8 + 8m = 0$

$4m^2 - 4m + 1 = 0$

$(2m - 1)^2 = 0$

$m = 1/2$

$y - 3 = x + \frac{2}{x} - 3 = m(x - 2)$

**Tangent to the curve $y = \frac{x^4}{16}$ at the point $(2, 1)$.**

$y - 1 = \frac{x^4}{16} - 1 = m(x - 2)$

$x^4 - 16 = 16mx - 32m$

$x^4 - 16mx + 32m - 16 = 0$

$(x - 2)\left(x^3 + 2x^2 + 4x - 16m + 8\right) = 0$

*Because x=2 is the only real solution,*

$x^3 + 2x^2 + 4x - 16m + 8$ *must be divisible by* x-2.

$\frac{x^3 + 2x^2 + 4x - 16m + 8}{x - 2} = x^2 + 4x + 12$ *with*

a *remainder of* $32 - 16m$

$32 - 16m$ *must* $= 0$

$m = 2$

$y - 1 = 2(x - 1)$

## 10.3 Chapter 6 Solutions

Multiplicative Magic Square with magic constant = 216

| | | |
|---|---|---|
| 2 | 36 | 3 |
| 9 | 6 | 4 |
| 12 | 1 | 18 |

Multiplicative Magic Square with magic constant = 6720

| | | | |
|---|---|---|---|
| 20 | 24 | 1 | 14 |
| 2 | 7 | 40 | 12 |
| 56 | 4 | 6 | 5 |
| 3 | 10 | 28 | 8 |

Prime Magic Square with magic constant = 411 (47, 137, and 227 ± 36)

| | | |
|---|---|---|
| 47 | 191 | 173 |
| 263 | 137 | 11 |
| 101 | 83 | 227 |

Another 4x4 Magic Square, but with fewer special properties

| | | | |
|---|---|---|---|
| 8 | 10 | 5 | 11 |
| 15 | 4 | 1 | 14 |
| 2 | 13 | 16 | 3 |
| 9 | 7 | 12 | 6 |