



Overview of Enriched ∞ -categories

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§1. Motivation: Enriched 1-Categories

Def: Let V be a monoidal category. A V -enriched category C consists of

- A set of objects $\text{ob}C$
- $\forall x, y \in \text{ob}C$: hom-object $C(x, y) \in V$
- Composition maps: $C(x, y) \otimes C(y, z) \rightarrow C(x, z)$
- Units: $I_x \rightarrow C(x, x)$

such that composition is associative and unital.

Examples of enriched 1-categories

- $\mathcal{V} = \mathbf{Set} \rightsquigarrow$ locally small categories
- $\mathcal{V} = \mathbf{Chain\ complexes} \rightsquigarrow$ DG-categories
- $\mathcal{V} = \mathbf{Cat} \rightsquigarrow$ Strict 2-categories
- $\mathcal{V} = \mathbf{sSet}$ or $\mathbf{Top} \rightsquigarrow$ $(\infty, 1)$ -categories

Q: How to generalize?

A: Multicategories!

§1.1: Multicategories - Also called non-symmetric coloured operads

Intuition: Category with morphisms having a list of objects as source

Def: A Multicategory M consists of

- Set of objects: $\text{ob } M$
- For $x_1, \dots, x_n, y \in \text{ob } M$, $0 \leq n$, a set $M(x_1, \dots, x_n; y)$ of multimorphisms from $(x_1, \dots, x_n)$ to y
- Identity multimorphism: $\text{id}_x: (x) \rightarrow x$
- Associative & unital composition laws

composition: $((z_1, \dots, z_{i_1}) \rightarrow y_1, \dots, (z_{i_{n-1}}, \dots, z_{i_n}) \rightarrow y_n)$ with

$$(y_1, \dots, y_n) \rightarrow x$$

$$\rightsquigarrow (z_1, \dots, z_n) \rightarrow x$$

Def: A multifunctor $F: M \rightarrow N$ between multicategories assigns

$$x \mapsto F(x)$$

$$(x_1, \dots, x_n) \rightarrow y \mapsto (F(x_1), \dots, F(x_n)) \rightarrow F(y)$$

in a way compatible with unit and composition

Ex: $\mathcal{V}$ monoidal category $\rightsquigarrow$ multicategory $\mathcal{V}^\otimes$ by

$$\mathcal{V}^\otimes(x_1, \dots, x_n; y) := \mathcal{V}^\otimes(x_1 \otimes \dots \otimes x_n, y)$$

$\uparrow$ Normal set of maps

§1: Motivation of enriched 1-Categories
§11: Multicategories

Ex: $\mathcal{V}$ monoidal category $\rightsquigarrow$ multicategory $\mathcal{V}^{\otimes}$ by

$$\mathcal{V}^{\otimes}(x_1, \dots, x_n; y) := \mathcal{V}(x_1 \otimes \dots \otimes x_n, y)$$

§1: Motivation of enriched 1-categories
§11: Multicategories

Def: An **Algebra** of a multicategory M in a monoidal category $\mathcal{V}$ is a multifunctor

$$M \rightarrow \mathcal{V}^{\otimes}$$

Ex: S a set $\rightsquigarrow$ Multicategory O_S defined by

- $\text{Ob } O_S := S \times S$
- Multimorphism set

$$O_S((x_0, y_1), (x_1, y_2), \dots, (x_{n-1}, y_n); (y_0, x_n)) := \begin{cases} \star & , y_i = x_i \quad 0 \leq i \leq n \\ \emptyset & \text{otw} \end{cases}$$

- $\text{ob } O_S := S \times S$

- $O_S((X_0, Y_1), (X_1, Y_2), \dots, (X_{n-1}, Y_n); (Y_0, X_n)) := \begin{cases} * & , Y_i = X_i \text{ for } 0 \leq i \leq n \\ \emptyset & , \text{ otherwise} \end{cases}$

O_S -algebra $C: O_S \rightarrow \mathcal{V}^{\otimes}$ gives the following structure:

- $(x, y) \in \text{ob } O_S \xrightarrow{e} C(x, y) \in \mathcal{V}$

- Unique map $() \rightarrow (x, x) \xrightarrow{e} \text{unit } I_x \rightarrow C(x, x)$

$\uparrow$ over case $n=0$

- Unique multimorphism $((x, y), (y, z)) \rightarrow (x, z)$

$\xrightarrow{e}$ Composition written $C(x, y) \otimes C(y, z) \rightarrow C(x, z)$

Looking at triples of pairs we get that this is associative & unital

O_S -algebras in $\mathcal{V}^{\otimes}$



$\mathcal{V}$ -enriched category
 C w. $\text{ob } C = S$

O_S - algebras in $\mathcal{V}^{\otimes}$



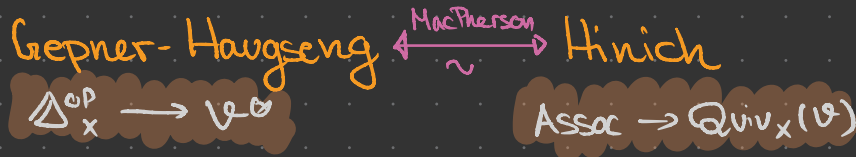
$\mathcal{V}$ -enriched category C with $\text{ob } C = S$

§1: Motivation of enriched 1-categories
§11: Multicategories

↗
This version we can generalize!

Idea: "Many object associative algebras"

2 ways of doing this:



§2: Many-object associative algebras

§2.1: Non-Symmetric ω -Operads

multicategories $\rightsquigarrow$ (generalized) non-symmetric ω -operads

Def. A non-symmetric ω -operad is an inner fibration $\pi: \mathcal{O} \rightarrow \Delta^{\text{op}}$ s.t.

1. $\varphi: [m] \rightarrow [n]$ inert, $x \in \mathcal{O}_{[n]}$ $\xrightarrow{\varphi}$ Exist π -cocartesian morphism
 $x \rightarrow \varphi_! x$
over φ

2. For every $[n] \in \Delta^{\text{op}}$, the functor

$$\mathcal{O}_{[n]} \rightarrow (\mathcal{O}_{[1]})^{\times n}$$

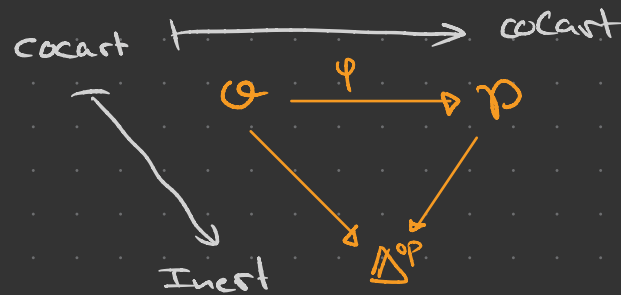
induced by cocartesian arrows over the inert maps $\overset{\text{standard inerts. } P_i: [1] \hookrightarrow [i-1, i] \hookrightarrow [n]}{\downarrow} p_i$ ($i=1, \dots, n$) is an equivalence

3. ...

Def. A **morphism** of non-symmetric ∞ -operads

§2: Many-object associative algebras
§2.1: Non-symmetric ∞ -Operads

$\mathcal{O} \rightarrow \mathcal{P}$:



$\hookrightarrow$ Also called **$\mathcal{O}$ -algebra** $\hookrightarrow \text{Alg}_{\mathcal{O}}(\mathcal{P}) \subseteq \text{Fun}_{\Delta^{\circ \mathcal{P}}}(\mathcal{O}, \mathcal{P})$

$\hookrightarrow$ Category **$\text{Op}_{\infty}^{\text{ns}}$**

§2.2: From multicategories to non-symmetric α -operads

M a multicategory $\rightsquigarrow N(M^\otimes)$ non-symmetric α -operad

Def: The category of operads $M^\otimes$ is defined by

- Objects: Lists $(x_1, \dots, x_n)$, $x_i \in M$, $n=0, 1, \dots$
- Morphism $(x_1, \dots, x_n) \rightarrow (y_1, \dots, y_m)$:

> $\psi: [m] \rightarrow [n]$ in Δ

> For each $j=1, \dots, m$ a multimorphism in M

$$(x_{\psi(j-1)+1}, x_{\psi(j-1)+2}, \dots, x_{\psi(j)}) \rightarrow y_j$$

Rem:

$$M^\otimes \longrightarrow \Delta^{\circ p}$$

$$(x_1, \dots, x_n) \mapsto [n]$$

$\hookrightarrow$ Non-symmetric α -operad

Def: Let $\mathcal{V}$ be a monoidal ∞ -category & S a set.

§2: Many object associative algebras
§2.2: From multicategories
to non-symmetric ∞ -operads

A $\mathcal{V}$ -enriched category with set of objects S is an

$N(O_S^{\otimes}) \rightarrow O_S^{\otimes}$ -algebra in $\mathcal{V}$ i.e. a morphism of non-symmetric ∞ -operads $O_S \rightarrow \mathcal{V}$

Rem: $\mathcal{C}: O_S^{\otimes} \rightarrow \mathcal{V}$ gives the desired structure.

- $(x, y) \in O_S^{\otimes} \xrightarrow{\mathcal{C}} \mathcal{C}(x, y) \in \mathcal{V}$
- $((x, y), (y, z)) \rightarrow (x, z)$ in $O_S^{\otimes}$ over $d_1: [2] \rightarrow [1]$

$$\mathcal{C}((x, y), (y, z)) \simeq \mathcal{C}(x, y) \otimes \mathcal{C}(y, z) \rightarrow \mathcal{C}(x, z) \text{ in } \mathcal{V}$$

- $() \rightarrow (x, x)$ in $O_S^{\otimes} \xrightarrow{\mathcal{C}} \mathcal{C}(x, x)$ in $\mathcal{V}$

! We want **Space** of objects

§2.3: Enriched with space of objects

'Virtual double categories' instead of multicategories

- objects
- + vertical & horizontal morphisms
- + Cells with list of vertical arrows as source

$\mathcal{O}_{[0]}$ non-trivial

$$\mathcal{O}_{[n]} \simeq \mathcal{O}_{[0]} \times_{\mathcal{O}_{[0]} \times \mathcal{O}_{[0]}} \cdots \times_{\mathcal{O}_{[0]} \times \mathcal{O}_{[0]}} \mathcal{O}_{[1]}$$

→ Generalized non-symmetric ∞ -operads

Generalize non-symmetric ∞ -operads so the fibre over $[0]$ is not necessarily contractible

Key: Consider cocartesian lifts of the inert maps

$[n] \rightarrow [0]$, $P_i: [n] \rightarrow [1]$, instead of just the standard inerts

Def: A **Generalized non-symmetric ∞ -op** is an inner fibration $\pi: \mathcal{C} \rightarrow \Delta^{\text{op}}$ s.t.

1. $\varphi: [m] \rightarrow [n]$ inert, $X \in \mathcal{C}_{[n]}$ $\xrightarrow{\varphi}$ Exist π -cocartesian morphism

$X \rightarrow \varphi_! X$
over φ

2. For every $[n] \in \Delta^{\text{op}}$, the functor

$$\mathcal{C}_{[n]} \rightarrow (\mathcal{C}_{[1]})^{\times n} \quad \mathcal{C}_{[n]} \rightarrow \lim_{\substack{[1] \rightarrow [i] \text{ inert} \\ i=0,1}} \mathcal{C}_{[i]} \simeq \mathcal{C}_{[1]}^{\times} \mathcal{C}_{[0]} \times \dots \times \mathcal{C}_{[0]} \mathcal{C}_{[1]}$$

induced by cocartesian arrows over the inert maps P_i ($i=1, \dots, n$)

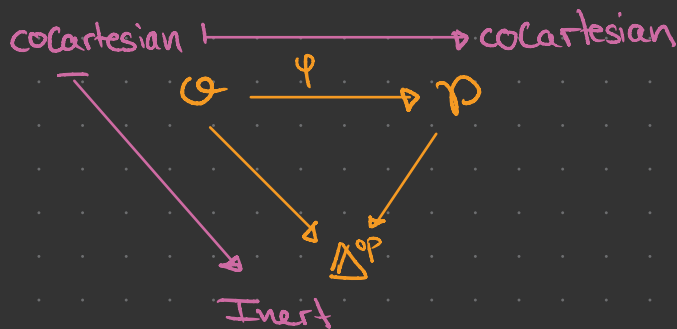
$[n] \rightarrow [0]$ is an equivalence

3. ...

Def. A **morphism** of **generalized non-symmetric ∞ -operad**

§2: Many-object associative algebras
§2.3: Enriched with space of objects

$\mathcal{O} \rightarrow \mathcal{P}$:



$\hookrightarrow$ Also called **$\mathcal{O}$ -algebra** $\hookrightarrow \text{Alg}_{\mathcal{O}}(\mathcal{P}) \subseteq \text{Fun}_{\Delta^{\text{op}}}(\mathcal{O}, \mathcal{P})$

$\hookrightarrow$ Category **$\text{Op}_{\infty}^{\text{ns, gen}}$**

Def: A double ∞ -category is a generalized non-symmetric ∞ -operad $\mathcal{O} \rightarrow \Delta^{\text{op}}$ which is also Cartesian

§2: Many object associative algebras
§2.3: Enriched with space of objects

Explanation of name:

$\mathcal{O} \rightarrow \Delta^{\text{op}}$
 double ∞ -category

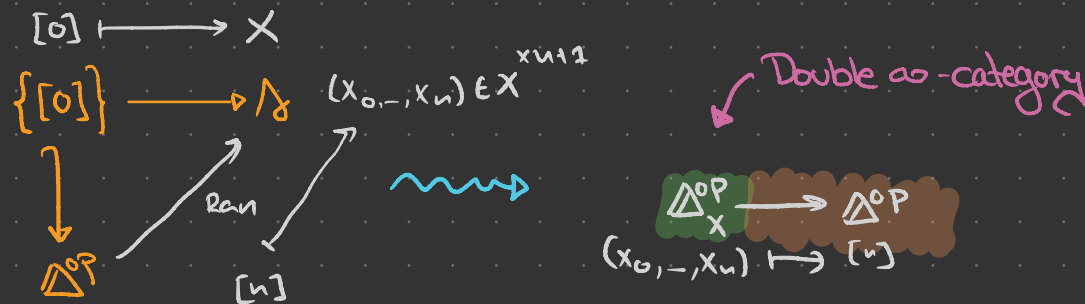
Straightening 

Simplicial category $\mathcal{C}_\bullet: \Delta^{\text{op}} \rightarrow \text{Cat}_\infty$
 satisfying Segal-Rezk condition:

$$\mathcal{C}_n \simeq \mathcal{C}_1 \times_{\mathcal{C}_0} \cdots \times_{\mathcal{C}_0} \mathcal{C}_1$$

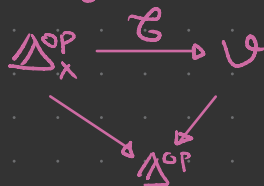
Exactly the internal categories in Cat_∞

Construction: Fix a space X .



Def: Let $\mathcal{V}$ be a monoidal category. A **categorical $\mathcal{V}$ -algebra** with space of objects is a **Δ_X^{op} -algebra** in $\mathcal{V}$.

$\rightsquigarrow$ i.e. a morphism of generalized non-symmetric ∞ -operads



Let $\mathcal{C}: \Delta_x^{\text{op}} \rightarrow \mathcal{V}$ be a categorical $\mathcal{V}$ -algebra

- $(x, y) \in X^{x^2} \rightsquigarrow \mathcal{C}(x, y)$ in $\mathcal{V}$
- $(x, y, z) \in X^{x^3}$, $d_1: [2] \rightarrow [1] \rightsquigarrow (x, y, z) \rightarrow (x, z)$ in Δ_x^{op}

Composition maps $\rightsquigarrow \mathcal{C}(x, y) \otimes \mathcal{C}(y, z) \rightarrow \mathcal{C}(x, z)$ in $\mathcal{V}$

- $(x) \in X$, $[0] \rightarrow [1] \rightsquigarrow (x) \rightarrow (x, x)$ in Δ_x^{op}

Unit $\rightsquigarrow I \rightarrow \mathcal{C}(x, x)$ in $\mathcal{V}$

Ex:

Associative $\mathcal{V}$ -algebra



Categorical $\mathcal{V}$ -algebra
 with contractible space
 of objects

Ex:

$$\begin{aligned} I_{\mathcal{V}}: \Delta_x^{\text{op}} &\rightarrow \mathcal{V} \\ * &\mapsto I_{\mathcal{V}} \end{aligned}$$

Construction:

§2: Many-object associative algebras
§2.3: Enriched with space of objects

$$\mathcal{A}^{\text{op}} \xrightarrow{\text{AOP}_{\mathcal{A},1}} (\text{OP}_{\infty}^{\text{ns,gen}})^{\text{op}} \xrightarrow{\text{Alg}_{\mathcal{A}}(U)} \text{Cat}_{\infty}$$

$\hookrightarrow \text{Alg}_{\text{cat}}(U) \rightarrow \mathcal{A}$ associated Cartesian fibration

Prop: $\text{Alg}_{\text{cat}}(\mathcal{A}) \simeq \text{Seq}_{\infty}$

! We want $\mathcal{A}$ -enriched $\simeq \text{Cat}_{\infty} \xrightarrow{\text{red}} \text{Need a completeness condition}$

Thm: $\text{Cat}_{\infty} \simeq \text{Alg}_{\text{cat}}(\mathcal{A})[\text{FFES}^{-1}]$

Def: $\mathcal{V}$ -enriched category $\text{Cat}_{\infty}^{\mathcal{V}} := \text{Alg}_{\text{cat}}(\mathcal{V})[\text{FFES}^{-1}]$

§3: Enrichment through closed action

§3.1: Motivation V a closed monoidal category

C a V -enriched category
with V -tensors

$C \rightarrow \Delta$
 $x \mapsto \text{Mor}_V(V, \text{Mor}_C(C, x))$
is corepresentable by some $U \in C$
for all $V \in V, C \in C$



The V -tensors endow
 C with a
closed V -action

Underlying V -enriched category
that admits V -tensors



Category with
closed V -action

$\mathcal{V}$ -enriched category
with $\mathcal{V}$ -tensors



Category with
closed $\mathcal{V}$ -action

V -enriched categories
with V -tensors



Categories with
closed V -action

~

V -enriched category

~

Relaxed version of categories
with closed V -action
in which V -tensors need
not exist

§3.2: Right-tensored ∞ -categories

Def: • $\mathbf{BM} := \Delta^{\text{op}} / [1]$

$$> [n] \rightarrow [1] \rightsquigarrow (0, -, 1, -)$$

• $\mathbf{RM} \subseteq \mathbf{BM}$ of the form $(0, 1, -, 1), (1, -, 1)$

Prop: $\mathbf{BM} \rightarrow \Delta^{\text{op}}, \mathbf{RM} \rightarrow \Delta^{\text{op}}$ are both double ∞ -categories

Def: For M a generalized non-symmetric ∞ -category

• $\mathbf{BMod}(M) := \text{Alg}_{\mathbf{BM}}(M)$

• $\mathbf{RMod}(M) := \text{Alg}_{\mathbf{RM}}(M)$

• $i: \Delta^{\text{op}}_{\{1\}} \hookrightarrow \Delta^{\text{op}} \rightsquigarrow i^*: \mathbf{RMod}(M) \rightarrow \text{Alg}_{\Delta^{\text{op}}}(M)$

$\rightsquigarrow \mathbf{RMod}_A(M) := \text{fibre of } i^* \text{ over } A \in \text{Alg}_{\Delta^{\text{op}}}(M)$

Rem: This agrees with Lurie's definition

Def: A **weakly right-tensored** ∞ -category is a morphism of generalized non-symmetric ∞ -categories

$$g: M \rightarrow RM$$

such that both $M_{(0,1)}$, $M_{(1,1)}$ are contractible.

- $M_m := M_{(0,1)}$
 - $M_r^\omega: i^*M \rightarrow \Delta^{\text{op}} \in \mathcal{O}_{\text{P}\infty}^{\text{ns, gen}}$
- } g exhibits M_m as weakly right-tensored over M_r

If g is furthermore **coCartesian** we say it exhibits M_m as **right-tensored** over M_r

Ex: $\mathcal{U}^\omega$ monoidal $\rightsquigarrow \pi^* \mathcal{U} \rightarrow RM$ exhibits $\mathcal{U}_{[1]}$ as **right-tensored** over $\mathcal{U}$

$\nwarrow \pi: RM \rightarrow \Delta^{\text{op}}$

Intuition: Let $g: M \rightarrow RM$ exhibit $\mathcal{C}$ as

right-tensored over $\mathcal{V}$

Unstraightening $\Rightarrow$

$F: RM \rightarrow \text{Cat}_{\mathcal{V}}$ such that

- $F(0, 1) \simeq \mathcal{C}$
- $i^* F$ is the associative algebra corresponding to $\mathcal{V}$

$$\left. \begin{array}{l} F \in \text{Mon}_{RM}(\text{Cat}_{\mathcal{V}}) \\ \simeq \text{RMod}(\text{Cat}_{\mathcal{V}}) \end{array} \right\} \rightsquigarrow$$



Can think of g as giving rise to an action map

$$\otimes: \mathcal{C} \times \mathcal{V} \rightarrow \mathcal{V}$$

which is well-defined up to homotopy & compatible with the monoidal structure on $\mathcal{V}$

§3.3: Closed right-tensored ∞ -categories

§3: Enrichment through closed action

Def. Assume $g: \mathcal{M} \rightarrow \mathbf{RM}$ exhibits $\mathcal{C}$ as **weakly right-tensored** over $\mathcal{V}$.

- For $v \in \mathcal{V}$, $c_0, c_1 \in \mathcal{C}$, we write

$$\mathrm{Map}_{\mathcal{C}}(c_0 \boxtimes v, c_1) := \mathrm{Map}_{\mathcal{M}}^{\varphi}((c_0, v), c_1) \subseteq \mathrm{Map}_{\mathcal{M}}((c_0, v), c_1) \text{ which is mapped to the unique active map } \varphi: [1] \rightarrow [2]$$

- A **morphism object** of $c_0, c_1 \in \mathcal{C}$

consists of

> An object $F^R(c_0, c_1) \in \mathcal{V}$

> A morphism $\alpha: c_0 \boxtimes F^R(c_0, c_1) \rightarrow c_1$ in $\mathcal{V}$, such that

$$\mathrm{Map}_{\mathcal{V}}(v, F^R(c_0, c_1)) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{C}}(c_0 \boxtimes v, c_1) \quad \forall v \in \mathcal{V}$$

- g is **closed right-tensored** if there exist a (right) **morphism object** $F^R(c_0, c_1)$ for all $c_0, c_1 \in \mathcal{C}$.

Def: $g: M \rightarrow RM$ exhibiting $\mathcal{C}$ as closed right-tensored over $\mathcal{V}$: $\forall c_0, c_1 \in \mathcal{C}, \exists F^R(c_0, c_1) \in \mathcal{V}$ st. $\text{Map}_{\mathcal{V}}(v, F^R(c_0, c_1)) \simeq \text{Map}_{\mathcal{C}}(c_0 \boxtimes v, c_1)$

- $\text{RTens}^{\mathcal{C}} \subseteq \text{Op}_{\infty}^{\text{ns, gen}} / RM$
- $\Delta^{\mathcal{C}} \hookrightarrow RM \rightsquigarrow \text{Projection } \text{RTens}^{\mathcal{C}} \xrightarrow{\pi} \text{Mon}_{\mathcal{V}}$
 $\rightsquigarrow \text{RTens}^{\mathcal{C}}(\mathcal{V}) = \text{Fibre of } \pi \text{ at } \mathcal{V}$

Ex: $g: M \rightarrow RM$ exhibits $\mathcal{C}$ as closed right-tensored over $\mathcal{V}$



For every $c \in \mathcal{C}$ the right-tensoring functor

$$c \boxtimes - : \mathcal{V} \rightarrow \mathcal{C}$$

admits a right adjoint $F^R(c, -)$

All of this could have been done with

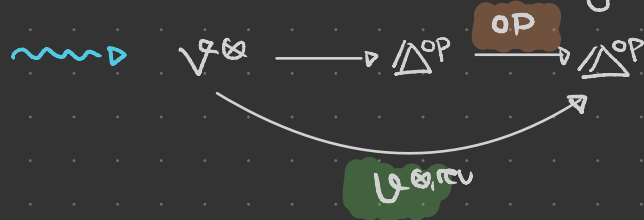
$LM \subseteq BM$ of the form $(0, -, 0, 1), (0, -, 0)$

§3: Enrichment through closed action
§3.3: closed right-tensorial ∞ -categories

~ Dual theory of left-tensored

- Right version: $F^R(a, b) \otimes F^R(b, c) \rightarrow F^R(a, c)$
- Left version: $F^L(b, a) \otimes F^L(c, b) \rightarrow F^L(a, c)$

Construction: Let $\mathcal{V}^{\otimes}$ be a monoidal category



- $op: BM \rightarrow BM$ ~
 $LM \mapsto RM$
 $RM \mapsto LM$

$$LTens^{\mathcal{V}}(U^{rev}) \simeq RTens(U)$$

$$G^{op} \mapsto G$$

§4: Comparison

Thm: (Heine) Let $\mathcal{V}$ be a monoidal category. Then

$$\mathrm{LTens}^{\mathrm{cl}}(\mathcal{V}^{\mathrm{rev}}) \simeq \mathrm{Cat}_{\infty}^{\mathcal{V}}$$

Prop: (R.) There exists a functor

$$\overline{(-)}: \mathrm{Cat}_{\infty}^{\mathrm{en}} \rightarrow \mathrm{Cat}_{\infty}^{\mathrm{sp}}$$

preserving the underlying ∞ -category.

Prop: (R.) $(-): \text{Cat}_{\infty}^{\text{ex}} \longrightarrow \text{Cat}_{\infty}^{\text{sp}}$

Proof: • $\mathcal{C} \in \text{Cat}_{\infty}^{\text{ex}} \rightsquigarrow \text{Ind}(\mathcal{C}) \in \text{Pr}^{\text{st}}$, $\mathcal{C} \subseteq \text{Ind}(\mathcal{C})$ full subcategory

• HA.4.8.2.18: $\text{Pr}^{\text{st}} \simeq \text{Mod}_{\text{Sp}}(\text{Pr}^{\text{L}})$ $\xrightarrow{\text{GH 7.4.13}}$ $\text{Ind}(\mathcal{C})$ both right- and left-tensored over Sp

• $\mathcal{C} \otimes -$ admits right-adjoint for all $\mathcal{C} \in \text{Ind}(\mathcal{C})$

$\rightsquigarrow \eta: M \rightarrow RM$ exhibiting $\text{Ind}(\mathcal{C})$ as closed right-tensored over Sp

$\rightsquigarrow \tilde{(-)}: \text{Cat}_{\infty}^{\text{ex}} \xrightarrow{\text{Ind}(-)} \text{Pr}^{\text{st}} \simeq \text{Mod}_{\text{Sp}}(\text{Pr}^{\text{L}}) \hookrightarrow \text{RTens}^{\text{cl}}(\text{Sp})$

• $\mathcal{C} \subseteq \tilde{\mathcal{C}}$ full subcategory s.t. for any $f: \mathcal{C} \rightarrow \mathcal{Q}$ in $\text{Cat}_{\infty}^{\text{ex}}$, $\mathcal{C} \hookrightarrow \tilde{\mathcal{C}} \xrightarrow{f} \mathcal{Q}$ factors through $\mathcal{Q}$

$\rightsquigarrow$ Subfunctor $(-): \text{Cat}_{\infty}^{\text{ex}} \rightarrow \text{RTens}^{\text{cl}}(\text{Sp}) \simeq \text{Cat}_{\infty}^{\text{sp}}$

Many-Object
Associative
Algebras



Closed
Tensor
Categories

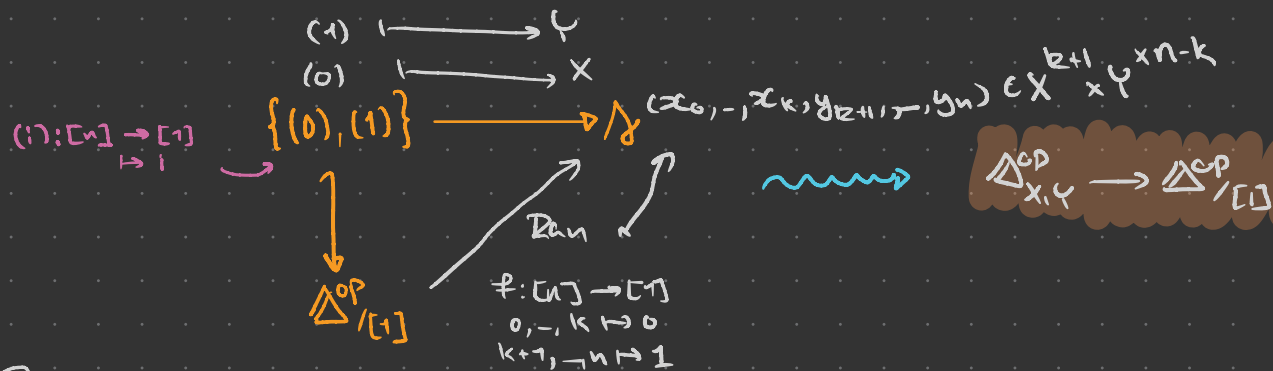
- + Intuitive how we get the desired structure
- + Follows 'higher algebra' intuition
- + Nicely "packaged" theory
- ÷ Hard to work with
- ÷ No Yoneda
- + Easy extension to "enriched algebra"

- + Geometric intuition
- + More "hands on"
- ÷ Book keeping
- + Yoneda

§5: Enriched Algebra

§5.1: Bimodules

Def: X, Y spaces



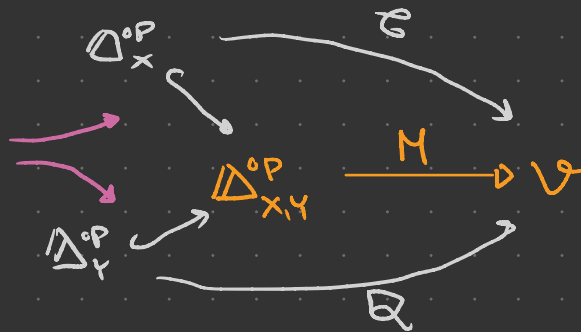
Prop: The composite

$$\Delta_{X,Y}^{op} \rightarrow \Delta_{X,Y}^{op}/[1] \rightarrow \Delta^{op}$$

is a double ∞ -category

Understanding $\Delta_{x,y}^{\text{op}}$ -algebra in $\mathcal{V}$

Laying over
the two inclusions
 $\Delta^{\text{op}} \hookrightarrow \Delta^{\text{op}}/[1]$
given by composing
with the two maps
 $[0] \rightarrow [1]$



- $x \in X, y \in Y \rightsquigarrow M(x, y) \in \mathcal{V}$
- $x_0, x_1 \in X, y \in Y, d_1: (0, 1, 1) \rightarrow (0, 1) \rightsquigarrow (x_0, x_1, y) \rightarrow (x_0, y)$
 $\rightsquigarrow C(x_0, x_1) \otimes M(x_1, y) \rightarrow M(x_0, y)$
- similarly for $x \in X, y_0, y_1 \in Y \rightsquigarrow M(x, y_0) \otimes Q(y_0, y_1) \rightarrow M(x, y_1)$

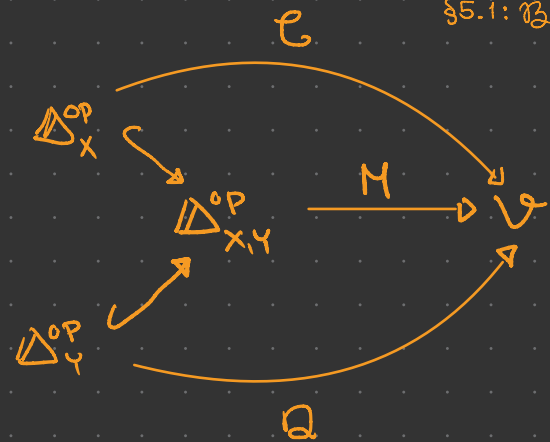
$$x_0, x_1 \in X, y_0, y_1 \in Y$$

$$(x_0, x_1, y_0, y_1) \longrightarrow (x_0, y_0, y_1)$$

~



$$(x_0, x_1, y_1) \longrightarrow (x_0, y_1)$$



$$C(x_0, x_1) \otimes M(x_1, y_0) \otimes Q(y_0, y_1) \longrightarrow M(x_0, y_0) \otimes Q(y_0, y_1)$$

~



$$C(x_0, x_1) \otimes M(x_1, y_1) \longrightarrow M(x_0, y_1)$$

~

$$M: \Delta_{X,Y}^{op} \rightarrow V$$

is what we want as

V -enriched C - Q -bimodule

Construction: Let $\mathcal{V}$ be a monoidal category.

$$\mathcal{S}^{x_2, op} \xrightarrow{\text{Alg}_{\Delta^{op}, -}(\mathcal{V})} \text{Cat}_{\infty} \rightsquigarrow \text{Bimodcat}(\mathcal{V}) \longrightarrow \mathcal{S}^{x_2} \quad \text{Cartesian fibration}$$

Recall that

$$\mathcal{S}^{op} \xrightarrow{\text{Alg}_{\Delta^{op}}(\mathcal{V})} \text{Cat}_{\infty} \rightsquigarrow \text{Algcat}(\mathcal{V}) \longrightarrow \mathcal{S} \quad \text{Cartesian fibration}$$

Using the two inclusions

$$\left. \begin{array}{l} \Delta_x^{op} \hookrightarrow \Delta_{x,y}^{op} \\ \Delta_y^{op} \hookrightarrow \Delta_{x,y}^{op} \end{array} \right\} \rightsquigarrow \text{Bimodcat}(\mathcal{V}) \rightarrow \text{Algcat}(\mathcal{V})^{x_2}$$

A $\mathcal{V}$ -enriched $\mathcal{C}$ - $\mathcal{Q}$ -bimodule is an object in the fibre over

$$(\mathcal{C}, \mathcal{Q}) \in \text{Algcat}(\mathcal{V})^{x_2} \rightsquigarrow \text{Bimod}_{\mathcal{C}, \mathcal{Q}}(\mathcal{V})$$

Thm: (Haugsgeng) Let $\mathcal{V}$ be a monoidal category and $\mathcal{A}, \mathcal{B}$ categorical $\mathcal{V}$ -algebras with space of objects X, Y, Z .

$$M: \Delta_{X,Y}^{\text{op}} \rightarrow \mathcal{V} \quad \mathcal{C} - \mathcal{A} - \text{bimodule}$$

$$N: \Delta_{Y,Z}^{\text{op}} \rightarrow \mathcal{V} \quad \mathcal{A} - \mathcal{C} - \text{bimodule}$$

$$\xrightarrow{\quad} M \otimes_{\mathcal{A}} N: \Delta_{X,Z}^{\text{op}} \rightarrow \mathcal{V} \quad \mathcal{C} - \mathcal{C} - \text{bimodule}$$

$M \otimes_{\mathcal{A}} N$ evaluated at $(x, z) \in X \times Z$ is

$$| [x] \mapsto \text{colim}_{(y_0, \dots, y_n) \in Y^{n+2}} M(x, y_0) \otimes \mathcal{A}(y_0, y_1) \otimes \dots \otimes \mathcal{A}(y_{n-1}, y_n) \otimes N(y_n, z) |$$

Thm: (Haugsgeng) There exists an $(\infty, 2)$ -category $\text{ALG}_{\text{cat}}(\mathcal{V})$, called the **Morita category**, which has

- Objects: Categorical $\mathcal{V}$ -algebras
- Morphisms: $\mathcal{V}$ -enriched bimodules
- Composition: $\otimes$ -product

Prop: (R.) $\mathcal{C}: \mathcal{A}_X^{\text{op}} \rightarrow \mathcal{U}$ categorical $\mathcal{U}$ -algebra, then $S_0^* \mathcal{C}$ is the $\otimes_{\mathcal{U}}$ -unit, i.e.

$$M \otimes_{\mathcal{C}} S_0^* \mathcal{C} \simeq M, \quad S_0^* \mathcal{C} \otimes_{\mathcal{U}} N \simeq N$$

§5: Enriched Algebra
§5.1: Bimodules

§5.2: Enriched Modules

Def: Let $\mathcal{V}$ be a monoidal category, $G: \Delta_{\mathcal{V}}^{\text{op}} \rightarrow \mathcal{V}$ a categorical $\mathcal{V}$ -algebra

- $\mathcal{V}$ -enriched right G -module : $I_{\mathcal{V}} - G$ -bimodule

$$> \text{RMod}_G^{\text{en}}(\mathcal{V}) := \text{Bimod}_{I_{\mathcal{V}}, G}(\mathcal{V})$$

- $\mathcal{V}$ -enriched left G -module : $G - I_{\mathcal{V}}$ -bimodule

$$> \text{LMod}_G^{\text{en}}(\mathcal{V}) := \text{Bimod}_{G, I_{\mathcal{V}}}(\mathcal{V})$$

- $\mathcal{S}^{\text{op}} \xrightarrow{\Delta_{I_{\mathcal{V}}}}^{\text{op}} (\mathcal{O}_{\mathcal{P}_{\infty}}^{\text{ns, gen}})^{\text{op}} \xrightarrow{\text{Alg}_{(-), (\mathcal{V})}}^{\text{op}} \text{Cat}_{\infty} \rightsquigarrow \text{RMod}_{\text{cat}}^{\text{en}}(\mathcal{V}) \longrightarrow \mathcal{S}$

- $\mathcal{S}^{\text{op}} \xrightarrow{\Delta_{-, I_{\mathcal{V}}}}^{\text{op}} (\mathcal{O}_{\mathcal{P}_{\infty}}^{\text{ns, gen}})^{\text{op}} \xrightarrow{\text{Alg}_{(-), (\mathcal{V})}}^{\text{op}} \text{Cat}_{\infty} \rightsquigarrow \text{LMod}_{\text{cat}}^{\text{en}}(\mathcal{V}) \longrightarrow \mathcal{S}$

Rem: $\text{RMod}_G^{\text{en}}(\mathcal{V}) \simeq \text{LMod}_{G^{\text{op}}}^{\text{en}}(\mathcal{V})$

- $\text{RMod}_{\text{cat}}^{\text{en}}(\mathcal{V}) \simeq \text{LMod}_{\text{cat}}^{\text{en}}(\mathcal{V}^{\text{rev}})$

Thm: (R.) Given a monoidal category $\mathcal{V}$, then there exists a functor

$$\text{Cat}_{\infty}^{\mathcal{V}} / \overline{\mathcal{V}} \rightarrow \text{RMod}_{\text{cat}}^{\mathcal{V}}(\mathcal{V})$$

$$G: \overline{\mathcal{G}} \rightarrow \overline{\mathcal{V}} \mapsto R_G$$

where

$$R_G(*, -, *, c_0, \dots, c_n) \cong G(c_0) \otimes F^2(c_0, c_1) \otimes \dots \otimes F^2(c_{n-1}, c_n)$$

Thank
you

