Overview of Enriched ∞-categories

Young-HoM 18.04.24

Julie Rasmusen

§1. Motivation: Enriched 1-Categories

<u>Def:</u> Let V be a monoidal category. A V-enriched category C consists of

- A set of objects $\mathrm{ob}C$
- $\forall x, y \in \mathrm{ob}C$: hom-object $C(x,y) \in V$
- Composition maps: $C(x,y) \otimes C(y,z) \to C(x,z)$
- Units: $I_x \to C(x,x)$

such that composition is associative and unital

Examples of enriched 1-categories

- $\mathcal{V}$ = Set ⇝ locally small categories
- $\mathcal{V}$ = Chain complexes ⇝ DG-categories
- $\mathcal{V}$ = Cat ⇝ Strict 2-categories
- $\mathcal{V}$ = sSet or Top ⇝ $(\infty,1)$-categories

Q: How to generalize?

A: Multicategories!

§11: Multicategories - Also called non-symmetric coloured operads

Intuition: Category with morphisms having a list of objects as source

Def: A Multicategory M consists of

- Set of objects: $\mathrm{ob}M$
- For $X_1, \ldots, X_n, Y \in \mathrm{ob}M$, $0 \leq n$, a set $M(X_1, \ldots, X_n; Y)$ of multimorphisms from $(X_0, \ldots, X_n)$ to Y
- Identity multimorphism: $\mathrm{id}_X : (X) \to Y$
- Associative & unital composition laws

composition: $((Z_1, \ldots, Z_{i_1}) \to Y_1, \ \_\_\_\_ , (Z_{i_{n-1}}, \ldots, Z_{i_n}) \to Y_n)$ with

$$(Y_1, \ldots, Y_n) \to X$$

$$\rightsquigarrow (Z_1, \ldots, Z_n) \to X$$

Def: A multifunctor $F: M \to N$ between multi-categories assigns

$$x \mapsto F(x)$$

$$(x_1, \_, x_n) \to y \mapsto (F(x_1), \_, F(x_n)) \to F(y)$$

in a way compatible with unit and composition

Ex: V monoidal category $\rightsquigarrow$ multicategory $V^{\otimes}$ by

$$V^{\otimes}(x_1, \_, x_n; Y) := V^{\otimes}(x_1 \otimes \_ \otimes x_n, Y)$$

↑ Normal set of maps

Ex: $\mathcal{V}$ monoidal category $\rightsquigarrow$ multicategory $\mathcal{V}^{\otimes}$ by

$$\mathcal{V}^{\otimes}(X_1, \dots, X_n; Y) := \mathcal{V}(X_1 \otimes \dots \otimes X_n, Y)$$

Def: An Algebra of a multicategory M in a monoidal category $\mathcal{V}$ is a multifunctor

$$M \longrightarrow \mathcal{V}^{\otimes}$$

Ex: S a set $\rightsquigarrow$ Multicategory O_S defined by

- $\mathrm{Ob}\, O_S := S \times S$
- Multimorphism set

$$O_S((X_0, Y_1), (X_1, Y_2), \dots, (X_{n-1}, Y_n); (Y_0, X_n)) := \begin{cases} * & , Y_i = X_i \quad 0 \leq i \leq n \\ \emptyset & \text{otw} \end{cases}$$

- $\mathrm{ob}\, O_S := S \times S$
- $O_S((x_0,y_1),(x_1,y_2),\_\_,(x_{n-1},y_n);(y_0,x_n)) := \begin{cases} * &, y_i = x_i \text{ for } 0 \leq i \leq n \\ \varnothing &, \text{otw} \end{cases}$

O_S-algebra $C : O_S \to \mathcal{V}^{\otimes}$ gives the following structure:

- $(x,y) \in \mathrm{ob}\, O_S \rightsquigarrow C(x,y) \in \mathcal{V}$
- Unique map $() \to (x,x) \rightsquigarrow$ unit $I_x \to C(x,x)$
  over case $n=0$
- Unique multimorphism $((x,y),(y,z)) \to (x,z)$
  $\rightsquigarrow$ Composition written $C(x,y) \otimes C(y,z) \to C(x,z)$

Looking at triples of pairs we get that this is associative & unital

$$\boxed{O_S\text{-algebras in } \mathcal{V}^{\otimes}} \longleftrightarrow \boxed{\begin{array}{l}\mathcal{V}\text{-enriched category} \\ C \text{ w. } \mathrm{ob}\, C = S\end{array}}$$

$\boxed{O_S\text{-algebras in } \mathcal{V}^{\otimes}} \longleftrightarrow \boxed{\mathcal{V}\text{-enriched category } C \text{ with } \mathrm{ob}\,C = S}$

This version we can generalize!

Idea: "Many object associative algebras"

2 ways of doing this:

$$\text{Gepner-Haugseng} \xleftrightarrow[\sim]{\text{MacPherson}} \text{Hinich}$$

$\Delta^{op}_X \longrightarrow \mathcal{V}^{\otimes}$ $\quad\quad$ $\mathrm{Assoc} \to \mathrm{Quiv}_X(\mathcal{V})$

§2: Many-object associative algebras

§2.1: Non-Symmetric ∞-Operads

multicategories $\rightsquigarrow$ (generalized) non-symmetric ∞-operads

Def.: A **non-symmetric ∞-operad** is an inner fibration $\pi : \mathcal{O} \to \Delta^{op}$ s.t.

1. $\varphi : [m] \to [n]$ inert, $X \in \mathcal{O}_{[n]}$ $\rightsquigarrow$ Exist π-cocartesian morphism $X \to \varphi_! X$ over φ

2. For every $[n] \in \Delta^{op}$, the functor

$$\mathcal{O}_{[n]} \to (\mathcal{O}_{[1]})^{\times n}$$

standard inerts $p_i : [1] \simeq \{i-1, i\} \hookrightarrow [n]$

induced by cocartesian arrows over the inert maps p_i $(i = 1, \dots, n)$ is an equivalence

3. ...

Def. A morphism of non-symmetric ∞-operads $\mathcal{O} \to \mathcal{P}$:

cocart $\longmapsto$ cocart

$$\begin{tikzcd} \mathcal{O} \arrow[rr, "\varphi"] \arrow[dr] & & \mathcal{P} \arrow[dl] \\ & \Delta^{op} & \end{tikzcd}$$

Inert

$\rightsquigarrow$ Also called $\mathcal{O}$-algebra $\rightsquigarrow$ $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P}) \subseteq \mathrm{Fun}_{\Delta^{op}}(\mathcal{O}, \mathcal{P})$

$\rightsquigarrow$ Category $\mathrm{Op}_{\infty}^{ns}$

§2.2: From multicategories to non-symmetric ∞-operads

$\mathcal{M}$ a multicategory $\rightsquigarrow$ $N(\mathcal{M}^{\otimes})$ non-symmetric ∞-operad

Def: The category of operads $\mathcal{M}^{\otimes}$ is defined by

- Objects: Lists $(X_1, \ldots, X_n)$, $X_i \in \mathcal{M}$, $n = 0, 1, \ldots$
- Morphism $(X_1, \ldots, X_n) \longrightarrow (Y_1, \ldots, Y_m)$:
  - $\varphi : [m] \longrightarrow [n]$ in Δ
  - For each $j = 1, \ldots, m$ a multimorphism in $\mathcal{M}$

$$(X_{\varphi(j-1)+1}, X_{\varphi(j-1)+2}, \ldots, X_{\varphi(j)}) \to Y_j$$

Rem: $\mathcal{M}^{\otimes} \longrightarrow \Delta^{op}$ $\rightsquigarrow$ Non-symmetric ∞-operad

$(X_1, \ldots, X_n) \mapsto [n]$

Def: Let $\mathcal{V}$ be a monoidal ∞-category & S a set.

A $\mathcal{V}$-enriched category with set of objects S is an

$N(\mathcal{O}_S^{\otimes}) \rightsquigarrow$

$\mathcal{O}_S^{\otimes}$-algebra in $\mathcal{V}$ $\rightsquigarrow$ i.e. a morphism of non-symmetric ∞-operads $\mathcal{O}_S \to \mathcal{V}$

Rem: $\mathcal{C}: \mathcal{O}_S^{\otimes} \longrightarrow \mathcal{V}$ gives the desired structure:

- $(X,Y) \in \mathcal{O}_S^{\otimes} \rightsquigarrow \mathcal{C}(X,Y) \in \mathcal{V}$
- $((X,Y),(Y,Z)) \to (X,Z)$ in $\mathcal{O}_S^{\otimes}$ over $d_1: [2] \longrightarrow [1]$

  $$\mathcal{C}((X,Y),(Y,Z)) \simeq \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Z) \longrightarrow \mathcal{C}(X,Z) \quad \text{in } \mathcal{V}$$
- $() \to (X,X)$ in $\mathcal{O}_S^{\otimes} \rightsquigarrow I_{\mathcal{V}} \to \mathcal{C}(X,X)$ in $\mathcal{V}$

! We want space of objects

§2.3: Enriched with space of objects

'Virtual double categories' instead of multicategories

- objects
- \+ vertical & horizontal morphisms
- \+ Cells with list of vertical arrows as source

$\mathcal{O}_{[0]}$ non-trivial

$\mathcal{O}_{[n]} \simeq \mathcal{O}_{[1]} \times_{\mathcal{O}_{[0]}} \cdots \times_{\mathcal{O}_{[0]}} \mathcal{O}_{[1]}$

$\rightarrow$ Generalized non-symmetric ∞-operads

Generalize non-symmetric ∞-operads so the fibre over $[0]$ is not necessarily contractible

Key: Consider cocartesian lifts of the inert maps $[n] \to [0]$, $p_i : [n] \to [1]$, instead of just the standard inerts

Def. A **Generalized non-symmetric ∞-op** is an inner fibration $\pi : \mathcal{O} \to \Delta^{op}$ s.t.

1. $\varphi : [m] \to [n]$ inert, $X \in \mathcal{O}_{[n]}$ $\rightsquigarrow$ Exist π-cocartesian morphism $X \to \varphi_! X$ over φ

2. For every $[n] \in \Delta^{op}$, the functor

$$\mathcal{O}_{[n]} \to (\mathcal{O}_{[1]})^{\times n}$$

$$\mathcal{O}_{[n]} \to \lim_{\substack{[n] \to [i] \text{ inert} \\ i = 0,1}} \mathcal{O}_{[i]} \simeq \mathcal{O}_{[1]} \times_{\mathcal{O}_{[0]}} \cdots \times_{\mathcal{O}_{[0]}} \mathcal{O}_{[1]}$$

induced by cocartesian arrows over the inert maps p_i $(i = 1, \dots, n)$ $[n] \to [0]$ is an equivalence

3. ...

Def: A **morphism** of generalized non-symmetric ∞-operad $\mathcal{O} \to \mathcal{P}$:

cocartesian $\longmapsto$ cocartesian

$$\begin{tikzcd} \mathcal{O} \arrow[rr, "\varphi"] \arrow[dr] & & \mathcal{P} \arrow[dl] \\ & \Delta^{op} & \end{tikzcd}$$

Inert

$\rightsquigarrow$ Also called **$\mathcal{O}$-algebra** $\rightsquigarrow$ $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P}) \subseteq \mathrm{Fun}_{\Delta^{op}}(\mathcal{O}, \mathcal{P})$

$\rightsquigarrow$ Category $\mathrm{Op}_\infty^{ns,gen}$

Def: A double ∞-category is a generalized non-symmetric ∞-operad $\mathcal{O} \longrightarrow \Delta^{op}$ which is also cocartesian

Explenation of name:

$\mathcal{O} \to \Delta^{op}$ $\xrightarrow{\text{Straightning}}$ Simplicial category $\mathcal{C}_\bullet : \Delta^{op} \to Cat_\infty$

double ∞-category

satisfying Segal-Rezk condition:

$$\underbrace{\mathcal{C}_n \simeq \mathcal{C}_1 \times_{\mathcal{C}_0} \cdots \times_{\mathcal{C}_0} \mathcal{C}_1}_{\text{Exactly the internal categories in } Cat_\infty}$$

Construction: Fix a space X.

$$\begin{tikzcd} \{[0]\} \arrow[r] \arrow[d] & \Delta \\ \Delta^{op} \arrow[ur, "\mathrm{Ran}"] & \end{tikzcd} \qquad [0] \longmapsto X, \qquad [n] \longmapsto (x_0,\dots,x_n) \in X^{\times n+1}$$

$\rightsquigarrow$ $\Delta^{op}_X \to \Delta^{op}$, $(x_0,\dots,x_n) \longmapsto [n]$

Double ∞-category

Def: Let $\mathcal{V}$ be a monoidal category. A categorical $\mathcal{V}$-algebra with space of objects is a Δ^{op}_X-algebra in $\mathcal{V}$.

$\rightsquigarrow$ i.e. a morphism of generalized non-symmetric ∞-operads

$$\begin{tikzcd} \Delta^{op}_X \arrow[rr, "\mathcal{C}"] \arrow[dr] & & \mathcal{V} \arrow[dl] \\ & \Delta^{op} & \end{tikzcd}$$

Let $\mathcal{C}: \Delta_X^{op} \to \mathcal{V}$ be a categorical $\mathcal{V}$-algebra

- $(x,y) \in X^{\times 2} \rightsquigarrow \mathcal{C}(x,y)$ in $\mathcal{V}$
- $(x,y,z) \in X^{\times 3}$, $d_1: [2] \to [1] \rightsquigarrow (x,y,z) \to (x,z)$ in Δ_X^{op}

  Composition maps $\rightsquigarrow \mathcal{C}(x,y) \otimes \mathcal{C}(y,z) \to \mathcal{C}(x,z)$ in $\mathcal{V}$
- $(x) \in X$, $[0] \to [1] \rightsquigarrow (x) \to (x,x)$ in Δ_X^{op}

  Unit $\rightsquigarrow I \to \mathcal{C}(x,x)$ in $\mathcal{V}$

Ex: $\boxed{\text{Associative } \mathcal{V}\text{-algebra}} \longleftrightarrow \boxed{\text{Categorical } \mathcal{V}\text{-algebra with contractible space of objects}}$

Ex: $$\begin{aligned} I_{\mathcal{V}}: \Delta_*^{op} &\longrightarrow \mathcal{V} \\ * &\longmapsto I_{\mathcal{V}} \end{aligned}$$

Construction:

$$\mathcal{S}^{op} \xrightarrow{\Delta^{op}_{(-)}} (\mathrm{Op}_\infty^{ns,gen})^{op} \xrightarrow{\mathrm{Alg}_{(-)}(\mathcal{V})} \mathrm{Cat}_\infty$$

$\rightsquigarrow$ $\mathrm{Alg}_{cat}(\mathcal{V}) \to \mathcal{S}$ associated Cartesian fibration

Prop: $\mathrm{Alg}_{cat}(\mathcal{S}) \simeq \mathrm{Seg}_\infty$

! We want $\mathcal{S}$-enriched $\simeq$ Cat_∞ $\rightsquigarrow$ Need a completness condition

Thm: $\mathrm{Cat}_\infty \simeq \mathrm{Alg}_{cat}(\mathcal{S})[\mathrm{FFES}^{-1}]$

Def: $\mathcal{V}$-enriched category $\mathrm{Cat}_\infty^{\mathcal{V}} := \mathrm{Alg}_{cat}(\mathcal{V})[\mathrm{FFES}^{-1}]$

§3: Enrichment through closed action

§3.1: Motivation

V a closed monoidal category

C a **V-enriched category** with **V-tensors** ⇝ The V-tensors endow C with a **closed V-action**

$C \to \mathcal{S}$
$X \mapsto \mathrm{Mor}_V(v, \mathrm{Mor}_C(c,x))$
is corepresentable by some $v \otimes c \in C$
for all $v \in V$, $c \in C$

Underlying **V-enriched category** that admits **V-tensors** ⇜ Category with **closed V-action**

$\boxed{\text{$V$-enriched category with V-tensors}} \longleftrightarrow \boxed{\text{Category with closed V-action}}$

V-enriched categories with V-tensors	$\longleftrightarrow$	Categories with closed V-action

$\rightsquigarrow$ V-enriched category $\sim$ Relaxed version of categories with closed V-action in which V-tensors need not exist

§ 3.2: Right-tensored ∞-categories

Def:
- $\mathrm{BM} := \Delta^{op}_{/[1]}$
  > $[n] \longrightarrow [1] \rightsquigarrow (0, \ldots, 0, 1, \ldots, 1)$
- $\mathrm{RM} \subseteq \mathrm{BM}$ of the form $(0, 1, \ldots, 1), (1, \ldots, 1)$

Prop: $\mathrm{BM} \to \Delta^{op}$, $\mathrm{RM} \to \Delta^{op}$ are both double ∞-categories

Def: For M a generalized non-symmetric ∞-category
- $\mathrm{BMod}(M) := \mathrm{Alg}_{\mathrm{BM}}(M)$
- $\mathrm{RMod}(M) := \mathrm{Alg}_{\mathrm{RM}}(M)$
- $i : \Delta^{op}_{\{1\}} \hookrightarrow \Delta^{op} \rightsquigarrow i^* : \mathrm{RMod}(M) \longrightarrow \mathrm{Alg}_{\Delta^{op}}(M)$

  $\rightsquigarrow \mathrm{RMod}_A(M) :=$ fibre of i^* over $A \in \mathrm{Alg}_{\Delta^{op}}(M)$

Rem: This agrees with Lurie's definition

Def: A weakly right-tensored ∞-category is a morphism of generalized non-symmetric ∞-categories

$$g : \mathcal{M} \to RM$$

such that both $\mathcal{M}_{(0)}, \mathcal{M}_{(1)}$ are contractible.

- $\mathcal{M}_m := \mathcal{M}_{(0,1)}$
- $\mathcal{M}_r^{\otimes} : i^*\mathcal{M} \longrightarrow \Delta^{op} \in \mathrm{Op}_\infty^{ns,gen}$

} g exhibits $\mathcal{M}_m$ as weakly right-tensored over $\mathcal{M}_r$

If g is furthermore coCartesian we say it exhibits $\mathcal{M}_m$ as right-tensored over $\mathcal{M}_r$

$\pi : RM \to \Delta^{op}$

Ex: $\mathcal{V}^{\otimes}$ monoidal $\rightsquigarrow$ $\pi^*\mathcal{V} \to RM$ exhibits $\mathcal{V}_{[\cdot]}$ as right-tensored over $\mathcal{V}$

Intuition: Let $g: \mathcal{M} \to \mathcal{RM}$ exhibit $\mathcal{C}$ as right-tensored over $\mathcal{V}$

Unstraightning $\rightsquigarrow$ $F: \mathcal{RM} \to \mathrm{Cat}_\infty$ such that

- $F(0,1) \simeq \mathcal{C}$
- $i^* F$ is the associative algebra corresponding to $\mathcal{V}$

$\rightsquigarrow$ $F \in \mathrm{Mon}_{\mathcal{RM}}(\mathrm{Cat}_\infty) \simeq \mathrm{RMod}(\mathrm{Cat}_\infty)$

$\rightsquigarrow$ Can think of g as giving rise to an action map

$$\otimes: \mathcal{C} \times \mathcal{V} \to \mathcal{V}$$

which is well-defined up to homotopy & compatible with the monoidal structure on $\mathcal{V}$

§3.3: Closed right-tensored ∞-categories

Def: Assume $g: \mathcal{M} \to RM$ exhibits $\mathcal{C}$ as weakly right-tensored over $\mathcal{V}$.

- For $v \in \mathcal{V}$, $c_0, c_1 \in \mathcal{C}$, we write

$$\mathrm{Map}_{\mathcal{C}}(c_0 \boxtimes v, c_1) := \mathrm{Map}^{\varphi}_{\mathcal{M}}((c_0, v), c_1) \subseteq \mathrm{Map}_{\mathcal{M}}((c_0, v), c_1)$$ which is mapped to the unique active map $\varphi: [1] \to [2]$

- A morphism object of $c_0, c_1 \in \mathcal{C}$ consists of
  > An object $F^R(c_0, c_1) \in \mathcal{V}$
  > A morphism $\alpha: c_0 \boxtimes F^R(c_0, c_1) \to c_1$ in $\mathcal{V}$, such that

$$\mathrm{Map}_{\mathcal{V}}(v, F^R(c_0, c_1)) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{C}}(c_0 \boxtimes v, c_1) \qquad \forall v \in \mathcal{V}$$

- g is closed right-tensored if there exist a (right) morphism object $F^R(c_0, c_1)$ for all $c_0, c_1 \in \mathcal{C}$.

Def: $g: \mathcal{M} \to \mathcal{RM}$ exhibiting $\mathcal{C}$ as closed right-tensored over $\mathcal{V}$: $\forall c_0, c_1 \in \mathcal{C}, \exists F^R(c_0, c_1) \in \mathcal{V}$ s.t. $\mathrm{Map}_{\mathcal{V}}(v, F^R(c_0,c_1)) \simeq \mathrm{Map}_{\mathcal{C}}(c_0 \boxtimes v, c_1)$

- $\mathrm{RTens}^{cl} \subseteq \mathrm{Op}_{\infty}^{ns,gen} / \mathcal{RM}$
- $\Delta^{op} \hookrightarrow \mathcal{RM}$ ⇝ Projection $\mathrm{RTens}^{cl} \xrightarrow{\pi} \mathrm{Mon}_{\infty}$
  ⇝ $\mathrm{RTens}^{cl}(\mathcal{V})$ = Fibre of π at $\mathcal{V}$

Ex:

$g: \mathcal{M} \to \mathcal{RM}$ exhibits $\mathcal{C}$ as closed right-tensored over $\mathcal{V}$	$\longleftrightarrow$	For every $c \in \mathcal{C}$ the right-tensoring functor $c \otimes - : \mathcal{V} \to \mathcal{C}$ admits a right adjoint $F^R(c, -)$

All of this could have been done with

$LM \subseteq BM$ of the form $(0,-,0,1)$, $(0,-,C)$

⇝ Dual theory of left-tensored

- Right version: $F^R(a,b) \otimes F^R(b,c) \to F^R(a,c)$
- Left version: $F^L(b,a) \otimes F^L(c,b) \to F^L(a,c)$

Construction: Let $\mathcal{V}^\otimes$ be a monoidal category

$$\rightsquigarrow \begin{tikzcd} \mathcal{V}^\otimes \arrow[r] \arrow[rr, bend right, "\mathcal{V}^{\otimes,\mathrm{rev}}"'] & \Delta^{op} \arrow[r, "op"] & \Delta^{op} \end{tikzcd}$$

- $op : BM \to BM$ ⇝ $LTens^{cl}(\mathcal{V}^{rev}) \simeq RTens(\mathcal{V})$
  - $LM \mapsto RM$
  - $RM \mapsto LM$
  - $\mathcal{C}^{op} \mapsto \mathcal{C}$

§4: Comparison

Thm: (Heine) Let $\mathcal{V}$ be a monoidal category. Then

$$\mathrm{LTens}^{cl}(\mathcal{V}^{rev}) \simeq \mathrm{Cat}_\infty^{\mathcal{V}}$$

Prop: (R.) There exists a functor

$$\overline{(-)} : \mathrm{Cat}_\infty^{ex} \to \mathrm{Cat}_\infty^{sp}$$

preserving the underlying ∞-category.

Prop: (R.) $\overline{(-)}: \mathrm{Cat}_\infty^{ex} \longrightarrow \mathrm{Cat}_\infty^{sp}$

Proof:

- $\mathcal{C} \in \mathrm{Cat}_\infty^{ex} \rightsquigarrow \mathrm{Ind}(\mathcal{C}) \in \mathrm{Pr}^{st}$, $\mathcal{C} \subseteq \mathrm{Ind}(\mathcal{C})$ full subcategory
- HA.4.8.2.18 : $\mathrm{Pr}^{st} \simeq \mathrm{Mod}_{Sp}(\mathrm{Pr}^L) \overset{\text{GH 7.4.13}}{\rightsquigarrow} \mathrm{Ind}(\mathcal{C})$ both right- and left-tensored over Sp
- $c \otimes -$ admits right-adjoint for all $c \in \mathrm{Ind}(\mathcal{C})$

  $\rightsquigarrow g : M \to RM$ exhibiting $\mathrm{Ind}(\mathcal{C})$ as closed right-tensored over Sp

  $\rightsquigarrow \widetilde{(-)} : \mathrm{Cat}_\infty^{ex} \xrightarrow{\mathrm{Ind}(-)} \mathrm{Pr}^{st} \simeq \mathrm{Mod}_{Sp}(\mathrm{Pr}^L) \hookrightarrow \mathrm{RTens}^{cl}(Sp)$
- $\mathcal{C} \subseteq \widetilde{\mathcal{C}}$ full subcategory s.t. for any $f: \mathcal{C} \to \mathcal{Q}$ in Cat_∞^{ex},

  $\mathcal{C} \hookrightarrow \widetilde{\mathcal{C}} \xrightarrow{\widetilde{f}} \widetilde{\mathcal{Q}}$ factors through $\mathcal{Q}$

  $\rightsquigarrow$ Subfunctor $\overline{(-)} : \mathrm{Cat}_\infty^{ex} \to \mathrm{RTens}^{cl}(Sp) \simeq \mathrm{Cat}_\infty^{sp}$

Many-Object Associative Algebras $\xleftrightarrow[\text{Heine}]{\sim}$ Closed Tensored Categories

+ Intuitive how we get the desired structure
+ Follows 'higher algebra' intuition
+ Nicely "packaged" theory
÷ Hard to work with
÷ No Yoneda
+ Easy extension to "enriched algebra"

+ Geometric intuition
+ More "hands on"
÷ Bookkeeping
+ Yoneda

§5: Enriched Algebra

§5.1: Bimodules

Def: X, Y spaces

$(1) \longmapsto Y$

$(0) \longmapsto X$

$(i) : [n] \to [1]$, $\mapsto i$

$$\begin{tikzcd}
\{(0),(1)\} \arrow[r] \arrow[d] & \gamma \\
\Delta^{op}_{/[1]} \arrow[ur, "\mathrm{Ran}"'] &
\end{tikzcd}$$

γ: $(x_0, \dots, x_k, y_{k+1}, \dots, y_n) \in X^{\times k+1} \times Y^{\times n-k}$

$f : [n] \to [1]$

$0, \dots, k \mapsto 0$

$k+1, \dots, n \mapsto 1$

$\rightsquigarrow \quad \Delta^{op}_{X,Y} \longrightarrow \Delta^{op}_{/[1]}$

Prop: The composite

$$\Delta^{op}_{X,Y} \to \Delta^{op}_{/[1]} \longrightarrow \Delta^{op}$$

is a double ∞-category

Understanding $\Delta^{op}_{X,Y}$-algebra in $\mathcal{V}$

Laying over the two inclusins $\Delta^{op} \hookrightarrow \Delta^{op}_{/[1]}$ given by composing with the two maps $[0] \to [1]$

$$\begin{tikzcd} \Delta^{op}_X \arrow[dr, hook] \arrow[drr, bend left, "\mathcal{C}"] & & \\ & \Delta^{op}_{X,Y} \arrow[r, "M"] & \mathcal{V} \\ \Delta^{op}_Y \arrow[ur, hook] \arrow[urr, bend right, "\mathcal{D}"'] & & \end{tikzcd}$$

- $x \in X, y \in Y \rightsquigarrow \boxed{M(x,y)} \in \mathcal{V}$
- $x_0, x_1 \in X, y \in Y, \ d_1 : (0,1,1) \to (0,1) \rightsquigarrow \boxed{(x_0, x_1, y) \to (x_0, y)}$

$$\rightsquigarrow \boxed{\mathcal{C}(x_0, x_1) \otimes M(x_1, y) \to M(x_0, y)}$$

- similarly for $x \in X, y_0, y_1 \in Y \rightsquigarrow \boxed{M(x, y_0) \otimes \mathcal{D}(y_0, y_1) \to M(x, y_1)}$

$x_0, x_1 \in X$, $y_0, y_1 \in Y$

$$\begin{tikzcd}
(x_0, x_1, y_0, y_1) \arrow[r] \arrow[d] & (x_0, y_0, y_1) \arrow[d] \\
(x_0, x_1, y_1) \arrow[r] & (x_0, y_1)
\end{tikzcd}$$

$$\begin{tikzcd}
\Delta^{op}_X \arrow[dr, hook] \arrow[drr, bend left, "\mathcal{C}"] & & \\
& \Delta^{op}_{X,Y} \arrow[r, "M"] & \mathcal{V} \\
\Delta^{op}_Y \arrow[ur, hook] \arrow[urr, bend right, "\mathcal{Q}"'] & &
\end{tikzcd}$$

$\rightsquigarrow$

$$\begin{tikzcd}
\mathcal{C}(x_0, x_1) \otimes M(x_1, y_0) \otimes \mathcal{Q}(y_0, y_1) \arrow[r] \arrow[d] & M(x_0, y_0) \otimes \mathcal{Q}(y_0, y_1) \arrow[d] \\
\mathcal{C}(x_0, x_1) \otimes M(x_1, y_1) \arrow[r] & M(x_0, y_1)
\end{tikzcd}$$

$\rightsquigarrow$ $M : \Delta^{op}_{X,Y} \to \mathcal{V}$ is what we want as $\mathcal{V}$-enriched $\mathcal{C}$-$\mathcal{Q}$-bimodule

Construction: Let $\mathcal{V}$ be a monoidal category.

$$\mathcal{S}^{\times 2, op} \xrightarrow{\mathrm{Alg}_{\Delta^{op}_{-,-}}(\mathcal{V})} \mathrm{Cat}_\infty \rightsquigarrow \mathrm{Bimod}_{cat}(\mathcal{V}) \longrightarrow \mathcal{S}^{\times 2} \quad \text{Cartesian fibration}$$

Recall that

$$\mathcal{S}^{op} \xrightarrow{\mathrm{Alg}_{\Delta^{op}_{-}}(\mathcal{V})} \mathrm{Cat}_\infty \rightsquigarrow \mathrm{Alg}_{cat}(\mathcal{V}) \longrightarrow \mathcal{S} \quad \text{Cartesian fibration}$$

Using the two inclusions

$$\left.\begin{array}{l} \Delta^{op}_X \hookrightarrow \Delta^{op}_{X,Y} \\ \Delta^{op}_Y \hookrightarrow \Delta^{op}_{X,Y} \end{array}\right\} \rightsquigarrow \mathrm{Bimod}_{cat}(\mathcal{V}) \to \mathrm{Alg}_{cat}(\mathcal{V})^{\times 2}$$

A $\mathcal{V}$-enriched $\mathcal{C}$-$\mathcal{D}$-bimodule is an object in the fibre over $(\mathcal{C}, \mathcal{D}) \in \mathrm{Alg}_{cat}(\mathcal{V})^{\times 2} \rightsquigarrow \mathrm{Bimod}_{\mathcal{C},\mathcal{D}}(\mathcal{V})$

Thm: (Haugseng) Let $\mathcal{V}$ be a monoidal category and categorical $\mathcal{V}$-algebras with space of objects X, Y, Z.

$M : \Delta^{op}_{X,Y} \to \mathcal{V}$ $\mathcal{C}$-$\mathcal{Q}$-bimodule

$N : \Delta^{op}_{Y,Z} \to \mathcal{V}$ $\mathcal{Q}$-$\mathcal{E}$-bimodule

$\rightsquigarrow$ $M \otimes_{\mathcal{Q}} N$: $\Delta^{op}_{X,Z} \longrightarrow \mathcal{V}$ $\mathcal{C}$-$\mathcal{E}$-bimodule

$M \otimes_{\mathcal{Q}} N$ evaluated at $(x, z) \in X \times Z$ is

$$\left| [n] \mapsto \operatorname*{colim}_{(y_0, \dots, y_n) \in Y^{\times n+1}} M(x, y_0) \otimes \mathcal{Q}(y_0, y_1) \otimes \dots \otimes \mathcal{Q}(y_{n-1}, y_n) \otimes N(y_n, z) \right|$$

Thm: (Haugseng) There exists an $(\infty, 2)$-category $\mathrm{ALG}_{cat}(\mathcal{V})$, called the Morita category, which has

- Objects: Categorical $\mathcal{V}$-algebras
- Morphisms: $\mathcal{V}$-enriched bimodules
- Composition: $\otimes$-product

Prop: (R.) $\mathcal{C}: \Delta^{op}_{\times} \to \mathcal{V}$ categorical $\mathcal{V}$-algebra, then $S^*_0\mathcal{C}$ is the $\otimes_{\mathcal{C}}$-unit, i.e.

$$M \otimes_{\mathcal{C}} S^*_0\mathcal{C} \cong M, \quad S^*_0\mathcal{C} \otimes_{\mathcal{C}} N \cong N$$

§5.2: Enriched Modules

Def: Let $\mathcal{V}$ be a monoidal category, $\mathcal{C}: \Delta^{op}_{X} \to \mathcal{V}$ a categorical $\mathcal{V}$-algebra

- $\mathcal{V}$-enriched right $\mathcal{C}$-module : $I_{\mathcal{V}}$–$\mathcal{C}$-bimodule
  > $\mathrm{RMod}^{en}_{\mathcal{C}}(\mathcal{V}) := \mathrm{Bimod}_{I_{\mathcal{V}},\mathcal{C}}(\mathcal{V})$
- $\mathcal{V}$-enriched left $\mathcal{C}$-module : $\mathcal{C}$–$I_{\mathcal{V}}$-bimodule
  > $\mathrm{LMod}^{en}_{\mathcal{C}}(\mathcal{V}) := \mathrm{Bimod}_{\mathcal{C},I_{\mathcal{V}}}(\mathcal{V})$
- $\mathcal{S}^{op} \xrightarrow{\Delta^{op}_{I_{\mathcal{V}},-}} (\mathrm{Op}^{ns,gen}_{\infty})^{op} \xrightarrow{\mathrm{Alg}_{(-)}(\mathcal{V})} \mathrm{Cat}_{\infty} \rightsquigarrow \mathrm{RMod}^{en}_{cat}(\mathcal{V}) \longrightarrow \mathcal{S}$
- $\mathcal{S}^{op} \xrightarrow{\Delta^{op}_{-,I_{\mathcal{V}}}} (\mathrm{Op}^{ns,gen}_{\infty})^{op} \xrightarrow{\mathrm{Alg}_{(-)}(\mathcal{V})} \mathrm{Cat}_{\infty} \rightsquigarrow \mathrm{LMod}^{en}_{cat}(\mathcal{V}) \longrightarrow \mathcal{S}$

Rem:
- $\mathrm{RMod}^{en}_{\mathcal{C}}(\mathcal{V}) \simeq \mathrm{LMod}^{en}_{\mathcal{C}^{op}}(\mathcal{V})$
- $\mathrm{RMod}^{en}_{cat}(\mathcal{V}) \simeq \mathrm{LMod}^{en}_{cat}(\mathcal{V}^{rev})$

Thm: (R.) Given a monoidal category $\mathcal{V}$, then there exists a functor

$$\mathrm{Cat}_\infty^{\mathcal{V}}/\overline{\mathcal{V}} \longrightarrow \mathrm{RMod}^{\mathrm{en}}_{\mathrm{cat}}(\mathcal{V})$$

$$G : \mathcal{C} \to \overline{\mathcal{V}} \longmapsto R_G$$

where

$$R_G(*, \dots, *, c_0, \dots, c_n) \simeq G(c_0) \otimes F^R(c_0, c_1) \otimes \dots \otimes F^R(c_{n-1}, c_n)$$

Thank you