KEY CONCEPT

Kinematics

Study of motion of objects without taking into account the factor which cause the motion (i.e. nature of force).

Motion

If a body changes its position with time, it is said to be moving else it is at rest. Motion is always relative to the observer.

Motion is a combined property of the object under study and the observer. There is no meaning of rest or motion without the viewer. In other words absolute motion or rest is meaningless.

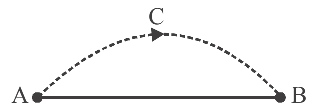
- To locate the position of a particle we need a reference frame. A commonly used reference frame is Cartesian coordinate system or simply coordinate system.

  The coordinates (x, y, z) of a particle specify the position of the particle with respect to origin of that frame. If all the three coordinates of the particle remain unchanged as time passes it means the particle is at rest w.r.t. this frame.
- The reference frame is chosen according to the problems.
- If the frame is not mentioned, then ground is taken as the reference frame.

DISTANCE AND DISPLACEMENT

Total length of path covered by the particle, in definite time interval is called distance.

Let a body moves from A to B via C. The length of path ACB is called the distance travelled by the body.



But overall, body is displaced from A to B. A vector from A to B, i.e. $\overrightarrow{AB}$ is its displacement vector or displacement that is the minimum distance and directed from initial position to final position.

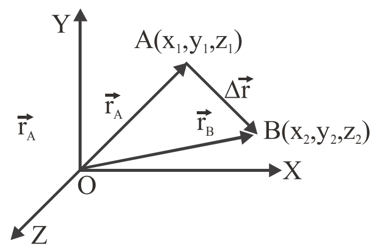
- Distance is a scalar while displacement is a vector.
- Distance depends on path while displacement is independent of path but depends only on final and initial position.
- For a moving body, distance can't have zero or negative values but displacement may be +ive, –ive or zero.
- For a moving/stationary object distance can't be decreasing.
- If motion is in straight line without change in direction then

  distance = |displacement| i.e. magnitude of displacement.
- Magnitude of displacement may be equal or less than distance but never greater than distance.

  distance $\geq$ |displacement|

Displacement in terms of position vector

Let a body is displaced from A (x_1, y_1, z_1) to B (x_2, y_2, z_2) then its displacement is given by vector $\vec{AB}$.



From ΔOAB $\vec{r}_A + \Delta\vec{r} = \vec{r}_B \Rightarrow \Delta\vec{r} = \vec{r}_B - \vec{r}_A$

$$\vec{r}_B = x_2\hat{i} + y_2\hat{j} + z_2\hat{k} \text{ and } \vec{r}_A = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$

$$\Delta\vec{r} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k} \Rightarrow \Delta\vec{r} = \Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}$$

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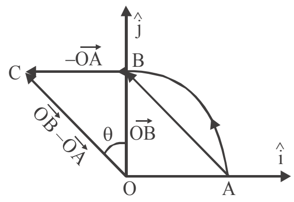
Ex. A particle goes along a quadrant from A to B of a circle radius 10m as shown in fig. Find the direction and magnitude of displacement and distance along path AB.

Sol. Displacement $\vec{AB} = \vec{OB} - \vec{OA} = 10\hat{j} - 10\hat{i}$



$|\vec{AB}| = \sqrt{10^2 + 10^2} = 10\sqrt{2}$ m

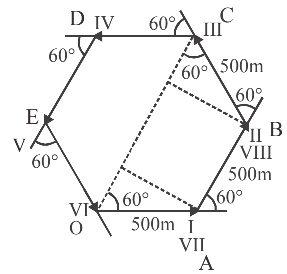
From ΔOBC $\tan\theta = \dfrac{OA}{OB} = \dfrac{10}{10} = 1 \Rightarrow \theta = 45°$

Angle between displacement vector $\vec{OC}$ and x–axis = 90° + 45° = 135°

Distance of path AB = $\dfrac{1}{4}$ (circumference) = $\dfrac{1}{4}(2\pi R)$ m = (5π) m

Ex. On an open ground a motorist follows a track that turns to his left by an angle of 60° after every 500 m. Starting from a given turn, specify the displacement of the motorist at the third, sixth and eighth turn. Compare the magnitude of displacement with the total path length covered by the motorist in each case.

Sol. **At III turn**



Displacement $= \overline{OA} + \overline{AB} + \overline{BC} = \overline{OC}$

= 500 cos 60° + 500 + 500 cos 60°

$= 500 \times \dfrac{1}{2} + 500 + 500 \times \dfrac{1}{2} = 1000$ m from O to C

Distance = 500 + 500 + 500 = 1500 m. So $\dfrac{\text{Displacement}}{\text{Distance}} = \dfrac{1000}{1500} = \dfrac{2}{3}$

At VI turn

$\because$ initial and final positions are same so displacement = 0 and distance = 500 × 6 = 3000 m

$\therefore \dfrac{\text{Displacement}}{\text{Distance}} = \dfrac{0}{3000} = 0$

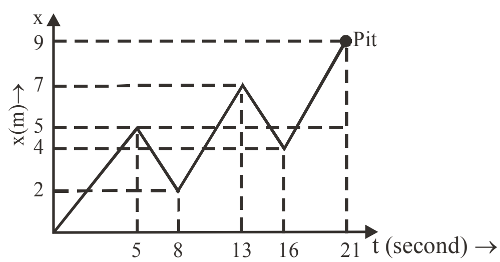
At VIII turn

$$\text{Displacement} = 2(500)\cos\left(\frac{60°}{2}\right) = 1000 \times \cos 30° = 1000 \times \frac{\sqrt{3}}{2} = 500\sqrt{3} \text{ m}$$

Distance = 500 × 8 = 4000 m

$\therefore \dfrac{\text{Displacement}}{\text{Distance}} = \dfrac{500\sqrt{3}}{4000} = \dfrac{\sqrt{3}}{8}$

Ex. A drunkard walking in a narrow lane takes 5 steps forward and 3 steps backward, followed again by 5 steps forward and 3 steps backward, and so on. Each step is 1m long and requires 1s. Plot the x–t graph of his motion. Determine graphically or otherwise how long the drunkard takes to fall in a pit 9m away from the start.



Sol. from x–t graph time taken $= 21$ s

or $(5m - 3m) + (5m - 3m) + 5m = 9m$

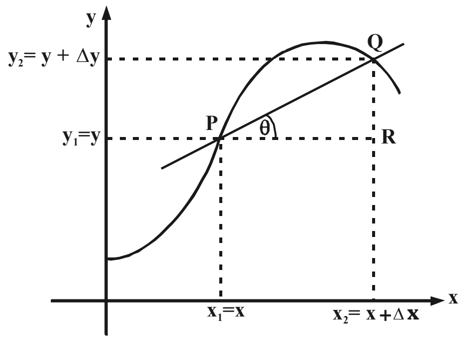
$\Rightarrow$ total steps $= 21 \Rightarrow$ time $=$ 21 s

DERIVATIVE OF A FUNCTION

Average Rate of Change

Let a function $y = f(x)$ be plotted by an arbitrary graph as shown in the figure. Average rate of change in y with respect to x in an interval $[x_1, x_2]$ is defined as

$$\text{Average rate of change} = \frac{\text{Change in y}}{\text{Change in x}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$



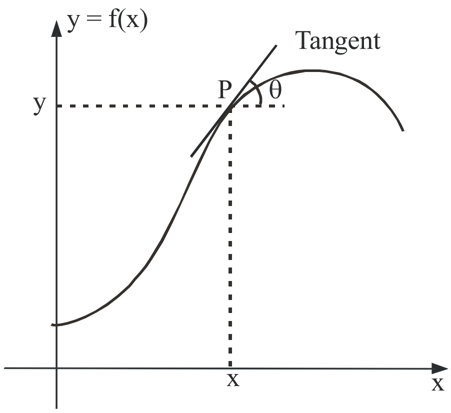
= slope of chord

If both the axes have equal scale

Average rate of change = slope of chord PQ = $\tan\theta$

Instantaneous Rate of Change : First derivative

It is defined as the rate of change in y with x at a particular value of x. It is measured graphically by the slope of the tangent drawn to the y-x graph at the point (x, y) and algebraically by the first derivative of the function $y=f(x)$.



$$\text{Instantaneous rate of change} = \frac{dy}{dx} = \text{Slope of the tangent}$$

If both the axes have equal scale then $\frac{dy}{dx} = \tan\theta$

$$\text{Instantaneous rate of change} = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Derivatives of Commonly Used Functions.

- $y = \text{constant} \Rightarrow \frac{dy}{dx} = 0$
- $y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}$
- $y = e^x \Rightarrow \frac{dy}{dx} = e^x$
- $y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x}$
- $y = \sin x \Rightarrow \frac{dy}{dx} = \cos x$
- $y = \cos x \Rightarrow \frac{dy}{dx} = -\sin x$
- $y = \tan x \Rightarrow \frac{dy}{dx} = \sec^2 x$
- $y = \cot x \Rightarrow \frac{dy}{dx} = -\text{cosec}^2 x$
- $y = \text{cosec}\, x \Rightarrow \frac{dy}{dx} = -\text{cosec}\, x \cot x$
- $y = \sec x \Rightarrow \frac{dy}{dx} = \sec x \tan x$

Method of Differentiation.

If y = f(x), let us denote $\frac{dy}{dx} = f'(x)$

- Sum or Subtraction of two functions $y = f(x) \pm g(x) \Rightarrow \frac{dy}{dx} = f'(x) \pm g'(x)$
- Product of two functions

  $$y = f(x)\cdot g(x) \Rightarrow \frac{dy}{dx} = g(x)f'(x) + f(x)g'(x)$$
- Division of two functions.

  $$y = \frac{f(x)}{g(x)} \Rightarrow \frac{dy}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{\{g(x)\}^2}$$
- Chain Rule

  $$y = f\{g(x)\} \Rightarrow \frac{dy}{dx} = g'(x)f'\{g(x)\}$$

Ex. Find $\frac{dy}{dx}$, when (i) $y = \sqrt{x}$ (ii) $y = x^5 + x^4 + 7$ (iii) $y = x^2 + 4x^{-1/2} - 3x^{-2}$

Sol. (i) $\frac{dy}{dx} = \frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

(ii) $\frac{dy}{dx} = \frac{d}{dx}(x^5 + x^4 + 7) = \frac{d}{dx}(x^5) + \frac{d}{dx}(x^4) + \frac{d}{dx}(7) = 5x^4 + 4x^3 + 0 = 5x^4 + 4x^3$

(iii) $\frac{dy}{dx} = \frac{d}{dx}(x^2 + 4x^{-1/2} - 3x^{-2}) = \frac{d}{dx}(x^2) + \frac{d}{dx}(4x^{-1/2}) - \frac{d}{dx}(3x^{-2})$

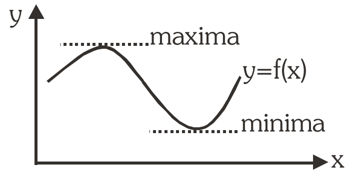
$$= \frac{d}{dx}(x^2) + 4\frac{d}{dx}(x^{-1/2}) - 3\frac{d}{dx}(x^{-2}) = 2x + 4\left(-\frac{1}{2}\right)x^{-3/2} - 3(-2)x^{-3} = 2x - 2x^{-3/2} + 6x^{-3}$$

Second Derivative and it's meaning

Second derivative of a function y = f(x) is defined as $\frac{d}{dx}\left[\frac{dy}{dx}\right]$. It is obtained by differentiating the function with respect to x two times successively. Geometrically it expresses rate of change in slope of graph of the function.

Maxima & Minima

Maxima & minima of a function y = f(x)



for maximum value $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} =$ negative

for minimum value $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} =$ positive

Ex. Find minimum value of $y = 25x^2 - 10x + 5$.

Sol. For maximum/minimum value $\frac{dy}{dx} = 0 \Rightarrow 50x - 10 = 0 \Rightarrow x = \frac{1}{5}$

Now at $x = \frac{1}{5}$, $\frac{d^2y}{dx^2} = 50$ which is positive. So $y_{min} = 25\left(\frac{1}{5}\right)^2 - 10\left(\frac{1}{5}\right) + 5 = 1 - 2 + 5 = 4$

INTEGRATION

- This operation enables us to find sum of infinite number of infinitely small quantities.

$$\lim_{\Delta x \to 0} \sum_{i=1}^{\infty} \Delta x_i = \int dx = x$$

- It is reverse operation of differentiation. If derivative, which is rate of change at a point, is given as a function $f(x) = \frac{dF(x)}{dx}$, operation of integration enables us to find original function F(x).

$$\int f(x)dx = F(x) + C$$

Here function f(x) is known as integrand, function F(x) as integral and C as constant of integration. Value of C is obtained by substituting initial, final or any other condition (known as boundary conditions) in the above equation.

This interpretation enables us to find integral of those functions whose derivative is known.

Integrand $f(x) = \frac{dF(x)}{dx}$	Integral $\int f(x)dx = F(x) + C$
k = Constant	kx+C
x^n	$\frac{x^{n+1}}{n+1} + C$ If $n \neq -1$
x^{-1}	$\ln x + C$
e^x	$e^x + C$
sin x	$-\cos x + C$
cos x	$\sin x + C$
f(ax+b)	$\frac{F(ax+b)}{a} + C$

Ex. Evaluate the following:

(i) $\int x^{-7}dx$ (ii) $\int x^{p/q}dx$

Sol. (i) $\int x^{-7}dx = \frac{x^{-7+1}}{-7+1} + c = -\frac{1}{6}x^{-6} + c$

(ii) $\int x^{\frac{p}{q}}dx = \frac{x^{\frac{p}{q}+1}}{\frac{p}{q}+1} + c = \frac{q}{p+q}x^{(p+q)/q} + c$

Ex. Evaluate $\int\left(x^2 - \cos x + \frac{1}{x}\right)dx$

Sol. $I = \int x^2dx - \int \cos x dx + \int \frac{1}{x}dx = \frac{x^{2+1}}{2+1} - \sin x + \log_e x + c = \frac{x^3}{3} - \sin x + \log_e x + c$

SPEED :

Speed is the rate of change of distance with respect to time.

Uniform speed :

An object is said to be moving with a uniform speed, if it covers equal distances in equal intervals of time, irrespective of duration of interval. The uniform speed is shown by straight line in distance–time graph.

For example, suppose a train travels 1000 m in 60 s. The train is said to be moving with uniform speed, if it travels 500 m. in 30s, 250 m in 15s, 125 m in 7.5s and so on.

Non Uniform speed :

An object is said to be moving with a variable speed if it covers equal distances in unequal intervals of time or unequal distances in equal intervals of time, irrespective of duration of interval.

For example, suppose a train travels first 1000 m in 60s next 1000 m in 120s and next 1000m in 50s, then the train is moving with variable speed.

Average speed :

Speed is distance travelled per unit time. Average speed of a trip $v_{av} = \dfrac{\text{Total travelled distance}}{\text{Total time taken}}$

If a particle travels a distance s in time t_1 to t_2, the average speed is $v_{av} = \dfrac{\Delta s}{\Delta t} = \dfrac{s}{t_2 - t_1}$

Instantaneous speed

The speed at a particular instant is defined as instantaneous speed (or speed) while average speed is defined for a time interval.

If Δt approaches zero, average speed becomes instantaneous speed. $v = \lim\limits_{\Delta t \to 0} \dfrac{\Delta s}{\Delta t} = \dfrac{ds}{dt}$

i.e. instantaneous speed is the time derivative of distance.

If a particle travels distances s_1, s_2, s_3 etc. with speeds v_1, v_2, v_3 etc. respectively, then total travelled distance

$$s = s_1 + s_2 + s_3 + ... + s_n$$

Total time taken $t = \dfrac{s_1}{v_1} + \dfrac{s_2}{v_2} + \dfrac{s_3}{v_3} + ... + \dfrac{s_n}{v_n}$

No sign is needed for distance or speed. They are always positive quantities.

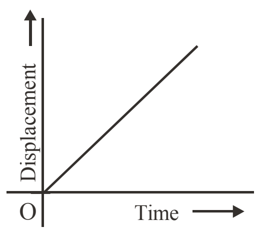
Average speed of the trip $= \dfrac{s_1 + s_2 + s_3 ...... + s_n}{\left(\dfrac{s_1}{v_1} + \dfrac{s_2}{v_2} + ... + \dfrac{s_n}{v_n}\right)}$

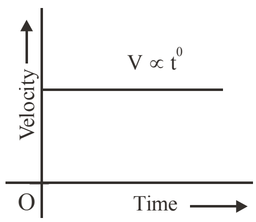
VELOCITY :

Velocity is the rate of change of displacement with respect to time.

Uniform Velocity :

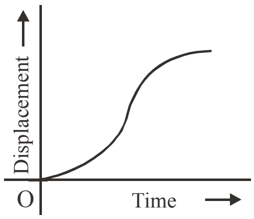
A body is said to move with uniform velocity, if it covers equal displacements in equal intervals of time, irrespective of duration of interval. When a body is moving with uniform velocity, then the magnitude and direction of the velocity of the body remains same at all points of its path.





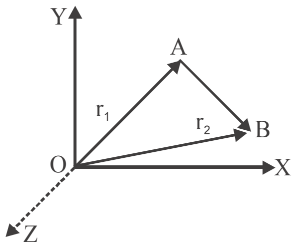
Non–uniform Velocity :

The particle is said to have non–uniform motion if it covers unequal displacements in equal intervals of time, irrespective of duration of interval. In this type of motion velocity does not remain constant.



Average Velocity

The average velocity of a particle in a time interval t_1 to t_2 is a defined as its displacement divided by the time interval. Let a particle is at a point A at time t_1 and B at time t_2. Position vectors of A and B are $\vec{r}_1$ and $\vec{r}_2$. The displacement in this time interval is the vector $\overrightarrow{AB} = (\vec{r}_2 - \vec{r}_1)$. The average velocity in this time interval is, $\vec{v}_{av} = \dfrac{\text{displacement vector}}{\text{time interval}}$

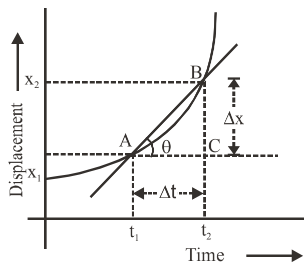


$$\vec{v}_{av} = \frac{\overrightarrow{AB}}{t_2 - t_1} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1} = \frac{\Delta\vec{r}}{\Delta t}$$

here $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{r}_2 - \vec{r}_1$ = change in position vector.

Instantaneous velocity

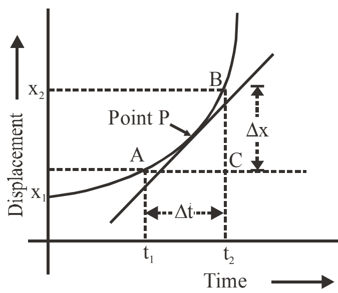
The velocity of the object at a given instant of time or at a given position during motion is called instantaneous velocity $\vec{v} = \lim\limits_{\Delta t \to 0} \dfrac{\Delta\vec{r}}{\Delta t} = \dfrac{d\vec{r}}{dt}$



From fig., the average velocity between points A and B is

$$v_{av} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t} = \text{slope of secant AB} = \tan\theta$$

Average velocity is equal to slope of straight line joining two points on displacement time graph. If $\Delta t \to 0$, then average velocity becomes instantaneous velocity.

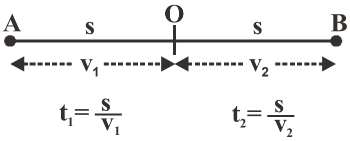


$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = \text{slope of tangent at P} = \tan\alpha$$

- Magnitude of instantaneous velocity is the instantaneous speed.

Note : When a particle moves with constant velocity, its magnitude of average velocity, its magnitude of instantaneous velocity and its speed all are equal.

Ex. If a particle travels the first half distance with speed v_1 and second half distance with speed v_2.Find its average speed during journey.

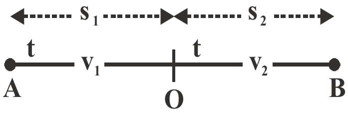


Sol. $$v_{avg.} = \frac{s+s}{t_1+t_2} = \frac{2s}{\frac{s}{v_1}+\frac{s}{v_2}} = \frac{2v_1v_2}{v_1+v_2}$$

Note :– Here $v_{avg.}$ is the harmonic mean of two speeds.

Ex. If a particle travels with speed v_1 during first half time interval and with v_2 speed during second half time interval. Find its average speed during its journey.

Sol. Total distance $= s_1 + s_2 = v_1t+v_2t = (v_1 + v_2)t$



Total time $= t + t = 2t$ $$v_{avg.} = \frac{s_1+s_2}{t+t} = \frac{(v_1+v_2)t}{2t} = \frac{v_1+v_2}{2}$$

Note :– here v_{avg} is arithmatic mean of two speeds.

Ex. A car travels a distance A to B at a speed of 40 km/h and returns to A at a speed of 30 km/h.
(i) What is the average speed for the whole journey?
(ii) What is the average velocity?

Sol. (i) Let AB = s, time taken to go from A to B, $t_1 = \frac{s}{40}$ h

and time taken to go from B to A, $t_2 = \frac{s}{30}$ h

$\therefore$ total time taken $= t_1 + t_2 = \frac{s}{40}+\frac{s}{30} = \frac{(3+4)\ s}{120} = \frac{7s}{120}$ h

Total distance travelled = s + s = 2s

$$\therefore \text{ Average speed } = \frac{\text{total distance travelled}}{\text{total time taken}} = \frac{2s}{\frac{7s}{120}} = \frac{120\times 2}{7} = 34.3 \text{ km/h.}$$

(ii) Total displacement = zero, since the car returns to the original position.

$$\text{Therefore, average velocity} = \frac{\text{total displacement}}{\text{time taken}} = \frac{0}{2t} = 0$$

Ex. A man walks on a straight road from his home to a market 2.5 km away with a speed of 5 km/h. On reaching the market he instantly turns and walks back with a speed of 7.5 km/h. What is the (a) magnitude of average velocity and (b) average speed of the man, over the interval of time (i) 0 to 30 min. (ii) 0 to 50 min (iii) 0 to 40 min.

Sol. Time taken by man to go from his home to market, $t_1 = \frac{\text{distance}}{\text{speed}} = \frac{2.5}{5} = \frac{1}{2}$ h

Time taken by man to go from market to his home, $t_2 = \frac{2.5}{7.5} = \frac{1}{3}$ h

$\therefore$ Total time taken $= t_1 + t_2 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}\text{h} = 50$ min.

(i) 0 to 30 min

$$\text{Average velocity} = \frac{\text{displacement}}{\text{time interval}} = \frac{2.5}{\frac{30}{60}} = 5 \text{ km/h towards market}$$

$$\text{Average speed} = \frac{\text{distance}}{\text{time interval}} = \frac{2.5}{\frac{30}{60}} = 5 \text{ km/h}$$

(ii) 0 to 50 min

Total displacement = zero so average velocity = 0

$$\text{So, average speed} = \frac{5}{50/60} = 6 \text{ km/h}$$

Total distance travelled = 2.5 + 2.5 = 5 km.

(iii) 0 to 40 min

Distance moved in 30 min (from home to market) = 2.5 km.

Distance moved in 10 min (from market to home) with speed 7.5 km/h $= 7.5 \times \frac{10}{60} = 1.25$ km

So, displacement = 2.5 – 1.25 = 1.25 km (towards market)

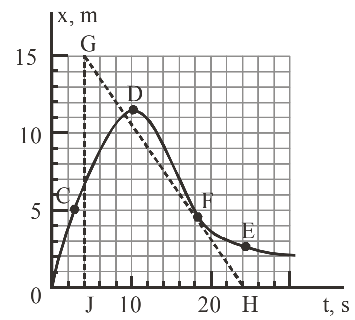
Distance travelled = 2.5 + 1.25 = 3.75 km

$$\text{Average velocity} = \frac{1.25}{\frac{40}{60}} = 1.875 \text{ km/h. (towards market)}$$

$$\text{Average speed} = \frac{3.75}{\frac{40}{60}} = 5.625 \text{ km/h.}$$

Note : Moving body with uniform speed may have variable velocity. e.g. in uniform circular motion speed is constant but velocity is non–uniform.

Ex. Refer to figure for the motion of an object along the x-axis.



What is the instantaneous velocity of the object at (a) F (b) D

Sol. **(a)** The tangent at F is the dashed line GH. Taking triangle GHJ,

$\Delta t = 24 - 4 = 20$ s $\quad \Delta x = 0 - 15 = -15$m

Hence slope at F is $v_F = \dfrac{\Delta x}{\Delta t} = \dfrac{-15\text{m}}{20\ \text{s}} = -0.75$ m/s

The negative sign tells us that the object is moving in the –x direction.

(b) At point D slope of curve is zero so $v = \dfrac{dx}{dt} = 0$.

ACCELERATION :

Acceleration

The acceleration is rate of change of velocity or change in velocity per unit time interval. Velocity is a vector quantity hence a change in its magnitude or in direction or in both, will change the velocity .

Uniform acceleration :

An object is said to be moving with a uniform acceleration if its velocity changes by equal amounts in equal intervals of time, irrespective of duration of intervals.

Variable acceleration :

An object is said to be moving with a variable acceleration if its velocity changes by unequal amounts in equal intervals of time, irrespective of duration of intervals..

Average Acceleration :

When an object is moving with a variable acceleration, then the average acceleration of the object for the given motion is defined as the ratio of the total change in velocity of the object during motion to the total time taken

$$\text{Average Acceleration} = \frac{\text{Total change in velocity}}{\text{total time taken}}$$

Suppose the velocity of a particle is $\vec{v}_1$ at time t_1 and $\vec{v}_2$ at time t_2. Then $\vec{a}_{av} = \dfrac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \dfrac{\Delta \vec{v}}{\Delta t}$

Instantaneous Acceleration :

The acceleration of the object at a given instant of time or at a given point of motion, is called its instantaneous acceleration. Suppose the velocity of a particle at time $t_1 = t$ is $\vec{v}_1 = \vec{v}$ and becomes $\vec{v}_2 = \vec{v} + \Delta\vec{v}$ at time $t_2 = t + \Delta t$, Then, $\vec{a}_{av} = \dfrac{\Delta \vec{v}}{\Delta t}$

If Δt approaches to zero then the rate of change of velocity will be instantaneous acceleration.

$$\text{Instantaneous acceleration } \vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}$$

Ex. An athlete takes 2 second to reach the maximum speed of 18 km/h from rest. What is the magnitude of his average acceleration.

Sol. Here, Initial velocity u = 0, $v = (v_{max}) = 18$ km/h $= 18 \times \dfrac{5}{18} = 5$ m/s, $t_1 = 0$s, $t_2 = 2$s.

$$a_{av} = \frac{v - u}{t_2 - t_1} = \frac{5.0}{2} = 2.5 \text{ m/s}^2$$

Ex. A car moving with a velocity of 20 ms^{-1} is brought to rest in 5 seconds by applying brakes. Calculate the retardation of the car.

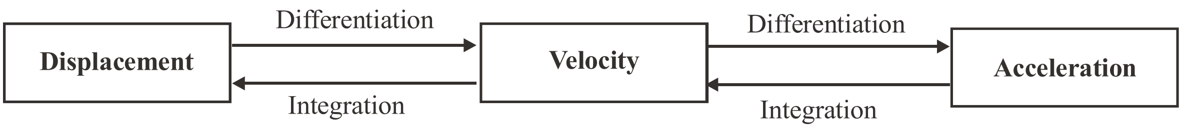
Sol. Here, $u = 20\ ms^{-1}$, $v = 0$, $t = 5s$. acceleration $a = \frac{v-u}{t} = \frac{(0-20)}{5} = -4 m/s^2$

–ve acceleration is known as retardation. Thus, retardation of the car = 4 ms^{-2}.

Use of Mathematical Tools in Solving Problems of One–Dimensional Motion

If displacement–time equation is given, we can get velocity–time equation with the help of differentiation. Again, we can get acceleration–time equation with the help of differentiation.

If acceleration–time equation is given, we can get velocity–time equation by integration. From velocity equation, we can get displacement–time equation by integration.



Ex. The velocity of any particle is related with its displacement as; $x = \sqrt{v+1}$, Calculate acceleration at $x = 5$ m.

Sol. $\therefore x = \sqrt{v+1}$ $\therefore x^2 = v + 1 \Rightarrow v = (x^2-1)$

Therefore $a = \frac{dv}{dt} = \frac{d}{dt}\left(x^2-1\right) = 2x\frac{dx}{dt} = 2x\ v = 2x\left(x^2-1\right)$

At x = 5 m, a = 2 × 5 (25 –1) = 240 m/s^2

Ex. The velocity of a particle moving in the positive direction of x–axis varies as $v = \alpha\sqrt{x}$ where α is positive constant. Assuming that at the moment $t = 0$, the particle was located at $x = 0$ find, (i) the time dependance of the velocity and the acceleration of the particle and (ii) the average velocity of the particle averaged over the time that the particle takes to cover first s metres of the path.

Sol. (i) Given that $v = \alpha\sqrt{x}$

$$\Rightarrow \frac{dx}{dt} = \alpha\sqrt{x} \therefore \frac{dx}{\sqrt{x}} = \alpha\, dt \Rightarrow \int_0^x \frac{dx}{\sqrt{x}} = \int_0^t \alpha\, dt \quad 2\sqrt{x} = \alpha t \Rightarrow x = \left(\alpha^2t^2/4\right)$$

Velocity $\frac{dx}{dt} = \frac{1}{2}\alpha^2 t$ and Acceleration $\frac{d^2x}{dt^2} = \frac{1}{2}\alpha^2$

(ii) Time taken to cover first s metres $s = \frac{\alpha^2t^2}{4} \Rightarrow t^2 = \frac{4s}{\alpha^2} \Rightarrow t = \frac{2\sqrt{s}}{\alpha}$;

$$\text{average velocity} = \frac{\text{total displacement}}{\text{total time}} = \frac{s\alpha}{2\sqrt{s}} = \frac{1}{2}\sqrt{s\alpha}$$

Ex. A particle moves in the plane xy with constant acceleration a directed along the negative y–axis. The equation of motion of the particle has the form $y = px - qx^2$ where p and q are positive constants. Find the velocity of the partcle at the origin of coordinates.

Sol. Given that $y = px - qx^2$

$$\therefore \frac{dy}{dt} = p\frac{dx}{dt} - q.2x\frac{dx}{dt} \text{ and } \frac{d^2y}{dt^2} = p\frac{d^2x}{dt^2} - 2qx\frac{d^2x}{dt^2} - 2q\left(\frac{dx}{dt}\right)^2 = (p-2qx)\frac{d^2x}{dt^2} - 2q\left(\frac{dx}{dt}\right)^2$$

$$\because \frac{d^2x}{dt^2} = 0 \text{ (no acceleration along x–axis) and } \frac{d^2y}{dt^2} = -a$$

$$\therefore v_x^2 = \frac{a}{2q} \Rightarrow v_x = \sqrt{\frac{a}{2q}} \text{ Further, } \left(\frac{dy}{dt}\right)_{x=0} = p\frac{dx}{dt} \Rightarrow v_y = p\sqrt{\left(\frac{a}{2q}\right)}$$

$$\text{Now } v = \sqrt{\left(v_x^2 + v_y^2\right)} = \sqrt{\left(\frac{a}{2q} + \frac{ap^2}{2q}\right)} \Rightarrow v = \sqrt{\left[\frac{a\left(p^2+1\right)}{2q}\right]}$$

Ex. A particle moves along a straight line path such that its magnitude of velocity is given by $v = (3t^2 - 6t)\ ms^{-1}$, where t is the time in seconds. If it is initially located at the origin O then determine the magnitude of particle's average velocity and average speed in time interval from t = 0 to t = 4s.

Sol. $$\text{Average velocity} = \frac{\int v\,dt}{\int dt} = \frac{\int_0^4 \left(3t^2 - 6t\right)dt}{\int_0^4 dt} = \frac{\left(t^3 - 3t^2\right)_0^4}{(t)_0^4} = 4\ ms^{-1}$$

$$\text{Average speed} = \frac{\int |v|\,dt}{\int dt} = \frac{\int_0^4 \left|3t^2 - 6t\right|dt}{\int_0^4 dt} = \frac{\int_0^2 \left(6t - 3t^2\right)dt + \int_2^4 \left(3t^2 - 6t\right)dt}{\int_0^4 dt}$$

$$= \frac{\left(3t^2 - t^3\right)_0^2 + \left(t^3 - 3t^2\right)_2^4}{(t)_0^4} = \frac{24}{4} = 6\ ms^{-1}$$

Ex. The coordinates a particle moving in a plane are given by x = 3 cos2t and y = 4sin2t.

(i) Find the equation of the path of the particle.

(ii) Find the angle between $\vec{r}$ and $\vec{v}$ at $t = \frac{\pi}{4}$.

(iii) Prove that acceleration of the particle is always directed towards a fixed point.

Sol. (i) Eliminating t from x = 3 cos2t & y = 4sin2t. We get $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1 \Rightarrow \frac{x^2}{9} + \frac{y^2}{16} = 1$

(ii) $\vec{r} = x\hat{i} + y\hat{j} = 3\cos 2t\,\hat{i} + 4\sin 2t\,\hat{j} \Rightarrow \vec{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} = -6\sin 2t\,\hat{i} + 8\cos 2t\,\hat{j}$

At $t = \frac{\pi}{4}$, $\vec{r} = 4\hat{j}$, $\vec{v} = -6\hat{i}$

Angle between $\vec{r}$ and $\vec{v}$ $\cos\theta = \frac{\vec{r}.\vec{v}}{rv} = \frac{(4\hat{j}).(-6\hat{i})}{(4)(6)} = 0 \Rightarrow \theta = \frac{\pi}{2}$

(iii) $\vec{a} = \frac{d\vec{v}}{dt} = -12\cos 2t\,\hat{i} - 16\sin 2t\,\hat{j} = -4\,(3\cos 2t\,\hat{i} + 4\sin 2t\,\hat{j}) = -4\vec{r}$

So acceleation is always directed toward origin (a fixed point)

Ex. A particle moves in a straight line according to the relation $x = \frac{t^3}{3} - \frac{5t^2}{2} + 6t$.

Find the displacement and distance travelled by the particle upto t = 4 sec.

Sol. $v = \frac{dx}{dt} = t^2 - 5t + 6$

The particle turns when $v = 0 = t^2 - 5t + 6 \Rightarrow (t-2)(t-3) = 0$ i.e. t = 2 sec, 3 secs

Displacement $= x(4) - x(0) = \frac{64}{3} - \frac{80}{2} + 24 = \frac{16}{3}$ m

Distance $= |x(2) - x(0)| + |x(3) - x(2)| + |x(4) - x(3)|$

$$= \left[\frac{8}{3} - 10 + 12\right] + \left|\left(9 - \frac{45}{2} + 18 - \frac{14}{3}\right)\right| + \left|\frac{16}{3} - \frac{9}{2}\right| = \frac{2}{3} + \frac{1}{6} + \frac{5}{6} = \frac{5}{3}\text{ m}$$

Alter : distance $= \int_0^4 |v|\,dt$

Equations of motion (motion with constant acceleration)

If a particle moves with acceleration $\vec{a}$, then by definition $\vec{a} = \frac{d\vec{v}}{dt} \Rightarrow d\vec{v} = \vec{a}dt$. Let at starting (t = 0)

initial velocity of the particle $\vec{u}$ and at time t its final velocity = $\vec{v}$ then $\int_{\vec{u}}^{\vec{v}} d\vec{v} = \int_0^t \vec{a}dt$

If acceleration is constant

$$\int_{\vec{u}}^{\vec{v}} d\vec{v} = \vec{a}\int_0^t dt \Rightarrow [\vec{v}]_{\vec{u}}^{\vec{v}} = \vec{a}[t]_0^t \Rightarrow \vec{v} - \vec{u} = \vec{a}\,t \Rightarrow \vec{v} = \vec{u} + \vec{a}t \qquad \text{........(1)}$$

Now by definition of velocity, equation (1) reduces to

$$\vec{v} = \frac{d\vec{s}}{dt} = \vec{u} + \vec{a}t \Rightarrow \int_0^{\vec{s}} d\vec{s} = \int_0^t (\vec{u} + \vec{a}t)\,dt \Rightarrow \vec{s} = \left[\vec{u}t + \frac{1}{2}\vec{a}t^2\right]_0^t \Rightarrow \vec{s} = \vec{u}t + \frac{1}{2}\vec{a}t^2 \qquad \text{........(2)}$$

Now substituting the value of t from equation (1) to equation (2)

$$s = u\frac{(v-u)}{a} + \frac{1}{2}a\left[\frac{v-u}{a}\right]^2 \Rightarrow 2as = 2uv - 2u^2 + v^2 + u^2 - 2uv \Rightarrow v^2 = u^2 + 2as \text{........(iii)}$$

vector form of equation (iii) $\boxed{v^2 = u^2 + 2\vec{a}.\vec{s}}$ ....(3)

These three equation are called equations of motion and are applicable only and only when acceleration is constant.

Distance travelled by the body in n^{th} second

$$s_{n^{th}} = s_n - s_{n-1} = un + \frac{1}{2}an^2 - u(n-1) - \frac{1}{2}a(n-1)^2 = un + \frac{1}{2}an^2 - un + u - \frac{1}{2}an^2 + an - \frac{a}{2}$$

vector form of equation (iv)

$$\boxed{s_{n^{th}} = u + \frac{a}{2}(2n-1)} \quad \text{........(4)}$$

Ex. A driver takes 0.20 s to apply the brakes after he sees a need for it. This is called the reaction time of the driver. If he is driving a car at a speed of 54 km/h and the brakes cause a deceleration of $6.0 m/s^2$, find the distance travelled by the car after he sees the need to put the brakes on

Sol. Distance covered by the car during the application of brakes by driver –

$$s_1 = ut = \left(54 \times \frac{5}{18}\right)(0.2) = 15 \times 0.2 = 3.0 \text{ meter}$$

After applying the brakes; $v = 0$ $u = 15$ m/s, $a = 6$ m/s^2 $s_2 = ?$

Using $v^2 = u^2 - 2as \Rightarrow 0 = (15)^2 - 2 \times 6 \times s_2 \Rightarrow 12\ s_2 = 225 \Rightarrow s_2 = \frac{225}{12} = 18.75$ metre

Distance travelled by the car after driver sees the need for it $s = s_1 + s_2 = 3 + 18.75 = 21.75$ metre.

Ex. A passenger is standing d distance away from a bus. The bus begins to move with constant acceleration a. To catch the bus, the passenger runs at a constant speed u towards the bus. What must be the minimum speed of the passenger so that he may catch the bus?

Sol. Let the passenger catch the bus after time t.

The distance travelled by the bus, $s_1 = 0 + \frac{1}{2}at^2$ ....(1)

and the distance travelled by the passenger $s_2 = ut + 0$ ....(2)

Now the passenger will catch the bus if $d + s_1 = s_2$ ....(3)

$$\Rightarrow d + \frac{1}{2}at^2 = ut \Rightarrow \frac{1}{2}at^2 - ut + d = 0 \Rightarrow t = \frac{[u \pm \sqrt{u^2 - 2ad}]}{a}$$

So the passenger will catch the bus if t is real, i.e., $u^2 \geq 2\,ad \Rightarrow u \geq \sqrt{2ad}$

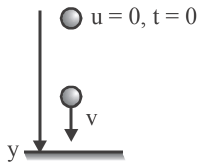
So the minimum speed of passenger for catching the bus is $\sqrt{2ad}$.

Vertical motion under gravity

If air resistance is neglected and a body is freely moving along vertical line near the earth surface then an acceleration downward which is $9.8 m/s^2$ or $980\ cm/s^2$ or $32\ ft/s^2$ is experienced by the body.

Freely falling bodies from a height h above the ground

Taking initial position as origin and direction of motion (i.e. downward direction) positive y axis, as body is just released/dropped u = 0



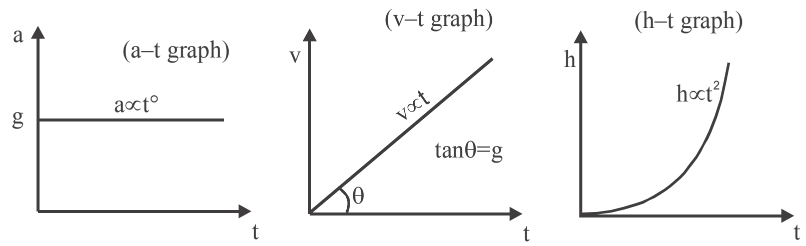
acceleration along +Y axis a = g

Use equations of motion to describe the motion, i.e.

$$v = u + at,\ \ y = ut + \frac{1}{2}at^2,\ v^2 = u^2 + 2ay$$

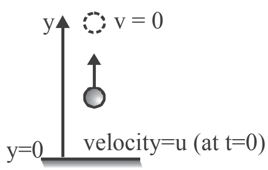
Let the body acquires velocity v(downward) after falling a distance h in time t, then

$$v = gt \Rightarrow t = v/g \because h = \frac{1}{2}gt^2 \Rightarrow t = \sqrt{\frac{2h}{g}},\ v^2 = 2gh \Rightarrow v = \sqrt{2gh}$$



Body is projected vertically upward : With velocity u take initial position as origin and direction of motion

(i.e. vertically upward) as positive y–axis.



v = 0 at maximum height, at t = T,

a = – g (because directed downward)

Put the values in equation of motion

$$v = u + at \Rightarrow 0 = u - gT \Rightarrow u = gT$$

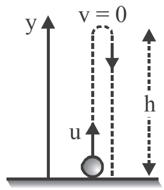
$$s = ut + \frac{1}{2}at^2 \qquad \Rightarrow h = ut - \frac{1}{2}gt^2$$

$$\Rightarrow h_{max} = uT - \frac{1}{2}gT^2 \qquad \Rightarrow h_{max} = (gT)T - \frac{1}{2}gT^2 = \frac{1}{2}gT^2$$

$$v^2 = u^2 + 2as \qquad \Rightarrow v^2 = u^2 - 2gh \Rightarrow 0 = u^2 - 2g\,h_{max} \Rightarrow u^2 = 2g\,h_{max} \Rightarrow u = \sqrt{2g\,h_{max}}$$

After attaining maximum height body turns and come back at ground. During complete flight acceleration is constant,

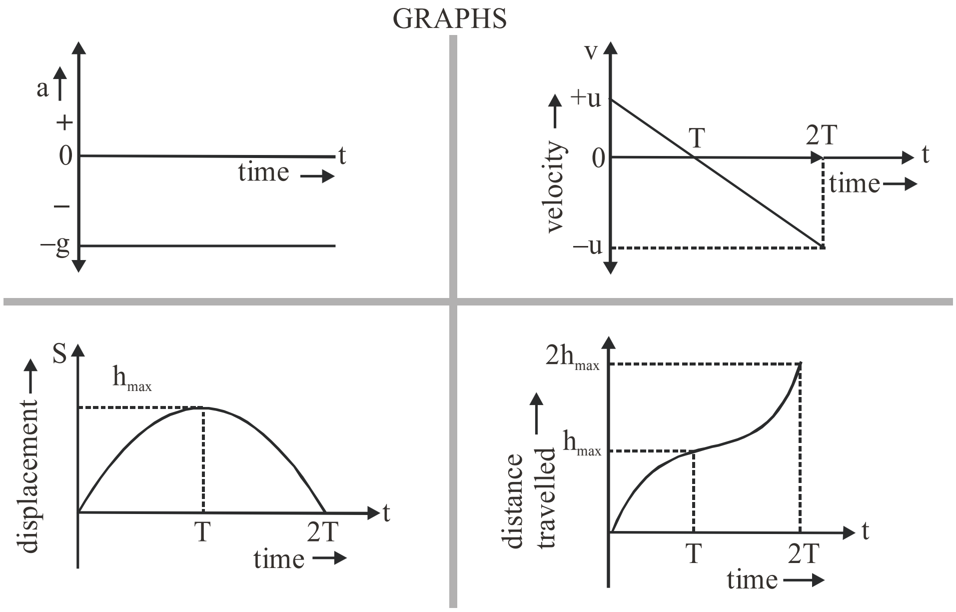
Time taken during up flight and down flight are equal



Time for one side $T = \frac{u}{g}$ and total flight time $= 2T = \frac{2u}{g}$

At each equal height from ground speed of body will be same either going up or coming down.

SOME RELATED GRAPHS FOR ABOVE MOTION'S



Ex. A body is freely dropped from a height h above the ground. Find the ratio of distances fallen in first one second, first two seconds, first three seconds, also find the ratio of distances fallen in 1^{st} second, in 2^{nd} second, in 3^{rd} second etc.

Sol. From second equation of motion, i.e. $h = \frac{1}{2}gt^2$ ($h = ut + \frac{1}{2}gt^2$ and $u = 0$)

$$h_1 : h_2 : h_3 \ldots\ldots = \frac{1}{2}g(1)^2 : \frac{1}{2}g(2)^2 : \frac{1}{2}g(3)^2 = 1^2 : 2^2 : 3^2 \ldots\ldots\ldots = 1 : 4 : 9 : \ldots\ldots .$$

Now from the of distance travelled in n^{th} second

$$s_n = u + \frac{1}{2}a(2n - 1) \text{ here } u = 0, a = g \Rightarrow s_n = \frac{1}{2}g(2n - 1)$$

$$\Rightarrow s_1 : s_2 : s_3 \ldots\ldots\ldots = \frac{1}{2}g(2 \times 1 - 1) : \frac{1}{2}g(2 \times 2 - 1) : \frac{1}{2}g(2 \times 3 - 1) = 1 : 3 : 5 \ldots\ldots\ldots$$

Ex. A rocket is fired vertically up from the ground with a resultant vertical acceleration of $10 m/s^2$. The fuel is finished in 1 minute and it continues to move up.

(a) What is the maximum height reached?

(b) After finishing fuel, calculate the time for which it continues its upwards motion. (Take $g = 10 m/s^2$)

Sol. (a) The distance travelled by the rocket during burning interval (1minute= 60s) in which resultant acceleration $10 m/s^2$ is vertically upwards will be $h_1 = 0 \times 60 + (1/2) \times 10 \times 60^2 = 18000$ m = 18 km and velocity acquired by it will be $v = 0 + 10 \times 60 = 600$ m/s

Now after 1 minute the rocket moves vertically up with initial velocity of 600 m/s and acceleration due to

gravity opposes its motion. So, it will go to a height h_2 from this point, till its velocity becomes zero such that

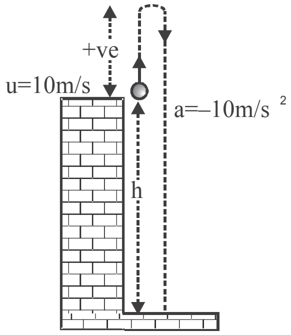
$$0 = (600)^2 - 2gh_2 \Rightarrow h_2 = 18000 \text{ m} = 18 \text{ km } [g = 10 \text{ ms}^{-2}]$$

So the maximum height reached by the rocket from the ground, $H = h_1 + h_2 = 18 + 18 = 36$ km

(b) As after burning of fuel the initial velocity 600m/s and gravity opposes the motion of rocket, so from 1st equation of motion time taken by it till it velocity v = 0

$0 = 600 - gt \Rightarrow t = 60$ s

Ex. A ball is thrown upwards from the top of a tower 40 m high with a velocity of 10 m/s, find the time when it strikes the ground ($g = 10$ m/s^2)



Sol. In the problem $u = +10$ m/s, $a = -10$ m/s^2 and $s = -40$m (at the point where ball strikes the ground)

Substituting in $s = ut + \frac{1}{2}at^2$

$-40 = 10t - 5t^2 \Rightarrow 5t^2 - 10t - 40 = 0 \Rightarrow t^2 - 2t - 8 = 0$

Solving this we have t = 4 s and – 2s. Taking the positive value t = 4s.

Ex. The acceleration of a particle moving in a straight line varies with its displacement as, a = 2s velocity of the particle is zero at zero displacement. Find the corresponding velocity displacement equation.

Sol. $a = 2s \Rightarrow \frac{dv}{dt} = 2s \Rightarrow \frac{dv}{ds}.\frac{ds}{dt} = 2s \Rightarrow \frac{dv}{ds}.v = 2s$

$\Rightarrow \int vdv = 2\int sds \Rightarrow \left(\frac{v^2}{2}\right)_0^v = 2\left(\frac{s^2}{2}\right)_0^s$

$\Rightarrow \frac{v^2}{2} = s^2 \Rightarrow v = s\sqrt{2}$

Ex. If a body travels half its total path in the last second of its fall from rest, find :

(a) The time and

(b) height of its fall. Explain the physically unacceptable solution of the quadratic time equation. ($g = 9.8$ m/s^2)

Sol. If the body falls a height h in time t, then

$h = \frac{1}{2}gt^2$ [u = 0 as the body starts from rest] ... (1)

Now, as the distance covered in (t – 1) second is $h' = \frac{1}{2}g(t-1)^2$ ... (2)

So from Equations (1) and (2) distance travelled in the last second.

$h - h' = \frac{1}{2}gt^2 - \frac{1}{2}g(t-1)^2$ i.e., $h - h' = \frac{1}{2}g(2t - 1)$

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But according to given problem as $(h - h') = \frac{h}{2}$

i.e., $\left(\frac{1}{2}\right)h = \left(\frac{1}{2}\right)g(2t-1)$ or $\left(\frac{1}{2}\right)gt^2 = g(2t-1)$ [as from equation (1) $h = \left(\frac{1}{2}\right)gt^2$]

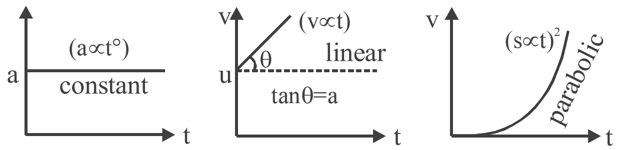
or $t^2 - 4t + 2 = 0$ or $t = [4 \pm \sqrt{(4^2 - 4\times 2)}]/2$ or $t = 2 \pm \sqrt{2} \Rightarrow t = 0.59$ s or 3.41 s

0.59 s is physically unacceptable as it gives the total time t taken by the body to reach ground lesser than one sec while according to the given problem time of motion must be greater than 1s.

so $t = 3.41$s and $h = \frac{1}{2} \times (9.8) \times (3.41)^2 = 57$ m

Graphs based on 1–D

For constant acceleration, a/t, v/t and s/t curve from equations of motion are –



In case of constant acceleration motion in a straight line, scalar form of equations of motion can be applied and problem becomes fairly simple.

As $d\vec{v} = \vec{a}dt$ or $[\vec{v}]_{\vec{u}}^{\vec{v}} = \vec{v} - \vec{u} = \int_{t_1}^{t_2} \vec{a}dt$ = Area between curve and time axis from t_1 to t_2.

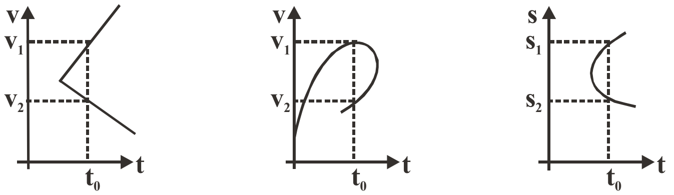
Area under the curve of a – t graph always gives the change in velocity.

Similarly $d\vec{s} = \int \vec{v}dt$ or $\vec{s} = \int_{t_1}^{t_2} \vec{v}dt$ = Area between curve and time axis from t_1 to t_2.

Here $\vec{s}$ is the displacement of particle in time interval t_1 to t_2, i.e. area under the curve of v/t graph always gives the displacement. If only magnitude of area is taken into account then sum of all area is the total distance travelled by the particle.

- Slopes of v–t or s–t graphs can never be infinite at any point, because infinite slope of v–t graph means infinite acceleration. Similarly, infinite slope of s–t graph means infinite velocity. Hence, the following graphs are not possible.

- At one time, two values of velocity or displacement are not possible Hence, the following graphs are not acceptable.



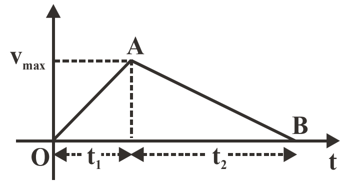
- The slope of velocity–time graph of uniform motion is zero.
- When a body is having uniform motion along a straight line in a given direction, the magnitude of the displacement of body is equal to the actual distance travelled by the body in the given time.
- The average and instantaneous velocity in a uniform motion are equal in magnitude.
- In a uniform motion along a straight line, the slope of position–time graph gives the velocity of the body.
- The position–time graph of a body moving along a straight line can never be a straight line parallel to position axis because it will indicate infinite velocity.
- The speed of a body can never be negative
- Medium effects the motion of a body falling freely under gravity due to thrust and viscous drag.

Ex. A car accelerates from rest at a constant rate α for some time, after which it decelerates at a constant rate β, to come to rest. If the total time elapsed is t evaluate (a) the maximum velocity attained and (b) the total distance travelled.

Sol. (a) Let the car accelerates for time t_1 and decelerates for time t_2 then

$$t = t_1 + t_2 \qquad \text{......(i)}$$

and corresponding velocity–time graph will be as shown in. fig.



From the graph α = slope of line AB $= \dfrac{v_{max}}{t_1} \Rightarrow t_1 = \dfrac{v_{max}}{\alpha}$

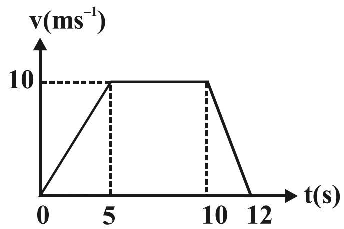
and β = – slope of line OB $= \dfrac{v_{max}}{t_2} \Rightarrow t_2 = \dfrac{v_{max}}{\beta}$

$$\Rightarrow \frac{v_{max}}{\alpha} + \frac{v_{max}}{\beta} = t \Rightarrow v_{max}\left(\frac{\alpha+\beta}{\alpha\beta}\right) = t \Rightarrow v_{max} = \frac{\alpha\beta t}{\alpha+\beta}$$

(b) Total distance = area under v–t graph $= \dfrac{1}{2} \times t \times v_{max} = \dfrac{1}{2} \times t \times \dfrac{\alpha\beta t}{\alpha+\beta} = \dfrac{1}{2}\left(\dfrac{\alpha\beta t^2}{\alpha+\beta}\right)$

Note: This problem can also be solved by using equations of motion (v = u + at, etc.).

Ex. Draw displacement time and acceleration – time graph for the given velocity–time graph



Sol. For $0 \le t \le 5$ $v \propto t \Rightarrow s \propto t^2$ and a_1 = constant $\frac{10}{5} = 2\ ms^{-2}$

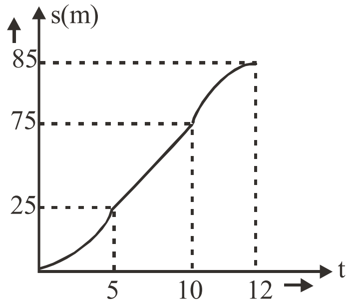
for whole interval s_1 = Area under the curve = $\frac{1}{2} \times 5 \times 10 = 25$ m

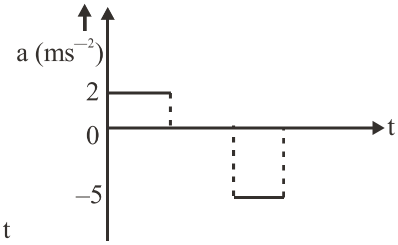
For $5 \le t \le 10$ $v = 10 ms^{-1}$ $\Rightarrow a = 0$

for whole interval s_2 = Area under the curve = $\frac{1}{2} \times 5 \times 10 = 50$ m

For $10 \le t \le 12$ v linearly decreases with time $\Rightarrow a_3 = -\frac{10}{2} = -5\ ms^{-1}$

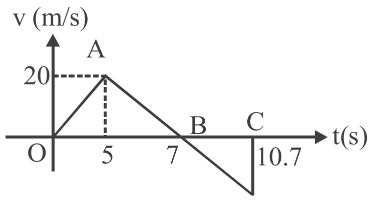
for whole interval s_3= Area under the curve = $\frac{1}{2} \times 2 \times 10 = 10$ m

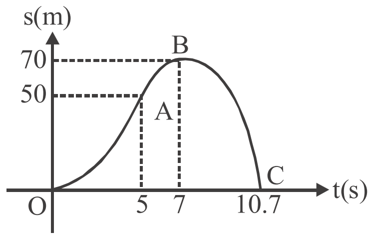




Ex. A rocket is fired upwards vertically with a net acceleration of 4 m/s² and initial velocity zero. After 5 seconds its fuel is finished and it decelerates with g. At the highest point its velocity becomes zero. Then it accelerates downwards with acceleration g and return back to ground. Plot velocity–time and displacement–time graphs for the complete journey. Take g = 10 m/s².

Sol.





In the graphs, $v_A = at_{OA} = (4)(5) = 20$ m/s $\qquad v_B = 0 = v_A - gt_{AB}$

$\therefore t_{AB} = \frac{v_A}{g} = \frac{20}{10} = 2s$ $\qquad \therefore t_{OAB} = (5 + 2)s = 7s$

Now, s_{OAB} = area under v–t graph between 0 to 7 s = $\frac{1}{2}(7)(20) = 70$ m

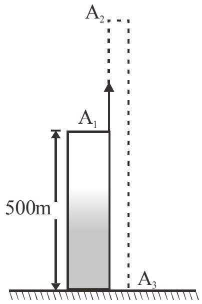
Now, $s_{OAB} = s_{BC} = \frac{1}{2} g t^2_{BC}$ $\qquad \therefore 70 = \frac{1}{2}(10)t^2_{BC}$

$\therefore t_{BC} = \sqrt{14} = 3.7s$ $\qquad \therefore t_{OAB} = 7 + 3.7 = 10.7s$

Also s_{OA} = area under v–t graph between OA = $\frac{1}{2}(5)(20) = 50$ m

Ex. At the height of 500m, a particle A is thrown up with $v = 75\ ms^{-1}$ and particle B is released from rest. Draw, accelearation –time, velocity–time, speed–time and displacement–time graph of each particle.

For particle A :



Time of flight

$$-500 = +75\,t - \frac{1}{2} \times 10t^2$$

$\Rightarrow t^2 - 15t - 100 = 0$

$\Rightarrow t = 20$ s

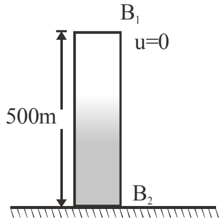
Time taken for A_1A_2

$= 75 - 10t \Rightarrow t = 7.5$ s

Velocity at A_3, $v = 75 - 10 \times 20 = -125\ ms^{-1}$

$$\text{Height } A_2A_1 = \frac{1}{2}(10)(7.5)^2 = 281.25 \text{ m}$$

For Particle B

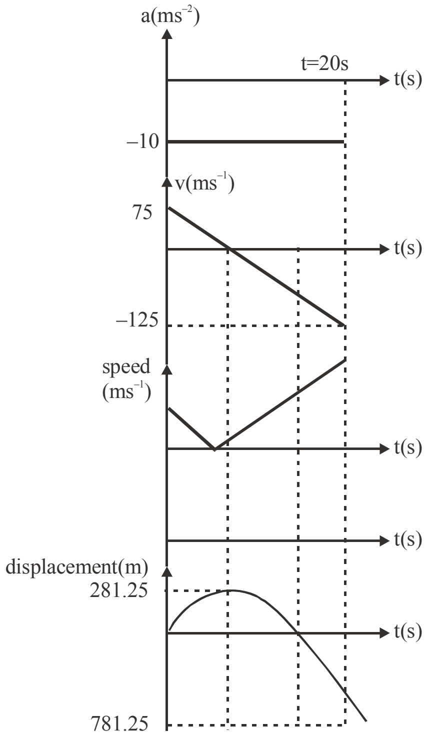


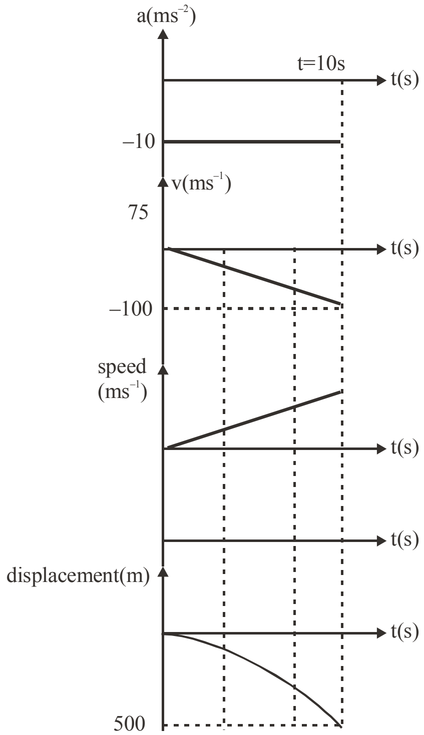
Time of flight

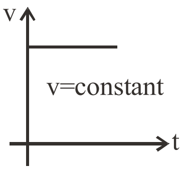
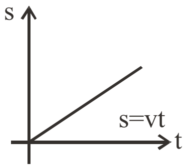
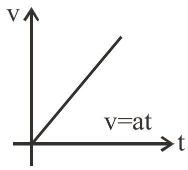
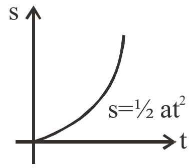
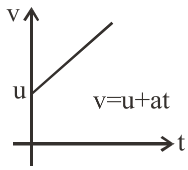
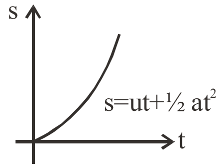
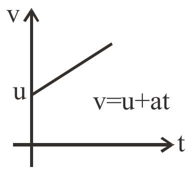
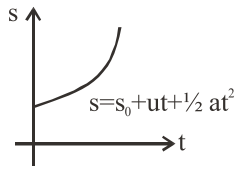
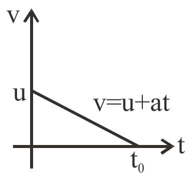
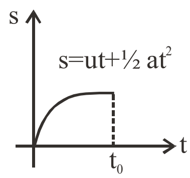
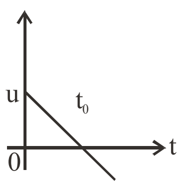
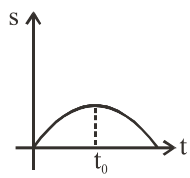
$$500 = \frac{1}{2}(10)t^2 \Rightarrow t = 10 \text{ s}$$

Velocity at B_2

$v = 0 - (10)(10) = -100\ ms^{-1}$





S.N.	Different Cases	v–t graph	s–t graph	Important Points
1.	Uniform motion			(i) Slope of s–t graph = v = constant (ii) In s–t graph s = 0 at t = 0
2.	Uniformly accelerated motion with u = 0 at t = 0			(i) u = 0, i.e. v = 0 at t = 0 (ii) u = 0, i.e., slope of s–t graph at t = u, should be zero (iii) a or slope of v – t graph is constant
3.	Uniformly accelerated with $u \neq 0$ at t = 0			(i) $u \neq 0$, i.e., v or slope motion of s – t graph at t = 0 is not zero (ii) v or slope of s – t graph gradually goes on increasing.
4.	Uniformly accelerated motion with $u \neq 0$ and $s = s_0$ at t = 0			(i) $s = s_0$ at t = 0
5.	Uniformly retarded motion till velocity becomes zero			(i) Slope of s – t graph at t = 0 gives u (ii) Slope of s – t graph at $t = t_0$ becomes zero (iii) In this case u can't be zero.
6.	Uniformly retarded then accelerated in opposite direction			(i) At time $t = t_o$, v = 0 or slope of s – t graph is zero (ii) In s – t graph slope or velocity first decreases then increases with opposite sign.

node06\B0BB-BC\Kota\JEE(Advanced)\Nurture\Phy\Module\Module#U & D, Basic Maths, Vector, NLM, Kinematics-1D & K-2D\Eng\03-K-1D & Calculus\01.Theory & Ex.p65

KEY CONCEPT

MOTION IN TWO AND THREE DIMENSIONS :

When a particle is moving in space then its motion can be broken up in three co-ordinate axis (x, y& z). The motion in these three directions is governed only by velocity & acceleration in that particular direction and is totally independent of the velocities and acceleration in other directions.

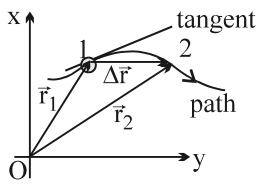
Lets say a particle is moving in space

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Gives position of particle in space.

VELOCITY

Using the language of calculus, we may write $\vec{v}$ as the derivative



$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(x\hat{i} + y\hat{j} + z\hat{k}\right) = \left(\frac{dx}{dt}\right)\hat{i} + \left(\frac{dy}{dt}\right)\hat{j} + \left(\frac{dz}{dt}\right)\hat{k}$$

where the scalar components of $\vec{v}$ are

$$v_x = \frac{dx}{dt},\ v_y = \frac{dy}{dt},\ v_z = \frac{dz}{dt}$$

Differentiating $\vec{r}$ w.r.t. time gives us velocity vector of particle at that time.

ACCELRATION

Similarly, if we differentiate $\vec{V}$ w.r.t. time we get acceleration of particle $\vec{a} = \frac{d\vec{V}}{dt}$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k}$$

where the scalar components of $\vec{a}$ are

$$a_x = \frac{dv_x}{dt},\ a_y = \frac{dv_y}{dt},\ a_z = \frac{dv_z}{dt}$$

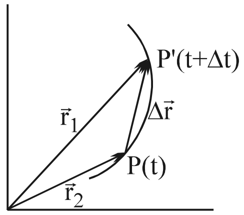
Now, collecting equations of motion relating to x & y axes separately

x-axis	y-axis
$V_x = \frac{dx}{dt}$	$V_y = \frac{dy}{dt}$
$a_x = \frac{dV_x}{dt}$	$a_y = \frac{dV_y}{dt}$

Thus we can see that motion in plane is composed of two straight line motions. **These motions are completely independent of each other**. Only thing connecting them is fact that they are occuring simultaneously.

Velocity is along tangent of path

The direction of the instantaneous velocity $\vec{v}$ of a particle is always tangent to the particle's path at the particle position. $\vec{V} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t}$



The result is the same in three dimensions:

Ex. A particle with velocity $\vec{v}_0 = -2\hat{i} + 4\hat{j}$ (in meters per second) at t = 0 undergoes a constant acceleration $\vec{a}$ of magnitude a = 3 m/s² at an angle θ = 127° from the positive direction of the x axis. What is the particle's velocity $\vec{v}$ at t = 5 sec, in unit vector notation ?

Sol. We know that $v = v_0 + at$

now $v_x = v_{0x} + a_x t$ and $v_y = v_{0y} + a_y t$

$a_x = a \cos\theta = (3\ m/s^2)(\cos 127°) = -1.80\ m/s^2$

$a_y = a \sin\theta = (3\ m/s^2)(\sin 127°) = +2.40\ m/s^2$

at time t = 5 sec

$v_x = -2\ m/s + (-1.80\ m/s^2)(5\ sec) = -11\ m/s$

$v_y = 4\ m/s + (2.40\ m/s^2)(5\ sec) = 16\ m/s$

Thus, at t = 5 sec,

$\vec{v} = (-11 m/s)\hat{i} + (16 m/s)\hat{j}$ **Ans.**

Ex. A particle moves in the x–y plane according to the law x = at; y = at(1–αt) where a and α are positive constants and t is time. Find the velocity and acceleration vector. Also find the moment t_0 at which the velocity vector forms angle of 90° with acceleration vector.

Sol. $V_x = a;\ V_y = a - 2a\alpha t \Rightarrow \vec{V} = a\hat{i} + (a - 2a\alpha t)\hat{j}$

$a_x = 0;\ a_y = -2a\alpha \Rightarrow \vec{a} = -2a\alpha\hat{j}$

for 90°, $\vec{V}\cdot\vec{a} = 0$

$-2a\alpha(a - 2a\alpha t_0) = 0$

$1 - 2\alpha t_0 = 0 \Rightarrow t_0 = 1/(2\alpha)$ sec.

PROJECTILE MOTION

We next consider a special case of two-dimensional motion: A particle moves in a vertical plane with some initial velocity $\vec{v}_0$ but its acceleration is always the freefall acceleration $\vec{g}$, which is downward. Such a particle is called a projectile (meaning that it is projected or launched) and its motion is called **projectile motion**.

Assumptions:–

Particle remains close to earth's surface, so acceleration due to gravity remains constant.

Air resistance is neglected.

Distance that projectile travels is small so that earth can be treated as plane surface.

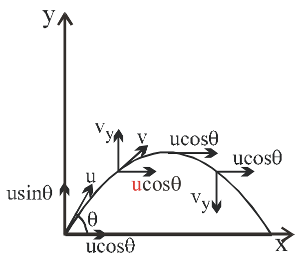
Two straight line motions:–

Our goal here is to analyse projectile motion using the tools for two dimensional motion. This feature allows us to break up a problem involving two dimensional motion into two separate and easier one-dimensional problems,

(a) The horizontal motion is motion with uniform velocity (no effect of gravity)

(b) The vertical motion is motion of uniform acceleration, or freely falling bodies.

Note: In projectile motion, the horizontal motion and the vertical motion are independent of each other, that is either motion does not affects the other.



Treating as two straight line motions:–

The horizontal Motion(x axis):

Because there is no accelration in the horizontal direction, the horizontal component v_x of the projectile velocity remains unchanged from its initial value v_{0x} throughout the motion,

The vertical motion(y axis):

The vertical motion is the motion we discussed for a particle in free fall.

As is illustrated in figure, the vertical component behaves just as for a ball thrown vertically upward. It is directed upward initially and its magnitude steadily decreasing to zero, at the maximum height of the path. The vertical velocity component then reverses direction, and its magnitude becomes larger with time.

x-axis

Initial velocity(u_x) = ucosθ

acceleration(a_x) = 0

Thus, velocity after time t

$v_x = u\cos\theta$

Displacement after time t

$x = u\cos\theta t$

y-axis

Initial velocity(u_y) = usinθ

acceleration(a_y) = –g

Thus, velocity after time t

$v_y = u\sin\theta - gt$

Displacement after time t

$y = u\sin\theta t - gt^2/2$

Resultant velocity

$$(\vec{V}_R) = (u\cos\theta)\hat{i} + (u\sin\theta - gt)\hat{j}$$

$$|\vec{V}_R| = \sqrt{u^2\cos^2\theta + (u\sin\theta - gt)^2}$$

$$\& \quad \tan\alpha = \frac{u\sin\theta - gt}{u\cos\theta}$$

where α is angle that velocity vector makes with horizontal. Also known as direction or angle of motion

Time of flight(T)

$$T = \frac{2u\sin\theta}{g}$$

Considering vertical motion

$s_y = 0;\ u_y = u\sin\theta;\ a_y = -g$

$$0 = u\sin\theta T - gT^2/2 \Rightarrow T = \frac{2u\sin\theta}{g}$$

Maximum Height(H)

$$H = \frac{u^2\sin^2\theta}{2g}$$

Vertical velocity at maximum height $v_y = 0$

$$0 = u^2\sin^2\theta - 2gH \Rightarrow H = \frac{u^2\sin^2\theta}{2g}$$

Horizontal Range(R)

$$R = \frac{u^2\sin 2\theta}{g} = \frac{2u_x u_y}{g}$$

Total time $T = \dfrac{2u\sin\theta}{g}$

Velocity in horizontal direction $u_x = u\cos\theta$

Total displacement in horizontal direction $R = u\cos\theta T$

$$R = \frac{u^2\sin 2\theta}{g}$$

Note:– *For complementry angles i.e. $\theta + \alpha = 90°$, the range is same for same projection speed but maximum height and time of flight are different.*

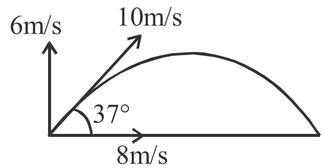
Ex. A body is thrown with initial velocity 10m/sec. at an angle 37° from horizontal. Find (Take : $g = 10 m/s^2$)

(i) Time of flight (ii) Maximum height.

(iii) Range (iv) Position vector at t = 1 sec.

Ans. (i) 1.2 sec, (ii) 1.8 m, (iii) 9.6 m, (iv) $(8\hat{i} + \hat{j})$

Sol. (i) Time of flight $T = \dfrac{2u\sin\theta}{g} = \dfrac{2\times 10\times\frac{3}{5}}{10} = \dfrac{6}{5} = 1.2$ sec



(ii) Maximum height $H = \dfrac{u^2\sin^2\theta}{2g} = \dfrac{100\left(\frac{9}{25}\right)}{2\times 10} = \dfrac{9}{5} = 1.8$ m

(iii) $R = \dfrac{u^2\sin 2\theta}{g} = \dfrac{100\times 2\times\frac{3}{5}\times\frac{4}{5}}{10} = \dfrac{240}{25} = 9.6$m

(iv) $x = 8 \times 1 = 8$ m

$$y = 6 \times 1 - \frac{1}{2} \times 10 \times (1)^2 = 1\text{m}$$

$$\vec{r} = 8\hat{i} + \hat{j}$$

Maximum Range

$$R = \frac{u^2 \sin 2\theta}{g}$$

for $\theta = 45°$, R is maximum

$$R_{max} = \frac{u^2}{g}$$

Ex. A person can throw a ball vertically upto maximum height of 20 mt. How far can he throw the ball.

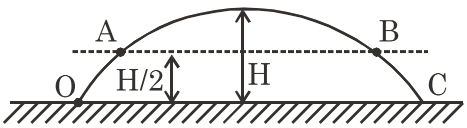
Sol. $H_{max} = \frac{u^2}{2g}$

$\therefore$ u = 20 m/s

$$R_{max} = \frac{u^2}{g} = 40 \text{ m}$$

Ex. A particle is projected with a speed u at an angle θ with horizontal. Find the average velocity of projectile for the period during which it crosses half of maximum height.

Sol.



avg. velocity is a vector

Let t_1 is time taken by particle to travel from A to B

First we will find vertical component.

For motion along AB

$$(\vec{V}_y)_{avg} = \frac{\text{Total Displacement along y direction}}{\text{Total time}} = \frac{0}{t_1} = 0$$

For Horizontal component

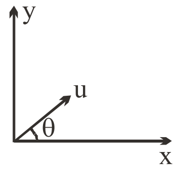
$$(\vec{V}_x)_{avg} = \frac{\text{Total Displacement along x direction}}{\text{Total time}}$$

$$(\vec{V}_x)_{avg} = \frac{(u\cos\theta)t_1}{t_1} = u\cos\theta$$

So average velocity for motion along AB is $u\cos\theta$

EQUATION OF TRAJECTORY

Lets say point of projection is our origin and horizontal direction is x-axis and vertically upwards is positive y-axis.



We know $x = u\cos\theta t$

$$\therefore \quad t = \frac{x}{u\cos\theta} \qquad .....(1)$$

also $y = u\sin\theta t - \frac{1}{2}gt^2$ .....(2)

Putting value of 't' from eq. (1) in eq. (2), we get

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

Ex. A particle is projected with a velocity 10 m/s at an angle 37° to the horizontal. Find the location at which the particle is at a height 1m from point of projection. ($g = 10\ m/s^2$)

Ans. 1.6 m, 8 m.

Sol. $y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$

for $y = 1$; $\theta = 37°$; $u = 10$ m/s

$$1 = \frac{3}{4}x - \frac{10x^2}{2\times100\times\left(\frac{16}{25}\right)}$$

$$1 = \frac{3}{4}x - \frac{5}{64}x^2$$

$5x^2 - 48x + 64 = 0$

$5x^2 - 40x - 8x + 64 = 0$

$x = 8m, 1.6m$

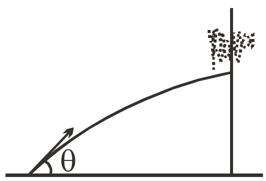
Ex. We have a hose pipe which disposes water at the speed of $10\ ms^{-1}$. The safe distance from a building on fire, on ground is 5 m. How high can this water go? (take : $g = 10\ ms^{-2}$)

Sol. Here we must understand that taking range of projectile as 10m and making projectile hit the building when it is at maximum height is wrong. By doing this we are not acheiving maximum y for given x = 5m. This just makes highest point of path to like on x = 5, But there may be other path for which y will be maximum for given x. This problem will be solved by using equation of trajectory by putting x = 5m and maximising y by varying θ.

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$

Putting x = 5m we get

$$y = 5\tan\theta - \frac{10\times25\sec^2\theta}{2\times100}$$



$5\tan^2\theta - 20\tan\theta + (4y + 5) = 0$

for real roots discriminant must be positive.

$400 - 4 \times 5(4y + 5) > 0$

Solving $3.75 \geq y$

hence maximum y = 3.75 m

If we have taken range as 10 m then angle of projection will be θ = 45° corsponding maximum hight H = 2.5m which is smaller than our answer.

Ex. A projectile is fired horizontaly with a velocity of 98 m/s from the top of a hill 490 m high. Find
(i) Time to reach ground
(ii) The horizontal distance from foot of hill to ground.
(iii) The speed with which it hits the ground.

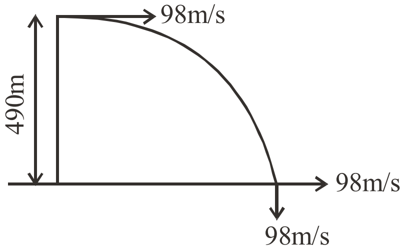
Ans. (i) 10 sec, (ii) 980 m, (iii) $98\sqrt{2}$ m/se

Sol. $u_y = 0; u_x = 98$ m/s
on y-axis

$$s = ut + \frac{1}{2}at^2$$

$$-490 = 0 - \frac{1}{2} \times 9.8 \times t^2$$

$$t = \sqrt{\frac{490 \times 2}{9.8}} = 10 \text{ sec}$$

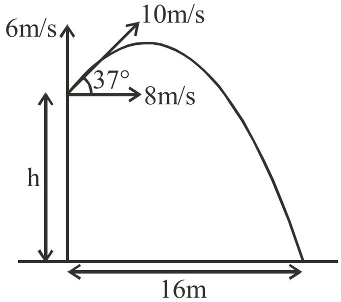


on x-axis for horizontal displacement,
$R = u_x t = 98 \times 10 = 980$ m
$v_x = 98$ m/s
$v_y = 0 - 9.8 \times 10 = -98$ m/s
So speed = $98\sqrt{2}$ m/s

Ex. A ball is thrown from the top of a tower with an initial velocity of 10m/s at an angle 37° above the horizontal, hits the ground at a distance 16m from the base of tower. Calculate height of tower. $[g = 10 \text{ m/s}^2]$

Ans. 8 m

Sol. Time to reach ground is $= \frac{16}{8} = 2$ sec



on y-axis (for height of tower)

$$-h = 6 \times 2 - \frac{1}{2} \times 10 \times (2)^2$$

$-h = 12 - 22$
$h = 8$ m

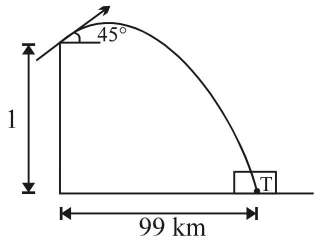
Ex. Prithvi missile is fired to destroy an enemy military base situated on same horizontal level, situated 99 km away. The missile rises vertically for 1 km & then for remainder of flight, it followes parbolic path like a free body under earth's gravity, at an angle of 45°. Calculate its velocity at beginning of parabolic path. $(g = 10 \text{ ms}^{-2})$

Sol. for horizontal motion time t

$$t = \frac{99 \times 10^3}{u \cos 45°}$$

for vertical

$$-1 \times 10^3 = u \sin 45° \, t - \frac{1}{2} \times 10 \times t^2$$



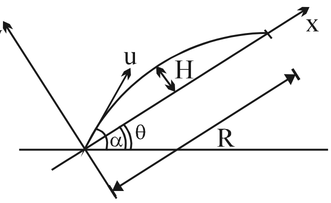
$$1 \times 10^3 + \frac{u \sin 45°}{u \sin 45°} \times 99 \times 10^3 = \frac{10}{2} \times \frac{(99 \times 10^3)^2 \times 2}{u^2}$$

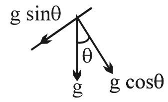
$$u^2 = \frac{(99\times10^3)^2\times10}{100\times10^3}$$

$$u = 99\times10^3\sqrt{\frac{1}{10^4}} = 990\ ms^{-1}$$

PROJECTION ON INCLINED PLANE

There is an inclined plane making an angle θ with horizontal. A particle is projected at an angle α from horizontal.





x-axis	y-axis
$u_x = u\cos(\alpha-\theta)$	$u_y = u\sin(\alpha-\theta)$
$a_x = -g\sin\theta$	$a_y = -g\cos\theta$
vel. at any time t	vel. at any time t
$v_x = u\cos(\alpha-\theta) - g\sin\theta\, t$	$v_y = u\sin(\alpha-\theta) - g\cos\theta\, t$

Time of flight

Displacement in y direction $s_y = 0$

$$0 = u\sin(\alpha-\theta)\,T - \frac{1}{2}g\cos\theta\,T^2$$

$$T = \frac{2u\sin(\alpha-\theta)}{g\cos\theta}$$

Maximum distance of particle from inclined plane

Point where $v_y = 0$ is max.-distance from inlcine plane

$$(0)^2 = u^2\sin^2(\alpha-\theta) - 2g\cos\theta\,H$$

$$H = \frac{u^2\sin^2(\alpha-\theta)}{2g\cos\theta}$$

Range along the inclined plane

$$s_x = u_xT + \frac{1}{2}a_xT^2$$

$$R = \frac{u\cos(\alpha-\theta)\times2u\sin(\alpha-\theta)}{g\cos\theta} - \frac{g\sin\theta\times2\times2u^2\sin^2(\alpha-\theta)}{2g^2\cos^2\theta}$$

$$= \frac{2u^2\sin(\alpha-\theta)[\cos(\alpha-\theta)\cos\theta - \sin\theta\sin(\alpha-\theta)]}{g\cos^2\theta}$$

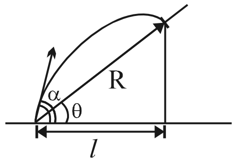
$$R = \frac{2u^2 \sin(\alpha - \theta)\cos\alpha}{g\cos^2\theta}$$

$$R = \frac{u^2[\sin(2\alpha - \theta) - \sin\theta]}{g\cos^2\theta}$$

Alternate Method

$l = u \cos \alpha \, T$

$$R = \frac{l}{\cos\theta}$$



$$R = \frac{u\cos\alpha}{\cos\theta} \times \frac{2u\sin(\alpha-\theta)}{g\cos\theta}$$

$$R = \frac{2u^2 \sin(\alpha - \theta)\cos\alpha}{g\cos^2\theta}$$

Note : Presence of incline plane does not affect the path of projectile in any way.

Maximum Range :

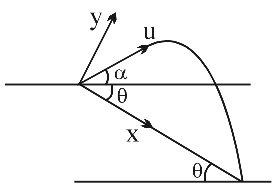
$$R = \frac{u^2[\sin(2\alpha - \theta) - \sin\theta]}{g\cos^2\theta}$$

For max. range $2\alpha - \theta = \frac{\pi}{2} \Rightarrow \alpha = \frac{\pi}{4} + \frac{\theta}{2}$

so $R_{max} = \frac{u^2}{g(1+\sin\theta)}$

Projection from top of incline plane:

Incline plane is at an angle θ with horizontal and a particle is projected at an angle α from horizontal. In all formulae replace θ with –θ



$$H = \frac{u^2 \sin^2(\alpha + \theta)}{2g\cos\theta}$$

$$T = \frac{2u\sin(\alpha + \theta)}{g\cos\theta}$$

$$R = \frac{2u^2 \sin(\alpha + \theta)\cos\alpha}{g\cos^2\theta}$$

$R_{max} = \frac{u^2}{g(1-\sin\theta)}$ and $\alpha = \frac{\pi}{4} - \frac{\theta}{2}$

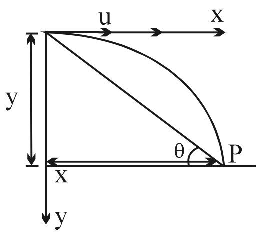
Note : If a particle strikes the incline plane ⊥ then its comp. of velocity along incline must be zero.

Ex. A particle is projected horizontally with a speed u from the top of a plane inclined at an angle θ with the horizontal. How far along the plane, from the point of projection will particle strike the plane?

Sol.

x-axis	**y-axis**
$u_x = u$	$u_y = 0$
$a_x = 0$	$a_y = g$
$x = ut$	$y = \dfrac{gt^2}{2}$

$$\Rightarrow \ y = \frac{gx^2}{2u^2}$$



also $\dfrac{y}{x} = \tan\theta \Rightarrow x\tan\theta = \dfrac{gx^2}{2u^2}$

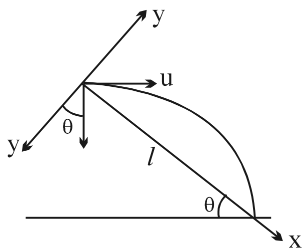
$$x = 0, \ \frac{2u^2\tan\theta}{g}$$

$$x = \frac{2u^2\tan\theta}{g}$$

$$\Rightarrow y = \frac{2u^2\tan^2\theta}{g}$$

dist. $l = \sqrt{x^2 + y^2}$

$$l = \frac{2u^2\tan\theta\sec\theta}{g}$$



<u>Alternate Method.</u>

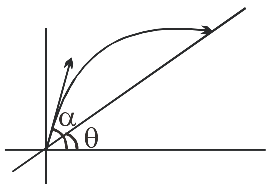
$$R = \frac{2u^2\sin(\alpha+\theta)\cos\alpha}{g\cos^2\theta} \quad \text{as } \alpha = 0$$

$R = 2\,u^2\tan\theta\sec\theta$

Ex. A particle is projected up an inclined plane. Plane is inclined at an angle θ with horizontal and particle is projected at an angle α with horizontal. If particle strikes the plane horizontally prove that $\tan\alpha = 2\tan\theta$

Sol. We know time of flight

$$T = \frac{2u\sin(\alpha+\theta)}{g\cos\theta}$$



considering vertical motion

$u = v\sin\alpha$

$a = -g$

$v = 0$

$$\therefore \ T = \frac{u\sin\alpha}{g} = \frac{2u\sin(\alpha-\theta)}{g\cos\theta}$$

$\sin\alpha\cos\theta = 2\sin\alpha\cos\theta - 2\cos\alpha\sin\theta$

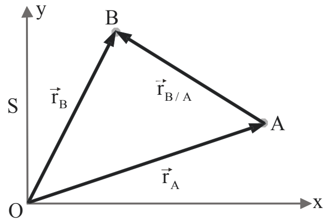
$2\cos\alpha\sin\theta = \sin\alpha\cos\theta$

$2\tan\theta = \tan\alpha$

Relative Motion

Motion of a body can only be observed, when it changes its position with respect to some other body. In this sense, motion is a relative concept. To analyze motion of a body say A, therefore we have to fix our reference frame to some other body say B. The result obtained is motion of body A relative to body B.

Relative position, Relative Velocity and Relative Acceleration



Let two bodies represented by particles A and B at positions defined by position vectors $\vec{r}_A$ and $\vec{r}_B$, moving with velocities $\vec{v}_A$ and $\vec{v}_B$ and accelerations $\vec{a}_A$ and $\vec{a}_B$ with respect to a reference frame S. For analyzing motion of terrestrial bodies the reference frame S is fixed with the ground.

The vectors $\vec{r}_{B/A}$ denotes position vector of B relative to A. Following triangle law of vector addition, we have

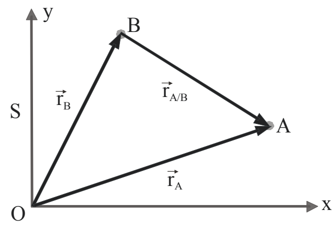
$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A} \qquad \text{...(i)}$$

First derivatives of $\vec{r}_A$ and $\vec{r}_B$ with respect to time equals to velocity of particle A and velocity of particle B relative to frame S and first derivative of $\vec{r}_{B/A}$ with respect to time defines velocity of B relative to A.

$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A} \qquad \text{...(ii)}$$

Second derivatives of $\vec{r}_A$ and $\vec{r}_B$ with respect to time equals to acceleration of particle A and acceleration of particle B relative to frame S and second derivative of $\vec{r}_{B/A}$ with respect to time defines acceleration of B relative to A.

$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A} \qquad \text{...(iii)}$$



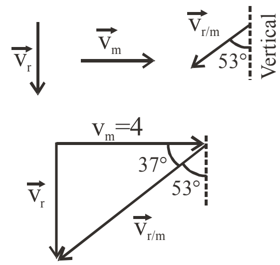
In similar fashion motion of particle A relative to particle B can be analyzed with the help of adjoining figure. You can observe in the figure that position vector of A relative to B is directed from B to A and therefore

$$\vec{r}_{B/A} = -\vec{r}_{A/B},\ \vec{v}_{B/A} = -\vec{v}_{A/B} \text{ and } \vec{a}_{B/A} = -\vec{a}_{A/B}.$$

The above equations elucidate that how a body A appears moving to another body B is opposite to how body B appears moving to body A.

Ex. A man when standstill observes the rain falling vertically and when he walks at 4 km/h he has to hold his umbrella at an angle of 53° from the vertical. Find velocity of the raindrops.

Sol. Assigning usual symbols $\vec{v}_m$, $\vec{v}_r$ and $\vec{v}_{r/m}$ to velocity of man, velocity of rain and velocity of rain relative to man, we can express their relationship by the following eq. $\vec{v}_r = \vec{v}_m + \vec{v}_{r/m}$



The above equation suggests that a standstill man observes velocity $\vec{v}_r$ of rain relative to the ground and while he is moving with velocity $\vec{v}_m$, he observes velocity of rain relative to himself $\vec{v}_{r/m}$. It is a common intuitive fact that umbrella must be held against $\vec{v}_{r/m}$ for optimum protection from rain. According to these facts, directions of the velocity vectors are shown in the adjoining figure.

The addition of velocity vectors is represented according to the above equation. From the figure we have

$v_r = v_m \tan 37° = 3$ km/h **Ans.**

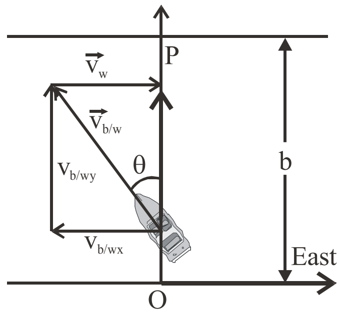
Ex. A boat can be rowed at 5 m/s on still water. It is used to cross a 200 m wide river from south bank to the north bank. The river current has uniform velocity of 3 m/s due east.

(a) In which direction must it be steered to cross the river perpendicular to current?

(b) How long will it take to cross the river in a direction perpendicular to the river flow?

(c) In which direction must the boat be steered to cross the river in minimum time? How far will it drift?

Sol. (a) Velocity of a boat on still water is its capacity to move on water surface and equals to its velocity relative to water.

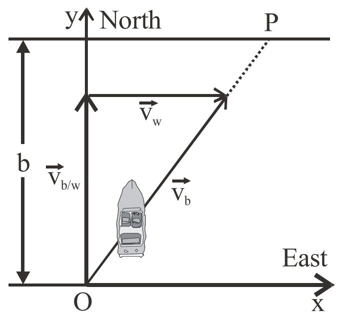


$\vec{v}_{b/w}$ = Velocity of boat relative to water = Velocity of boat on still water

On flowing water, the water carries the boat along with it. Thus velocity $\vec{v}_b$ of the boat relative to the ground equals to vector sum of $\vec{v}_{b/w}$ and $\vec{v}_w$. The boat crosses the river with the velocity $\vec{v}_b$.

$$\vec{v}_b = \vec{v}_{b/w} + \vec{v}_w$$

(b) To cross the river perpendicular to current the boat must be steered in a direction so that one of the components of its velocity ($\vec{v}_{b/w}$) relative to water becomes equal and opposite to water flow velocity $\vec{v}_w$ to neutralize its effect. It is possible only when velocity of boat relative to water is grater than water flow velocity. In the adjoining figure it is shown that the boat starts from the point O and moves along the line OP (y-axis) due north relative to ground with velocity $\vec{v}_b$. To achieve this it is steered at an angle θ with the y-axis.



$$v_{b/w} \sin\theta = v_w \rightarrow 5\sin\theta = 3 \Rightarrow \theta = 37° \textbf{ Ans.}$$

(c) The boat will cover river width b with velocity

$v_b = v_{b/wy} = v_{b/w} \sin 37° = 4$ m/s in time t, which is given by

$$t = b / v_b \rightarrow t = 50\text{s} \textbf{ Ans.}$$

(d) To cross the river in minimum time, the component perpendicular to current of its velocity relative to ground must be kept to maximum value. It is achieved by steering the boat always perpendicular to current as shown in the adjoining figure. The boat starts from O at the south bank and reaches point P on the north bank. Time t taken by the boat is given by

$$t = b / v_{b/w} \rightarrow t = 40\text{s} \textbf{ Ans.}$$

Drift is the displacement along the river current measured from the starting point. Thus, it is given by the following equation. We denote it by x_d.

$$x_d = v_{bx} t$$

Substituting $v_{bx} = v_w = 3$ m/s, from the figure, we have

x_d = 120 m **Ans.**