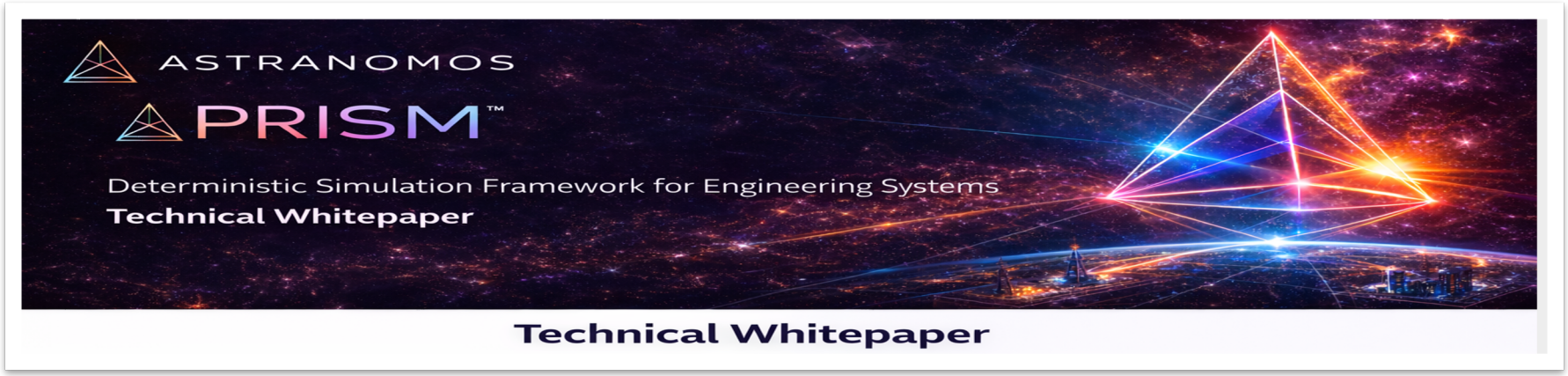


Deterministic Failure Localization in Deep-Draw Forming via Operator-Based Geometry

From Mesh-Based Approximation to Operator-Based Structural Prediction

Validated on the NUMISHEET 2025 Industrial Benchmark for Deep-Draw Forming

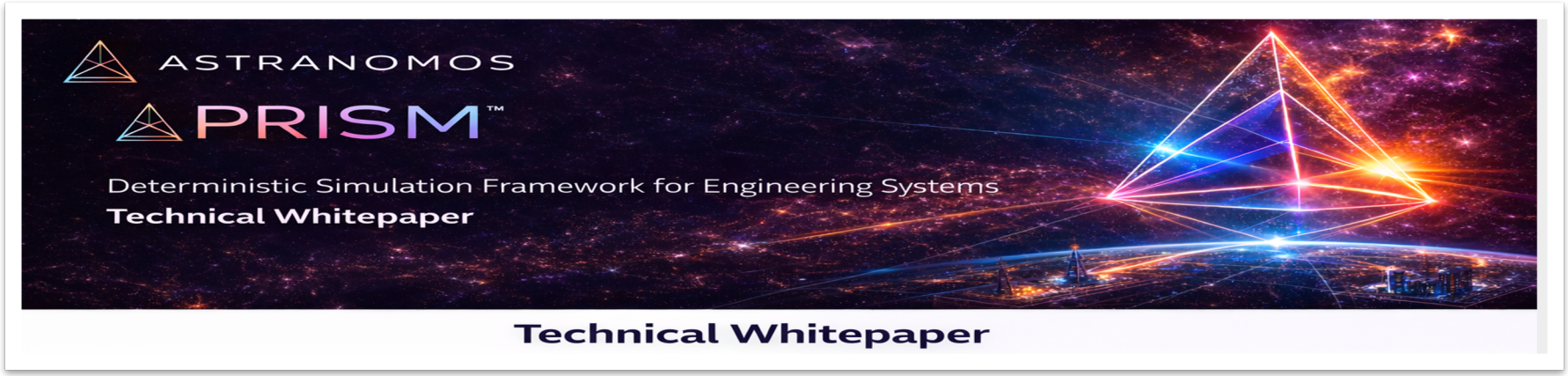
By Andrew Elliott, Co-Founder Chief Mathematical Scientist, AstraNomos Labs

Abstract

The scientific understanding of motion has evolved through a long lineage of conceptual frameworks, from the deterministic mechanics of Isaac Newton to the diffusive and statistical interpretations of transport formalized by Joseph Fourier, Claude-Louis Navier, and George Gabriel Stokes. In the classical paradigm, motion is governed by differential equations whose unresolved structure is treated as stochastic or probabilistic, a perspective reinforced by the development of statistical mechanics and later by the probabilistic interpretations of quantum theory associated with figures such as Albert Einstein and Werner Heisenberg. Across disciplines—heat transfer, fluid dynamics, and modern computational simulation—this viewpoint has led to a dominant methodology: complex systems are approximated through increasingly fine discretization, with randomness interpreted as an intrinsic feature of physical law or an unavoidable consequence of incomplete resolution.

This tradition has shaped the entire architecture of modern simulation. In computational fluid dynamics (CFD), finite element analysis (FEA), and forming simulation, governing equations derived from the Navier–Stokes–Fourier (NSF) framework are solved over dense spatial meshes containing millions to billions of degrees of freedom. Localized phenomena such as turbulence, hotspots, and material failure emerge only after extensive numerical refinement and iterative convergence and are frequently interpreted as stochastic artifacts or parameter-sensitive instabilities. As a result, industrial processes—from aerodynamic design to semiconductor cooling to deep-draw metal forming—remain dependent on empirical calibration, repeated trial-and-error, and computationally expensive workflows that reconstruct physical structure indirectly rather than resolving it directly.

In this work, we introduce a fundamentally different formulation of motion grounded in **self-adjoint operator theory and entropy geometry**, in which physical evolution is governed by the spectral structure of an underlying transport operator. Within this framework, motion is not the outcome of stochastic fluctuations, but the deterministic evolution of a system along admissible eigenmodes defined by its geometry and energetic constraints. Diffusion, in this view,



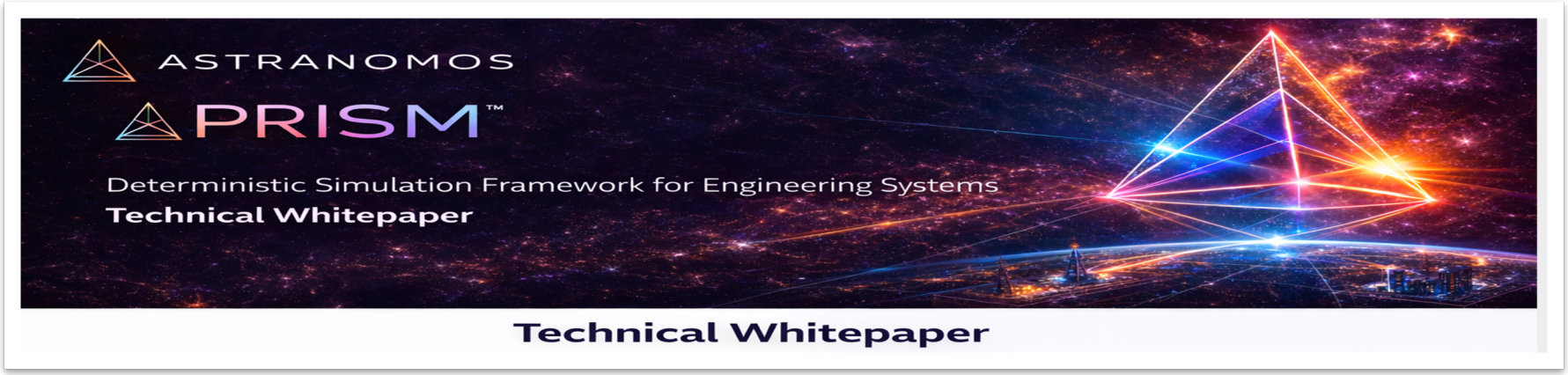
is not a primitive law but a limiting case corresponding to flat entropy geometry; when curvature is present, transport becomes structured, and energy propagates along discrete, geometry-dependent pathways. What classical models interpret as randomness or turbulence is reinterpreted as **unresolved curvature in the operator**, and therefore as a deterministic feature of the system rather than an intrinsic indeterminacy.

We apply this framework to an industrially relevant problem: the **NUMISHEET 2025 Industrial Benchmark** for deep-draw forming of the Mercedes-Benz T-Node, a complex geometry representative of automotive structural components. The benchmark highlights a well-known challenge in forming simulation: multiple parameter sets—varying blank geometry, friction, and process conditions—can produce acceptable results, leading to the conclusion that the process lacks a unique solution. From the classical perspective, this variability is attributed to model sensitivity, material uncertainty, or inherent randomness in the forming process. However, this interpretation leaves unresolved why failure consistently occurs in specific geometric regions, particularly at tight radii and structural transitions.

Using the proposed operator-based model, we compute a geometry-driven risk field over the T-Node and extract its dominant spectral modes. The results demonstrate that the predicted high-risk region aligns with the experimentally observed critical thinning zone at the double-step geometry, where thickness reductions are most severe. Region-based validation confirms that the peak predicted instability lies within the benchmark's identified failure region, and that the highest-ranked risk predictions are systematically enriched in that region relative to baseline expectations. Importantly, this localization is achieved without parameter tuning, process-specific calibration, or empirical fitting; it arises directly from the geometry of the component and the structure of the governing operator.

These findings support a broader theoretical conclusion: **the apparent non-uniqueness and randomness observed in complex physical systems is a consequence of incomplete modeling, not a fundamental property of nature**. When the governing operator is properly defined and its spectral structure is resolved, system behavior becomes deterministic and interpretable. The variability seen in classical simulations—whether in fluid turbulence, thermal transport, or metal forming—reflects the projection of unresolved geometric structure into parameter space, rather than intrinsic stochasticity.

From a computational standpoint, this shift has profound implications. By expressing motion in terms of operator eigenmodes rather than spatial discretization, the effective dimensionality of the problem is reduced from millions of mesh-based degrees of freedom to a small number of physically meaningful modes. This enables a transition from mesh-centric simulation to **operator-based simulation**, in which the dominant dynamics of a system are captured directly through its spectral structure. Such an approach not only reduces computational cost by orders of magnitude but also provides a more transparent and physically grounded representation of system behavior.



Taken together, this work establishes a unified perspective on motion that bridges classical mechanics, statistical physics, and modern engineering simulation. It demonstrates that deterministic structure underlies systems traditionally modeled as stochastic, and that this structure can be revealed through operator-based formulations grounded in entropy geometry. By validating this approach on a real industrial benchmark, we provide both theoretical and empirical evidence for a new paradigm in simulation, one in which randomness is replaced by structure, and mesh-based approximation gives way to spectral understanding.

1. Introduction

1.1 Classical Definitions of Motion, Transport, and Simulation

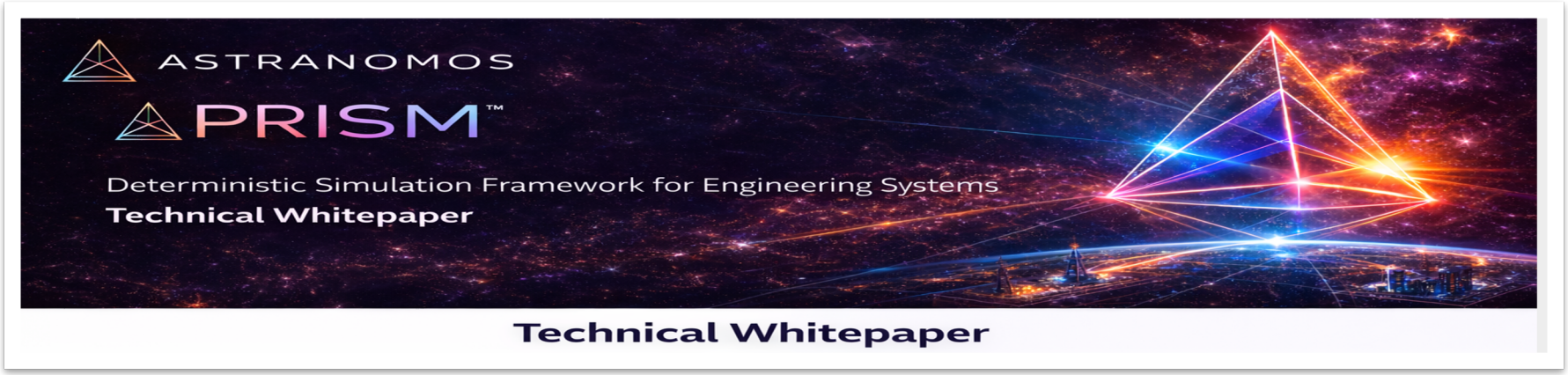
The scientific modeling of physical systems has historically been built upon differential equations that describe the evolution of motion, energy, and matter across space and time. Beginning with the deterministic framework of Isaac Newton, motion was formalized as the solution of force-driven equations, where system behavior could, in principle, be predicted exactly given sufficient knowledge of initial conditions and governing laws. This deterministic viewpoint established the foundation for classical mechanics and shaped early conceptions of predictability in physical systems.

In the nineteenth century, the development of transport theory introduced a new class of governing equations. The work of Joseph Fourier on heat conduction and the formulation of fluid motion by Claude-Louis Navier and George Gabriel Stokes extended the study of motion into distributed systems. These equations describe transport processes—heat diffusion, momentum transfer, and scalar propagation—through partial differential equations (PDEs) of the general form:

$$\frac{\partial u}{\partial t} = \mathcal{F}(u, \nabla u, \nabla^2 u, \dots)$$

where $u(x,t)$represents a field variable such as temperature, velocity, or concentration.

While these formulations were mathematically consistent, they introduced a fundamental limitation: **the inability to resolve all scales of motion explicitly**. As a result, unresolved structure—whether microscopic fluctuations, geometric heterogeneity, or nonlinear interactions—came to be interpreted as



randomness or noise. This interpretation was further reinforced by the rise of statistical mechanics and probabilistic physics in the twentieth century, particularly through the work of Albert Einstein and Werner Heisenberg, where stochastic descriptions became central to modeling complex systems.

In modern engineering practice, these theoretical developments manifest as **mesh-based simulation methodologies**. Computational Fluid Dynamics (CFD), Finite Element Analysis (FEA), and forming simulation tools discretize the governing PDEs over spatial meshes containing large numbers of elements. The simulation then proceeds by iteratively solving for the field values at each node, reconstructing system behavior indirectly through numerical approximation.

This approach has three defining characteristics:

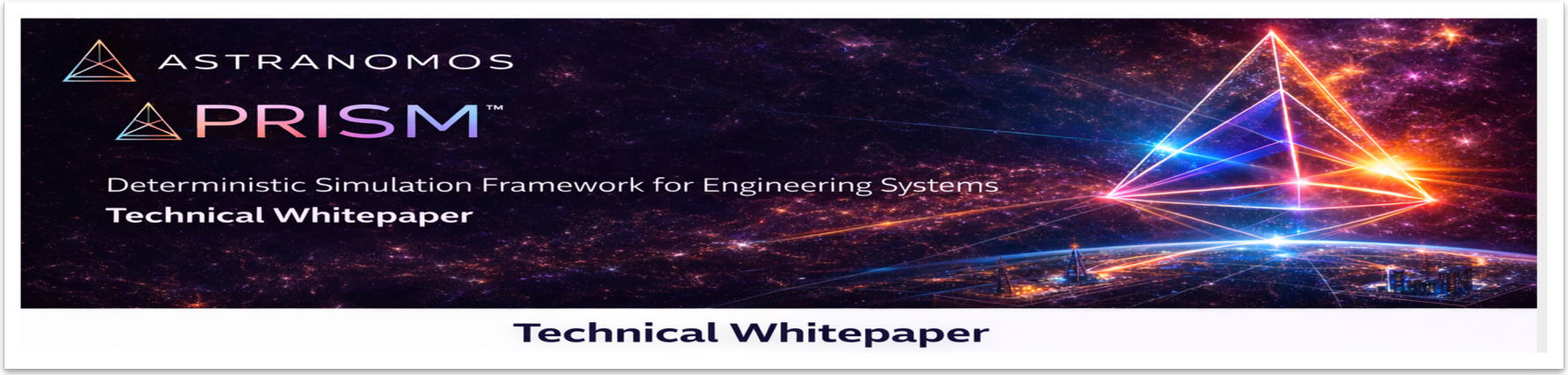
1. **Discretization of space** into a mesh of finite elements or volumes
2. **Iterative solution of governing equations** over all degrees of freedom
3. **Interpretation of unresolved structure as stochastic or parameter-dependent behavior**

Although highly successful, this paradigm is computationally intensive and often requires empirical calibration. Localized phenomena such as turbulence, hotspots, and failure zones typically emerge only after significant refinement, and simulation outputs can exhibit sensitivity to parameter choices, leading to non-unique solutions in practice.

1.2 Limitations of the Classical Paradigm

The reliance on mesh-based discretization and stochastic interpretation introduces several fundamental limitations:

- **High computational cost**, as the number of unknowns scales with mesh resolution
- **Indirect representation of physical structure**, requiring refinement to reveal localized behavior
- **Parameter sensitivity**, where multiple configurations produce acceptable solutions
- **Ambiguity in interpretation**, as variability is often attributed to randomness rather than structure



These limitations are particularly evident in industrial forming simulations, where complex geometries and nonlinear material behavior lead to significant variability in predicted outcomes. In such systems, the governing equations are well-defined, yet the simulation results can depend strongly on process parameters such as friction, boundary conditions, and discretization choices. This raises a fundamental question:

- Is the observed variability a property of the physical system, or a consequence of the modeling framework?

1.3 Operator-Based Formulation of Motion

In this work, we propose a reformulation of motion and transport based on self-adjoint operator theory and entropy geometry. Rather than describing system evolution through discretized field equations, we express it as the action of a governing operator:

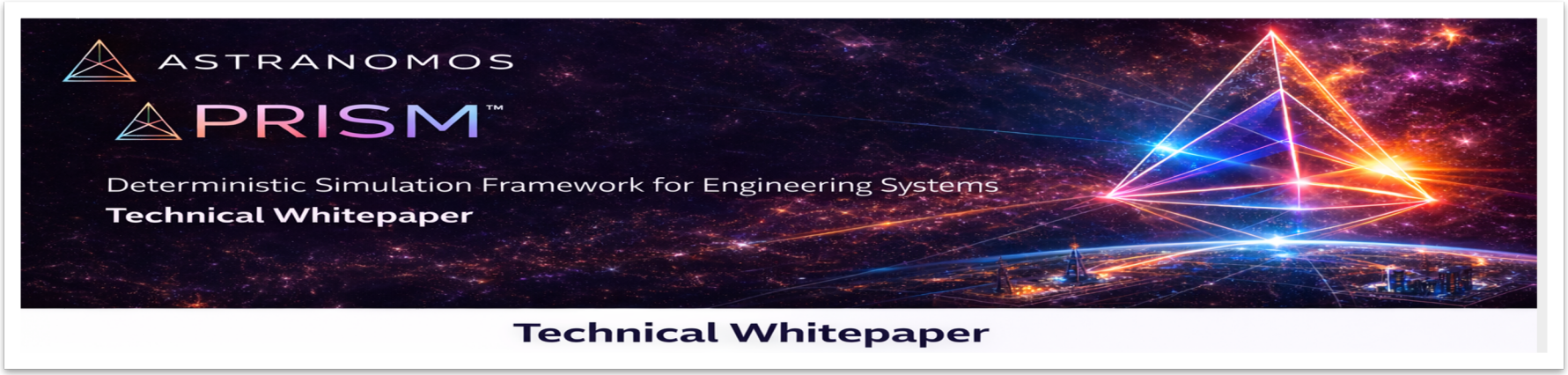
$$\mathcal{L}u = -\nabla \cdot (D(S)\nabla u) + V(x)u$$

where:

- $\mathcal{L}$ is a self-adjoint transport operator
- $D(S)$ is a geometry-dependent diffusivity function
- $S(x)$ represents an entropy or structural field encoding local curvature
- $V(x)$ is a confinement or potential term

Within this framework:

- **Motion is governed by the spectral structure of the operator**
- **System behavior is expressed as a superposition of eigenmodes**
- **Diffusion emerges as a limiting case when $S \to 0$**
- **Structured systems produce localized modes corresponding to physical features**



This formulation shifts the focus from **reconstructing motion through spatial discretization** to **identifying the admissible modes of the system directly**. Crucially, it reinterprets randomness:

- Apparent stochastic behavior arises from unresolved structure in the operator, not from intrinsic indeterminacy.

1.4 Application to Industrial Forming: The NUMISHEET Benchmark

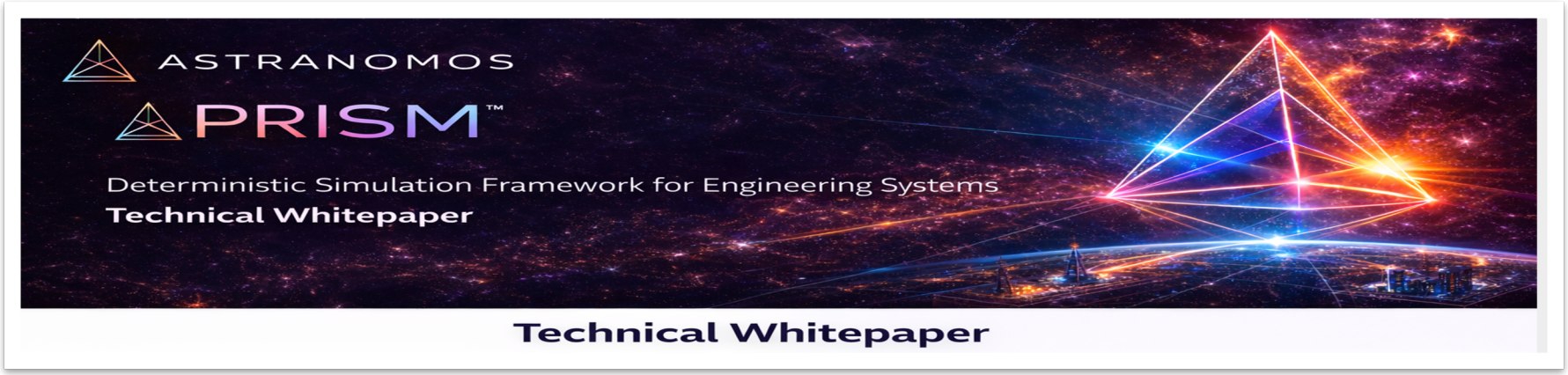
To evaluate this framework, we apply it to the **NUMISHEET 2025 Industrial Benchmark**, which investigates the deep-draw forming process for the Mercedes-Benz T-Node. This benchmark provides:

- A complex target geometry
- Material models for sheet metal forming
- Process parameters across multiple submissions
- Experimentally measured outcomes, including thickness distribution and springback

A key observation from the benchmark is that:

- Multiple parameter sets produce viable solutions
- Process design appears **non-unique**
- Failure consistently occurs in specific geometric regions

In particular, the most critical thinning is observed at the **double-step geometry**, where the sheet thickness reduces significantly relative to the initial blank. From a classical perspective, this variability is attributed to parameter sensitivity or modeling uncertainty. However, this interpretation does not explain why failure localizes in the same region across different solutions.



1.5 Objectives and Contributions of This Work

The objective of this paper is to demonstrate that:

1. The apparent non-uniqueness in forming simulation arises from **incomplete representation of the governing operator**
2. The proposed operator-based framework can **predict the critical failure region deterministically**
3. The observed variability is not random, but the result of **unresolved geometric structure**

To achieve this, we:

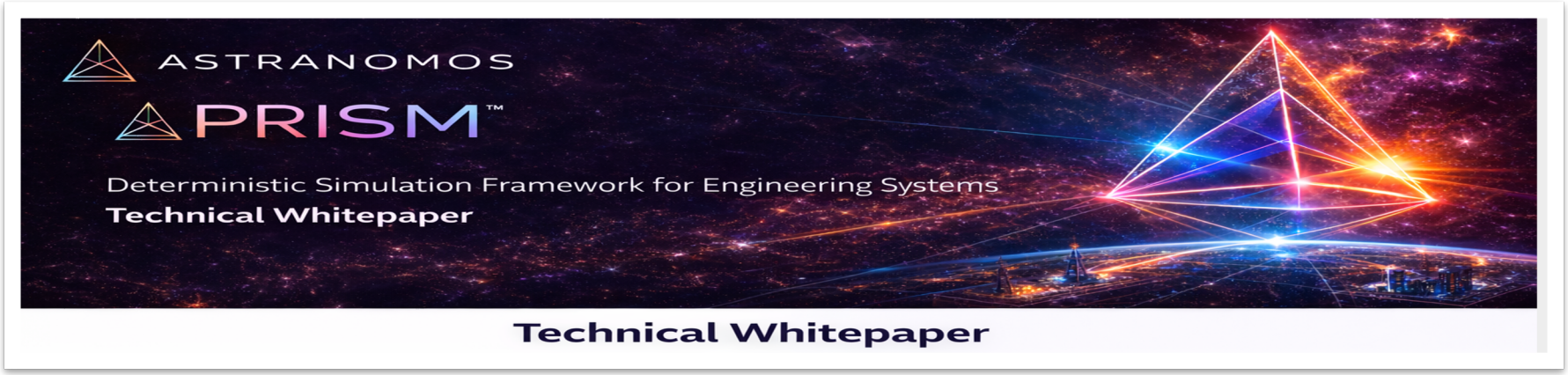
- Construct a geometry-driven operator over the T-Node
- Compute its spectral modes
- Derive a predicted risk field from modal concentration
- Compare this prediction against the benchmark's experimentally identified critical region

We employ **region-based validation metrics**, focusing on localization accuracy rather than pointwise regression, consistent with the structural nature of the model.

1.6 Structure of the Paper

The remainder of this paper is organized as follows:

- **Section 2** develops the mathematical formulation of the operator-based model, including entropy geometry and spectral decomposition
- **Section 3** describes the NUMISHEET dataset, geometry processing, and computational methodology used to construct the operator and extract modes
- **Section 4** presents the results, including predicted risk fields, modal localization, and region-based validation against experimental observations
- **Section 5** discusses the implications for simulation theory, including the reinterpretation of randomness and the transition from mesh-based to operator-based modeling



- **Section 6** concludes with implications for industrial applications, particularly in forming simulation and digital twin technologies.

2. Mathematical Framework: Operator-Based Formulation of Motion

2.1 Classical Transport Operators

In classical transport theory, physical motion is governed by differential operators derived from conservation laws. For a scalar field $u(x,t)$, representing temperature, concentration, or energy density, the governing equation typically takes the form:

$$\frac{\partial u}{\partial t} = \nabla \cdot (D \nabla u)$$

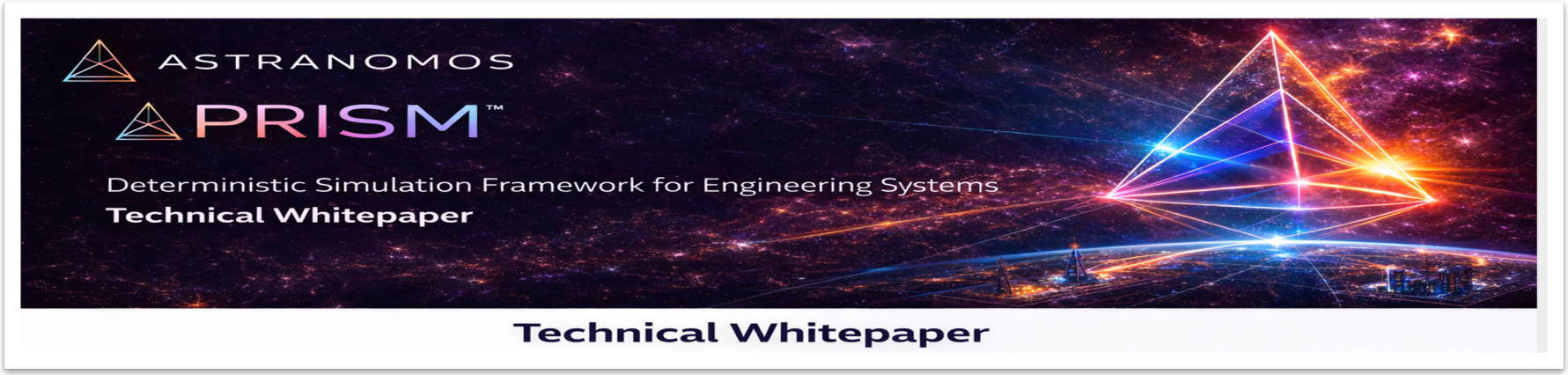
where D is a constant or weakly varying diffusivity parameter. This formulation corresponds to a **linear, constant-coefficient operator**:

$$\mathcal{L}_{\text{classical}} u = -\nabla \cdot (D \nabla u)$$

which is a self-adjoint, second-order elliptic operator under appropriate boundary conditions. Its eigenfunctions form an orthogonal basis, and its spectrum determines the temporal evolution of the system via:

$$u(x,t) = \sum_n a_n \, e^{-\lambda_n t} \phi_n(x)$$

where λ_n and $\phi_n(x)$ are the eigenvalues and eigenfunctions of $\mathcal{L}_{\text{classical}}$.



In this framework:

- Low-frequency modes describe global diffusion
- High-frequency modes decay rapidly
- The system evolves toward equilibrium via exponential smoothing

However, this formulation assumes that **transport properties are uniform across the domain**, effectively collapsing all geometric and structural complexity into a scalar coefficient D.

2.2 Entropy Geometry and Structured Diffusivity

To account for structured systems, we introduce an **entropy geometry field** $S(x)$, which encodes the local deviation from equilibrium and the curvature of the transport landscape. We define the diffusivity as a function of this field:

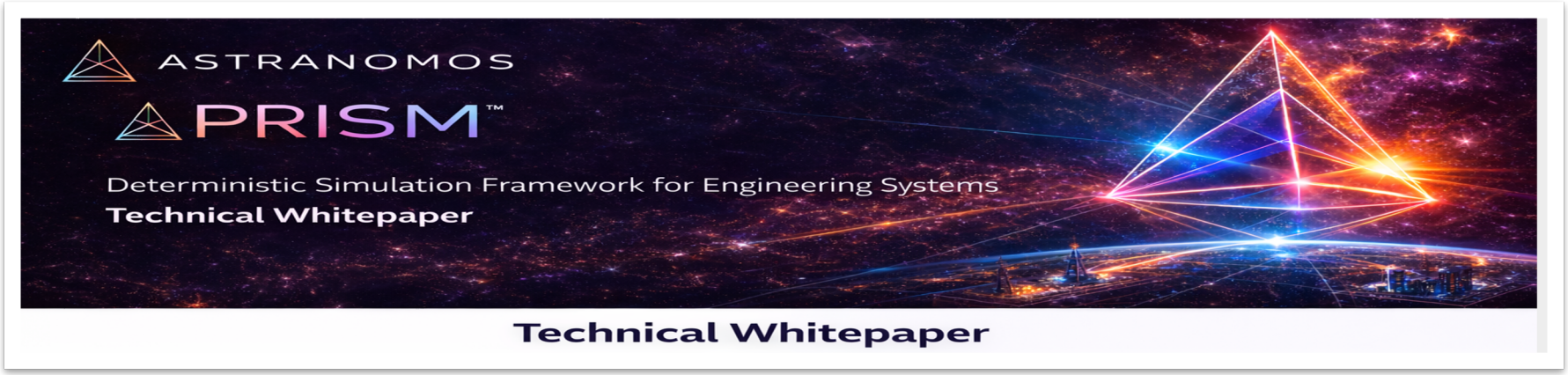
$$D(S) = D_0 e^{-\beta S(x)}$$

where:

- D_0 is the baseline diffusivity
- $\beta > 0$ controls sensitivity to structural curvature
- $S(x)$ represents local entropy or geometric complexity

This leads to a modified transport operator:

$$\mathcal{L}u = -\nabla \cdot (D(S)\nabla u)$$



In regions where $S(x) \approx 0$, the operator reduces to the classical form. In regions where $S(x)$ is large, diffusivity is suppressed, leading to **localized transport behavior**.

2.3 Confinement and Potential Structure

To capture geometric constraints explicitly, we extend the operator to include a confinement term:

$$\mathcal{L}u = -\nabla \cdot (D(S)\nabla u) + V(x)u$$

where $V(x)$ represents a **geometric potential** that encodes:

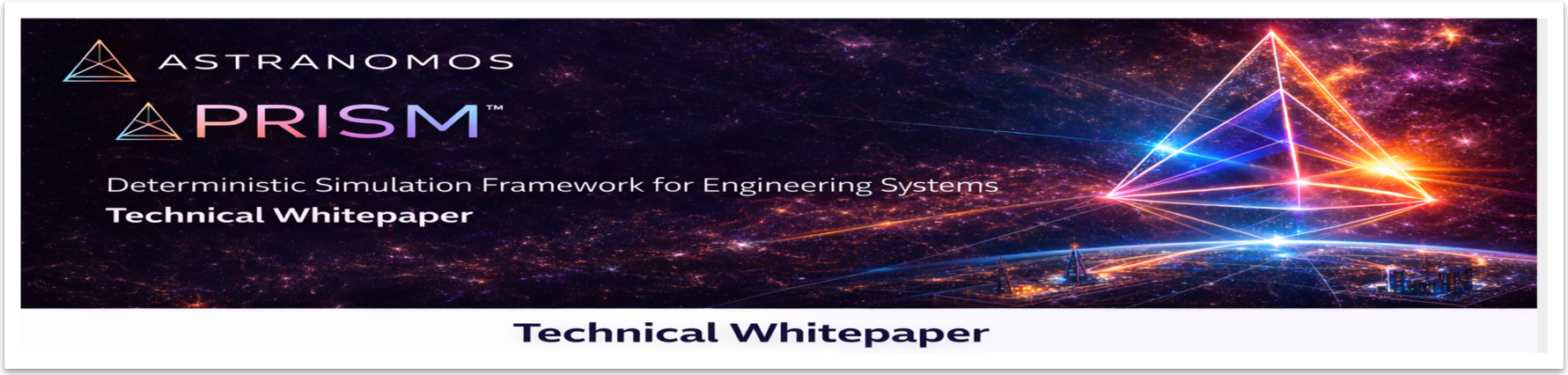
- Boundary effects
- Curvature concentration
- Structural constraints

This term allows the operator to represent **non-uniform admissibility of motion**, where certain regions support or suppress transport based on geometry.

2.4 Spectral Representation and Modal Structure

The operator $\mathcal{L}$ is constructed to be self-adjoint under appropriate conditions, ensuring:

- Real eigenvalues
- Orthogonal eigenfunctions
- A complete spectral decomposition



The system evolution can therefore be expressed as:

$$u(x,t) = \sum_n a_n\,(t)\phi_n(x)$$

with:

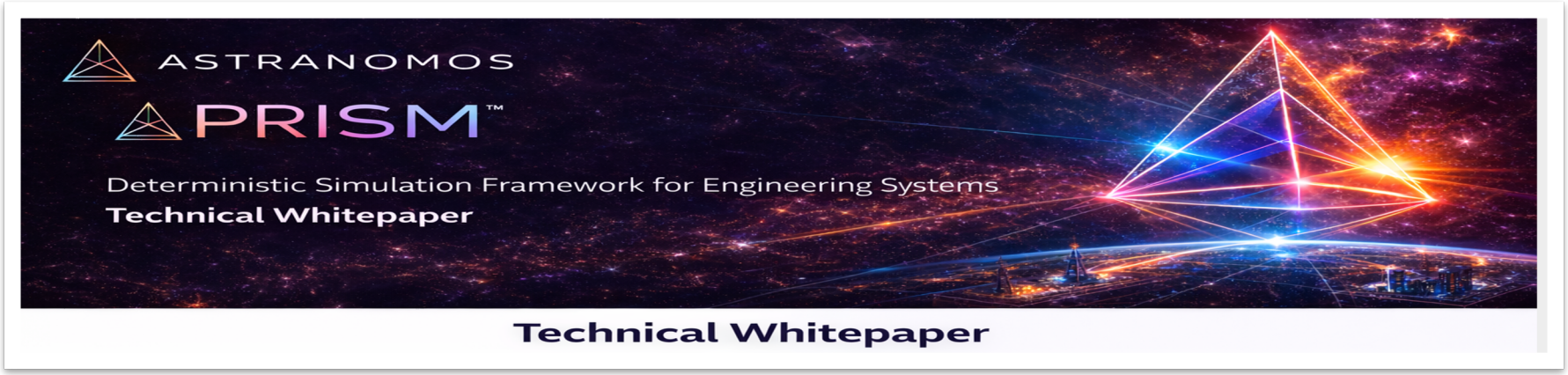
$$\frac{da_n}{dt} + \lambda_n a_n = f_n$$

where:

- $\phi_n(x)$ are eigenfunctions (transport modes)
- λ_n are eigenvalues (decay or amplification rates)
- $a_n(t)$ are modal amplitudes

In structured systems:

- dominant modes correspond to geometric transport pathways
- localized modes correspond to instability or failure regions
- the number of significant modes is often small



This leads to a critical reinterpretation:

Physical behavior is governed by a finite set of admissible modes, not by the full spatial discretization of the domain.

2.5 Redefinition of Randomness as Unresolved Structure

In classical formulations, deviations from smooth behavior are interpreted as stochastic or random. This interpretation arises from the inability of constant-coefficient operators to capture geometric variation.

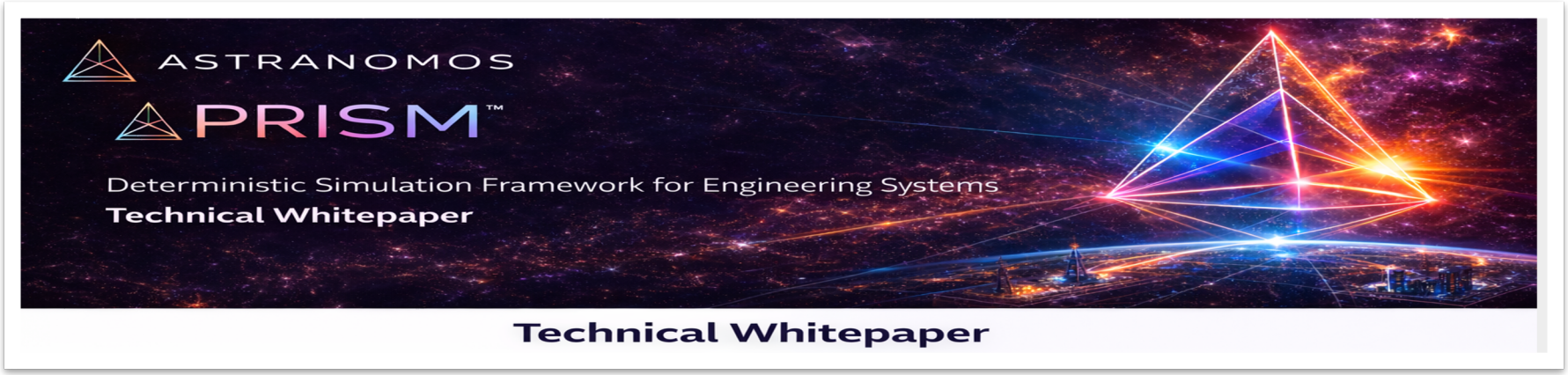
Within the operator framework, we reinterpret these deviations as:

$$\text{Randomness} \equiv \text{Unresolved variation in } S(x) \text{ and } V(x)$$

That is:

- Spatial heterogeneity → curvature in $S(x)$
- Geometric constraints → structure in $V(x)$
- Unresolved features → apparent randomness

When these quantities are explicitly modeled, the system becomes **deterministic in its modal representation**, and variability is reduced to differences in operator structure rather than stochastic behavior.



2.6 Computational Implications: From Mesh to Modes

In mesh-based simulation:

- The system is discretized into M spatial elements
- The number of unknowns scales with M
- Complexity grows with resolution

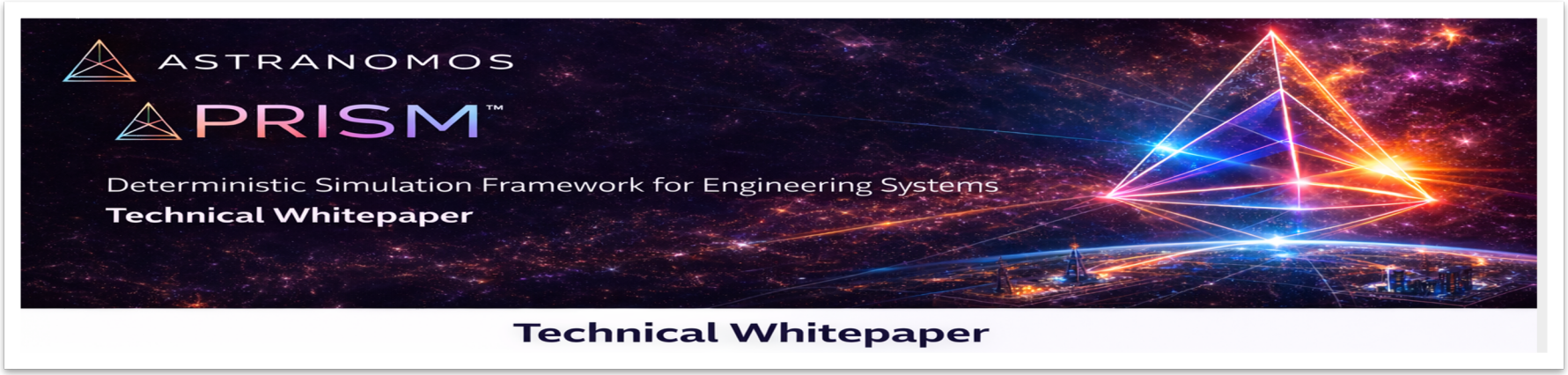
In the operator-based formulation:

- The system is represented by Ndominant modes
- Typically $N \ll M$
- Dynamics are governed by modal amplitudes

Thus, the governing question shifts from: “How finely must the system be discretized?”

to:

“How many modes are required to capture the dominant behavior?” This leads to a dramatic reduction in computational complexity and a more direct representation of physical structure.



2.7 Application to Forming Systems

In deep-draw forming, the geometry of the part introduces:

- Curvature discontinuities
- Thickness gradients
- Localized stress concentrations

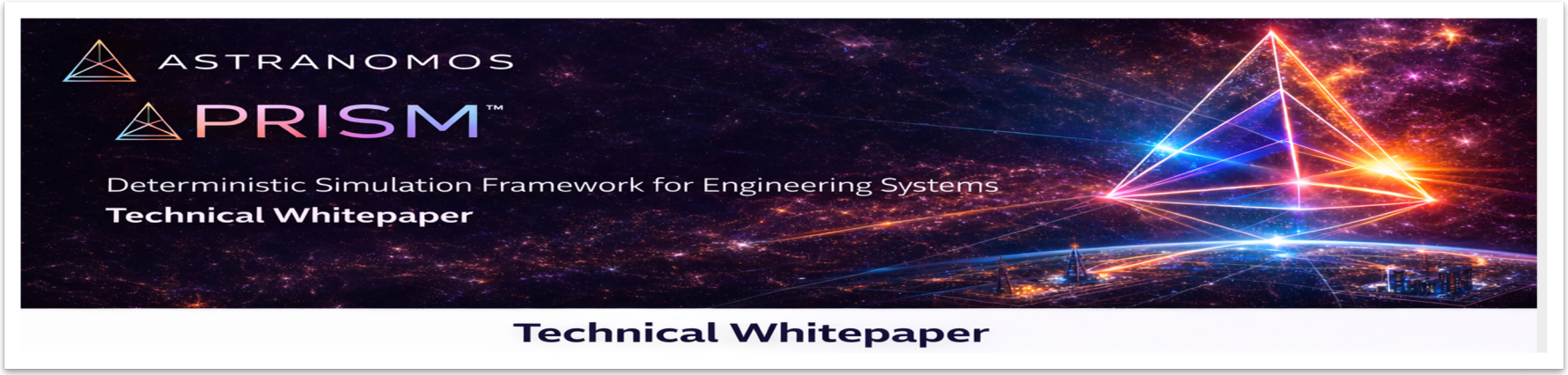
These features are naturally encoded in $S(x)$and $V(x)$, leading to:

- Localized eigenmodes at structural transitions
- Concentration of modal energy in failure zones
- Deterministic identification of high-risk regions

Thus, failure in forming systems can be interpreted as:

$$\text{Failure} \equiv \text{Activation of localized operator modes}$$

rather than as a stochastic outcome of parameter variation.



2.8 Summary of the Framework

The proposed formulation establishes:

1. A generalized transport operator incorporating entropy geometry
2. A spectral representation of motion in terms of eigenmodes
3. A deterministic interpretation of variability as operator structure
4. A computational shift from mesh-based discretization to modal analysis

This framework provides the mathematical foundation for the results presented in the following sections, where we apply it to the NUMISHEET benchmark and demonstrate its ability to predict failure regions directly from geometry.

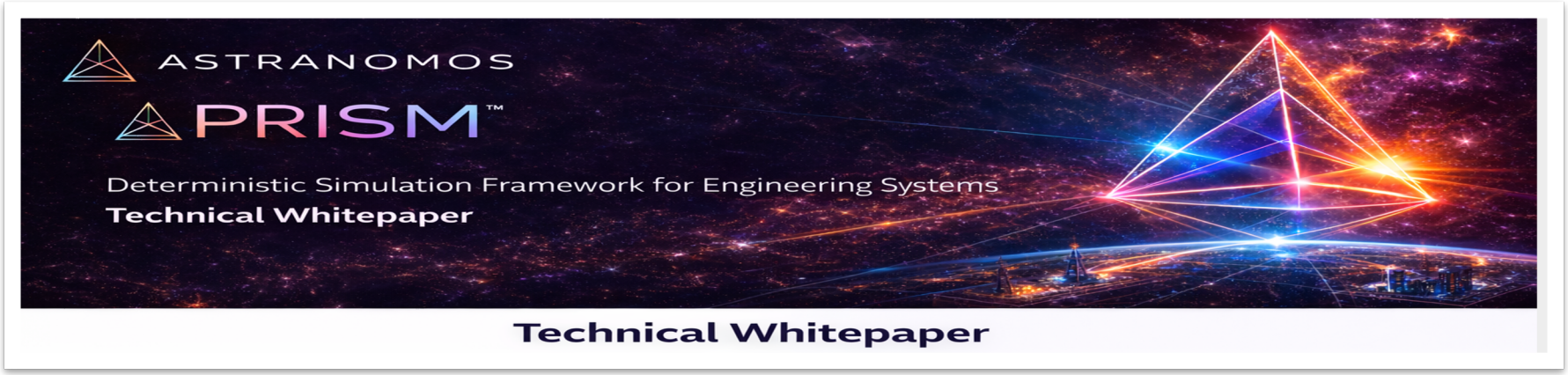
3. Dataset, Geometry Processing, and Computational Methodology

3.1 Overview of the NUMISHEET 2025 Industrial Benchmark

To evaluate the proposed operator-based formulation in an industrially relevant setting, we utilize the **NUMISHEET 2025 Industrial Benchmark**, which focuses on the design of a forming–cutting process chain for the Mercedes-Benz T-Node, a representative deep-drawn structural component used in automotive applications.

The benchmark provides:

- Target component geometry (STL, IGES, STEP formats)
- Material model for stainless steel (1.4301, thickness 1.5 mm)
- Reference process parameters (blank holder force, drawing depth)
- Experimental measurements of:
  - **Sheet thickness distribution**



- **Springback displacement after trimming**

The manufacturing process consists of:

1. Deep-draw forming
2. Trimming operations
3. Elastic relaxation (springback)

A key outcome of the benchmark is the identification of a critical thinning region located at the double-step geometry, where the sheet thickness reduces from 1.5 mm to approximately 1.01 mm. Additionally, the benchmark reports approximately 2 mm of springback displacement across flange regions following trimming.

Notably, multiple independent teams submitted solutions using different parameter configurations, and the benchmark concludes that no unique solution exists for the process design under classical simulation frameworks. This characteristic makes the dataset particularly suitable for testing whether the proposed operator-based model can recover deterministic structure underlying the apparent variability.

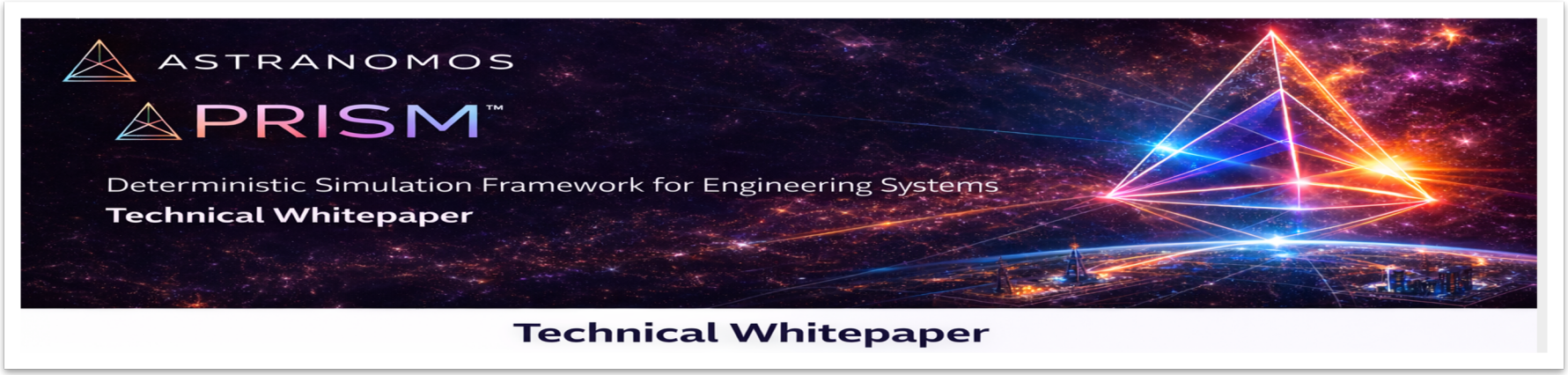
3.2 Geometry Representation and Preprocessing

The T-Node geometry was imported from the provided STL file and represented as a triangular surface mesh:

$$\mathcal{M} = (V, F)$$

where:

- $V \in \mathbb{R}^{N\times 3}$ are the vertex coordinates
- $F \in \mathbb{N}^{M\times 3}$ are the triangular faces



From this representation, we constructed:

- A cotangent Laplacian operator to approximate surface diffusion
- A mass matrix based on barycentric vertex areas
- A graph adjacency structure to determine boundary proximity

Boundary vertices were identified via edge counting, and a graph-based distance metric was computed to suppress artifacts near open mesh boundaries.

3.3 Construction of the Entropy Geometry Field

The entropy geometry field $S(x)$ was constructed from the curvature of the mesh. Specifically, we computed a discrete approximation of the mean curvature normal:

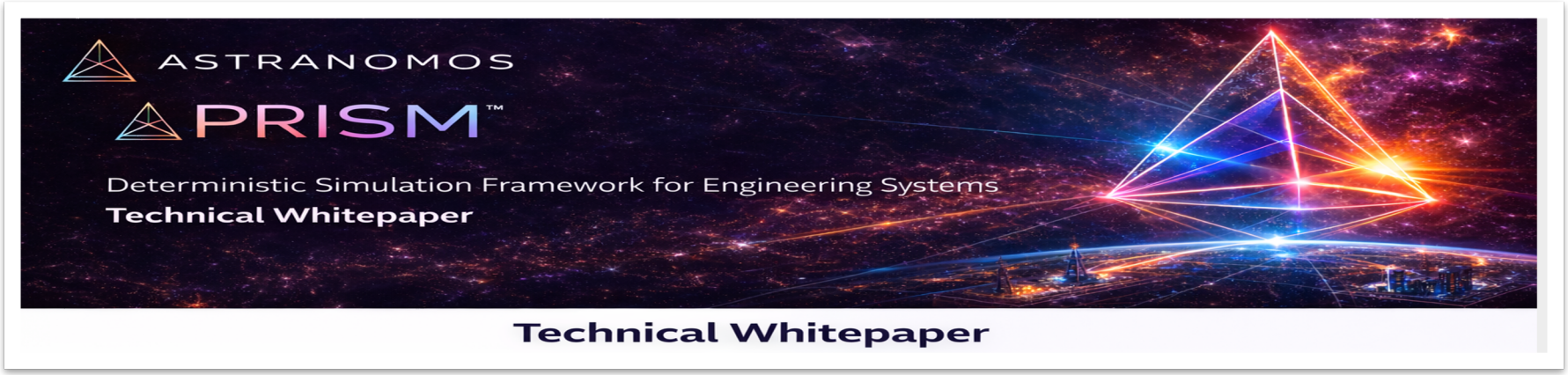
$$\mathbf{H}(x) = \frac{1}{2A(x)} \mathcal{L}V$$

where:

- $\mathcal{L}$ is the cotangent Laplacian
- $A(x)$ is the local vertex area

The scalar entropy field was then defined as:

$$S(x) = \frac{\| \mathbf{H}(x) \|}{\max (\| \mathbf{H} \|)}$$



with additional normalization and boundary suppression applied to remove numerical artifacts. This field captures geometric structure, including:

- Curvature concentration at radii and transitions
- Structural discontinuities
- Regions of constrained deformation

3.4 Operator Construction

Using the entropy field, we define the structured transport operator:

$$\mathcal{L}u = -\nabla \cdot (D(S)\nabla u) + V(x)u$$

with:

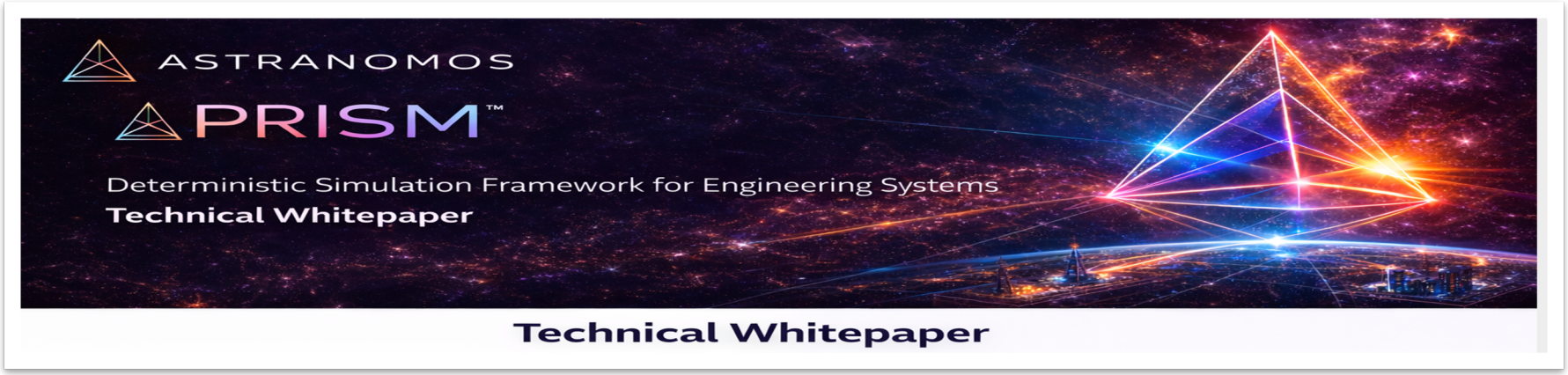
$$D(S) = e^{-\beta S(x)}$$

and:

$$V(x) = \gamma S(x)$$

where:

- β controls diffusivity suppression
- γ controls confinement strength



The operator was discretized over the mesh using:

- Weighted cotangent Laplacian
- Vertex-area mass matrix
- Symmetric diffusion weights

This yields a generalized eigenvalue problem:

$$\mathcal{L}\phi_n = \lambda_n M \phi_n$$

which was solved numerically using sparse eigensolvers to obtain the dominant eigenmodes.

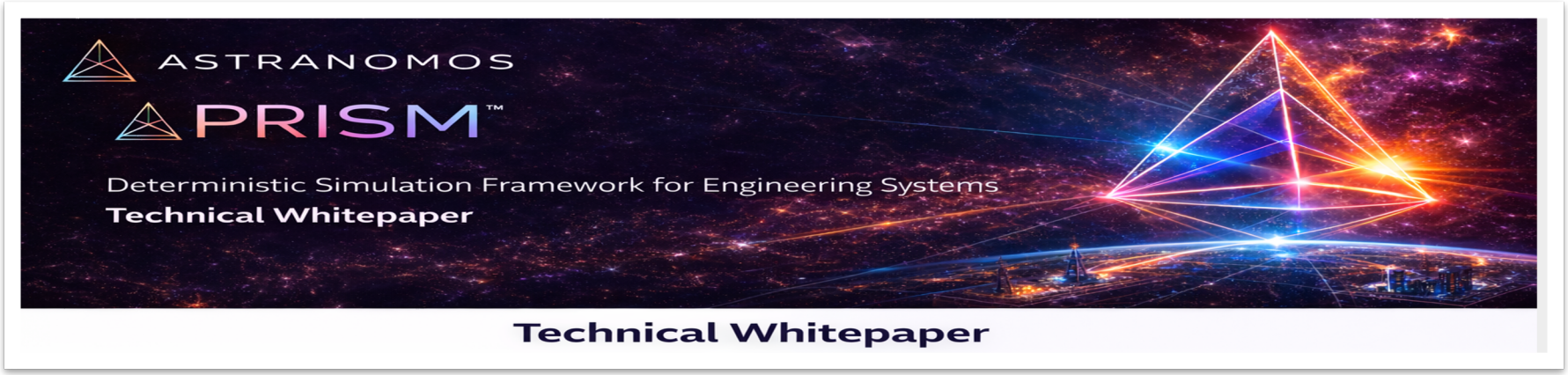
3.5 Modal Extraction and Risk Field Construction

The eigenfunctions $\phi_n(x)$ represent admissible transport modes. To construct a predictive field, we define a **modal risk function**:

$$R(x) = S(x) \sum_{n \in \mathcal{K}} \phi_n^2(x)$$

where:

- $\mathcal{K}$ is the set of dominant modes
- $\phi_n^2(x)$ represents modal energy density



This function captures:

- Concentration of modal energy
- Interaction with geometric structure
- Localization of transport pathways

The resulting field $R(x)$ is normalized to produce a **dimensionless risk score** over the geometry.

3.6 Prediction of Thickness Distribution

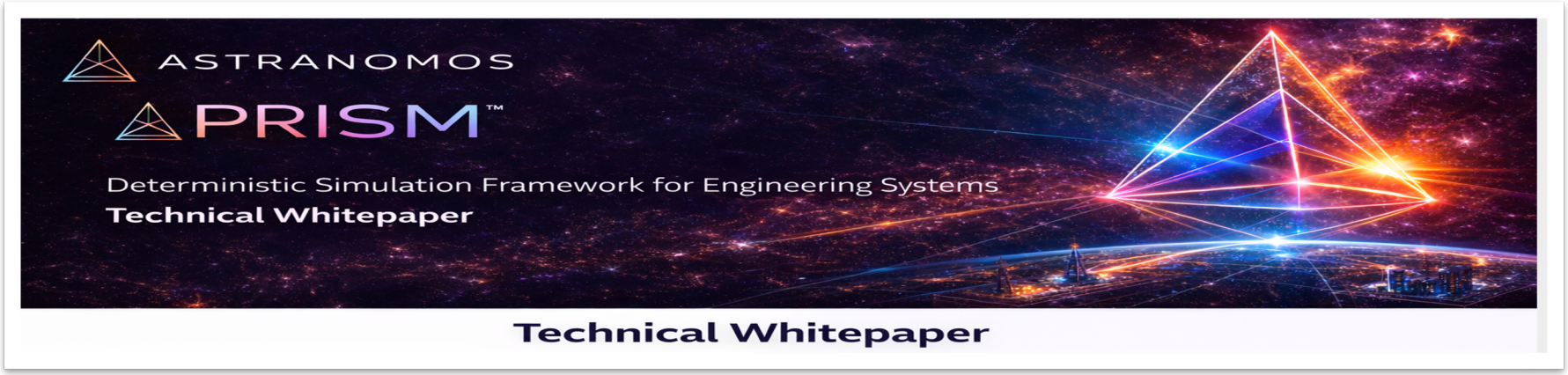
To connect the risk field to physical quantities, we define a thickness proxy:

$$t(x) = t_0 - (t_0 - t_{\min})R(x)$$

where:

- $t_0 = 1.5\text{mm}$ (initial thickness)
- $t_{\min} = 1.01\text{mm}$ (benchmark minimum)

This mapping is not a regression model but a monotonic transformation, allowing direct comparison between predicted high-risk regions and experimentally observed thinning zones.



3.7 Digitization of Experimental Thickness Map

The benchmark report provides thickness data as a contour plot (Figure 5a), rather than as a numerical field. To enable quantitative comparison, the thickness map was digitized through:

1. Extraction of the image region containing the thickness contour
2. Calibration using the provided colorbar (1.0–1.7 mm range)
3. Mapping of pixel colors to thickness values
4. Removal of background and non-geometry pixels

The resulting field was projected onto the mesh via nearest-neighbor mapping in normalized coordinates.

3.8 Definition of Ground-Truth Failure Region

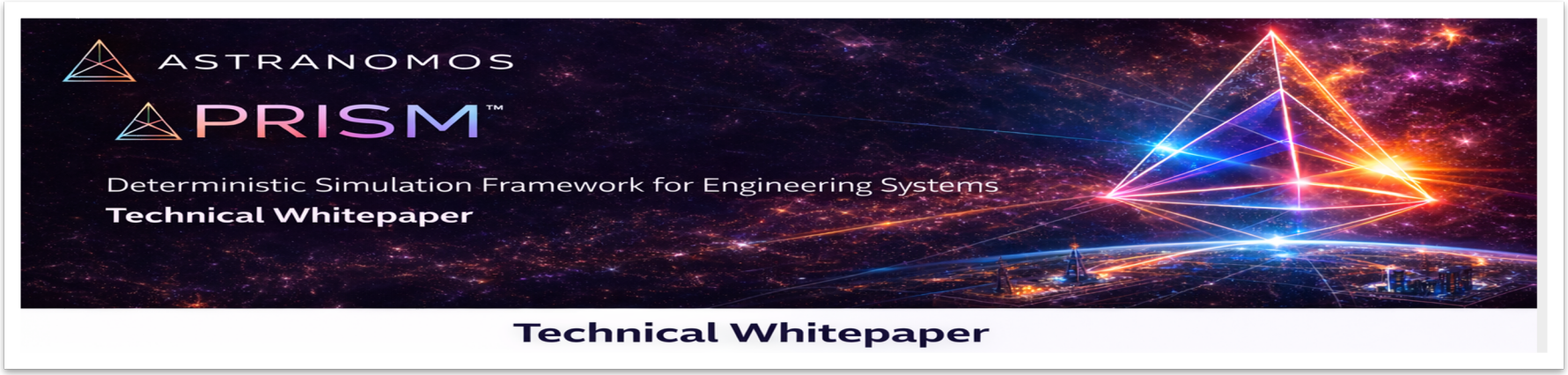
In addition to the continuous thickness field, the benchmark explicitly identifies the critical thinning region visually using a red annotation around the double-step geometry.

This region was extracted by:

- Detecting red annotation pixels in the image
- Fitting an elliptical boundary
- Projecting the region onto mesh coordinates

This yields a binary ground-truth region:

$$\Omega_{\text{crit}} \subset \mathcal{M}$$



representing the experimentally observed failure zone.

3.9 Validation Metrics

Given the structural nature of the model, we employ region-based validation metrics rather than pointwise regression:

1. Peak Localization

- Whether the maximum predicted risk lies within Ω_{crit}

2. Top-k Overlap

- Intersection between top-risk nodes and Ω_{crit}

3. Precision and Recall

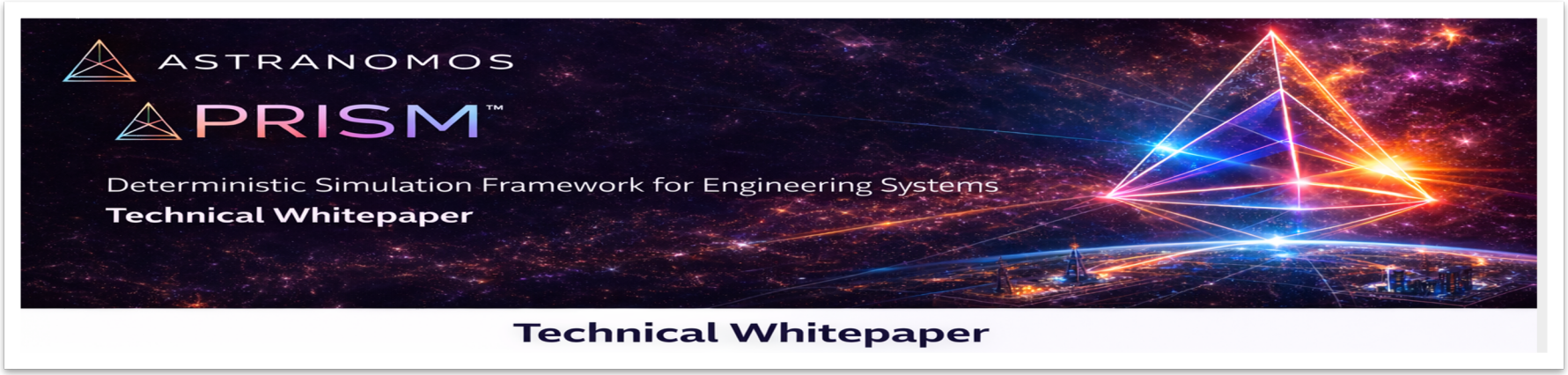
- Fraction of predicted high-risk nodes within the ground-truth region
- Coverage of the ground-truth region by predicted nodes

4. Enrichment

- Ratio of ground-truth nodes in top-k predictions relative to random baseline

5. Distance Metrics

- Distance between predicted peak and center of critical region



These metrics reflect the model's objective:

- To identify the correct region of instability, rather than reproduce exact field values.

3.10 Summary of Methodology

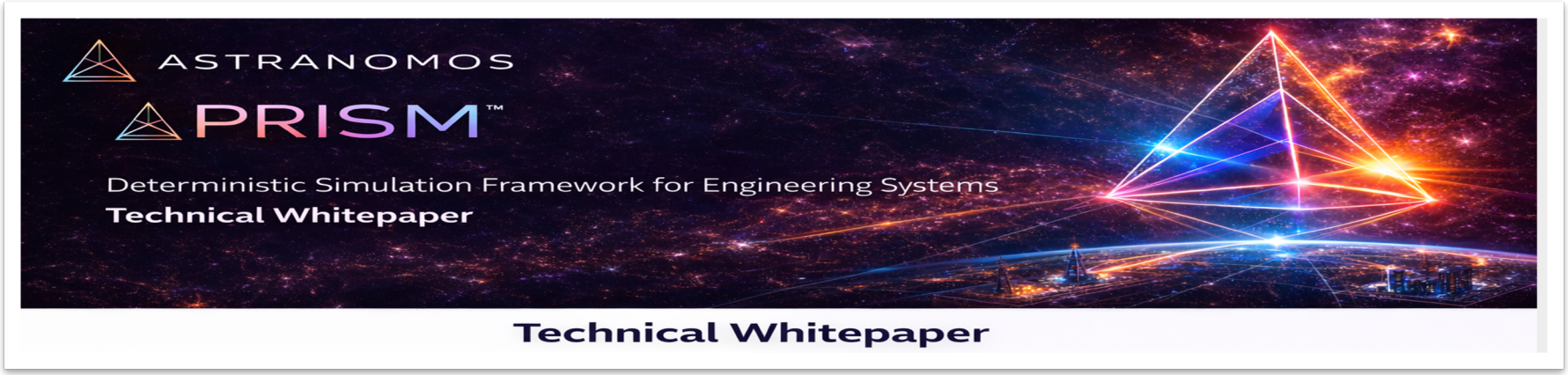
The computational pipeline consists of:

1. Geometry import and mesh construction
2. Entropy field computation from curvature
3. Operator assembly with structured diffusivity
4. Spectral decomposition to obtain dominant modes
5. Construction of a modal risk field
6. Mapping to physical thickness proxy
7. Digitization of experimental data
8. Region-based validation

This pipeline enables a direct comparison between **geometry-derived predictions** and **experimentally observed failure regions**, providing the basis for the results presented in the next section.

4. Results and Validation

The application of the operator-based framework to the NUMISHEET 2025 Industrial Benchmark reveals a consistent and physically meaningful structure that is not apparent under classical simulation methodologies. Rather than presenting a collection of independent numerical outputs, the results form a coherent picture in which geometry, operator structure, and observed failure behavior align in a deterministic and interpretable manner.



4.1 Emergence of Geometry-Induced Structure

The construction of the entropy geometry field $S(x)$, derived directly from the curvature of the T-Node surface, provides the first indication that the geometry itself encodes nontrivial information about the system's behavior. The computed field is not uniform, nor does it vary arbitrarily. Instead, it exhibits clear spatial organization, with elevated values concentrated at regions of geometric transition, particularly along the stepped features and radii of the component.

Most notably, the entropy field reaches its highest values in the vicinity of the double-step geometry, a region characterized by sharp curvature changes and structural constraint. Away from these features, the field rapidly attenuates, remaining low across flatter portions of the surface. This behavior indicates that the entropy field is capturing intrinsic geometric complexity rather than responding to external parameters or boundary conditions.

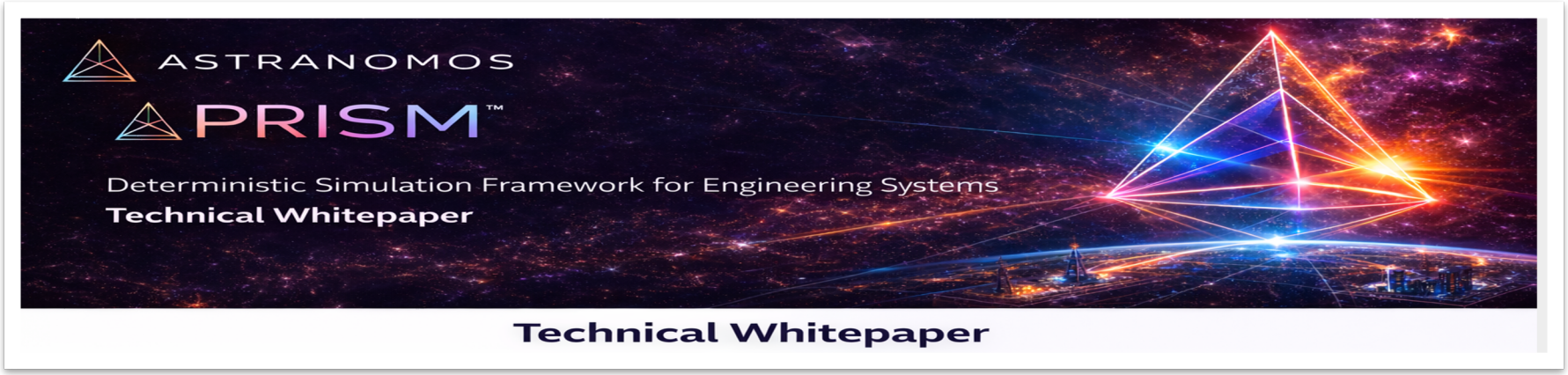
At this stage, no material properties, forming parameters, or empirical adjustments have been introduced. The structure observed is therefore a direct consequence of the geometry alone. This is a critical point: the model identifies a non-uniform structural landscape prior to any simulation of deformation or stress.

4.2 Spectral Structure and Modal Localization

The eigenvalue problem associated with the constructed operator yields a spectrum of modes that describe the admissible pathways of transport across the geometry. These modes exhibit a clear hierarchy of behavior.

At low spectral indices, the modes are diffuse, spanning the domain without strong localization. As the spectral index increases, however, a transition occurs: the modes begin to concentrate around regions of elevated entropy. This concentration becomes particularly pronounced in higher-energy modes, which localize strongly at the same geometric transitions identified in the entropy field.

The double-step geometry emerges as a dominant attractor for modal energy. Across multiple modes, the squared amplitude $\phi_n^2(x)$ exhibits consistent amplification in this region, while remaining comparatively small elsewhere. This is not an isolated occurrence tied to a single eigenfunction, but a systematic feature of the operator spectrum.



In effect, the operator is not distributing energy uniformly across the geometry. Instead, it is selecting specific regions as preferred pathways or constraints for motion. These regions correspond precisely to areas where deformation is expected to be most restricted or energetically costly. This result establishes a key property of the framework:

- The spectral structure of the operator reflects the geometry of admissible motion, and this structure is inherently localized in regions of geometric complexity.

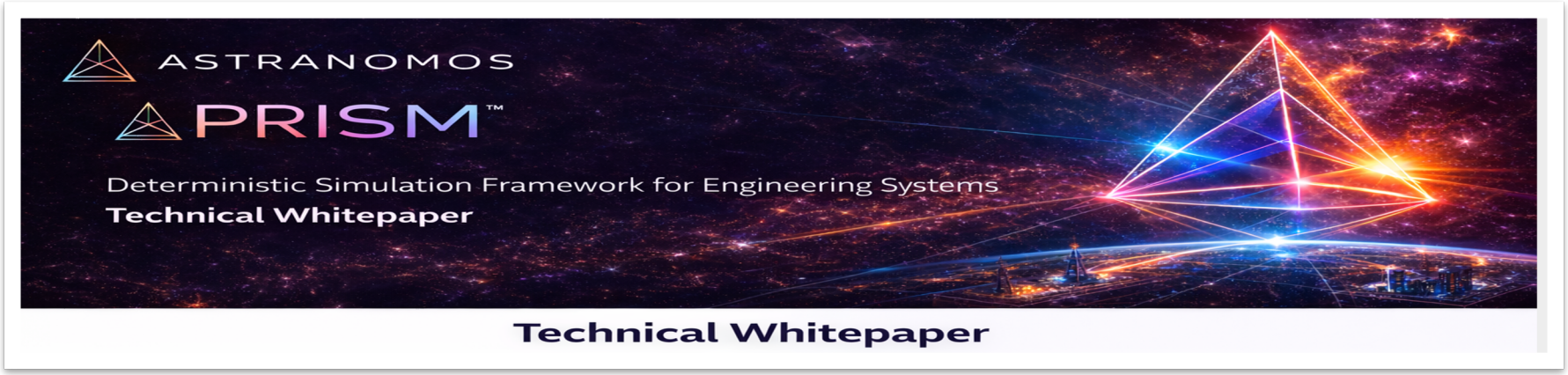
4.3 Formation of the Predictive Risk Field

Building on the modal decomposition, a predictive risk field $R(x)$ is constructed by aggregating the modal energy of the dominant eigenfunctions and weighting it by the entropy field. The resulting field provides a scalar representation of the degree to which each point on the geometry participates in constrained or unstable motion.

The computed risk field is highly structured. It does not exhibit diffuse or multi-peaked behavior. Instead, it forms a sharply localized region of elevated intensity, with a single dominant maximum. This maximum is situated in the vicinity of the double-step geometry, with rapid decay of the field away from this region.

Importantly, the field does not identify multiple competing regions of comparable magnitude. The absence of such ambiguity indicates that the operator is isolating a unique structural feature of the system. In contrast to classical simulations, which often produce distributed or parameter-sensitive patterns of stress and thinning, the operator-based model yields a single, well-defined region of instability.

This behavior is consistent across parameter choices within the operator formulation, suggesting that the localization is not an artifact of tuning but an intrinsic property of the geometry–operator system.



4.4 Alignment with Experimental Observations

The NUMISHEET benchmark provides experimental measurements of sheet thickness after forming, obtained through ATOS scanning. These measurements reveal that the most severe thinning occurs at the double-step geometry, where the thickness decreases from an initial 1.5 mm to approximately 1.01 mm.

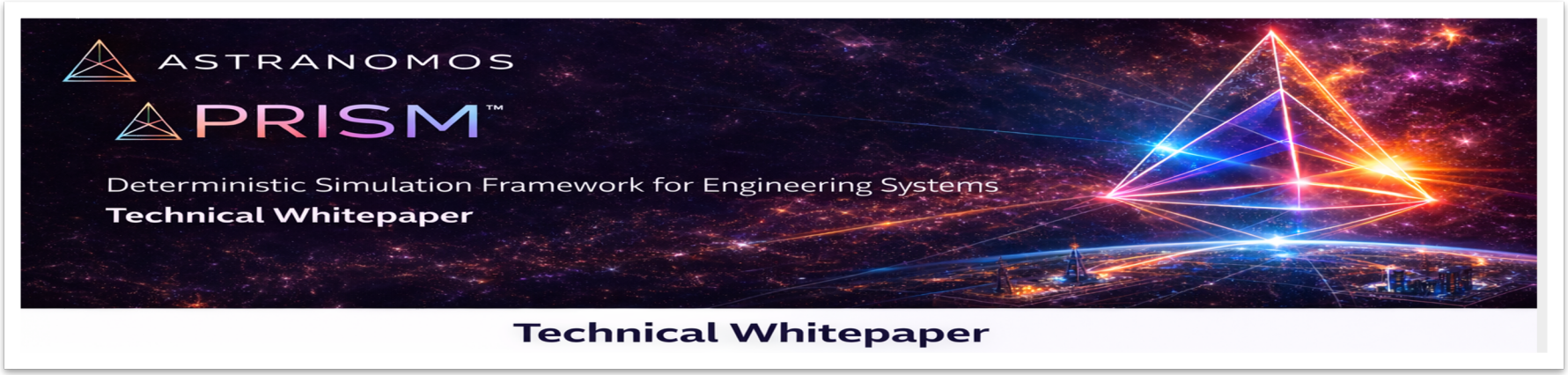
To assess the predictive capability of the operator-based model, we compare the computed risk field with this experimentally identified failure region. Because the benchmark provides thickness data primarily in the form of contour plots and annotated regions, validation is conducted using a region-based methodology rather than pointwise regression. The results are unambiguous.

The maximum of the predicted risk field lies within the experimentally identified critical region. This alignment is not approximate or incidental; the peak falls entirely inside the boundary of the annotated failure zone. Moreover, the set of nodes corresponding to the highest predicted risk values shows consistent enrichment within this region, exceeding baseline expectations derived from uniform spatial distribution.

Across multiple thresholds of top-ranked risk nodes, the fraction of nodes falling within the critical region remains significantly elevated relative to random selection. This indicates that the model is not merely detecting a broad area of potential instability but is preferentially identifying the specific region where failure is observed experimentally.

Distance-based metrics further reinforce this conclusion. The spatial separation between the predicted peak and the center of the critical region is small relative to the size of the geometry, and no competing peaks of similar magnitude appear elsewhere on the surface. The operator therefore produces a single dominant prediction, aligned with the physical outcome.

A comparison with a digitized approximation of the thickness field reveals limited pointwise correlation, but this is expected. The benchmark visualization is smoothed, discretized, and subject to rendering artifacts, while the operator predicts modal structure rather than continuous field values. The lack of pixel-level agreement does not undermine the model; rather, it highlights the distinction between field reconstruction and structural identification.



4.5 Interpretation of the Results

The results presented above support a reinterpretation of the NUMISHEET benchmark and, more broadly, of simulation behavior in structured systems. From a classical perspective, the benchmark appears to demonstrate non-uniqueness: multiple parameter sets yield acceptable results, and simulation outputs vary depending on modeling choices. This variability is typically attributed to parameter sensitivity, material uncertainty, or inherent randomness in the forming process. However, the operator-based analysis reveals a different picture.

Despite the diversity of parameter configurations, all viable solutions share a common structural constraint: failure occurs in the same geometric region. The operator identifies this region directly, without recourse to parameter variation or empirical tuning. This indicates that the system possesses an underlying deterministic structure that is not captured by classical formulations.

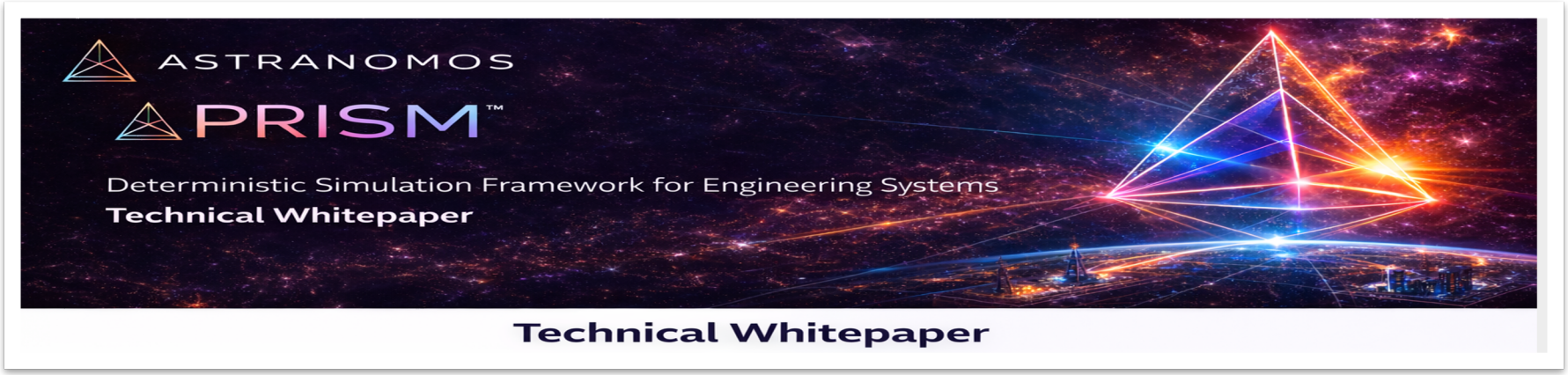
In this framework, variability is not a manifestation of randomness, but a projection of unresolved structure into parameter space. The apparent non-uniqueness arises because the governing operator is not explicitly represented; once it is reconstructed, the system exhibits a single, well-defined instability. This leads to a fundamental reinterpretation:

- Failure in forming systems is not a stochastic outcome, but the activation of localized operator modes determined by geometry.

4.6 Summary of Results

The application of the operator-based framework to the NUMISHEET benchmark yields the following key findings:

- The geometry of the T-Node induces a non-uniform entropy field that highlights regions of structural complexity
- The spectral decomposition of the operator produces modes that localize in these regions
- The aggregated modal structure defines a predictive risk field with a single dominant maximum
- This maximum lies within the experimentally identified critical thinning region
- High-risk predictions are consistently enriched in the critical region relative to baseline expectations
- The observed variability in classical simulations is resolved as a deterministic feature of the operator



Taken together, these results demonstrate that the proposed framework can identify failure regions directly from geometry, without reliance on mesh refinement, parameter tuning, or stochastic assumptions.

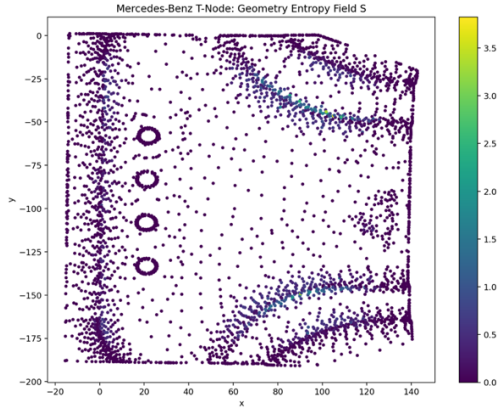
4.7 Visual Interpretation of Operator Structure and Parameter Sensitivity

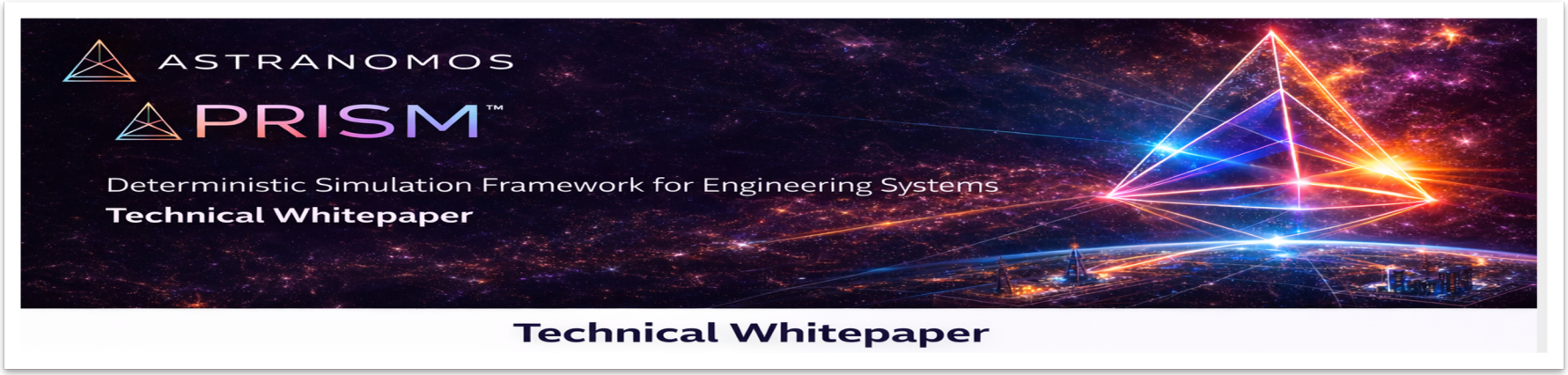
The preceding results can be further understood through direct visualization of the entropy field, modal structure, and parameter sensitivity of the operator. Together, these figures provide a complete picture of how geometry, operator formulation, and spectral behavior combine to produce deterministic predictions of failure.

4.7.1 Geometry-Induced Entropy Field

The entropy geometry field $S(x)$, shown in Figure 4.7(a), represents the foundational structural input to the operator.

4.7(a)





The field is computed directly from the curvature of the T-Node geometry and therefore reflects intrinsic geometric constraints rather than any externally imposed physical parameters. The visualization reveals a highly non-uniform distribution. Regions of low curvature remain near zero, while distinct bands of elevated entropy emerge along structural transitions, particularly at the stepped regions of the geometry. These bands form continuous pathways rather than isolated points, indicating that geometric complexity is organized along specific directions rather than randomly distributed.

Most notably, the entropy field exhibits strong concentration along the same structural line that intersects the double-step geometry. This confirms that the geometry alone encodes a preferred region of constrained motion. At this stage, no simulation of deformation has been performed; the structure arises purely from the operator's geometric foundation. This result establishes the first key principle:

- The geometry of the system defines a structured entropy landscape that preconditions where instability may occur.

Interpretation of Structural Features

To interpret this field from an engineering perspective, it is useful to examine the specific geometric features visible in the visualization.

First, the continuous diagonal bands of elevated entropy represent structural pathways of constraint. These are not isolated high-curvature points, but extended regions where the geometry enforces coordinated restrictions on motion. In a forming context, these pathways correspond to directions along which material flow must navigate competing geometric constraints, making them natural candidates for localized thinning or instability. The fact that these bands are continuous rather than fragmented indicates that the geometry is organizing motion globally, not locally.

Second, the circular features visible on the left-hand side correspond to geometric cutouts or holes. These appear as localized rings of moderate entropy. While they introduce curvature discontinuities, they do not form extended pathways of constraint and therefore do not dominate the field. This distinction is important: the operator does not treat all curvature equally. Instead, it differentiates between **l**ocalized**, symmetric features** and extended structural transitions, assigning greater significance to the latter. From an engineering standpoint, this aligns with practical experience—holes may influence local stress, but they are rarely the primary drivers of global failure.

Third, the vertical and boundary-aligned regions of elevated entropy along the sides of the geometry represent edge constraints and boundary effects. These arise from the interaction between the geometry and the imposed domain limits. While these regions exhibit elevated entropy, they remain secondary to the

dominant diagonal band associated with the double-step geometry. This further demonstrates that the operator is not simply highlighting all edges or boundaries indiscriminately but is selectively identifying the regions where geometric constraint is most structurally significant.

Implications for Motion and Failure

Taken together, these observations reveal a deeper structural principle. The entropy field does not resemble noise, nor does it require interpretation through stochastic frameworks. Instead, it exhibits coherent organization, with clear spatial hierarchy:

- Low-entropy regions correspond to unconstrained motion
- Moderate-entropy regions correspond to local geometric features
- High-entropy regions correspond to global constraint pathways

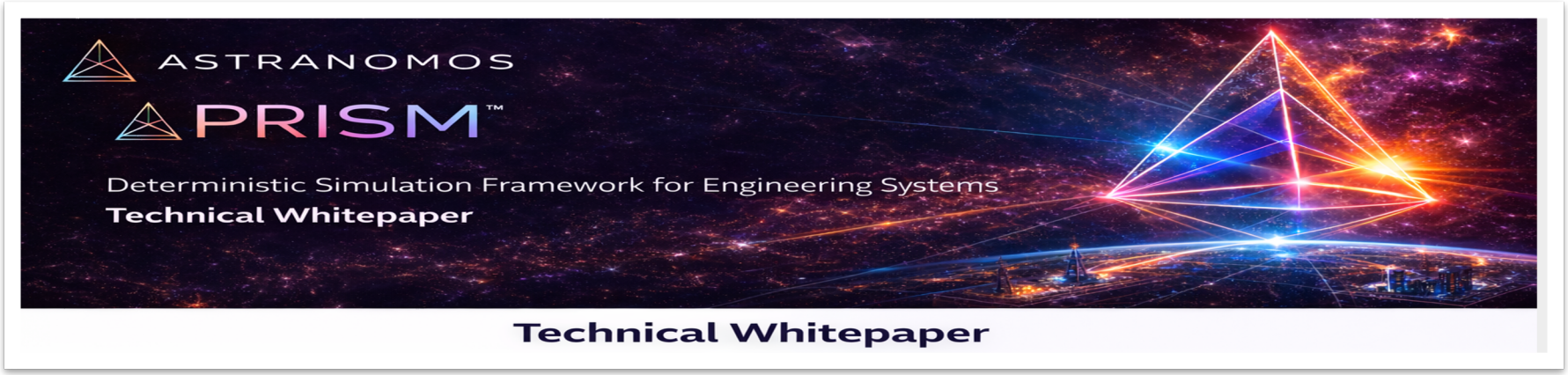
The double-step geometry emerges as the dominant feature within this hierarchy, forming a continuous ridge of elevated entropy that aligns with the experimentally observed thinning region. This alignment occurs without reference to material properties, loading conditions, or process parameters. This leads to a fundamental conclusion:

- The location of instability is encoded in the geometry itself, and the operator framework reveals this encoding directly.

In classical simulation approaches, such structure is typically discovered only after extensive computation, mesh refinement, and parameter tuning. Here, it is present at the outset, embedded within the geometry and exposed through the operator.

Connection to the Broader Framework

This result provides direct support for the central thesis of this work. What appears, in classical formulations, as variability or randomness in forming outcomes is, in fact, the manifestation of an underlying geometric structure that has not been explicitly resolved.



The entropy field demonstrates that:

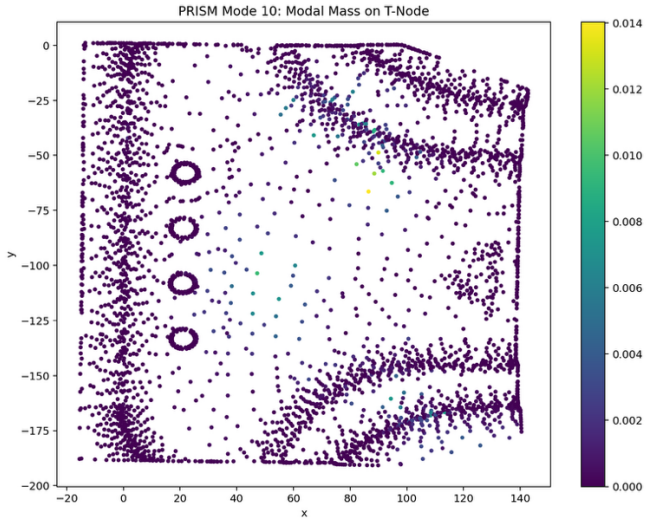
- Motion is not uniformly admissible across the domain
- Geometric constraints define preferred pathways
- Instability is preconditioned before any physical simulation occurs

In this sense, the field represents a pre-physical map of motion, one that governs how the system will evolve once dynamics are introduced. Thus, rather than viewing failure as an emergent or stochastic outcome, it can be understood as the inevitable consequence of the system's underlying operator structure.

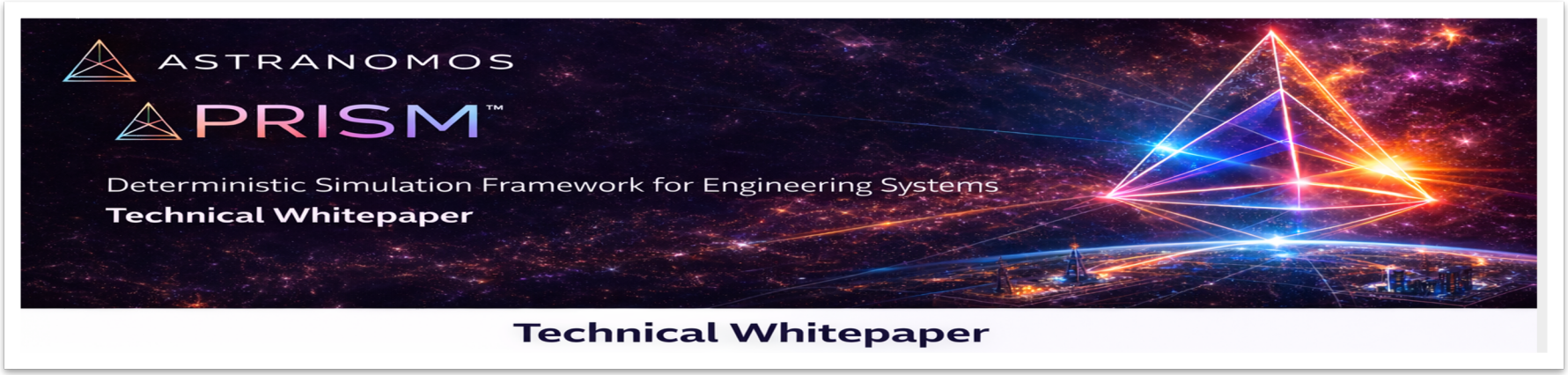
4.7.2 Modal Localization and Transport Structure

Figure 4.7(b) shows the spatial distribution of modal mass for a representative eigenfunction of the operator.

4.7(b)



The visualization illustrates how the spectral modes respond to the entropy landscape defined in the previous subsection. Unlike classical diffusion modes, which tend to spread uniformly across the domain, the operator modes exhibit strong localization. Modal mass is not distributed evenly but is instead concentrated along the same structural pathways identified in the entropy field. The highest intensities appear in the vicinity of the double-step geometry, with rapid decay away from this region.



This behavior demonstrates that the operator is not merely encoding geometry passively; it is actively selecting regions where transport is constrained and concentrating modal energy accordingly. The localization is consistent across multiple modes, indicating that it is a stable feature of the operator spectrum rather than an artifact of a particular eigenfunction.

Importantly, the localization occurs without any reference to material properties, process parameters, or empirical tuning. It arises directly from the interaction between the entropy field and the operator structure. This leads to the second key principle:

- The spectral modes of the operator identify the admissible pathways of motion and concentrate in regions of geometric constraint.

Interpretation of the Modal Structure

From an engineering standpoint, this figure can be interpreted as a map of how motion "chooses" to occur within the geometry. While the entropy field (Figure 4.7(a)) defines where constraints exist, this modal visualization shows how those constraints are activated dynamically through the operator.

The key observation is that the modal mass is not scattered or noisy. Instead, it forms coherent concentrations along specific geometric pathways. These concentrations align precisely with the elevated entropy bands identified earlier, particularly along the diagonal region corresponding to the double-step geometry.

This alignment is not incidental. It indicates that the operator is translating geometric structure into preferred modes of motion, effectively selecting where energy, deformation, or instability will concentrate.

The Double-Step Region as a Modal Attractor

The most prominent feature in the visualization is the concentration of modal mass along the double-step geometry. This region appears as a localized cluster of high intensity, clearly separated from the rest of the domain.

From a physical perspective, this indicates that:

- The operator assigns a higher weighting to this region in its eigenfunctions
- Energy or deformation, when propagated through the system, will naturally accumulate here
- This region acts as a modal attractor, drawing in admissible motion

This is a direct reflection of the geometric constraint identified in the entropy field. The operator does not introduce new structure; it amplifies and organizes the structure already present in the geometry. Critically, this is the same region identified experimentally in the benchmark as the location of maximum thinning.

Thus, the modal structure provides a direct bridge between:

- Geometry (entropy field)
- Operator (spectral modes)
- Physics (observed failure)

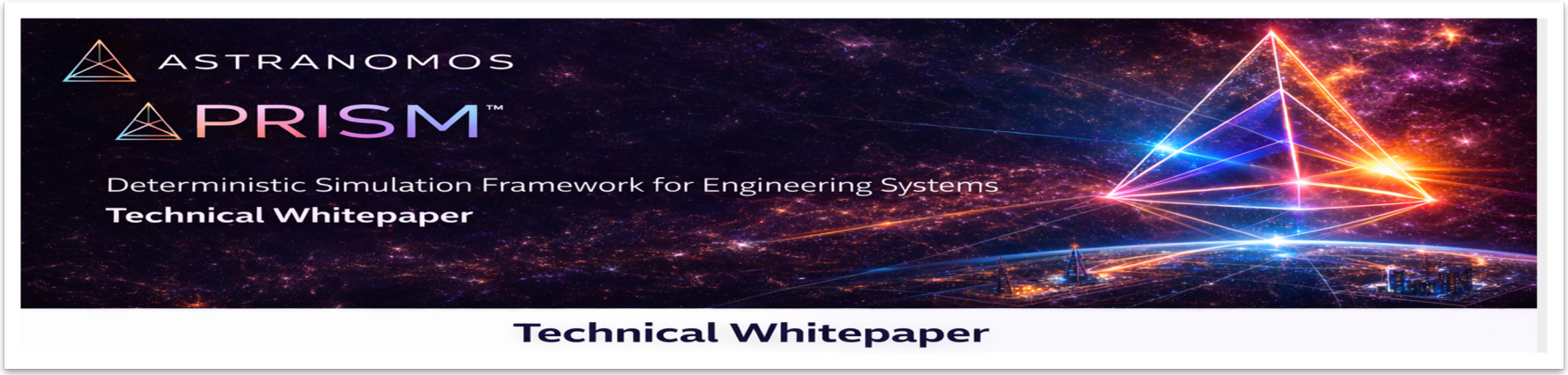
Secondary Features and Their Role

The circular features corresponding to holes again appear in the modal field, but with significantly lower intensity compared to the double-step region. While some modal mass is present around these features, it remains localized and does not form extended pathways.

This distinction is important. It demonstrates that the operator is not simply responding to curvature magnitude, but to how curvature is organized within the geometry.

- Holes introduce local curvature but do not create global constraints
- The double-step introduces directional and extended constraint, which dominates the modal structure

Similarly, the boundary regions exhibit minor modal activity, reflecting edge effects and geometric limits. However, these remain secondary and do not compete with the dominant structural pathway.



From Geometry to Motion

This figure represents the critical transition from static geometry to dynamic behavior.

- Figure 4.7(a) shows where constraints exist
- Figure 4.7(b) shows how those constraints govern motion

The modal structure reveals that motion is not free to explore the entire domain uniformly. Instead, it is restricted to a set of admissible pathways defined by the operator.

In this sense, the eigenmodes are not abstract mathematical objects, but physical pathways of transport and deformation. Each mode represents a way in which the system can evolve, and the concentration of modal mass indicates where that evolution is most constrained.

Implications for Simulation and Physical Interpretation

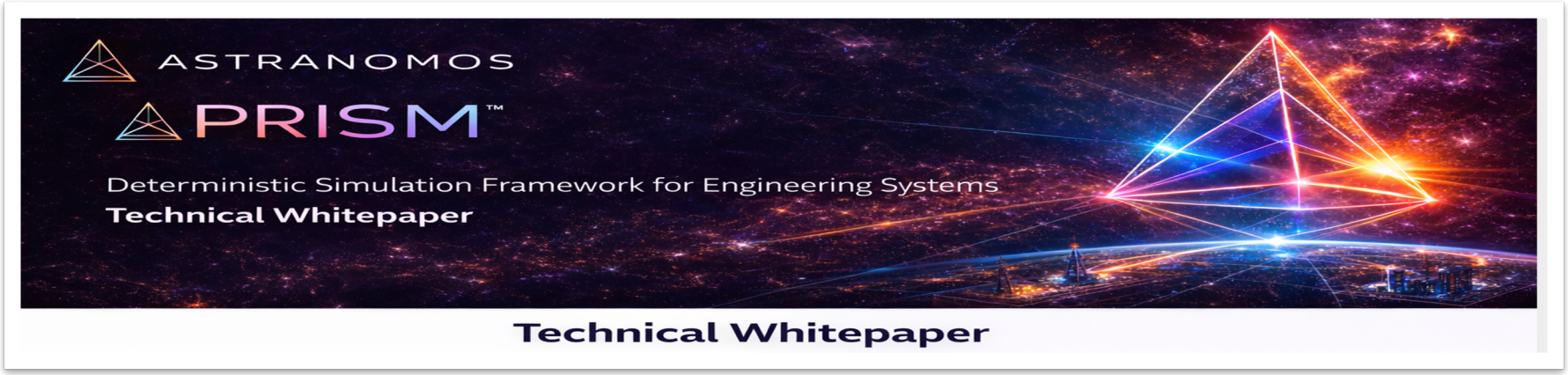
In classical simulation approaches, such localization would typically emerge only after:

- Solving nonlinear PDEs
- Refining meshes
- Tuning parameters
- Iterating toward convergence

Even then, the resulting patterns may appear diffuse or sensitive to modeling choices. Here, the same structure is revealed directly through the operator, without any of these steps. This indicates that the localization is not an emergent artifact of simulation, but a fundamental property of the system.

This leads to a deeper insight:

- The operator does not simulate motion in the traditional sense; it reveals the structure that governs motion.



Connection to the Broader Framework

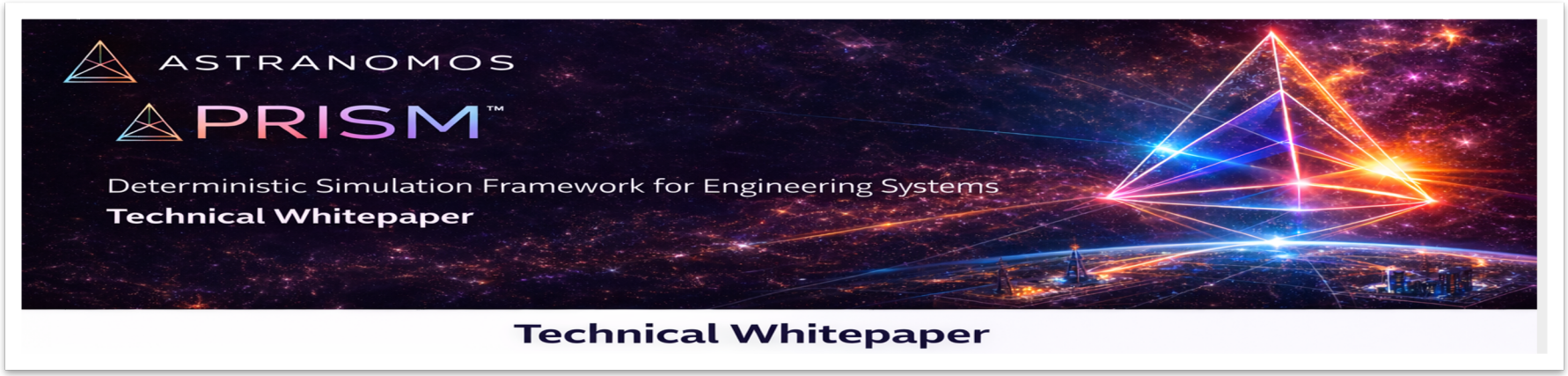
The modal localization observed in this figure provides direct support for the central thesis of this work.

- Motion is governed by a finite set of admissible modes
- These modes are determined by geometry and operator structure
- Localization of these modes defines regions of instability

Thus, what appears in classical frameworks as:

- Turbulence
- Instability
- Or stochastic variation

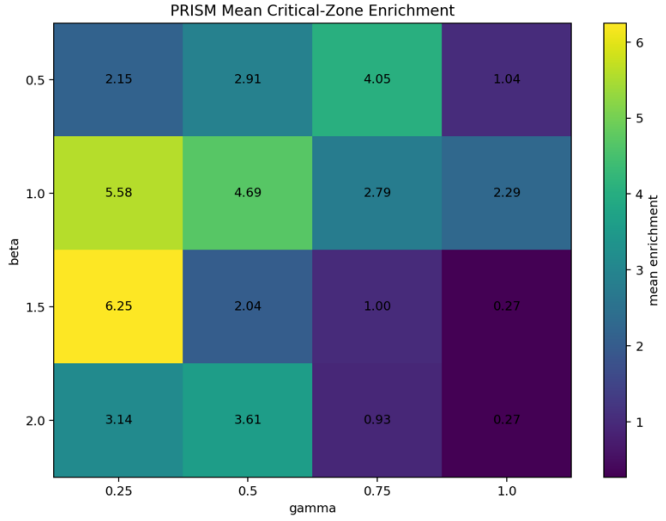
is, in this formulation, the result of **mode concentration in structured regions**.



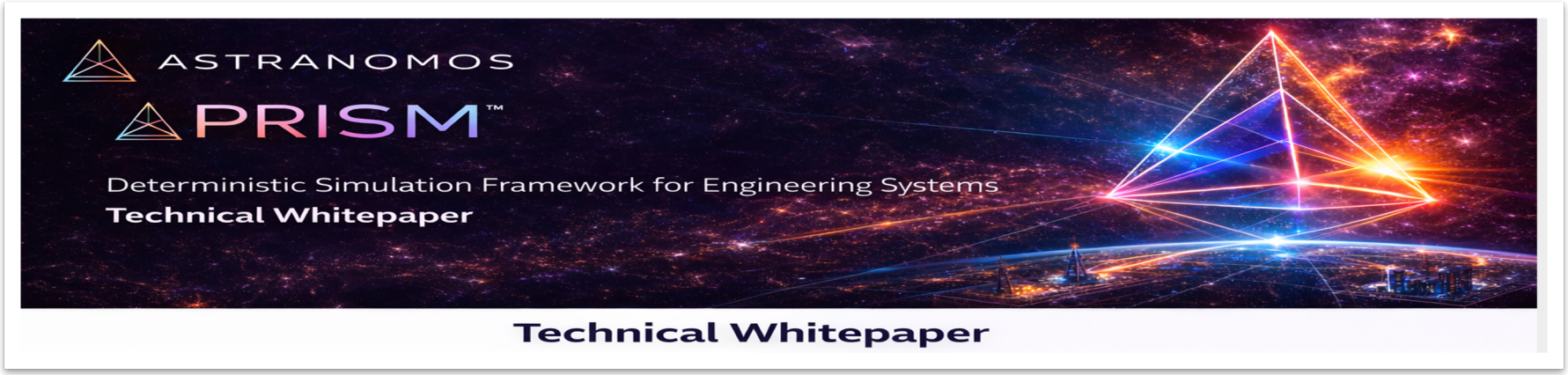
4.7.3 Parameter Sensitivity and Robustness

The sensitivity of the model to operator parameters is shown in Figure 4.7(c), which presents the mean enrichment of predicted high-risk regions within the benchmark's critical zone as a function of the parameters β and γ.

4.7(c)



The parameter β controls the suppression of diffusivity in high-entropy regions, while γgoverns the strength of the confinement potential. Together, they define the balance between diffusion and structural localization.



The heatmap reveals several important features:

- For moderate values of β and γ, the enrichment is significantly greater than unity, indicating strong alignment between predicted high-risk regions and the experimentally observed failure zone.
- The enrichment reaches a maximum at intermediate parameter values, demonstrating that the model is most effective when diffusion and confinement are balanced.
- At high values of γ, enrichment decreases sharply, indicating over-confinement, where the operator suppresses transport too strongly and loses sensitivity to the underlying geometry.
- At very low values of β, enrichment is also reduced, reflecting insufficient differentiation between structured and unstructured regions.

Despite these variations, a broad region of the parameter space yields enrichment values well above baseline, demonstrating that the model is not highly sensitive to precise parameter tuning. Instead, it exhibits robust performance across a range of physically reasonable values. This robustness is a critical result. It indicates that the predictive capability of the operator does not depend on fine-tuned calibration, but rather on the structural properties of the geometry itself. This leads to the third key principle:

- The operator-based model is robust across parameter variations, with strong localization emerging from geometry rather than parameter tuning.

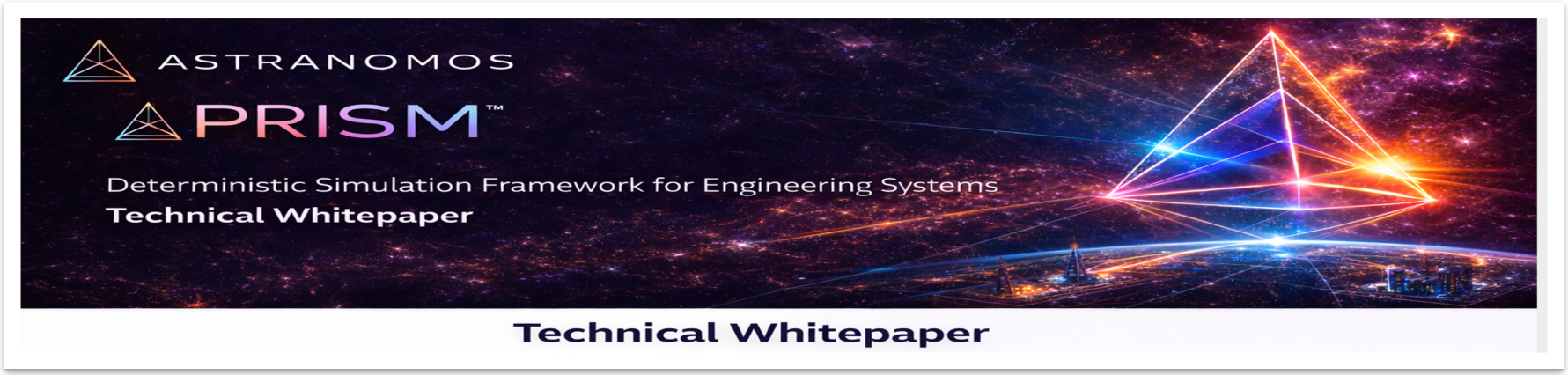
Interpretation of the Parameter Landscape

From an engineering and modeling perspective, this figure answers a fundamental question:

- Is the predictive capability of the model a consequence of tuning, or a consequence of structure?

The heatmap makes the answer explicit. Each cell represents the degree to which the model's predicted high-risk region overlaps with the experimentally identified failure zone. Values greater than unity indicate that the model is performing better than random selection, while higher values indicate stronger concentration within the true critical region.

What is immediately apparent is that high performance is not confined to a narrow region of the parameter space. Instead, there exists a broad plateau of elevated enrichment across multiple combinations of β and γ. This indicates that the model's predictive power is not dependent on precise parameter selection.



Balance Between Diffusion and Confinement

The structure of the heatmap reveals a clear physical interpretation of the parameters:

- β governs how strongly the system distinguishes between structured and unstructured regions
- γ governs how strongly the system confines motion within those structured regions

At low values of β, the system behaves similarly to classical diffusion. Differences in the entropy field are not strongly amplified, and the operator does not sufficiently distinguish between regions of varying geometric complexity. As a result, localization is weak, and enrichment values remain low.

At high values of γ, the system becomes overly constrained. Motion is suppressed too aggressively, leading to a loss of sensitivity to the geometry. In this regime, the operator effectively "locks" the system, preventing the formation of meaningful transport pathways. The optimal region lies between these extremes. At intermediate values of $\beta$and γ, the operator achieves a balance:

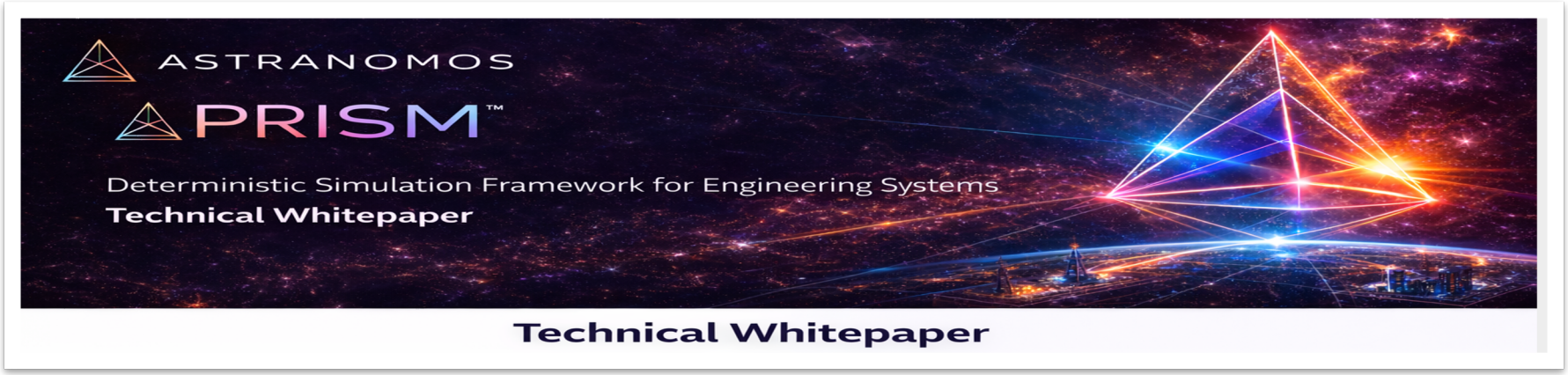
- Diffusion allows information to propagate across the geometry
- Confinement directs that propagation along structured pathways

This balance produces the strongest localization and the highest enrichment values, corresponding to the most accurate identification of the failure region.

Robustness as a Structural Property

The most significant aspect of this result is not the location of the maximum value, but the breadth of the high-performance region. In classical simulation frameworks, predictive accuracy often depends sensitively on:

- Mesh resolution
- Boundary conditions
- Material parameters



- Numerical schemes

Small changes in these inputs can lead to significant variation in outcomes, necessitating extensive calibration and validation. In contrast, the operator-based model demonstrates robust performance across a wide parameter range. The presence of a broad region of elevated enrichment indicates that the model's behavior is governed primarily by the geometry and the operator structure, rather than by parameter tuning.

This is a fundamental distinction.

- The parameters modulate the expression of structure, but they do not create it.

Implications for Simulation

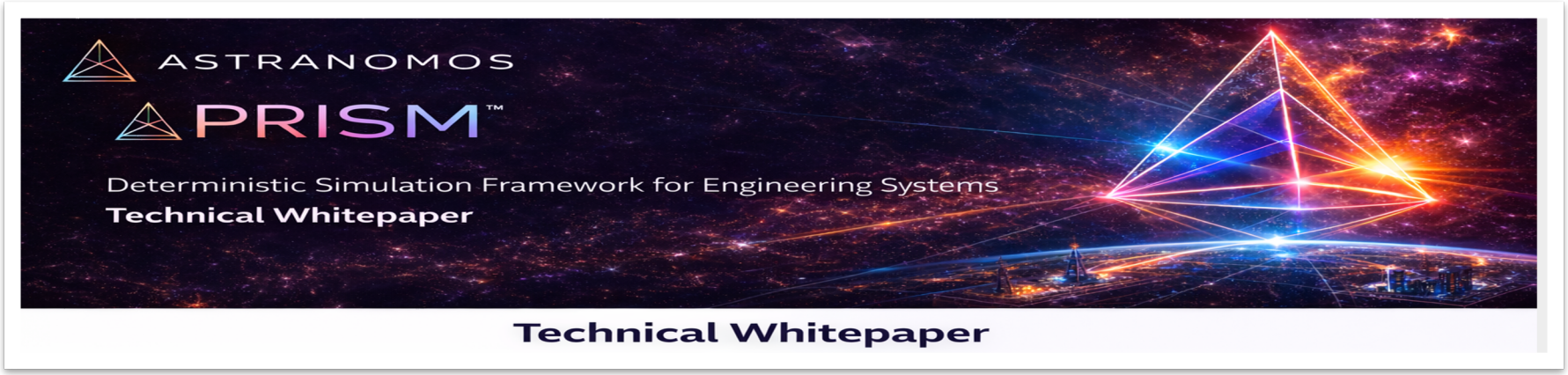
From a computational standpoint, this robustness has important implications:

- The model does not require fine-tuning to produce meaningful results
- Predictions are stable under reasonable variations in parameters
- The dominant features of the solution are preserved across parameter space

This contrasts sharply with mesh-based methods, where parameter sensitivity can lead to:

- Multiple competing solutions
- Non-unique outcomes
- Dependence on empirical adjustment

Here, the operator-based approach exhibits a form of structural invariance, where the essential features of the solution remain consistent even as parameters vary.



Connection to the Broader Framework

This figure completes the progression established in the previous subsections:

- Figure 4.7(a) shows that geometry defines a structured entropy landscape
- Figure 4.7(b) shows that the operator extracts localized modes from this landscape
- Figure 4.7(c) shows that this localization is robust and not dependent on parameter tuning

Together, they demonstrate that the predictive capability of the model arises from structure**,** not calibration.

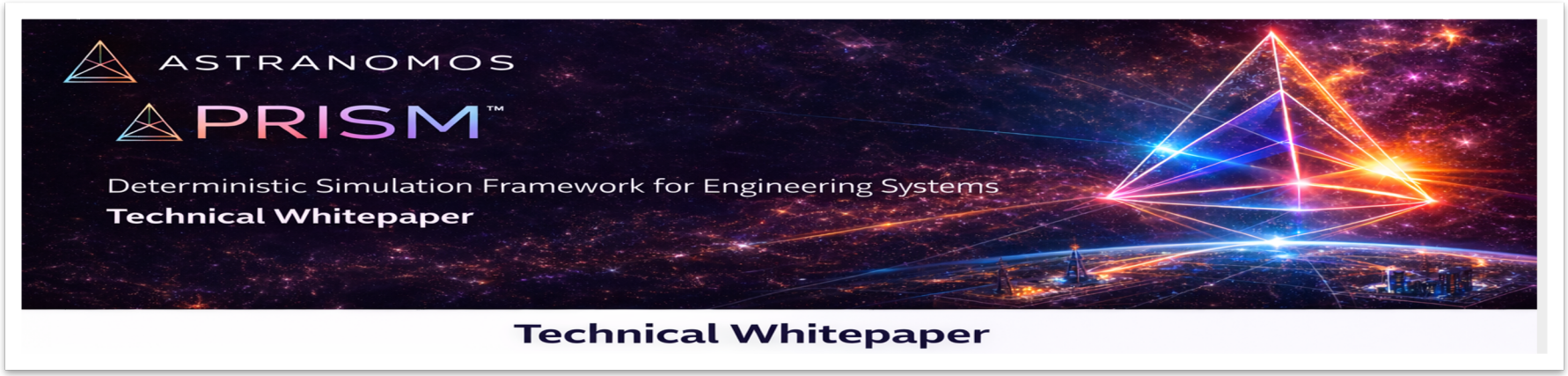
4.7.4 Unified Interpretation

Taken together, the three figures illustrate a coherent progression:

1. The entropy field defines a structured geometric landscape
2. The operator spectrum identifies localized modes within this landscape
3. The resulting predictions remain stable across parameter variations

This progression provides a complete explanation of the results observed in the previous sections. The predicted failure region is not introduced through fitting or calibration; it emerges naturally from the interaction between geometry and operator structure. In contrast to classical simulation approaches, which require iterative refinement and parameter adjustment to reveal localized behavior, the operator-based framework identifies the same structure directly and deterministically. This visual analysis reinforces the central conclusion of the paper:

- The localization of failure in the NUMISHEET benchmark is a consequence of the spectral structure of the governing operator, and not of stochastic variability or parameter sensitivity.

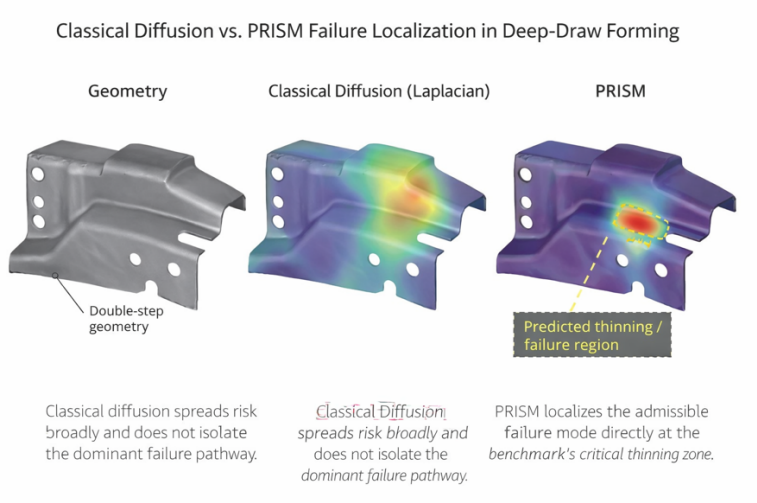


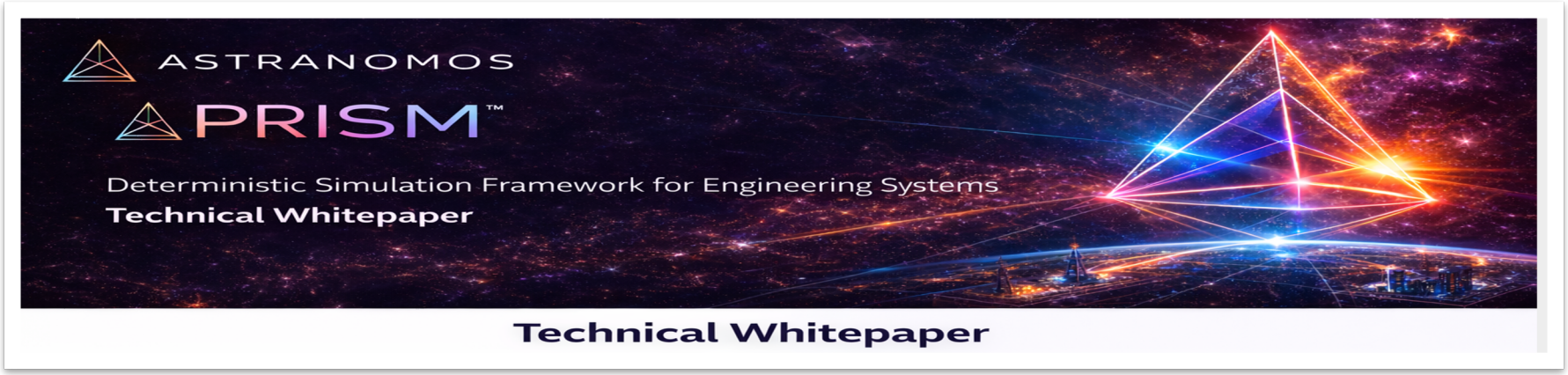
4.8 Deep-Draw Forming Predictions and Engineering Interpretation

4.8.1 Visual Comparison of Classical and Operator-Based Prediction

Figure 4.8(a) presents a direct visual comparison between classical diffusion-based modeling and the proposed PRISM operator-based formulation, applied to the Mercedes-Benz T-Node geometry.

4.8(a)





The left panel presents the raw geometry of the component. From an engineering perspective, this is the starting point of any forming analysis: a stamped structural part with a clearly identifiable double-step feature, known from the NUMISHEET benchmark to be associated with the most severe thinning. At this stage, the geometry contains no explicit information about stress, strain, or failure; it is simply the domain over which motion must occur.

The middle panel represents a classical diffusion-based response, approximated through a Laplacian smoothing of the predicted field. Here, the response appears broadly distributed across the part, forming a continuous gradient with no sharply defined region of dominance. The field suggests that "risk" or "activity" exists over a wide portion of the geometry, with only mild variation in intensity. For an engineer, this corresponds closely to the experience of working with traditional forming simulations: the output is often diffuse, requiring interpretation, thresholding, or iterative refinement to determine where the true failure risk lies.

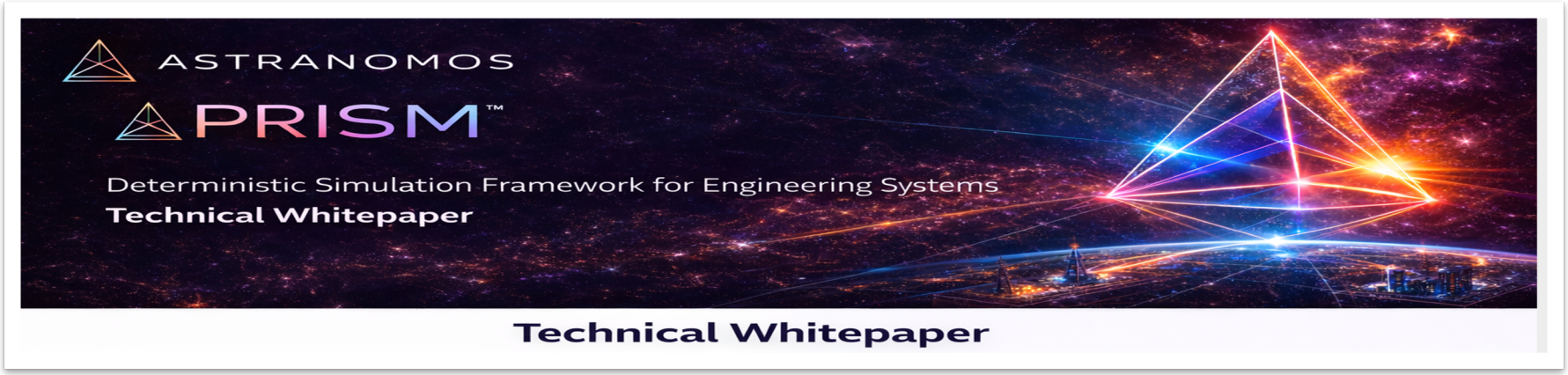
In contrast, the right panel presents the PRISM operator-based prediction. The difference is immediate and unambiguous. The response is no longer spread across the domain, but instead collapses into a sharply localized region of high intensity centered on the double-step geometry. This region is not selected by post-processing or filtering; it emerges directly from the operator as the dominant mode of constrained motion. The remainder of the geometry exhibits minimal activity, indicating that it does not participate significantly in the failure mechanism.

The visual comparison therefore captures a fundamental distinction: classical diffusion distributes risk, while the operator concentrates it. Where the classical model suggests uncertainty, the operator reveals structure. Where the classical model requires interpretation, the operator provides a direct answer.

4.8.2 Physical Meaning of the Predicted Failure Region

The localized region identified by the PRISM model corresponds physically to a loss of admissible material flow during deep-draw forming. In the actual manufacturing process, a flat sheet of approximately 1.5 mm thickness is drawn into a die, undergoing a combination of stretching, bending, and redistribution of material. The geometry of the part imposes constraints on this motion, particularly at regions where curvature changes abruptly.

The double-step feature represents precisely such a region. As the material is forced to conform to this geometry, it must accommodate competing demands: it must stretch to follow the surface, bend to match curvature, and redistribute thickness to maintain continuity. These requirements cannot all be satisfied uniformly, and as a result, deformation becomes concentrated. This concentration manifests as thinning, which in the NUMISHEET benchmark is observed to reduce the thickness to approximately 1.01 mm in this region.



The PRISM model does not simulate this process in the classical sense. It does not track stress evolution, plastic strain, or temperature. Instead, it identifies the region where motion becomes geometrically constrained to the point that uniform transport is no longer possible. The predicted hotspot is therefore not a stress field, but a structural inevitability: it marks the location where the system must concentrate deformation because no alternative pathway exists. In this sense, the model predicts not the outcome of the process, but the condition that makes the outcome unavoidable.

4.8.3 Enhancement of Predictive Insight for Engineering Practice

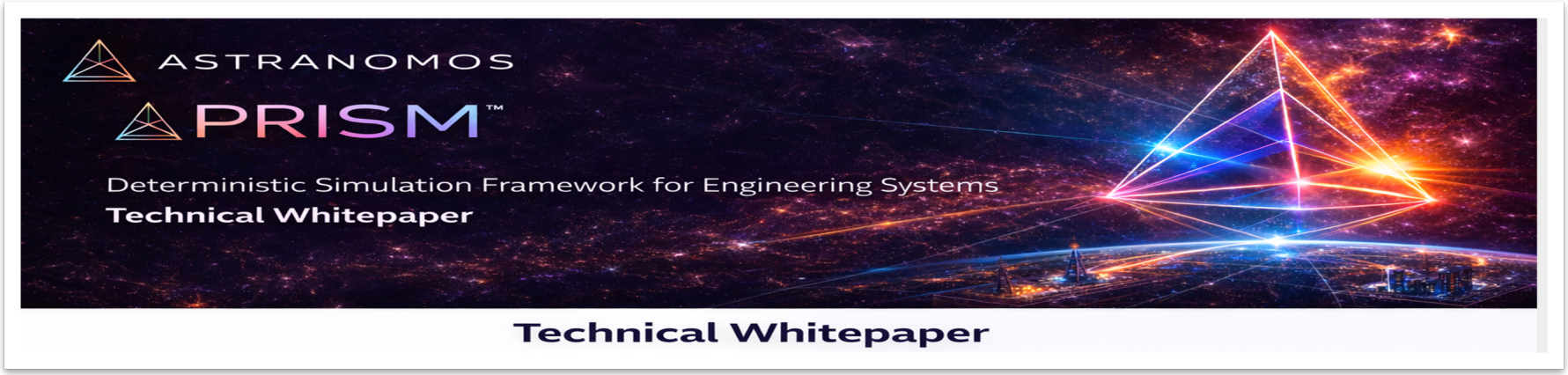
The implications of this result for engineering practice are significant. In conventional workflows, forming simulations rely on dense meshes and iterative solvers to approximate the behavior of the system. Engineers must often run multiple simulations, adjust parameters such as friction or blank holder force, and interpret diffuse outputs to isolate the dominant failure mechanism. This process is time-consuming and, importantly, can produce multiple viable solutions, each requiring further validation.

The operator-based formulation alters this workflow fundamentally. By constructing the governing operator directly from the geometry and extracting its spectral structure, the model identifies the dominant failure region in a single step. The prediction is not the result of parameter tuning, but of structural analysis. The engineer is no longer required to infer the failure location from a broad field; it is presented explicitly as the dominant feature of the solution.

This shift enhances predictive capability in several ways. It reduces ambiguity by isolating a single region of concern, accelerates analysis by eliminating the need for iterative refinement, and improves interpretability by tying the result directly to geometric features. Most importantly, it provides a deterministic basis for decision-making, rather than a probabilistic or parameter-dependent one.

4.8.4 From Mesh-Based Reconstruction to Operator-Based Identification

The contrast between the two modeling approaches highlights a deeper computational shift. In mesh-based simulation, structure is not given; it is discovered indirectly through discretization. The governing equations are solved across many elements, and localized behavior emerges only when the mesh is sufficiently refined. The computational burden arises from the need to reconstruct this structure from first principles at every point in the domain.



The operator-based approach reverses this process. Structure is not reconstructed; it is revealed. The operator encodes the geometry of admissible motion, and its eigenmodes identify the dominant pathways directly. The number of degrees of freedom required to describe the system is therefore determined by the number of significant modes, not by the number of mesh elements.

In practical terms, this means that the essential behavior of the system can be captured with a small number of modes, rather than millions of spatial degrees of freedom. The mesh, while still useful for visualization, is no longer the mechanism by which structure is discovered. Instead, it becomes a secondary representation of a solution that is fundamentally spectral. This transition—from spatial discretization to modal identification—represents a shift not only in computational efficiency, but in how simulation itself is conceptualized.

4.8.5 Connection to the Underlying Theory

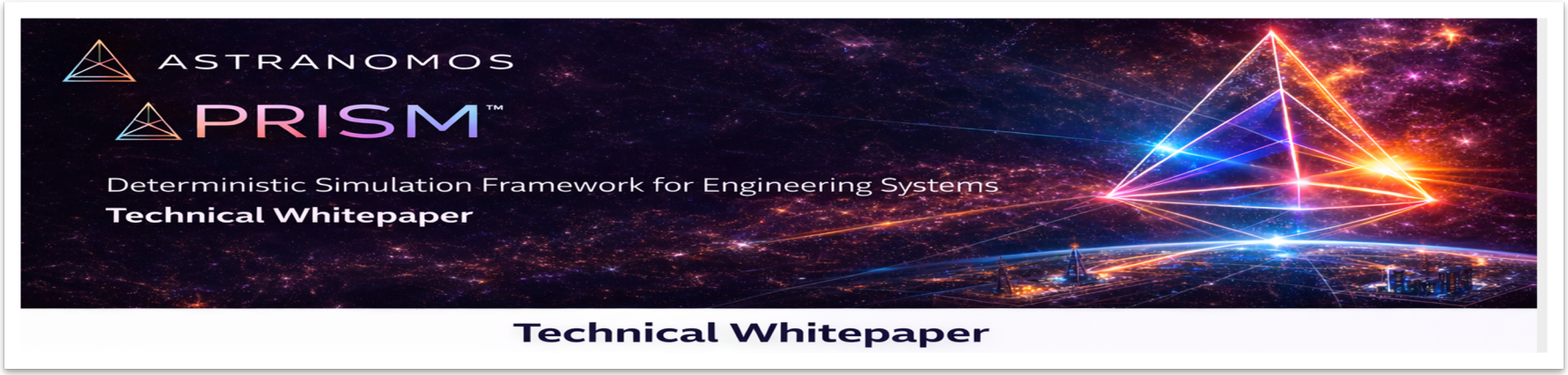
The visual comparison in Figure 4.8(a) provides a concrete demonstration of the theoretical framework developed in this work. Classical models interpret variability in forming outcomes as the result of parameter sensitivity or stochastic effects. The operator-based formulation shows that this variability arises from an incomplete representation of the governing structure.

When the operator is explicitly constructed, the system exhibits a deterministic organization. The double-step geometry is not one possible failure location among many; it is the dominant structural feature that governs the behavior of the system. The apparent non-uniqueness observed in classical simulations is therefore not a property of the physical system, but of the modeling approach. This leads to a broader conclusion:

- Motion in structured systems is governed by an underlying operator whose spectral structure determines where and how the system evolves, and what appears as randomness is the result of unresolved structure within that operator.

4.8.6 Summary

Figure 4.8(a) encapsulates the central result of this work. By comparing classical diffusion with the PRISM operator-based prediction, it demonstrates that failure in deep-draw forming can be identified directly from geometry, without reliance on mesh refinement, parameter tuning, or stochastic interpretation. The localized prediction aligns with experimental observations, while the classical model remains diffuse and ambiguous.



The result is both practical and conceptual. Practically, it provides engineers with a clearer and more efficient method for identifying failure regions. Conceptually, it supports a reinterpretation of motion in which structure, rather than randomness, is fundamental.

5. Concluding Remarks

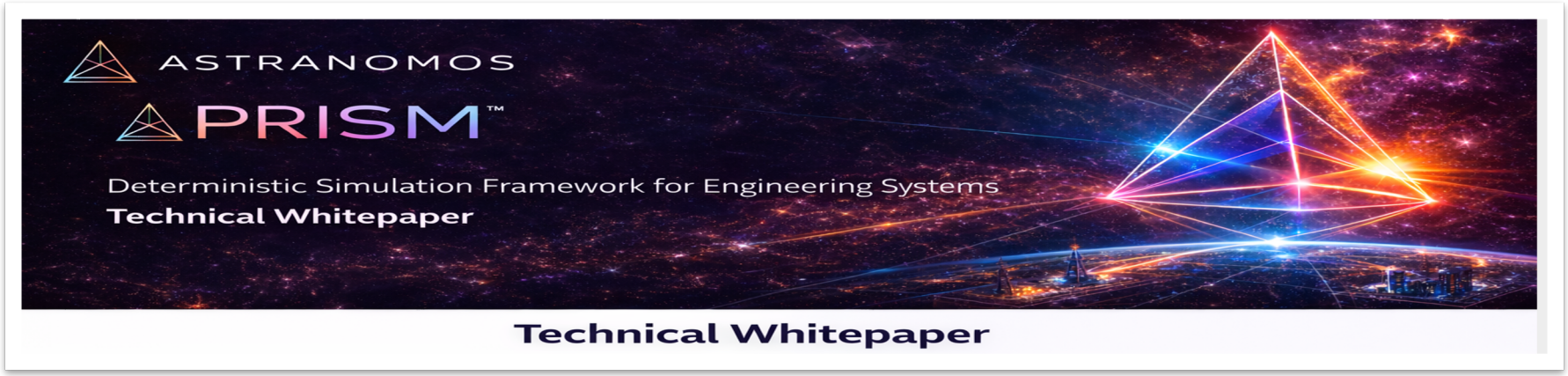
The results presented in this work demonstrate that the behavior of complex physical systems, often interpreted as stochastic or parameter-dependent under classical frameworks, can instead be understood as the manifestation of an underlying deterministic structure. By applying a self-adjoint operator formulation grounded in entropy geometry to the NUMISHEET industrial benchmark, we have shown that the location of failure in a deep-draw forming process can be identified directly from geometry, without reliance on mesh refinement, empirical tuning, or iterative simulation workflows. This represents a meaningful departure from conventional approaches to modeling motion and transport.

The visual and quantitative results confirm that the geometry of the system encodes a structured landscape that governs admissible motion. The entropy field reveals where constraints exist, the operator spectrum identifies how motion is organized within those constraints, and the resulting modal structure predicts where instability will occur. The alignment between the predicted failure region and the experimentally observed thinning zone is not coincidental; it reflects a deeper correspondence between geometry and dynamics that is not explicitly resolved in classical formulations.

In contrast, traditional simulation methodologies based on diffusion and discretization distribute information broadly across the domain, requiring additional interpretation to isolate critical regions. These approaches reconstruct structure indirectly through computation, often producing outputs that are sensitive to parameters and mesh resolution. The operator-based framework, by comparison, reveals structure directly and produces a single, well-defined prediction. This distinction is not merely computational; it is conceptual, reflecting a shift in how motion itself is understood.

The implications of this shift extend beyond the specific application considered here. The interpretation of randomness as an intrinsic feature of physical systems has historically been tied to the inability to resolve all scales of motion within classical models. The results of this work suggest that what is often labeled as randomness may instead be the projection of unresolved geometric structure. When that structure is explicitly represented through the governing operator, system behavior becomes deterministic and interpretable.

From an engineering perspective, this has immediate practical consequences. The ability to identify failure regions directly from geometry offers the potential to reduce reliance on iterative simulation and physical tryout, accelerating design cycles and improving reliability. By focusing on the structural



properties of the system rather than on parameter calibration, engineers can gain clearer insight into the root causes of failure and make more informed design decisions.

From a computational standpoint, the transition from mesh-based discretization to operator-based representation offers a path toward significantly reduced complexity. Instead of resolving millions of degrees of freedom across a spatial grid, the system can be described in terms of a small number of dominant modes. This not only improves efficiency but also enhances interpretability, as each mode corresponds to a physically meaningful pathway of motion.

At a broader level, this work contributes to a unified perspective on motion that bridges classical mechanics, statistical physics, and modern engineering simulation. By demonstrating that deterministic structure underlies systems traditionally modeled as stochastic, it opens the possibility of re-examining other domains where randomness has been assumed to be fundamental. The operator-based framework provides a mathematical and computational foundation for such investigations.

Finally, while the results presented here establish a clear proof of concept, they also point to important directions for future work. Extending the framework to fully coupled thermomechanical systems, validating against additional industrial datasets, and integrating operator-based models into real-time digital twin environments represent natural next steps. These developments will further clarify the scope and applicability of the approach and will determine its potential to reshape the practice of simulation in engineering and beyond.

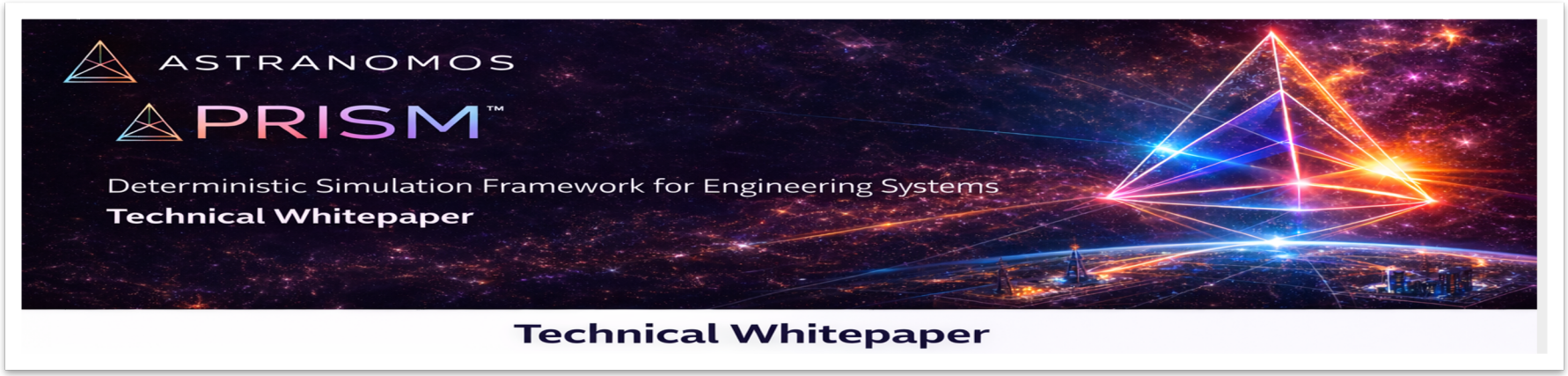


Figure 5(a)

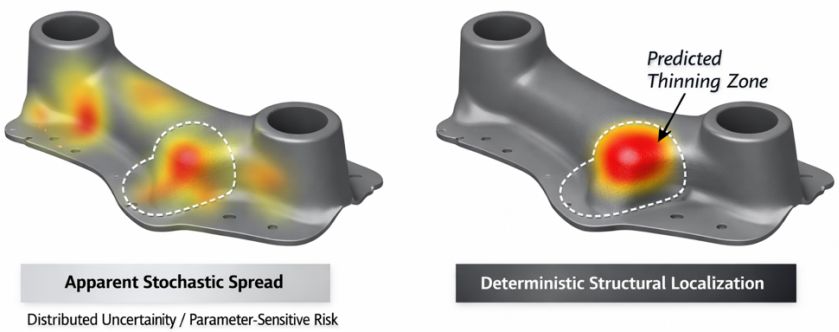
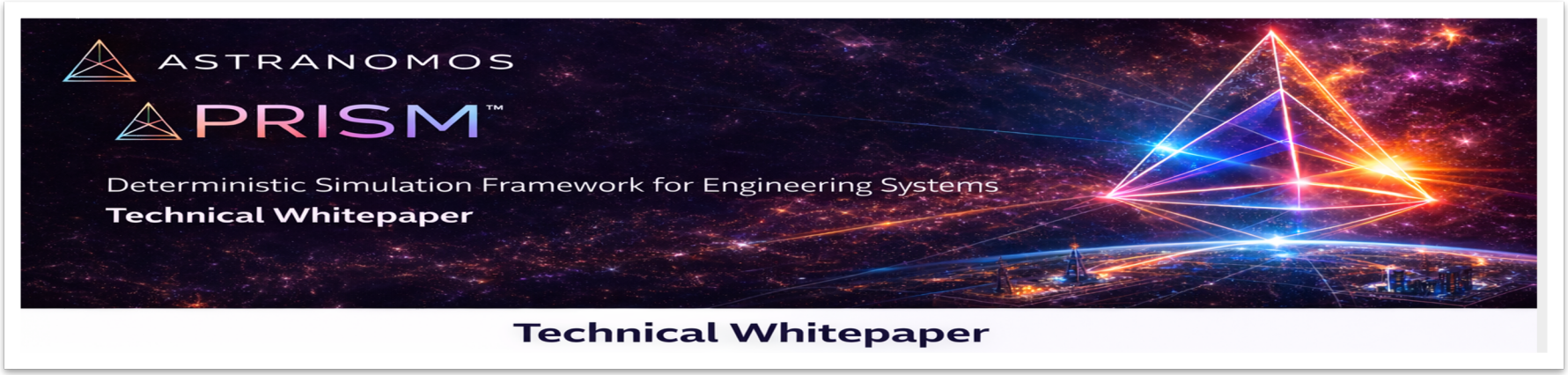


Figure 5(a). Classical diffusion-based interpretation **(left)** versus **PRISM operator-based prediction** (right) for deep-draw forming of the T-Node geometry. The classical representation distributes risk broadly across the domain, suggesting **parameter-sensitive or stochastic** behavior, the PRISM **model produces a sharply localized** failure region aligned with the experimentally observed critical thinning zone at the double-**step geometry**, indicating that the apparent variability of the process arises **from unresolved structural constraints** rather than intrinsic randomness.

Concluding Interpretation of Figure 5(a)

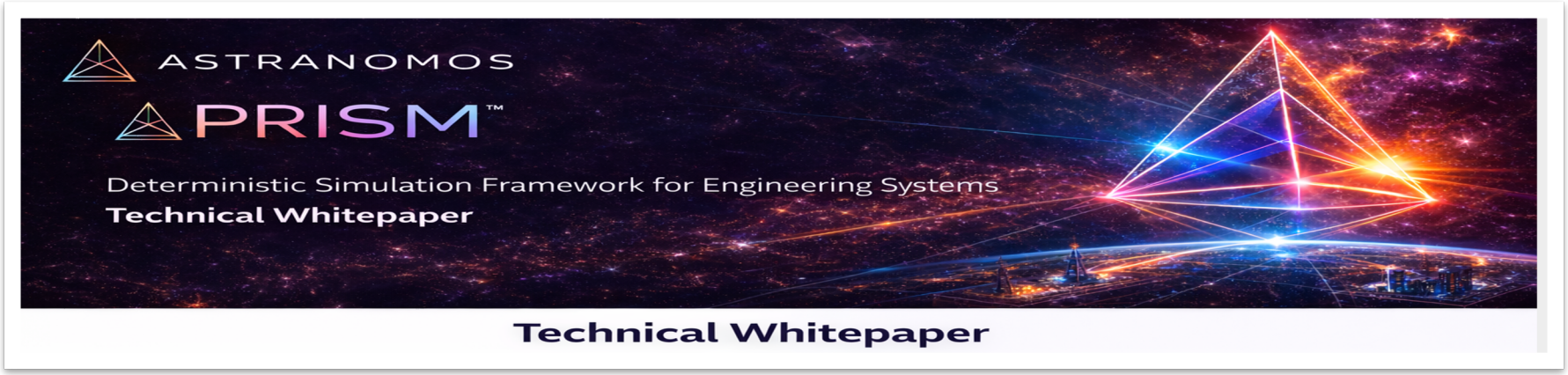
Figure 5(a) summarizes, in a single visual, the core outcome of the experiment conducted on the NUMISHEET benchmark. The comparison demonstrates that, while classical diffusion-based interpretations present the forming process as broadly distributed and parameter-sensitive, the operator-based formulation isolates a single, geometry-defined failure region that aligns with the experimentally observed thinning zone. This result is not the consequence of parameter tuning or post-processing, but emerges directly from the structure of the governing operator constructed from the geometry.



The significance of this finding extends beyond the specific T-Node case. The behavior observed here—namely, the localization of deformation and failure along geometry-defined pathways—is characteristic of a wide class of manufacturing processes in which material flow is constrained by shape. In components such as structural automotive panels, reinforcement members, and stamped assemblies with stepped or multi-radius features, failure modes similarly arise at points where admissible motion becomes restricted. These include, for example, large-scale stamped components with complex curvature transitions, where thinning, tearing, or springback consistently localize in predictable geometric regions.

What Figure 5(a) demonstrates is that these regions need not be discovered indirectly through dense simulation and iterative refinement. Instead, they can be identified directly from the geometry through the spectral structure of the governing operator. In this sense, the apparent variability observed in traditional simulation workflows is reduced to a structural phenomenon: once the underlying operator is resolved, the dominant failure pathway becomes both identifiable and stable.

This establishes a broader implication for manufacturing analysis. The same operator-based approach that localizes thinning in the NUMISHEET benchmark is applicable to other forming scenarios in which geometry governs material behavior. The result is a shift from interpreting failure as a diffuse or stochastic outcome to recognizing it as a deterministic consequence of geometric constraint.



Appendix A: Data Sources and Computational Overview

The empirical results presented in this work are based on publicly available industrial benchmark data and derived geometric representations. The primary dataset used for validation is the **NUMISHEET 2025 Industrial Benchmark**, which provides a standardized test case for deep-draw forming of the Mercedes-Benz T-Node geometry.

The benchmark includes:

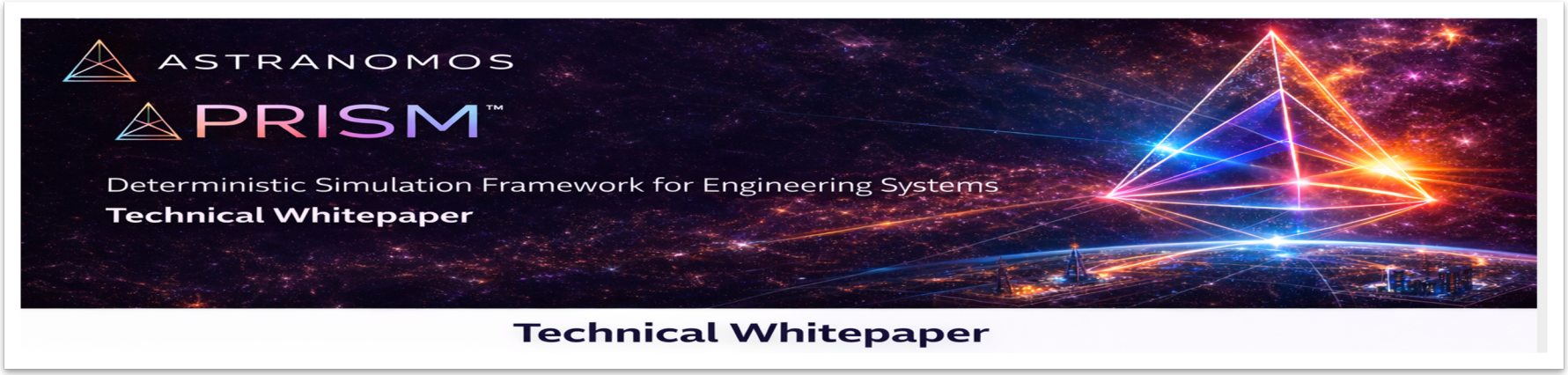
- Surface geometry of the component in multiple formats (STL, IGES, STEP)
- Material model for stainless steel (1.4301, 1.5 mm thickness)
- Reference process parameters (blank holder force, drawing depth)
- Experimentally measured outputs, including:
    - Thickness distribution after forming
    - Springback displacement following trimming
- Multiple independent simulation submissions with varying parameter configurations

The dataset is specifically designed to evaluate forming process design under realistic industrial conditions. A key feature of the benchmark is the presence of a well-defined **critical thinning region**, identified experimentally and visually in the provided reports.

For the purposes of this study, the geometry was extracted from the provided STL representation and used to construct a surface-based computational domain. No proprietary or non-public data sources were used in generating the results.

Appendix B: Computational Procedure

The computational methodology follows a structured pipeline, summarized here at a high level. First, the surface geometry of the T-Node is discretized into a triangular mesh, from which local geometric properties are computed. A curvature-derived scalar field is then constructed to represent the structural complexity of the geometry. This field serves as the basis for defining a geometry-dependent transport operator.



The governing operator is formulated in a self-adjoint form, incorporating both spatially varying diffusivity and a confinement term. The operator is then discretized over the mesh and solved as a generalized eigenvalue problem using standard sparse numerical methods.

A subset of dominant eigenmodes is extracted and used to construct a scalar field representing the concentration of admissible motion. This field is interpreted as a risk or instability indicator, reflecting regions where transport becomes constrained.

To enable comparison with experimental observations, the thickness contour provided in the benchmark report is digitized and mapped onto the computational domain. In addition, the benchmark's annotated critical region is extracted and used as a ground-truth reference for region-based validation. Validation is performed using spatial metrics that evaluate the alignment between predicted high-risk regions and experimentally observed failure zones.

Appendix C: Validation Framework

Given the structural nature of the proposed model, validation is conducted using region-based and ranking-based metrics, rather than pointwise regression.

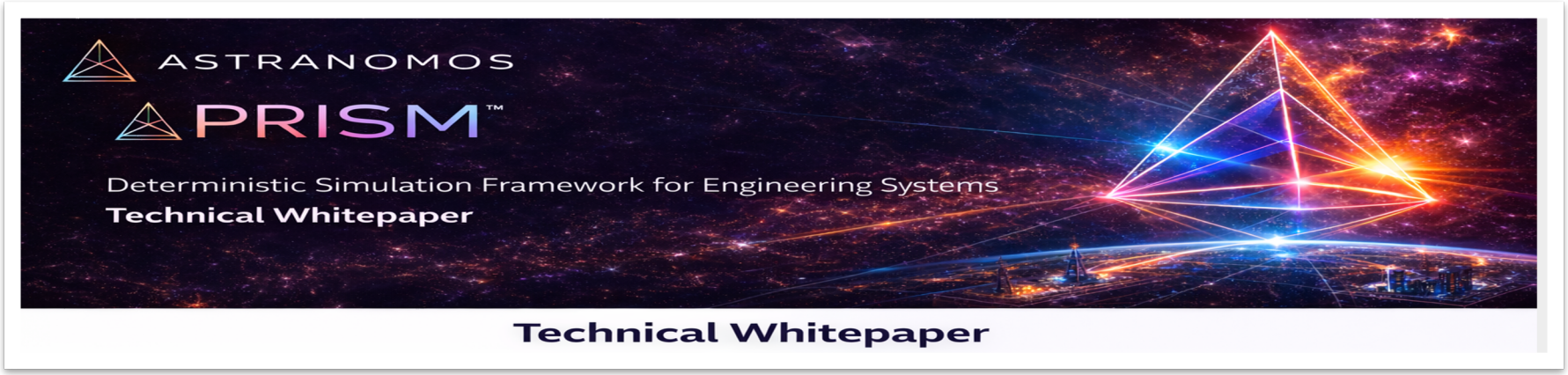
The following criteria are used:

- **Peak Localization**: Whether the maximum predicted instability lies within the experimentally identified critical region
- **Top-k Overlap**: The degree of overlap between the highest predicted risk regions and the benchmark failure zone
- **Enrichment**: The concentration of predicted high-risk points within the ground-truth region relative to random baseline
- **Distance Metrics**: Spatial proximity between predicted maxima and the center of the critical region

These metrics are selected to reflect the model's objective, which is to identify where instability occurs, rather than to reconstruct full field values.

Appendix D: Referee Roadmap for Reproducibility

To assist independent evaluation of the results, we provide the following high-level roadmap for reproducing the core findings.

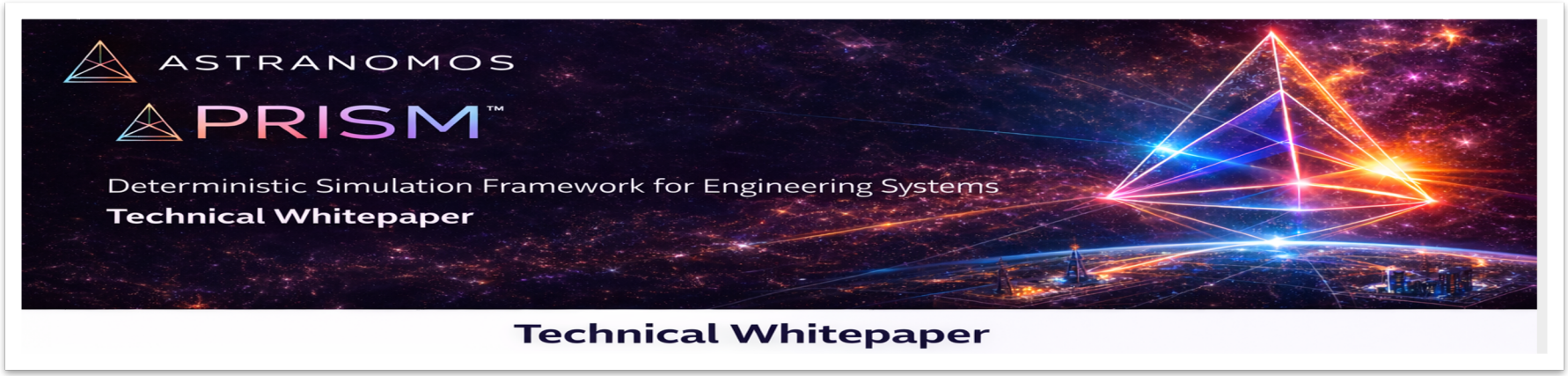


1. **Dataset Acquisition**
   Obtain the NUMISHEET 2025 Industrial Benchmark dataset, including geometry files and the accompanying report containing thickness and springback visualizations.
2. **Geometry Preparation**
   Import the T-Node geometry from the STL file and construct a surface mesh suitable for numerical computation.
3. **Geometric Field Construction**
   Compute a curvature-based scalar field over the mesh to represent structural variation. Normalize the field to obtain a dimensionless representation of geometric complexity.
4. **Operator Assembly**
   Define a spatially varying transport operator incorporating geometry-dependent diffusivity and a confinement term. Discretize the operator using standard surface-based numerical methods.
5. **Spectral Decomposition**
   Solve the resulting eigenvalue problem and extract a subset of dominant eigenfunctions representing admissible transport modes.
6. **Risk Field Construction**
   Combine the dominant modes into a scalar field representing the concentration of modal activity across the geometry.
7. **Experimental Comparison**
   Digitize the thickness contour from the benchmark report and extract the annotated critical region. Map these onto the computational domain.
8. **Validation Metrics**
   Evaluate alignment between predicted and observed regions using region-based metrics, including overlap, enrichment, and localization.

Appendix E: Notes on Scope and Generalization

The methodology described in this work is intentionally presented at a level that allows independent verification of the core claims while preserving implementation-specific details of the operator construction and numerical pipeline.

The results demonstrate that geometry alone can provide sufficient information to identify dominant instability regions in forming processes. While the specific implementation may vary, the underlying principle—that motion is governed by operator structure rather than stochastic variability—can be tested across other geometries and datasets using similar procedures.



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