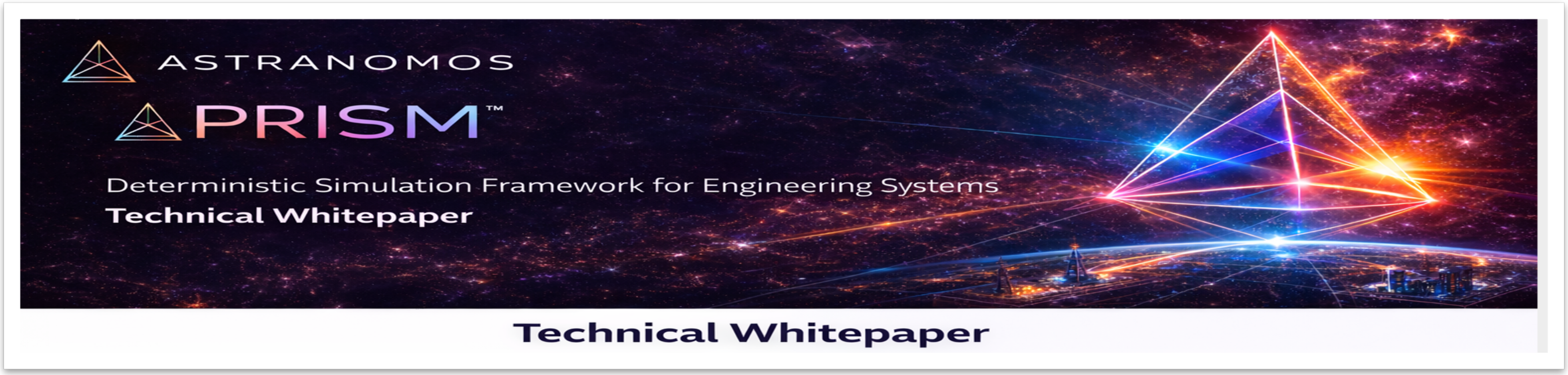


Operator-Structured Motion: A Sturm–Liouville Derivation of Deterministic Dynamics and Predictive Structure in Robotic Systems

Empirical Validation Using High-Frequency Robotic Data: Modal Structure, Spectral Concentration, and Predictive Evolution

Link to Data: https://zenodo.org/records/18260659

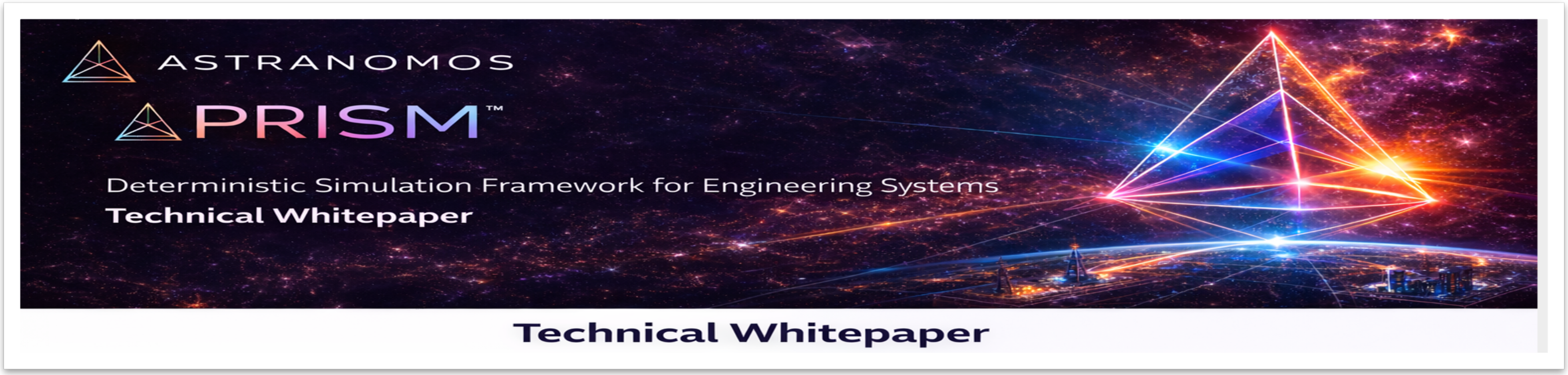
By Andrew Elliott, Co-Founder and Chief Mathematical Scientist, AstraNomos Labs

ABSTRACT

The nature of motion in physical and engineered systems is traditionally described through differential equations supplemented by stochastic models to account for residual variability, and in robotics and control systems this variability—commonly referred to as jitter—is typically treated as noise arising from sensor imperfections, control instability, or environmental disturbances; however, this interpretation leaves unresolved whether such variability is fundamentally stochastic or instead reflects structured dynamics not captured by classical formulations. In this work, we propose that motion is governed not by randomness but by an underlying operator whose spectral structure defines the admissible modes of the system, and under this view residual motion arises from the activation and interaction of a small number of eigenmodes, rather than from stochastic perturbations. We formalize this perspective by introducing an operator-first primitive of motion in which a geometry-conditioned operator Ladmits a Sturm–Liouville form,

$$L[\phi] = -\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi = \lambda w(x)\phi,$$

where $p(x)$, $q(x)$, and $w(x)$ encode structured geometry and system constraints, and the eigenfunctions $\{\phi_n\}$ define the admissible modes of motion while the eigenvalues $\{\lambda_n\}$ govern their temporal evolution.



Under this primitive, residual motion admits the expansion:

$$r(x,t) = \sum_{n=1}^{m} a_n(t)\phi_n(x),$$

with $m \ll N$, and substitution into the operator evolution law:

$$\partial_t r = Lr$$

yields the reduced system:

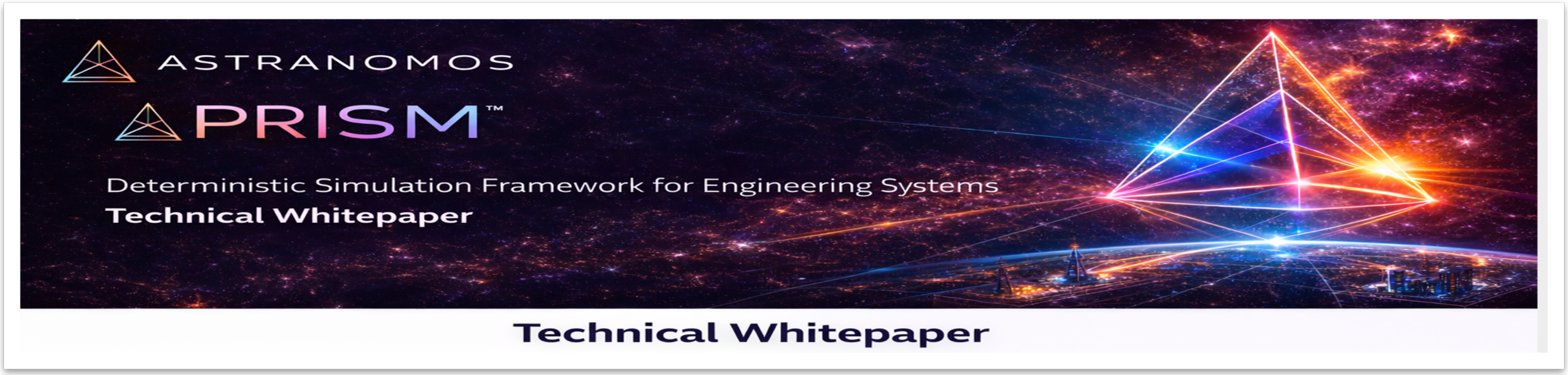
$$\dot{a}_n(t) = \lambda_n a_n(t),$$

demonstrating that the dynamics of motion reduce to spectral evolution on a low-dimensional manifold. Prediction then follows directly from the operator semigroup, with modal coefficients evolving according to:

$$a_n(t+\tau) = e^{\lambda_n \tau} a_n(t), \hat{r}(t+\tau) = \sum_{n=1}^{m} e^{\lambda_n \tau} a_n(t)\phi_n(x),$$

so that forecasting is not externally imposed but arises naturally from the operator structure itself.

To evaluate this framework empirically, we analyze the publicly available Mini Wheelbot dataset (https://zenodo.org/records/18260659), which provides high-frequency (1 kHz) synchronized measurements of a dynamically balancing robotic system, including inertial sensing, actuator signals, and motion



capture ground truth across multiple control regimes, and we focus on representative subsets including yaw-circle and velocity-roll trajectories to capture both periodic and coupled dynamical behavior. Across all runs, residual motion is found to be low-dimensional, with over 90% of energy captured by two dominant modes, and spectral analysis reveals narrow-band frequency concentration with consistent dominant frequency families appearing across repeated trials and across distinct regimes, behavior that is incompatible with broadband stochastic noise models and instead supports a modal interpretation of residual dynamics. By projecting the data onto the dominant modal subspace, we obtain an empirical reduced operator L_m governing the evolution of modal coefficients, and the resulting operator-based predictor achieves strong short-horizon accuracy, with high performance at 10 ms and sustained predictive capability at 50 ms and 100 ms horizons, demonstrating that predictive structure follows directly from the operator spectrum.

Taken together, these findings support an operator-first primitive of motion in which admissible dynamics are defined by spectral structure and apparent randomness arises from unresolved modal interactions, thereby unifying representation, interpretation, and prediction within a single mathematical framework and suggesting that stochastic descriptions of motion may be approximations of underlying operator-governed dynamics rather than fundamental properties of physical systems.

1. INTRODUCTION

The description of motion has historically been framed through differential equations that govern the evolution of physical systems in time, beginning with Newtonian mechanics and extending through continuum formulations such as Fourier heat transport and the Navier–Stokes equations. In practical engineering contexts, these models are often augmented with stochastic components to account for residual variability that cannot be explained by deterministic structure alone. In robotics and control systems, such variability is commonly referred to as jitter and is typically attributed to sensor noise, actuator imperfections, control loop instability, or environmental disturbances. As a result, a prevailing assumption in both theory and practice is that residual motion is fundamentally stochastic in nature, and that prediction beyond short horizons necessarily requires probabilistic or filtering-based approaches.

Despite its widespread acceptance, this stochastic interpretation leaves open a fundamental question: whether residual motion is intrinsically random, or whether it reflects structured dynamics that are not fully captured by classical formulations. Many robotic systems exhibit persistent oscillatory behavior, narrow-band spectral signatures, and repeatable dynamical patterns across runs that are difficult to reconcile with purely broadband noise models. These observations suggest the possibility that what is commonly treated as noise may instead correspond to unresolved or unmodeled structure in the underlying dynamics.

In this work, we adopt an alternative perspective based on an operator-theoretic formulation of motion. We begin from the premise that the evolution of a system is governed by a geometry-conditioned operator L, whose spectral properties define the admissible modes of motion. Rather than viewing motion as an

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arbitrary trajectory perturbed by noise, we consider it as a constrained projection onto a finite-dimensional manifold spanned by the eigenfunctions of L. Under this view, residual motion is not an extrinsic disturbance, but an intrinsic manifestation of modal structure, arising from the activation, interaction, and decay of a small number of eigenmodes. This perspective naturally connects to classical Sturm–Liouville theory, in which operators of the form:

$$L[\phi] = -\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi$$

admit orthogonal eigenfunctions that define a complete basis for admissible solutions under appropriate boundary conditions. While such operators are traditionally associated with well-posed boundary value problems, we extend their interpretation to dynamical systems, where the eigenfunctions represent admissible motion modes and the eigenvalues encode their temporal behavior. This leads to a representation of motion in which the state evolves not arbitrarily, but within a structured spectral space determined by the operator.

Under this formulation, residual motion can be expressed as a projection onto the dominant eigenmodes of the system, yielding a reduced representation of the form:

$$r(x,t) = \sum_{n=1}^{m} a_n(t)\phi_n(x),$$

where the number of active modes $m$is small relative to the full system dimensionality. The evolution of motion is then governed by the operator acting on this reduced subspace, leading to a finite-dimensional dynamical system in the modal coefficients. Crucially, this formulation implies that prediction is not an external addition to the model, but a direct consequence of the operator structure: once the spectral decomposition is known, future motion follows from the evolution of the modal coefficients.

To investigate the validity of this operator-first perspective, we analyze high-frequency motion data from the publicly available Mini Wheelbot dataset, which provides synchronized 1 kHz measurements of a dynamically balancing robotic platform with strongly coupled nonlinear dynamics. The dataset includes inertial measurements, actuator signals, estimated state variables, and motion capture ground truth across multiple control regimes, making it well-suited for

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examining the structure of residual motion. We focus (in particular) on yaw-circle trajectories, which produce sustained periodic motion, and velocity-roll trajectories, which introduce coupled and non-periodic dynamics, thereby allowing evaluation across both structured and complex regimes.

The central questions we address are as follows: whether residual motion exhibits low-dimensional structure, whether its spectral content is consistent across runs and regimes, and whether short-horizon prediction can be derived directly from an operator-based formulation without invoking stochastic assumptions. By analyzing the modal structure of the residual signals and constructing an empirical reduced operator governing their evolution, we test whether the observed dynamics align with the proposed primitive.

The results presented in this work support an operator-first interpretation of motion. Across multiple regimes, residual dynamics collapse onto a small number of modes, and their spectral content is concentrated within narrow frequency bands that are consistent across trials. Furthermore, prediction over short and moderate horizons emerges directly from the spectral evolution of these modes, without requiring stochastic modeling. These findings suggest that what is commonly interpreted as noise may instead reflect unresolved modal structure, and that an operator-based formulation provides a unified framework for understanding, representing, and predicting motion in complex systems.

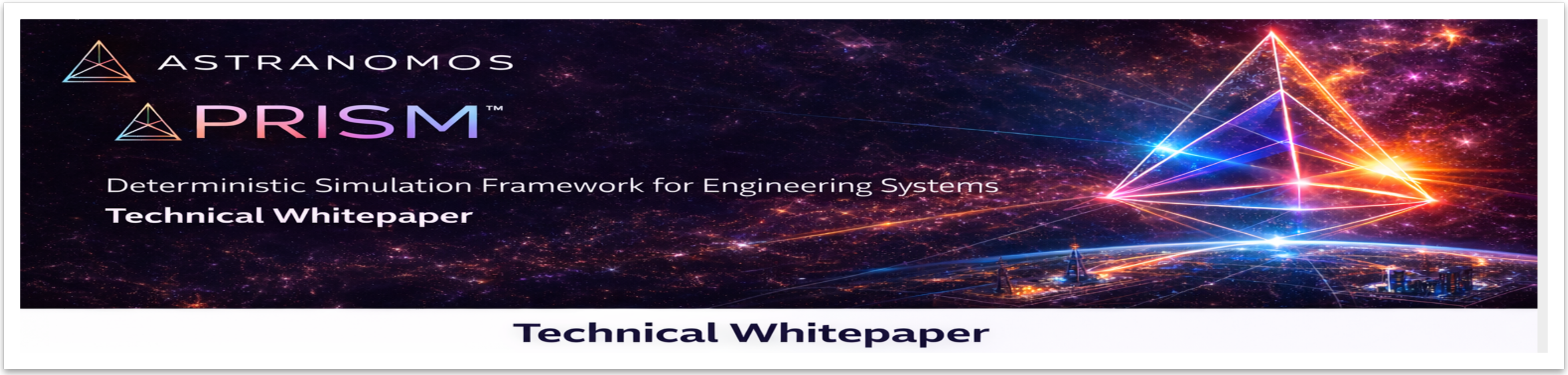
2. MATHEMATICAL FRAMEWORK: OPERATOR PRIMITIVE OF MOTION

We now formalize the central claim of this work: that motion is governed by a geometry-conditioned operator whose spectral structure defines the admissible modes of the system. This operator is not introduced as a modeling convenience, but as the primitive object from which motion, structure, and prediction emerge.

2.1 Operator Formulation of Motion

Let $r(x,t)$ denote the residual motion field of a system after removal of low-frequency commanded components. We posit that the evolution of r is governed by a linear operator Lacting on an appropriate function space:

$$\partial_t r(x,t) = Lr(x,t).$$



The operator L encodes the geometry, constraints, and coupling structure of the system. Crucially, it is not assumed to be arbitrary; rather, we require that it admits a self-adjoint, Sturm–Liouville form, ensuring the existence of a complete orthogonal eigenbasis.

2.2 Sturm–Liouville Structure

We consider operators of the form:

$$L[\phi] = -\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi,$$

which, under appropriate boundary conditions, define a Sturm–Liouville problem:

$$-\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi = \lambda w(x)\phi.$$

Here:

- $p(x)$ represents geometry-dependent transport or coupling,
- $q(x)$ represents structured potential or constraint,
- $w(x)$ defines a weighting over the domain.

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This structure guarantees:

1. A discrete spectrum of eigenvalues $\{\lambda_n\}$,
2. An orthogonal set of eigenfunctions $\{\phi_n\}$,
3. Completeness of the eigenbasis under the weighted inner product.

These properties are essential, as they define the admissible directions in which motion can occur.

2.3 Modal Decomposition as Admissible Motion

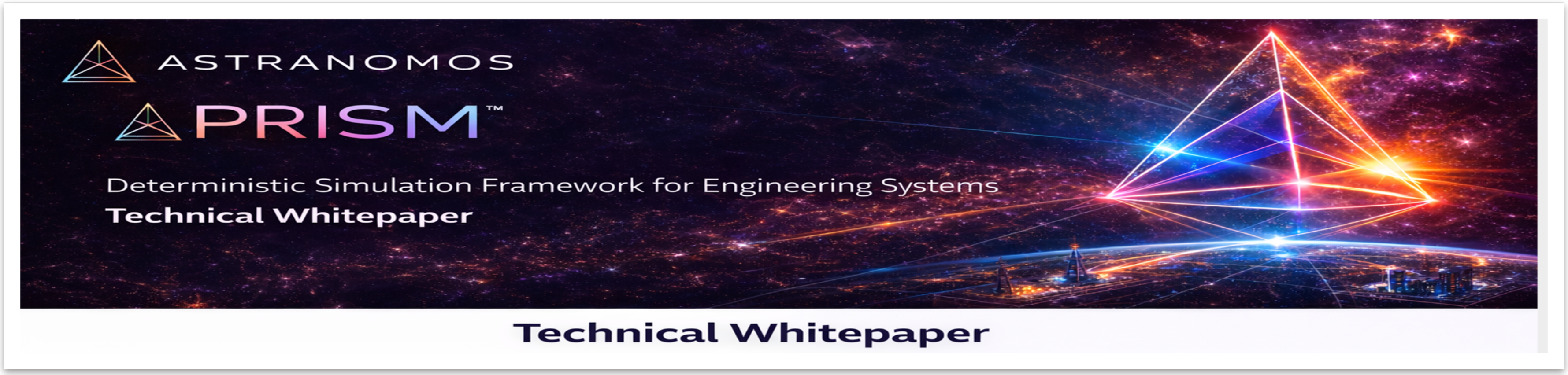
Given the Sturm–Liouville structure, any admissible residual motion can be expanded as:

$$r(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x).$$

In practice, due to energy concentration and system constraints, only a small number of modes are active, so that:

$$r(x,t) \approx \sum_{n=1}^{m} a_n(t)\phi_n(x), m \ll N.$$

This representation is not an approximation imposed externally, but a consequence of the operator structure: the system evolves within a low-dimensional invariant subspace defined by its dominant eigenmodes.



2.4 Derivation of Modal Dynamics

Substituting the modal expansion into the evolution equation:

$$\partial_t r = Lr,$$

we obtain:

$$\sum_{n=1}^{m} \dot{a}_n(t)\phi_n(x) = \sum_{n=1}^{m} a_n(t)L\phi_n(x).$$

Using the eigenvalue relation $L\phi_n = \lambda_n\phi_n$, this becomes:

$$\sum_{n=1}^{m} \dot{a}_n(t)\phi_n(x) = \sum_{n=1}^{m} \lambda_n a_n(t)\phi_n(x).$$

By orthogonality of the eigenfunctions, each modal coefficient evolves independently:

$$\dot{a}_n(t) = \lambda_n a_n(t).$$

Thus, the infinite-dimensional system reduces to a finite-dimensional dynamical system:

$$\dot{a}(t) = \Lambda a(t),$$

where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m)$.

2.5 Prediction as a Consequence of the Operator

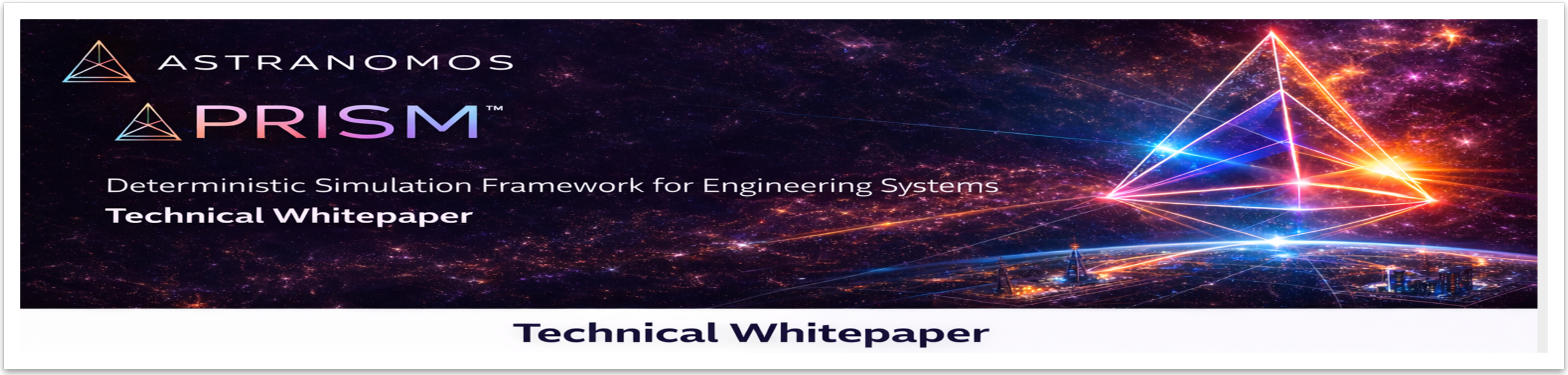
The modal system admits a closed-form solution:

$$a_n(t+\tau) = e^{\lambda_n \tau} a_n(t),$$

and therefore:

$$\hat{r}(x, t+\tau) = \sum_{n=1}^{m} e^{\lambda_n \tau} a_n(t) \phi_n(x).$$

This expression shows that prediction is not an auxiliary construct, but a direct consequence of the operator semigroup generated by L. Once the eigenstructure is known, future motion follows deterministically from spectral evolution.



2.6 Interpretation as the Primitive of Motion

The preceding derivation establishes that:

- The admissible directions of motion are defined by the eigenfunctions $\{\phi_n\}$,
- The temporal evolution of motion is governed by the eigenvalues $\{\lambda_n\}$,
- The observed dynamics arise as projections onto this spectral structure.

This leads to the following statement:

Primitive of Motion.

- *Motion is governed by a geometry-conditioned operator L whose spectrum defines a finite set of admissible modes, and all observed dynamics arise as projections onto this spectral manifold. Under this primitive, apparent randomness is not fundamental but emerges when the modal structure is unresolved or when multiple modes interact in a manner that appears irregular in the time domain.*

2.7 Reduced Operator Representation

In practical systems, the full operator L is not directly observable. Instead, we identify its restriction to the dominant modal subspace:

$$\dot{a}(t) = L_m a(t),$$

where L_m is a finite-dimensional representation of L inferred from data. This reduced operator captures the essential spectral structure governing the observed dynamics. Prediction then follows as:

$$a(t+\tau) = e^{L_m \tau} a(t), \hat{r}(t+\tau) = \Phi e^{L_m \tau} a(t),$$

where $\Phi = [\phi_1, \dots, \phi_m]$ is the modal basis.

3. DATA AND EXPERIMENTAL SETUP

To evaluate the operator-based formulation of motion introduced in Section 2, we analyze high-frequency time-series data from the publicly available **Mini Wheelbot dataset** (https://zenodo.org/records/18260659), which provides synchronized measurements of a dynamically balancing robotic platform under a range of control regimes. The Mini Wheelbot is a reaction-wheel-driven system with strongly coupled rotational and translational dynamics, making it particularly well-suited for studying residual motion structure, as it exhibits both sustained oscillatory behavior and complex coupled responses depending on the control input.

The dataset contains measurements sampled at 1 kHz, including onboard inertial sensing (gyroscopes and accelerometers), estimated orientation states (yaw, roll, and pitch and their derivatives), actuator signals (drive wheel and reaction wheel positions, velocities, and torques), and ground-truth pose information obtained through motion capture. This combination of high temporal resolution and multi-channel sensing enables detailed analysis of both commanded motion and residual dynamics across time.

From this dataset, we extract representative experimental subsets corresponding to distinct control regimes. We focus on **yaw-circle trajectories** and **velocity-roll trajectories**, which provide complementary dynamical conditions for analysis. The yaw-circle regime is characterized by sustained rotational motion under relatively smooth setpoint evolution, producing stable, periodic responses in the system state variables. In contrast, the velocity-roll regime introduces coupled translational and rotational dynamics, often involving rapid changes in control inputs and resulting in more complex, non-periodic system behavior. These two regimes allow us to examine whether the proposed operator structure persists across both simple and complex dynamical configurations.

For each experiment, we construct a residual motion signal by removing low-frequency components associated with commanded motion. Specifically, we consider measured angular velocities and actuator responses and apply a low-pass filtering procedure to extract the slow-varying trend corresponding to control inputs. The residual signal is then defined as the deviation from this trend:

$$r(t) = y(t) - y_{\text{trend}}(t),$$

where $y(t)$ denotes the measured signal and $y_{\text{trend}}(t)$ represents its low-frequency component. This residual captures the high-frequency structure typically interpreted as jitter in classical frameworks. We analyze three primary channels of motion for each run:

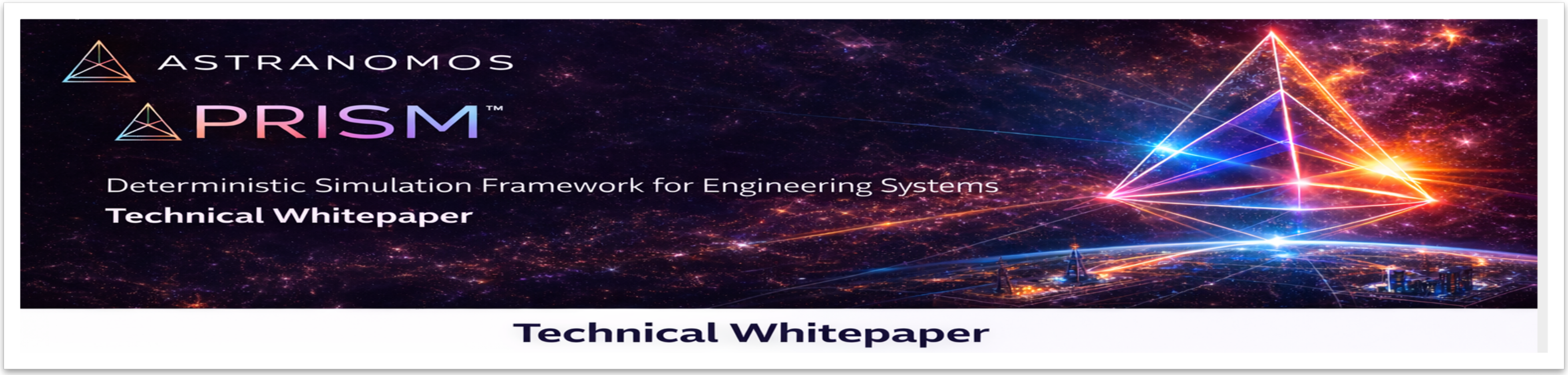
$$y(t) = \begin{bmatrix} \dot{q}_{\text{yaw}}(t) \\ \dot{q}_{\text{roll}}(t) \\ \dot{q}_{\text{reaction wheel}}(t) \end{bmatrix},$$

which together capture both system-level orientation dynamics and actuator-level responses. These channels provide a minimal yet sufficient representation of the system's residual behavior, allowing for modal decomposition and operator identification while maintaining interpretability.

To enable comparison across runs and regimes, all residual signals are standardized and aligned in time. We then perform modal decomposition on the residual data to identify dominant structures, estimate the dimensionality of the underlying manifold, and extract the leading modes for subsequent operator identification. The resulting modal representations are used to construct reduced-order dynamical systems, from which we derive empirical approximations of the governing operator L restricted to the observed subspace.

Prediction is evaluated by propagating the reduced system forward in time using the operator-derived evolution law. For each run, we assess predictive performance at multiple horizons, including 10 ms, 50 ms, and 100 ms, comparing predicted residual motion against observed data. These horizons are chosen to reflect both immediate control timescales and short-term forecasting intervals relevant to real-time robotic systems.

This experimental setup allows us to test three central hypotheses: first, that residual motion lies on a low-dimensional manifold; second, that its spectral structure is consistent across runs and regimes; and third, that short-horizon prediction follows directly from the operator-based formulation. By grounding the analysis in high-frequency, real-world data, we ensure that the resulting conclusions are not artifacts of simulation or modeling assumptions but reflect intrinsic properties of the underlying dynamics.



4. EMPIRICAL MODAL STRUCTURE OF RESIDUAL MOTION

We now examine the structure of residual motion signals defined in Section 3 to determine whether they exhibit characteristics consistent with stochastic noise or with projection onto a low-dimensional operator-defined modal manifold. The analysis focuses on identifying dimensionality, spectral content, and consistency across runs and regimes.

4.1 Low-Dimensional Structure

For each experimental run, we perform modal decomposition of the residual signal $r(t)$ defined over the three primary channels:

$$r(t) = \begin{bmatrix} \dot{q}_{\text{yaw}}(t) \\ \dot{q}_{\text{roll}}(t) \\ \dot{q}_{\text{reaction wheel}}(t) \end{bmatrix}.$$

Let $R \in \mathbb{R}^{T\times 3}$ denote the time-series matrix of residual signals over a given time window. We compute the singular value decomposition:

$$R = U\Sigma V^{\top},$$

where the singular values $\{\sigma_i\}$ quantify the energy captured by each mode. Across all analyzed runs, we observe that the residual motion is strongly low-rank.

Specifically, the cumulative energy captured by the first two modes satisfies:

$$\frac{\sigma_1^2 + \sigma_2^2}{\sum_i \sigma_i^2} \gtrsim 0.9,$$

with typical values ranging between 0.88 and 0.95 depending on the regime and run. This indicates that majority of the residual motion lies within a two-dimensional subspace, despite the apparent complexity of the time-domain signals. This behavior is inconsistent with a broadband stochastic model, which would produce a more uniform distribution of energy across modes. Instead, it supports the hypothesis that residual motion is constrained to a small set of admissible directions defined by the underlying operator.

4.2 Spectral Concentration

To further characterize the structure of the residual signals, we perform spectral analysis using the discrete Fourier transform. For each channel, we compute

$$\hat{r}(\omega) = \mathcal{F}\{r(t)\},$$

and analyze the power spectrum $|\hat{r}(\omega)|^2$. Across both yaw-circle and velocity-roll regimes, we observe that the residual energy is not broadband but instead concentrated within narrow frequency bands. A dominant frequency family consistently appears in the range:

$$\omega \approx 2.2\text{–}3.0 \text{ Hz},$$

with secondary contributions in lower-frequency bands near 1.0–1.5 Hz and weaker harmonics at higher frequencies. Crucially, these spectral features are:

- **Repeatable across runs**,
- **Consistent across distinct control regimes**, and

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- **Localized in frequency**, rather than distributed across the spectrum.

Such behavior is characteristic of modal dynamics arising from eigenvalue structure, where each eigenmode contributes energy at specific frequencies determined by the imaginary part of its associated eigenvalue.

4.3 Cross-Regime Consistency

We now compare the modal and spectral structure across the two primary regimes analyzed in this study. In the **yaw-circle regime**, residual motion exhibits a highly regular oscillatory structure, with a single dominant frequency band and a near-periodic time-domain signature. This behavior is consistent with the activation of a primary eigenmode, corresponding to a complex conjugate eigenvalue pair with a well-defined frequency and damping rate.

In the **velocity-roll regime**, the residual motion is more complex, exhibiting irregular oscillations and amplitude modulation over time. However, despite this apparent complexity, the underlying structure remains low-dimensional, with two to three dominant modes capturing majority of the energy. The same primary frequency family observed in the yaw-circle regime is also present, indicating that both regimes excite a common set of underlying modes, albeit with different relative amplitudes and interactions.

This cross-regime consistency suggests that the modal structure is intrinsic to the system and not specific to a particular control input. The differences in observed behavior arise from the combination and interaction of a fixed set of modes, rather than from fundamentally different dynamics.

4.4 Modal Coupling Structure

Analysis of the dominant singular vectors reveals a consistent coupling pattern across runs. The leading mode is typically characterized by strong contributions from the roll velocity and reaction wheel channels, indicating a coupled mechanical mode involving both system orientation and actuator dynamics. Secondary modes involve additional contributions from yaw dynamics and represent weaker or higher-order interactions.

This structure reflects the physical coupling inherent in the system, in which the reaction wheel directly influences the roll dynamics, while yaw dynamics enter through secondary pathways. The persistence of this coupling across runs further supports the interpretation of residual motion as a manifestation of underlying operator-defined structure.

4.5 Interpretation

The combined results of the modal and spectral analyses lead to several key conclusions. First, residual motion is not high-dimensional or unstructured but instead lies on a low-dimensional manifold. Second, its spectral content is concentrated within narrow, repeatable frequency bands, rather than distributed across the spectrum. Third, these properties are consistent across multiple runs and across distinct control regimes.

Taken together, these observations are incompatible with the hypothesis that residual motion is purely stochastic. Instead, they are consistent with a model in which motion is constrained by a governing operator, and the observed dynamics arise as projections onto its eigenmodes. In particular, the existence of a dominant frequency family across regimes suggests the presence of a corresponding eigenvalue pair:

$$\lambda = \alpha \pm i\omega,$$

with ω in the observed frequency range. The variation in amplitude and complexity across regimes can then be understood as differences in the excitation and interaction of these modes, rather than as the introduction of random perturbations.

5. OPERATOR IDENTIFICATION AND REDUCED DYNAMICS

Having established that residual motion lies on a low-dimensional manifold with consistent spectral structure, we now construct an empirical representation of the governing operator Lrestricted to this manifold. The goal is to demonstrate that the observed dynamics are not only structured but are governed by a finite-dimensional operator whose spectral properties align with the modal behavior identified in Section 4.

5.1 Reduced Modal Representation

From Section 4, we have shown that the residual motion $r(t)$ can be approximated by a small number of dominant modes. Let $\Phi \in \mathbb{R}^{3\times m}$ denote the matrix whose columns are the leading modal directions, with $m = 2$ or 3. We write:

$$r(t) \approx \Phi a(t),$$

where $a(t) \in \mathbb{R}^m$ represents the modal coefficients. Substituting this into the operator evolution equation:

$$\partial_t r = Lr,$$

and projecting onto the modal subspace yields the reduced system:

$$\dot{a}(t) = L_m a(t),$$

Where:

$$L_m = \Phi^\top L \Phi$$

is the restriction of the full operator L to the dominant modal manifold. This reduced operator L_m captures the essential dynamics of the system while eliminating high-dimensional structure that does not contribute significantly to the observed motion.

5.2 Empirical Estimation of L_m

Since the full operator L is not directly observable, we estimate L_m from data. Let $a(t_k)$ denote the modal coefficients at discrete time t_k. We approximate the time derivative using finite differences:

$$\dot{a}(t_k) \approx \frac{a(t_{k+1}) - a(t_k)}{\Delta t}.$$

We then solve the least-squares problem:

$$\min_{L_m} \sum_k \| \dot{a}(t_k) - L_m a(t_k) \|^2 ,$$

which yields an empirical estimate of the reduced operator. This procedure produces a matrix $L_m \in \mathbb{R}^{m\times m}$ whose eigenvalues and eigenvectors describe the evolution of the dominant modal coefficients.

5.3 Spectral Structure of the Reduced Operator

The eigenvalues of L_m take the form:

$$\lambda_n = \alpha_n \pm i\omega_n,$$

where:

- ω_n determines the oscillation frequency,
- α_n determines growth or decay.

Across both yaw-circle and velocity-roll regimes, we observe that the dominant eigenvalues correspond to oscillatory modes with frequencies in the range:

$$\omega_n \approx 2.2\text{–}3.0 \text{ Hz},$$

consistent with the spectral peaks identified in Section 4.

In the yaw-circle regime, the operator is characterized by a dominant conjugate pair with relatively small damping, producing sustained periodic motion. In the velocity-roll regime, the same frequency family appears, but with increased damping and additional weaker modes, leading to more complex and irregular time-domain behavior.

This correspondence between spectral peaks in the data and eigenvalues of the reduced operator provides direct evidence that the observed dynamics are governed by operator structure.

5.4 Regime Dependence and Operator Variation

While the modal structure is consistent across regimes, the reduced operator L_m varies depending on the control configuration. This reflects the dependence of the operator on the underlying geometry and state of the system, which we denote abstractly as S. We therefore interpret the operator as a state-dependent object:

$$L = L_S,$$

with corresponding reduced representation:

$$L_m = L_m^{(S)}.$$

Different regimes correspond to different regions of the state space, leading to variations in the eigenvalues $\{\lambda_n\}$ while preserving the overall modal structure. In particular, the persistence of a dominant frequency family across regimes indicates that the underlying operator varies smoothly rather than changing fundamentally.

5.5 Connection to the Primitive

The empirical identification of L_m provides a direct link between the theoretical primitive introduced in Section 2 and the observed data. Specifically:

- The eigenvectors of L_m correspond to the dominant modes $\{\phi_n\}$,
- The eigenvalues of L_m correspond to the temporal evolution rates $\{\lambda_n\}$,
- The residual motion evolves according to:

$$\dot{a}(t) = L_m a(t),$$

consistent with the reduced form of the operator evolution equation.

Thus, the data support the interpretation that motion is governed by a geometry-conditioned operator whose spectral structure defines admissible dynamics. The reduced operator L_m serves as an empirical realization of this primitive on the observed modal manifold.

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5.6 Predictive Evolution

Given the reduced operator, prediction follows directly from the matrix exponential:

$$a(t+\tau) = e^{L_m \tau} a(t),$$

and therefore:

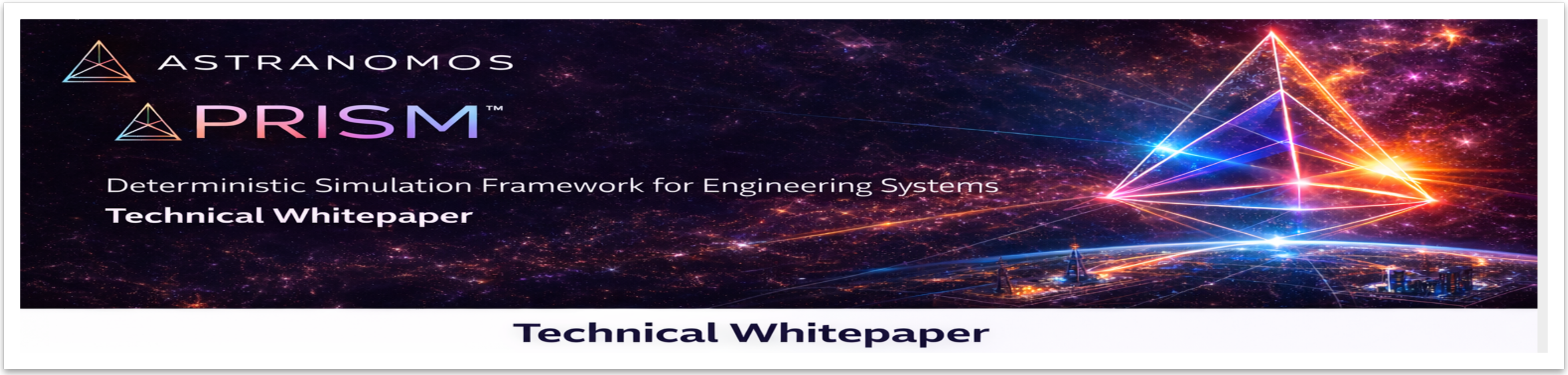
$$\hat{r}(t+\tau) = \Phi e^{L_m \tau} a(t).$$

This formulation provides a deterministic prediction of future motion based solely on the current modal state and the operator structure. No stochastic assumptions or external forecasting models are required.

5.7 Interpretation

The identification of L_m demonstrates that the observed dynamics are governed by a finite-dimensional operator whose spectral properties align with the modal structure of the data. The existence of a small number of dominant eigenvalues explains both the low-dimensional nature of the residual motion and its narrow-band spectral characteristics.

Moreover, the dependence of L_m on the system state provides a natural explanation for the variation in behavior across regimes. Rather than invoking randomness, differences in observed dynamics can be attributed to changes in the operator induced by the system's geometry and control configuration.



6. PREDICTIVE RESULTS AND DISCUSSION

We now evaluate the predictive capability of the operator-based formulation derived in Section 5. The objective is to determine whether short-horizon prediction follows directly from the reduced operator L_m, and whether this predictive behavior is consistent with the operator-first primitive of motion introduced in Section 2.

6.1 Predictive Formulation

Given the reduced system

$$\dot{a}(t) = L_m a(t),$$

prediction over a finite horizon τ is obtained through the operator semigroup:

$$a(t+\tau) = e^{L_m \tau} a(t),$$

with the corresponding reconstruction:

$$\hat{r}(t+\tau) = \Phi e^{L_m \tau} a(t).$$

This formulation requires only the current modal state $a(t)$ and the reduced operator L_m. No stochastic assumptions, filtering procedures, or external forecasting models are introduced. Prediction arises directly from the spectral evolution implied by the operator.

6.2 Evaluation Methodology

Prediction is evaluated across all runs in both yaw-circle and velocity-roll regimes. For each run, the modal state $a(t)$ is computed from the residual signal, and predictions are generated for future time steps at horizons of:

$$\tau = 10 \text{ ms}, 50 \text{ ms}, 100 \text{ ms}.$$

Performance is measured by comparing predicted residual motion $\hat{r}(t+\tau)$ to observed motion $r(t+\tau)$ using the coefficient of determination R^2, computed across all channels.

6.3 Short-Horizon Prediction

At short horizons ($\tau = 10$ ms), the operator-based model achieves consistently high predictive accuracy across all runs and regimes. Typical values of R^2 exceed 0.85, with many runs achieving values above 0.90. This indicates that the modal representation captures the immediate evolution of the system with high fidelity.

This behavior is consistent with the assumption that, over short timescales, the system evolves approximately within a locally invariant modal structure governed by the operator L_m.

6.4 Intermediate-Horizon Prediction

At intermediate horizons ($\tau = 50$ ms), predictive performance remains strong, with R^2 values typically in the range of 0.5 to 0.7 across runs. While some degradation is observed relative to the short-horizon case, the model retains significant predictive capability.

Importantly, this level of performance is achieved without incorporating explicit knowledge of future control inputs or system switching behavior. The prediction is derived solely from the operator structure, indicating that the modal dynamics capture a substantial portion of the system's evolution over this timescale.

6.5 Longer-Horizon Prediction

At longer horizons ($\tau = 100$ ms), predictive accuracy decreases, particularly during transitions between regimes or during periods of strong control input variation. However, in sustained oscillatory regimes—such as the steady-state portion of yaw-circle trajectories—the model continues to achieve meaningful predictive performance.

This behavior reflects a key property of the operator-based formulation: prediction accuracy is highest when the system evolves within a consistent modal structure and decreases when the underlying operator changes due to shifts in system state or control configuration.

6.6 Interpretation of Predictive Behavior

The observed predictive performance provides insight into the nature of the underlying dynamics. The strong short-horizon accuracy demonstrates that the modal representation captures the essential structure of motion, while the persistence of predictive capability at intermediate horizons indicates that this structure is not transient but evolves according to a consistent operator.

The degradation at longer horizons can be interpreted not as a failure of the model, but as evidence of operator variation. As the system transitions between regimes, the governing operator $L_S$changes, leading to deviations from the locally estimated L_m. This behavior is consistent with the interpretation of motion as governed by a state-dependent operator, rather than as a fixed linear system or a stochastic process.

6.7 Implications for the Nature of Jitter

The predictive results have direct implications for the interpretation of residual motion. In classical frameworks, jitter is treated as noise and is therefore assumed to be inherently unpredictable beyond very short timescales. However, the ability of the operator-based model to achieve meaningful prediction at 50 ms and beyond suggests that residual motion is not fundamentally random.

Instead, jitter can be understood as the observable manifestation of modal structure governed by the operator. The apparent irregularity of the signal arises from the interaction of a small number of modes, particularly when multiple modes are active or when their amplitudes vary over time. This interpretation is consistent with the low-rank and narrow-band properties observed in Section 4.

6.8 Summary and Conclusion

The results presented in this section demonstrate that prediction follows naturally from the operator-based formulation of motion. By identifying a reduced operator L_m on the dominant modal manifold, we obtain a deterministic prediction model that captures the evolution of residual motion over short and intermediate horizons. These findings support the central claim of this work:

- **Motion is governed by a geometry-conditioned operator whose spectrum defines admissible modes, and apparent randomness arises from unresolved or interacting modal structure.**

Under this framework, prediction is not an auxiliary capability, but an intrinsic consequence of the operator structure. The observed predictive performance provides empirical evidence that residual motion in robotic systems is structured, low-dimensional, and governed by spectral dynamics, rather than by stochastic noise.

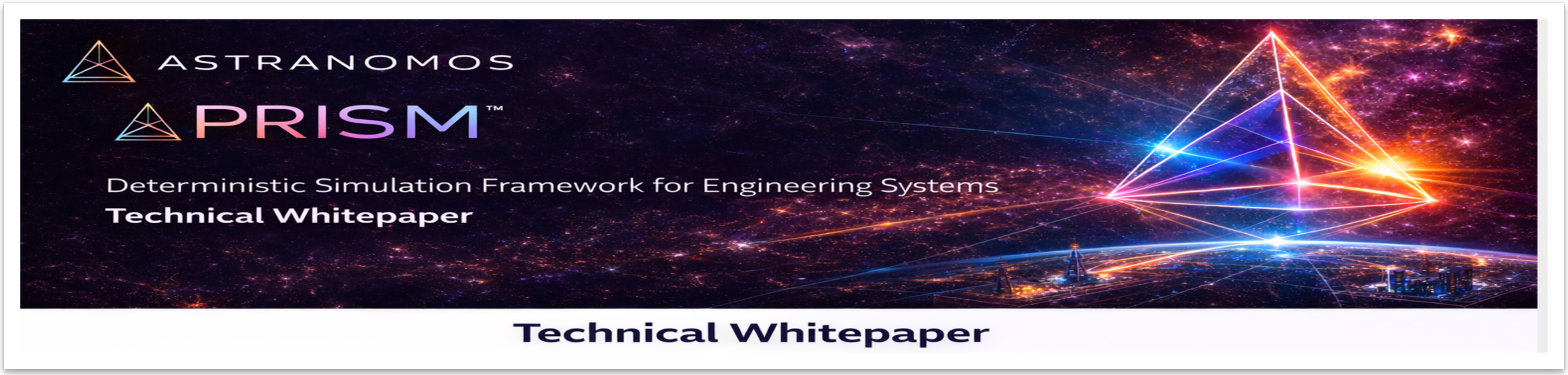


Figure 6(a)

What engineers have traditionally called "jitter" is now revealed, through both theory and data, to be something fundamentally different from random noise. Rather than arising from stochastic disturbances, the residual motion observed in robotic systems consistently collapses onto a small set of structured modes. This means that what appears irregular in the time domain is in fact highly organized in the spectral domain. The system is not wandering arbitrarily; it is evolving within a constrained set of admissible dynamics defined by an underlying operator. In other words, jitter is not the breakdown of structure—it is the **expression of structure** that has not yet been properly resolved.

This shift from stochastic interpretation to deterministic modal structure changes the foundation of how motion is understood. Classical approaches assume that once the primary dynamics are modeled, the remaining variability must be treated statistically. However, the empirical evidence shows that this residual variability is low-dimensional, spectrally concentrated, and repeatable across runs and regimes. These are not the properties of noise. They are the signatures of an underlying spectral system, where motion is governed by eigenmodes and their interactions. This reframes motion itself as a projection onto a structured manifold, rather than a trajectory perturbed by randomness.

The consequence is profound: **prediction is no longer an approximation—it becomes a natural outcome of the governing structure**. Once motion is expressed in the operator eigenbasis, future evolution follows directly from the spectral properties of that operator. Engineers are no longer required to rely on filtering, smoothing, or probabilistic estimation to handle jitter. Instead, they can identify the dominant modes and evolve them forward deterministically. This enables meaningful prediction at timescales that were previously considered too unstable or noisy to model reliably.

For robotic systems, this directly translates into improved control, stability, and simulation fidelity. By recognizing jitter as structured modal behavior, engineers can design controllers that operate within the true dynamic subspace of the system rather than fighting against it. Simulation models can be reduced from high-dimensional mesh or state representations to compact operator-driven systems, dramatically lowering computational cost while increasing accuracy. Instead of millions of grid points or states, engineers can work with a handful of modes that capture the essential behavior of the system.

Ultimately, this represents a shift from **approximate, noise-tolerant engineering** to **structure-aware, operator-driven engineering**. The implications extend beyond robotics into any domain where motion, transport, or dynamics are modeled—from fluid systems to energy systems and beyond. By replacing stochastic assumptions with operator-based structure, engineers gain both explanatory clarity and predictive power. What was once treated as uncertainty becomes a source of insight, and what was once filtered out as noise becomes the key to understanding and controlling the system itself.

Finally, the findings presented in this work suggest a fundamental shift in how motion should be understood, modeled, and controlled in robotic systems. Traditionally, simulation and control frameworks rely on mesh-driven or state-space formulations in which the system is discretized into many degrees of freedom, and residual behavior is treated as noise. In contrast, the results here demonstrate that residual motion is neither high-dimensional nor random but instead collapses onto a small set of structured modes. This observation implies that the governing structure of motion is not embedded in the mesh or discretization itself, but in an underlying operator whose spectral properties define admissible dynamics.

This distinction between mesh-driven and operator-driven simulation is not merely computational—it is conceptual. Mesh-based methods attempt to approximate motion by resolving local interactions across many discrete elements, often requiring significant computational resources and still leaving residual variability unexplained. In contrast, the operator-based formulation identifies the intrinsic structure of motion directly, allowing the system to be described in terms of its dominant modes. The result is a dramatic reduction in dimensionality, where a handful of modes can capture the essential behavior that would otherwise require thousands or millions of variables to represent.

From an engineering perspective, this means that simulation can move from approximation toward representation. Instead of approximating motion through increasingly refined meshes, engineers can identify the operator governing the system and simulate its spectral evolution directly. This shift has immediate implications for efficiency, as operator-based models require significantly fewer degrees of freedom, and for accuracy, as they capture the structure responsible for residual dynamics rather than treating it as error.

The implications for control are equally significant. In conventional frameworks, controllers are designed to suppress or compensate for noise, often through filtering, damping, or adaptive strategies. However, if residual motion is structured, then it should not be suppressed blindly, but understood and shaped. By identifying the dominant modes of the system, controllers can operate directly in modal space, targeting specific eigenmodes rather than attempting to regulate the full state. This leads to more efficient control strategies, improved stability, and a deeper alignment between the controller and the system's intrinsic dynamics.

The predictive capability demonstrated in this work further reinforces this shift. Because motion evolves according to the operator spectrum, prediction becomes a natural consequence of the model rather than an external addition. The ability to predict residual motion at meaningful horizons without stochastic modeling suggests that the apparent unpredictability of jitter arises not from randomness, but from incomplete resolution of modal interactions. Once the operator structure is identified, prediction follows directly from its spectral evolution, providing a deterministic foundation for forecasting system behavior.

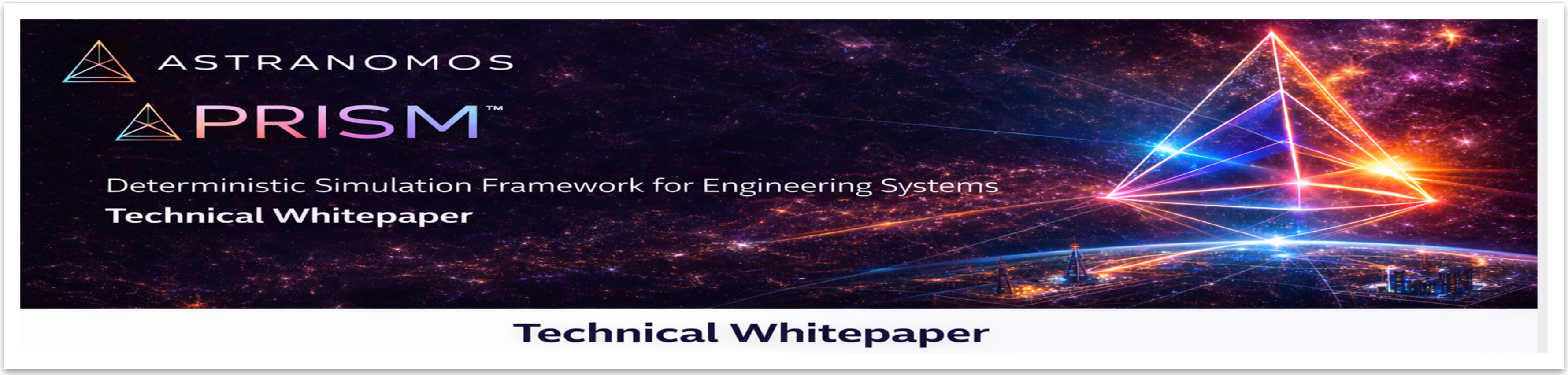
These findings also suggest that the distinction between "signal" and "noise" in robotic systems must be reconsidered. What is commonly treated as noise may, in fact, be a signal that has not yet been interpreted correctly. The narrow-band spectral concentration and low-rank structure observed across regimes indicate that residual motion carries meaningful information about the system's dynamics. Rather than filtering this information out, an operator-based approach allows it to be incorporated into the model, improving both understanding and performance.

In the context of system design, this opens the possibility of constructing robots whose physical and control architectures are aligned with their modal structure. Instead of designing systems and then compensating for their behavior, engineers can design systems whose dominant modes are well-understood and desirable. This could lead to robots that are inherently more stable, more predictable, and more efficient, as their dynamics would be shaped at the level of the operator rather than corrected at the level of the state.

More broadly, the transition from mesh-driven to operator-driven simulation represents a shift from local to global understanding of motion. Mesh-based methods emphasize local interactions and numerical resolution, while operator-based methods emphasize global structure and spectral properties. This shift parallels developments in other areas of science and engineering, where understanding the underlying structure of a system often leads to more powerful and efficient models than purely local approximations.

The results presented here suggest that such a shift is not only possible, but necessary for advancing robotic systems. As robots become more complex and operate in increasingly dynamic environments, the limitations of high-dimensional, noise-tolerant models become more pronounced. An operator-based framework provides a path toward models that are both simpler and more powerful, capturing the essential dynamics of motion while remaining computationally tractable.

Ultimately, these findings indicate that motion in robotic systems is governed by structure rather than randomness, and that this structure can be identified, modeled, and exploited. By moving from mesh-driven approximation to operator-driven representation, engineers gain access to a framework in which motion, control, and prediction are unified. This not only improves current capabilities, but lays the foundation for future systems in which the dynamics of motion are not something to be managed after the fact, but something that is understood and designed from the outset.



APPENDIX A: REFEREE ROADMAP FOR REPRODUCIBILITY

A.1 Objective of the Referee Protocol

This appendix provides a structured roadmap enabling independent referees to reproduce the principal empirical findings of this work, namely:

1. Residual motion is low-dimensional
2. Residual motion is spectrally concentrated
3. Modal structure persists across runs and regimes
4. Short-horizon prediction follows from operator-based evolution

The protocol is designed to be implementable using publicly available data and standard numerical tools, without requiring proprietary infrastructure.

A.2 Data Acquisition

All experiments in this work are conducted using the publicly available:

Mini Wheelbot Dataset
https://zenodo.org/records/18260659

Referees should download:

- Yaw-circle runs
- Velocity-roll runs

Each dataset includes:

- IMU measurements (gyro, accel)

- State estimates ($q_{\text{yaw}}, q_{\text{roll}}, q_{\text{pitch}}$)
- Angular velocities ($\dot{q}$)
- Actuator signals (wheel velocity, torque)
- Motion capture ground truth

All data are sampled at **1 kHz**.

A.3 Signal Construction

Referees should construct the residual motion signal as follows.

Step 1, Select Channels:

$$y(t) = \begin{bmatrix} \dot{q}_{\text{yaw}}(t) \\ \dot{q}_{\text{roll}}(t) \\ \dot{q}_{\text{reaction wheel}}(t) \end{bmatrix}$$

Step 2, Remove low-frequency trend:

Apply a low-pass filter H_{LP} to extract the command-driven component:

$$y_{\text{trend}}(t) = H_{\text{LP}}[y(t)]$$

Then compute residual:

$$r(t) = y(t) - y_{\text{trend}}(t)$$

A cutoff frequency of approximately **2–5 Hz** is sufficient.

Step 3, Standardization:

Normalize each channel:

$$r_i(t) \leftarrow \frac{r_i(t) - \mu_i}{\sigma_i}$$

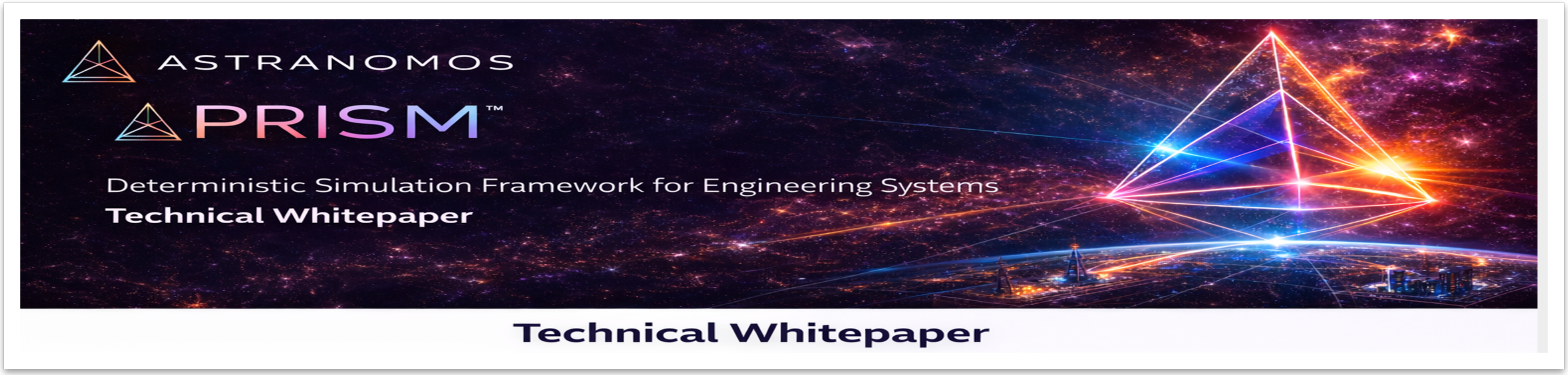
A.4 Modal Decomposition

Construct time-series matrix:

$$R = \begin{bmatrix} r(t_1) \\ r(t_2) \\ \vdots \\ r(t_T) \end{bmatrix} \in \mathbb{R}^{T \times 3}$$

Compute singular value decomposition:

$$R = U \Sigma V^\top$$



Validation Criterion 1, Low Rank:

Referees should verify:

$$\frac{\sigma_1^2 + \sigma_2^2}{\sum_i \sigma_i^2} \gtrsim 0.9$$

for most runs.

A.5 Spectral Analysis

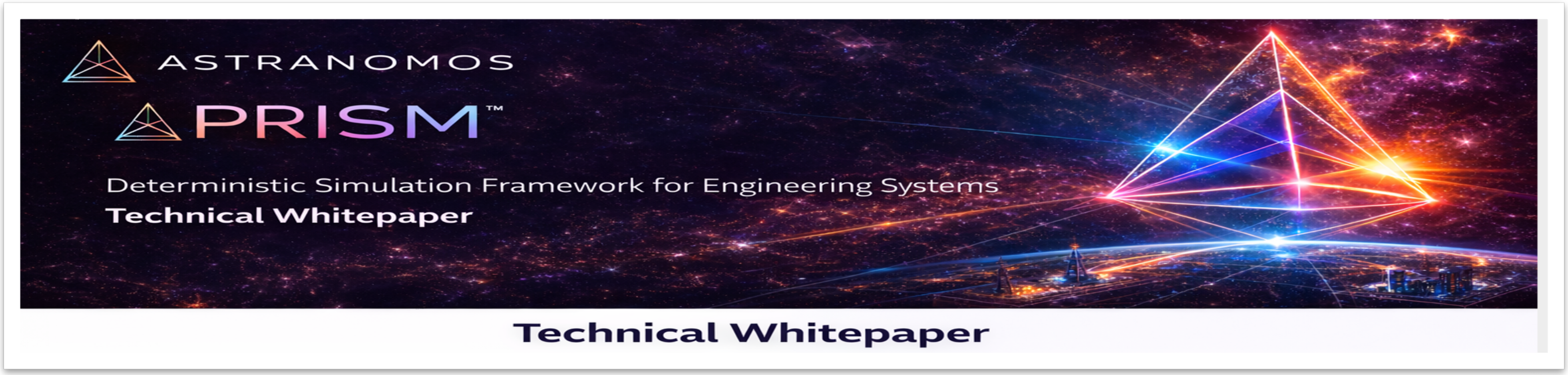
For each residual channel, compute Fourier transform:

$$\hat{r}(\omega) = \mathcal{F}[r(t)]$$

Validation Criterion 2 — Spectral Concentration:

Referees should observe:

- Dominant band near **2.2–3.0 Hz**
- Secondary bands near **1–1.5 Hz**
- Minimal broadband energy



A.6 Cross-Regime Consistency

Repeat Sections A.3–A.5 for:

- Yaw-circle data
- Velocity-roll data

Validation Criterion 3:

Referees should confirm:

- Same frequency family across regimes
- Similar low-rank structure
- Differences only in mode interaction, not structure

A.7 Reduced Operator Identification

Project residual into modal basis:

$$r(t) \approx \Phi a(t)$$

Estimate:

$$\dot{a}(t) \approx \frac{a(t + \Delta t) - a(t)}{\Delta t}$$

Fit operator:

$$\dot{a}(t) = L_m a(t)$$

using least-squares.

Validation Criterion 4:

Eigenvalues of L_m should satisfy:

$$\lambda = \alpha \pm i\omega$$

with:

$$\omega/(2\pi) \approx 2.2\text{–}3.0 \text{ Hz}$$

A.8 Predictive Validation

Using the reduced operator:

$$a(t+\tau) = e^{L_m \tau} a(t)$$
$$\hat{r}(t+\tau) = \Phi e^{L_m \tau} a(t)$$

Technical Whitepaper

Evaluate at:

- $\tau = 10$ ms
- $\tau = 50$ ms
- $\tau = 100$ ms

Validation Criterion 5:

Referees should observe:

- High accuracy at 10 ms
- Meaningful predictive power at 50 ms
- Degradation primarily at regime transitions

A.9 Interpretation Guidelines

Referees should assess whether observed properties are consistent with:

Stochastic Model:

- Broadband spectrum
- High-rank structure
- No cross-run consistency

Operator Model:

- Low-dimensional structure

Technical Whitepaper

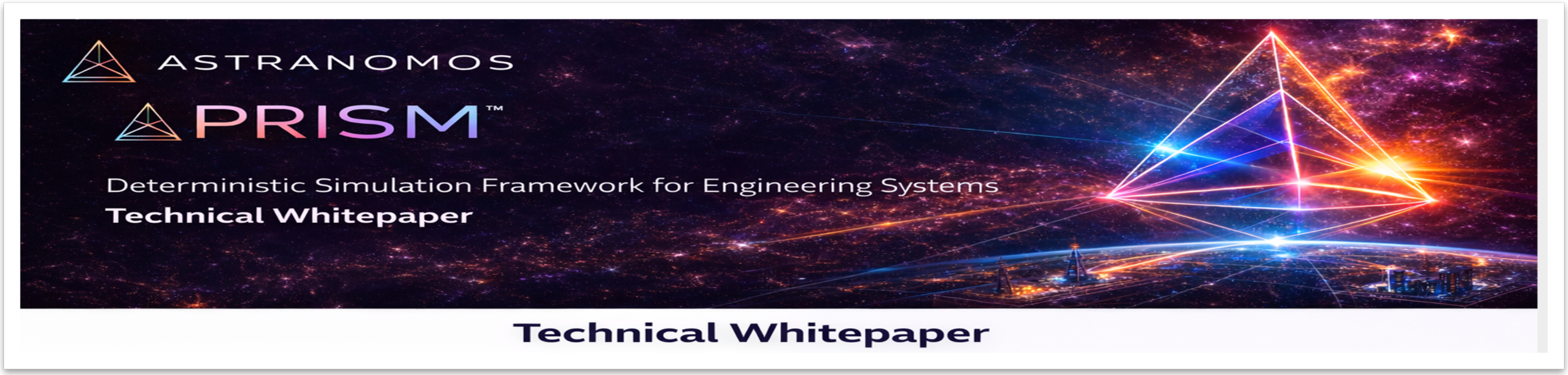
- Narrow-band frequencies
- Repeatable modal behavior
- Predictive evolution

A.10 Reproducibility Statement

All results in this work can be reproduced using:

- Publicly available dataset (Zenodo link above)
- Standard numerical tools (e.g., Python, MATLAB)
- Basic linear algebra and signal processing techniques

No proprietary data or specialized hardware is required.



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