K-theory by the Q-construction and through localization

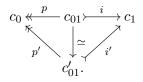
June 29, 2019

1 K-theory by the Q-construction

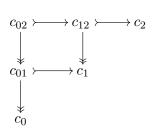
Throughout let \mathcal{C} denote an exact category. First of all we wish to define a new category \mathcal{QC} which has the same objects as \mathcal{C} , i.e ob $\mathcal{C} = \text{ob}\mathcal{QC}$. To define the morphisms let $c_0, c_1 \in \text{ob}\mathcal{C}$ and consider diagrams of the form

$$c_0 \stackrel{p}{\longleftarrow} c_{01} \stackrel{i}{\longmapsto} c_1, \tag{1}$$

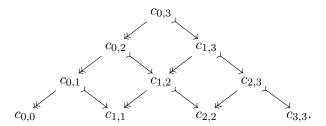
in C where p is an admissible epimorphism and i an admissible monomorphisms. We say that two such diagrams between c_0 and c_1 are equivalent if there exist an isomorphism making the following diagram commute



A morphism $f: c_0 \to c_1$ in \mathcal{QC} is all diagrams (1) up to this above equivalence. We define the composition of two such morphisms $f: c_1 \to c_2$ and $g: c_0 \to c_1$ as the pullback:



with $c_{02} = c_{01} \times_{c_1} c_{12}$. It can be checked that this indeed defines a category, which can be depicted as



We will now introduce the definition of higher algebraic K-theory using the Q-construction for small, exact categories. So let C be a small exact category and 0 a given zero-object.

Definition 1.1. For an exact category C, we define the K-theory space by $K(C) := \Omega B \mathcal{Q} C$. The K-groups are then given by $K_i C := \pi_n K C = \pi_{i+1}(B(\mathcal{Q}C), 0)$.

here $B(\mathcal{QC})$ denotes the classifying space of \mathcal{QC} , i.e $|N(\mathcal{QC})[-]|$. This definition of the K-groups are independent of the choice of basepoint 0, since if we are given another zero-object 0', then there would be a unique map $0 \to 0'$ in \mathcal{QC} hence a canonical path from 0 to 0' in $B(\mathcal{QC})$. Therefore we will not denote the basepoint from now on. One of the fundemental properties we wish satisfied for our K-groups is that K_0 canonically isomorphic to the Groethendick group of \mathcal{C} .

2 Localization of cofibered functors

We wish to construct an alternative definition of the K-theory space for the category of finitely generated projective modules over an integral domain, which we denote by \mathcal{P} . This will be done by introducing cofibered functors and localization of a category with respect to the action of a monoidal category. Let \mathscr{S} be a monoidal category and \mathscr{X} a category.

Definition 2.1. A left action of a monoidal category $\mathscr S$ on a category $\mathscr X$ is a functor

$$+: \mathscr{L} \times \mathscr{X} \to \mathscr{X}$$

together with natural isomorphisms

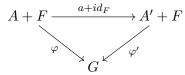
$$A + (B + F) \cong (A \square B) + F, e + F \cong F$$

where \square is the monoidal product of \mathscr{S} , e the unit object, $A, B \in \text{ob}\mathscr{S}$, $F \in \text{ob}\mathscr{X}$ such that some appropriate associtivity and unit diagrams commutes.

Definition 2.2. Let \mathscr{S} be a monoidal category acting on \mathscr{X} . The *orbit category* $\langle \mathscr{S}, \mathscr{X} \rangle$ is the following category:

• The objects are the same as the objects of \mathscr{X} ,

• A morphism $F \to G$ is an isomorphism class $[A, \varphi]$ where $A \in \text{ob}\mathscr{S}$ and $\varphi : A + F \to G$ is a morphism in \mathscr{X} . We say that two such pairs (A, φ) and (A', φ') are isomorphic if there exists an isomorphism $a : A \to A'$ in \mathscr{S} such that the triangle



is a commutative triangle.

Definition 2.3. Let \mathscr{S} be a monoidal category acting on a category \mathscr{X} . Then we define the *localization* $\mathscr{S}^{-1}\mathscr{X}$ to be the category $\langle \mathscr{S}, \mathscr{S} \times \mathscr{X} \rangle$ where the action of \mathscr{S} on $\mathscr{S} \times \mathscr{X}$ is given by

$$\mathscr{S} \times \mathscr{S} \times \mathscr{X} \to \mathscr{S} \times \mathscr{X}$$
$$(a,b,c) \mapsto (a \square b, a+c).$$

This definition is not related to Bousfield localization, but is a form of a generalization of localizing a commutative ring with respect to the action of a monoid.

Definition 2.4. A functor $f: \mathcal{C} \to \mathcal{D}$ is said to be *cofibered* if the following two conditions hold:

- 1. For every $c \in \text{ob}\mathcal{C}$ and $\varphi \in \mathcal{D}(f(c), d)$ there exists a morphism $\widehat{\varphi} \in \mathcal{C}(c, c')$ such that $f(\widehat{\varphi}) = \varphi$.
- 2. $\widehat{\varphi}$ is the universal such morphism, i.e. if $\widehat{\varphi'} \in \mathcal{C}(c, c'')$ is another morphism satisfying $f(\widehat{\varphi'}) = \varphi$, then there exists a unique morphism $\gamma \in \mathcal{C}(c', c'')$ such that

$$c \xrightarrow{\widehat{\varphi}} c' \downarrow \exists ! \gamma \\ c''$$

is a commutative triangle.

In this case we will say that $\widehat{\varphi}$ is the cofiber lift of φ .

So a cofibered functor $f: \mathcal{C} \to \mathcal{D}$ lets us describe a lift of any morphism $\varphi \in \mathcal{D}(f(c_0), d_1)$ by a morphism between the fibers $\hat{\varphi} \in \mathcal{D}(c_0, f^{-1}(d_1))$. When f is cofibered we will consider the category $f^{-1}(Y)$ for fixed $Y \in \text{ob}\mathcal{D}$ as a subcategory of \mathcal{C} . This category reminds us a lot about $f \setminus Y$ and there is a good reason for that, the classifying space of these two are equivalent:

Theorem 2.5. $B(f^{-1}(Y)) \simeq B(f \setminus Y)$ when f is cofibered.

This gives us a reformulation of Quillen's theorem B in the setting of cofibered functors:

Theorem 2.6 (Quillen's Theorem B - for cofibered functors). Let $f: \mathcal{C} \to \mathcal{D}$ be a cofibered functor and assume that for all $\varphi \in \mathcal{D}(Y,Y')$ the cofiber lift $\widehat{\varphi}: f^{-1}(Y) \to f^{-1}(Y')$ of φ , satisfying $f(\widehat{\varphi}) = \varphi$, is a homotopy equivalence. Then

$$B(f^{-1}(Y)) \to B\mathcal{C} \xrightarrow{B(f)} B\mathcal{D}$$

is a quasifibration.

Now, we turn to our specific case again. Let $i\mathcal{P}$ denote the subcategory of \mathcal{P} with isomorphisms and let $\tau\mathcal{P}$ denote the exact subcategory of \mathcal{P} with injective morphisms with cokernels in \mathcal{P} . Recall that the Segal subdivision $sd(\tau\mathcal{P})$ with objects $(c_0 \mapsto c_1)$ and morphisms $(c_0 \mapsto c_1) \to (c'_0 \mapsto c'_1)$ commutative squares

$$\begin{array}{ccc}
c_0 & \longrightarrow c_1 \\
\downarrow & & \downarrow \\
c'_0 & \longrightarrow c'_1.
\end{array}$$

We wish to consider a specific functor $f: sd(\tau P) \to \mathcal{QP}^{op}$, which maps objects $(K \to L)$ in $sd(\tau P)$ to a fixed L/K in P. Further we let f map the morphism (a,b) described by

$$\begin{array}{ccc}
K & & \downarrow L \\
a & & \downarrow b \\
K' & & \downarrow L'
\end{array}$$

to the morphism $h: L/K \to L'/K'$ in $\mathcal{Q}(\mathcal{P})^{op}$ represented by the diagram

$$L'/K' \longleftarrow L'/K \rightarrowtail L/K.$$