K-theory by the Q-construction and through localization

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1 K-theory by the Q-construction

Throughout let $\mathcal{C}$ denote an exact category. First of all we wish to define a new category $\mathcal{QC}$ which has the same objects as $\mathcal{C}$, i.e $\mathrm{ob}\mathcal{C} = \mathrm{ob}\mathcal{Q}C$. To define the morphisms let $c_0, c_1 \in \mathrm{ob}\mathcal{C}$ and consider diagrams of the form

$$c_0 \overset{p}{\twoheadleftarrow} c_{01} \overset{i}{\rightarrowtail} c_1, \tag{1}$$

in $\mathcal{C}$ where p is an admissible epimorphism and i an admissible monomorphisms. We say that two such diagrams between c_0 and c_1 are equivalent if there exist an isomorphism making the following diagram commute

$$\begin{tikzcd}
c_0 & c_{01} \arrow[l, two heads, "p"'] \arrow[r, tail, "i"] \arrow[d, "\simeq"] & c_1 \\
& c'_{01}. \arrow[ul, two heads, "p'"] \arrow[ur, tail, "i'"'] &
\end{tikzcd}$$

A morphism $f : c_0 \to c_1$ in $\mathcal{QC}$ is all diagrams (1) up to this above equivalence. We define the composition of two such morphisms $f : c_1 \to c_2$ and $g : c_0 \to c_1$ as the pullback:

$$\begin{tikzcd}
c_{02} \arrow[r, tail] \arrow[d, two heads] & c_{12} \arrow[r, tail] \arrow[d, two heads] & c_2 \\
c_{01} \arrow[r, tail] \arrow[d, two heads] & c_1 & \\
c_0 & &
\end{tikzcd}$$

with $c_{02} = c_{01} \times_{c_1} c_{12}$. It can be checked that this indeed defines a category, which can be depicted as

$$\begin{tikzcd}
 & & & c_{0,3} \arrow[dl, two heads] \arrow[dr, tail] & & & \\
 & & c_{0,2} \arrow[dl, two heads] \arrow[dr, tail] & & c_{1,3} \arrow[dl, two heads] \arrow[dr, tail] & & \\
 & c_{0,1} \arrow[dl, two heads] \arrow[dr, tail] & & c_{1,2} \arrow[dl, two heads] \arrow[dr, tail] & & c_{2,3} \arrow[dl, two heads] \arrow[dr, tail] & \\
c_{0,0} & & c_{1,1} & & c_{2,2} & & c_{3,3}.
\end{tikzcd}$$

We will now introduce the definition of higher algebraic K-theory using the Q-construction for small, exact categories. So let $\mathcal{C}$ be a small exact category and 0 a given zero-object.

Definition 1.1. For an exact category $\mathcal{C}$, we define the *K-theory space* by $K(\mathcal{C}) := \Omega B\mathcal{QC}$. The *K-groups* are then given by $K_i\mathcal{C} := \pi_n K\mathcal{C} = \pi_{i+1}(B(\mathcal{QC}), 0)$.

here $B(\mathcal{QC})$ denotes the classifying space of $\mathcal{QC}$, i.e $|N(\mathcal{QC})[-]|$. This definition of the K-groups are independent of the choice of basepoint 0, since if we are given another zero-object $0'$, then there would be a unique map $0 \to 0'$ in $\mathcal{QC}$ hence a canonical path from 0 to $0'$ in $B(\mathcal{QC})$. Therefore we will not denote the basepoint from now on. One of the fundemental properties we wish satisfied for our K-groups is that K_0 canonically isomorphic to the Groethendick group of $\mathcal{C}$.

2 Localization of cofibered functors

We wish to construct an alternative definition of the K-theory space for the category of finitely generated projective modules over an integral domain, which we denote by $\mathcal{P}$. This will be done by introducing cofibered functors and localization of a category with respect to the action of a monoidal category. Let $\mathscr{S}$ be a monoidal cateogory and $\mathscr{X}$ a category.

Definition 2.1. A *left action* of a monoidal category $\mathscr{S}$ on a category $\mathscr{X}$ is a functor

$$+ : \mathscr{S} \times \mathscr{X} \to \mathscr{X}$$

together with natural isomorphisms

$$A + (B + F) \cong (A \Box B) + F, \; e + F \cong F$$

where $\Box$ is the monoidal product of $\mathscr{S}$, e the unit object, $A, B \in \mathrm{ob}\mathscr{S}$, $F \in \mathrm{ob}\mathscr{X}$ such that some appropiate associtivity and unit diagrams commutes.

Definition 2.2. Let $\mathscr{S}$ be a monoidal category acting on $\mathscr{X}$. The *orbit category* $\langle \mathscr{S}, \mathscr{X} \rangle$ is the following category:

- The objects are the same as the objects of $\mathscr{X}$,

- A morphism $F \to G$ is an isomorphism class $[A, \varphi]$ where $A \in \mathrm{ob}\mathscr{S}$ and $\varphi : A + F \to G$ is a morphism in $\mathscr{X}$. We say that two such pairs (A, φ) and (A', φ') are isomorphic if there exists an isomorphism $a : A \to A'$ in $\mathscr{S}$ such that the triangle

$$\begin{tikzcd} A + F \arrow[rr, "a+id_F"] \arrow[dr, "\varphi"'] & & A' + F \arrow[dl, "\varphi'"] \\ & G & \end{tikzcd}$$

is a commutative triangle.

Definition 2.3. Let $\mathscr{S}$ be a monoidal category acting on a category $\mathscr{X}$. Then we define the *localization* $\mathscr{S}^{-1}\mathscr{X}$ to be the category $\langle \mathscr{S}, \mathscr{S} \times \mathscr{X} \rangle$ where the action of $\mathscr{S}$ on $\mathscr{S} \times \mathscr{X}$ is given by

$$\begin{aligned} \mathscr{S} \times \mathscr{S} \times \mathscr{X} &\to \mathscr{S} \times \mathscr{X} \\ (a, b, c) &\mapsto (a \Box b, a + c). \end{aligned}$$

This definition is not related to Bousfield localization, but is a form of a generalization of localizing a commutative ring with respect to the action of a monoid.

Definition 2.4. A functor $f : \mathcal{C} \to \mathcal{D}$ is said to be *cofibered* if the following two conditions hold:

1. For every $c \in \mathrm{ob}\mathcal{C}$ and $\varphi \in \mathcal{D}(f(c), d)$ there exists a morphism $\widehat{\varphi} \in \mathcal{C}(c, c')$ such that $f(\widehat{\varphi}) = \varphi$.
2. $\widehat{\varphi}$ is the universal such morphism, i.e. if $\widehat{\varphi'} \in \mathcal{C}(c, c'')$ is another morphism satisfying $f(\widehat{\varphi'}) = \varphi$, then there exists a unique morphism $\gamma \in \mathcal{C}(c', c'')$ such that

$$\begin{tikzcd} c \arrow[r, "\widehat{\varphi}"] \arrow[dr, "\widehat{\varphi'}"'] & c' \arrow[d, "\exists!\gamma"] \\ & c'' \end{tikzcd}$$

is a commutative triangle.

In this case we will say that $\widehat{\varphi}$ is the *cofiber lift* of φ.

So a cofibered functor $f : \mathcal{C} \to \mathcal{D}$ lets us describe a lift of any morphism $\varphi \in \mathcal{D}(f(c_0), d_1)$ by a morphism between the fibers $\hat{\varphi} \in \mathcal{D}(c_0, f^{-1}(d_1)$. When f is cofibered we will consider the category $f^{-1}(Y)$ for fixed $Y \in \mathrm{ob}\mathcal{D}$ as a subcategory of $\mathcal{C}$. This category reminds us a lot about $f \backslash Y$ and there is a good reason for that, the classifying space of these two are equivalent:

Theorem 2.5. $B(f^{-1}(Y)) \simeq B(f \backslash Y)$ *when f is cofibered.*

This gives us a reformulation of Quillen's theorem B in the setting of cofibered functors:

Theorem 2.6 (Quillen's Theorem B - for cofibered functors)**.** *Let $f : \mathcal{C} \to \mathcal{D}$ be a cofibered functor and assume that for all $\varphi \in \mathcal{D}(Y, Y')$ the cofiber lift $\widehat{\varphi} : f^{-1}(Y) \to f^{-1}(Y')$ of φ, satisfying $f(\widehat{\varphi}) = \varphi$, is a homotopy equivalence. Then*

$$B(f^{-1}(Y)) \to B\mathcal{C} \xrightarrow{B(f)} B\mathcal{D}$$

is a quasifibration.

Now, we turn to our specific case again. Let $i\mathcal{P}$ denote the subcategory of $\mathcal{P}$ with isomorphisms and let $\tau\mathcal{P}$ denote the exact subcategory of $\mathcal{P}$ with injective morphisms with cokernels in $\mathcal{P}$. Recall that the Segal subdivision $sd(\tau\mathcal{P})$ with objects $(c_0 \rightarrowtail c_1)$ and morphisms $(c_0 \rightarrowtail c_1) \to (c_0' \rightarrowtail c_1')$ commutative squares

$$\begin{tikzcd} c_0 \arrow[r, tail] \arrow[d, tail] & c_1 \\ c_0' \arrow[r, tail] & c_1'. \arrow[u, tail] \end{tikzcd}$$

We wish to consider a specific functor $f : sd(\tau\mathcal{P}) \to \mathcal{QP}^{op}$, which maps objects $(K \rightarrowtail L)$ in $sd(\tau\mathcal{P})$ to a fixed L/K in $\mathcal{P}$. Further we let f map the morphism (a, b) described by

$$\begin{tikzcd} K \arrow[r, tail] \arrow[d, tail, "a"'] & L \\ K' \arrow[r, tail] & L' \arrow[u, tail, "b"'] \end{tikzcd}$$

to the morphism $h : L/K \to L'/K'$ in $\mathcal{Q}(\mathcal{P})^{op}$ represented by the diagram

$$L'/K' \twoheadleftarrow L'/K \rightarrowtail L/K.$$