INTRODUCTION TO HIGHER ALGEBRA

§1: SYMMETRIC MONOIDAL → OPERADS → ∞-LAND

The basic algebraic structure we want to generalise is

commutative monoid: Set M + multiplication $M \times M \to M$, unit $1 \in M$ s.t.

$$1x = x, \quad xy = yx, \quad x(yz) = (xy)z \quad \forall x,y,z \in M$$

$\rightsquigarrow$ For categories this is a <u>symmetric monoidal category</u>: Category C + $1 \in C$ unit + $\otimes : C \times C \to C$

When working with categories its unnatural to ask for $X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$, instead we want this structure to be given by extra data in form isomorphisms

$$\alpha_X : 1 \otimes X \simeq X \qquad \beta_{X,Y} : X \otimes Y \simeq Y \otimes X$$

Just called monoidal if we do not assume this

$$\gamma_{X,Y,Z} : X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z$$

+ coherence data!

It is clear that this approach can not be used to generalize to ∞-categories. Lets spell out how to get an equivalent description of symmetric monoidal: The idea is that instead of giving the bifunctor $\otimes : C \times C \to C$, we instead for each n-tuple $c_1, \ldots, c_n$ and $d \in C$ we specify the collection of maps

$$c_1 \otimes \ldots \otimes c_n \to d \qquad + \quad \text{composition data (and coherence)}$$

This is done by colored operads (what I called multicategories in an earlier talk):

<u>Def:</u> A <u>coloured operad</u> $\mathcal{O}$ consists of

- A set of objects $\mathrm{ob}\,\mathcal{O}$ (sometimes called the colours)
- $\forall X_0, \ldots, X_n, Y$ in $\mathcal{O}$: A set of multimorphisms $\mathrm{Mul}_{\mathcal{O}}(X_0, \ldots, X_n; Y)$
- Composition: Given

  $$(X_1, \ldots, X_{i_1}) \to Y_1, \ldots, (X_{i_{n-1}}, \ldots, X_{i_n}) \to Y_n$$

  can compose this with $(Y_1, \ldots, Y_n) \to Z$ to obtain

  $$(X_1, \ldots, X_{i_n}) \to Z$$

- Unit: $\mathrm{id}_X : (X) \to X$

s.t. the composition law is unital and associative

<u>Rem:</u> Every colored operad $\mathcal{O}$ has an underlying category by setting:

- objects $= \mathrm{ob}\,\mathcal{O}$
- $\mathrm{Hom}(X,Y) = \mathrm{Mul}_{\mathcal{O}}(\{X\}, Y)$.

$\rightsquigarrow$ So can view a coloured operad as a category + extra data in form of the collection of multimorphisms.

<u>Rem:</u> Given a (symmetric) monoidal category $(\mathcal{C}, \otimes)$ we can obtain a coloured operad $\mathcal{C}^{\otimes}$ w. underlying category $\mathcal{C}$ by assigning

$$\mathrm{Mul}_{\mathcal{C}}(X_0, \ldots, X_n; Y) = \mathrm{Hom}_{\mathcal{C}}(X_0 \otimes \ldots \otimes X_n; Y).$$

We can recover the symmetric monoidal structure of $\mathcal{C}^{\otimes}$ (up to canonical isomorphism) by Yoneda's Lemma. E.g. the tensor product $X \otimes Y$ is characterized by the fact that it corepresents the functor $Z \mapsto \mathrm{Mul}_{\mathcal{C}}(X, Y; Z)$.

$\rightsquigarrow$ Can consider symmetric monoidal categories as a special case of coloured operads.

To identify which assumptions it is on a colored operad that makes it into a symmetric monoidal.

First recall:

<u>Def:</u> $\mathrm{Fin}_* = \{\langle n \rangle\}$ pointed sets

$f: \langle n \rangle \to \langle m \rangle$ is <u>inert</u> if it takes some point to the basepoint and injective (isomorphic) on the rest.

$\rho_i: \langle n \rangle \to \langle 1 \rangle$ takes $\begin{matrix} j \mapsto 0 & \text{if } j \neq i \\ i \mapsto i & \end{matrix}$, f is <u>active</u> if $f^{-1}(0) = 0$

<u>Construction:</u> Let $\mathcal{O}$ be a colored operad $\rightsquigarrow$ $\mathcal{O}^{\otimes}$ category:
2.1.1.7

- Objects = sequences of objects of $\mathcal{O}$ $X_0, \ldots, X_n$
- $\{X_i\}_{1 \leq i \leq m} \to \{Y_j\}_{1 \leq j \leq n}$ is given by a map $\alpha: \langle m \rangle \to \langle n \rangle$ in Fin_* together with a collection of multimorphisms

$$\{\varphi_j \in \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in \alpha^{-1}\{j\}}; Y_j)\}_{0 \leq j \leq n}$$

in $\mathcal{O}$.

- Composition of morphisms in $\mathcal{O}^{\otimes}$ is determined by composition laws on Fin_* and on $\mathcal{O}$.

By construction, $\mathcal{O}^{\otimes}$ comes equipped with a forgetful functor

$$\pi: \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$$
$$(X_0, \ldots, X_n) \mapsto \langle n \rangle.$$

Using π we can reconstruct the operad structure:

- Write $\mathcal{O}^{\otimes}_{\langle n \rangle} = \pi^{-1}\{\langle n \rangle\}$

- $p^i : \langle n \rangle \to \langle 1 \rangle \rightsquigarrow p^i_! : \mathcal{O}^{\otimes}_{\langle n \rangle} \to \mathcal{O}^{\otimes}_{\langle 1 \rangle} \simeq \mathcal{O}$ ← underlying category of $\mathcal{O}$

  which induces equivalences: $\mathcal{O}^{\otimes}_{\langle n \rangle} \simeq \mathcal{O}^{\times n}$.
  $$(x_0, -, x_n) \mapsto \overline{X}.$$

- $\mathrm{Mul}_{\mathcal{O}}(x_0, -, x_n; Y) \Leftrightarrow \{ f : \overline{X} \to Y \text{ in } \mathcal{O}^{\otimes} \text{ s.t. } \pi(f) : \langle n \rangle \to \langle 1 \rangle \text{ satisfies } \pi(f)^{-1}(0) = \{0\} \}$

  ↳ $\pi : \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ determines $\mathrm{Mul}_{\mathcal{O}}(x_0, -, x_n; Y)$ in the colored operad $\mathcal{O}$

- Can show that composition law for morphisms in $\mathcal{O}$ can be recovered from the one in $\mathcal{O}^{\otimes}$.

$\rightsquigarrow$ Can think of a coloured operad as an ordinary category $\mathcal{O}^{\otimes}$ together with a forgetful functor $\pi : \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ s.t. $\mathcal{O}^{\otimes}_{\langle n \rangle} \simeq (\mathcal{O}^{\otimes}_{\langle 1 \rangle})^{\times n}$.

It turns out that if π is further an "op-fibration" then this is exactly the symmetric monoidal categories.

So this is what we generalise to ∞-land:

<u>Def:</u> An <u>∞-operad</u> is a functor of ∞-categories $p : \mathcal{O}^{\otimes} \to N(\mathrm{Fin}_*)$ s.t.

1) coCart. lift of every inert $f : \langle n \rangle \to \langle m \rangle$

   ↳ In particular it induces $f_! : \mathcal{O}^{\otimes}_{\langle n \rangle} \to \mathcal{O}^{\otimes}_{\langle m \rangle}$.

2) For each $n \geq 0$, the functors $\{ p^i_! : \mathcal{O}^{\otimes}_{\langle n \rangle} \to \mathcal{O} \}_{1 \leq i \leq n}$ determines an equivalence of categories $\mathcal{O}^{\times n} \simeq \mathcal{O}^{\otimes}_{\langle n \rangle}$.

$+ \dots$

So we in particular get that $(x_0, -, x_n) \in \mathcal{O}^{\times n}$ corresponds to an object of $\mathcal{O}^{\otimes}_{\langle n \rangle}$

<u>Def:</u> A <u>symmetric monoidal ∞-category</u> is an ∞-operad $\mathcal{C}^{\otimes} \to \mathrm{Fin}_*$ which is also coCartesian fibration.

Let's understand why this gives the desired structure:

Note that: $\langle n \rangle \to \langle m \rangle$ in $\mathrm{Fin}_* \rightsquigarrow \mathcal{C}^{\otimes}_{\langle n \rangle} \longrightarrow \mathcal{C}^{\otimes}_{\langle m \rangle}$

so the active morphisms $\langle 0 \rangle \to \langle 1 \rangle$, $\langle 2 \rangle \to \langle 1 \rangle$

$$\rightsquigarrow \underbrace{\Delta^0 \to \mathcal{C}}_{\substack{\text{An object} \\ \text{we denote by } 1}}, \quad \underbrace{\mathcal{C} \times \mathcal{C} \to \mathcal{C}}_{\text{Our tensor}}$$

The reason we went through all this work introducing ∞-operads is because they are exactly what we need to get nice algebraic structures, and symmetric monoidal categories is just one of many 'monoidal'-ish categories.

In general:

Def: We say that $p: \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ exhibits $\mathcal{C}$ as $\mathcal{O}$-monoidal ∞-category if

1) p is coCartesian
2) $\mathcal{O}^{\otimes} \xrightarrow{q} \mathrm{Fin}_*$ is an ∞-operad and $\mathcal{C}^{\otimes} \xrightarrow{p} \mathcal{O}^{\otimes} \xrightarrow{q} \mathrm{Fin}_*$ exhibits $\mathcal{C}^{\otimes}$ as an ∞-operad.

Ex: Symmetric monoidal $\leftrightarrow$ Fin_*-monoidal.

Rem: If $\mathcal{C}$ is $\mathcal{O}$-monoidal by $p: \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$, then any

$$X \in \mathcal{O}^{\otimes}_{\langle n \rangle} \longleftrightarrow \{X_0, \dots, X_n\} \in \mathcal{O}$$

Given

$$f \in \mathrm{Mul}_{\mathcal{O}}(X_0, \dots, X_n; Y) \rightsquigarrow \underbrace{\prod_{0 \le i \le n} \mathcal{C}_{X_i} \simeq \mathcal{C}_X}_{} \xrightarrow{\otimes f} \mathcal{C}_Y$$

$\mathcal{C}_X$: fiber over X

It can be shown that one of the key elements is that p in particular is a morphism of ∞-operads:

ALGEBRA OBJECTS

Def: A morphism of ∞-operads:

$$\begin{tikzcd} \mathcal{O}^{\otimes} \arrow[rr, "f"] \arrow[dr, "p"'] & \circlearrowleft & \mathcal{O}'^{\otimes} \arrow[dl, "q"] \\ & \mathrm{Fin}_* & \end{tikzcd}$$

p-coCart $\longmapsto$ q-coCart, inert

$$\rightsquigarrow \mathrm{Alg}_{\mathcal{O}}(\mathcal{O}') \subseteq \mathrm{Fun}_{\mathrm{Fin}_*}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes})$$

Def: Assume $p: \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads (i.e. morphism + fibration on underlying). and given $\alpha: \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \subseteq \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})$$

$$\begin{tikzcd} \mathcal{O}'^{\otimes} \arrow[rr] \arrow[dr, "\alpha"'] & & \mathcal{C}^{\otimes} \arrow[dl, "p"] \\ & \mathcal{O}^{\otimes} & \end{tikzcd}$$

Equivalently:

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) = \text{fib. of } \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}) \xrightarrow{\circ p} \mathrm{Alg}_{\mathcal{O}'}(\mathcal{O}) \quad \text{at } \alpha$$

$$\mathcal{O}'^{\otimes} \to \mathcal{C}^{\otimes} \longmapsto \mathcal{O}'^{\otimes} \to \mathcal{C}^{\otimes} \xrightarrow{p} \mathcal{O}^{\otimes}$$

Case $\mathcal{O}^{\otimes} = \mathcal{O}'^{\otimes}$, $\alpha = \mathrm{id}$: $\mathrm{Alg}_{\mathcal{O}/\mathcal{O}}(\mathcal{C}) =: \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$

$$\begin{tikzcd} \mathcal{O}^{\otimes} \arrow[r] \arrow[d, equal, "\mathrm{id}"'] & \mathcal{C}^{\otimes} \arrow[dl, "p"] \\ \mathcal{O} & \end{tikzcd}$$

Case $\mathcal{O}'^{\otimes} = \mathcal{O}^{\otimes} = \mathrm{Fin}_*$:

$$\mathrm{CAlg}(\mathcal{C}) := \mathrm{Alg}_{\mathcal{O}/\mathcal{O}}(\mathcal{C}) = \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$$

$$\begin{tikzcd} \mathrm{Fin}_* \arrow[r] \arrow[d, equal] & \mathcal{C}^{\otimes} \arrow[dl] \\ \mathrm{Fin}_* & \end{tikzcd}$$

It's specific cases which gives us algebras!

Base case is <u>associative algebra</u>

Classically: An associative algebra object in a monoidal category C is an object $A \in \mathcal{C}$ + unit map $e: 1 \to A$ and multiplication $m: A \times A \to A$ s.t.

$$\begin{tikzcd} 1 \otimes A \arrow[r, "e \otimes id"] \arrow[dr] & A \otimes A \arrow[d, "m"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} A \otimes 1 \arrow[r] \arrow[dr] & A \otimes A \arrow[d, "m"] \\ & A \end{tikzcd}$$

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "m \otimes id"] \arrow[d, "id \otimes m"'] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"'] & A \end{tikzcd}$$

We will apply the formalisme of ∞-operads to introduce the notion of monoidal ∞-category, and to each monoidal ∞-category associate another ∞-category $\mathrm{Alg}(\mathcal{C})$ of associative algebra objects of $\mathcal{C}$.

<u>Def:</u> Colored operad Assoc (associative algebra)

- $\mathrm{ob}\,\mathrm{Assoc} = \{a\}$
- $\mathrm{Mul}_{\mathrm{Assoc}}(\{a\}_{i \in I}, a) =$ set of linear orderings on I.

  any finite set

+ composition..

$\rightsquigarrow$ Obtain a category $\mathrm{Assoc}^{\otimes}$ by applying construction 2.1.1.7

Unwinding the def of $\mathrm{Assoc}^{\otimes}$:

- $\mathrm{ob}\,\mathrm{Assoc}^{\otimes} = \mathrm{ob}\,\mathrm{Fin}_*$
- $\langle m \rangle \to \langle n \rangle$ in Fin_* $\rightsquigarrow$ $\langle m \rangle \to \langle n \rangle$ in $\mathrm{Assoc}^{\otimes}$ consists of a pair $(\alpha, \{\leq_i\}_{1 \leq i \leq n})$ where $\alpha: \langle m \rangle \to \langle n \rangle$ is a map in Fin_* and $\leq_i$ is a linear ordering on the inverse image $f^{-1}\{i\} \subseteq \langle m \rangle$ for $1 \leq i \leq n$

$\rightsquigarrow$ $\mathrm{Assoc}^{\otimes} = N(\mathrm{Assoc}^{\otimes}) \in \mathrm{Cat}_{\infty}$.

<u>Notation:</u> We write $\mathrm{Assoc} := \mathrm{Assoc}^{\otimes} \times_{\mathrm{Fin}_*} \{\langle 1 \rangle\}$

Note that as a simplicial set, Assoc is isomorphic to the 0-simplex Δ^0; But we use the notation Assoc to emphasize the role of the simplicial set as the underlying ∞-category for the ∞-operad $\mathrm{Assoc}^{\otimes}$.

Def: A <u>monoidal ∞-category</u> is a coCartesian fibration of ∞-operads $\mathcal{C}^{\otimes} \to \mathrm{Assoc}^{\otimes}$.

Def: $\mathcal{C}^{\otimes} \in \mathrm{Op}^{\infty}$ equipped w. a fibration $q: \mathcal{C}^{\otimes} \to \mathrm{Assoc}^{\otimes}$. Then the <u>$\infty$-category of associative algebra objects of $\mathcal{C}$</u> is $\mathrm{Alg}(\mathcal{C}) := \mathrm{Alg}_{/\mathrm{Assoc}}(\mathcal{C})$ (∞-operad sections of q)

$$\begin{tikzcd} \mathrm{Assoc}^{\otimes} \arrow[r, "A"] \arrow[dr, equal, "id"'] & \mathcal{C}^{\otimes} \arrow[d, "q"] \\ & \mathrm{Assoc}^{\otimes} \end{tikzcd}$$

Let's understand what this means: Let $\mathcal{C}^{\otimes} \to \mathrm{Assoc}^{\otimes}$ be a monoidal ∞-category. Then

Rem 4.1.1.12-13

- $\mathcal{C}^{\otimes}_{\langle n \rangle} \simeq \mathcal{C}^{\otimes} \times_{\mathrm{Assoc}^{\otimes}} \{\langle n \rangle\} \simeq \mathcal{C}^n$, and for every linear ordering on $\{1, \dots, n\}$, the corresponding map $\langle n \rangle \to \langle 1 \rangle$ in $\mathrm{Assoc}^{\otimes}$ induces a functor $\mathcal{C}^n \to \mathcal{C}$. In particular
  - $n = 0 \rightsquigarrow$ unit object $1 \in \mathcal{C}$
  - $n = 2$, standard ordering on $\{1,2\} \rightsquigarrow \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
- Evaluating on $a \in \mathrm{Assoc}^{\otimes}$ determines a forgetful functor

$$\Theta : \mathrm{Alg}(\mathcal{C}) \longrightarrow \mathcal{C}$$

By abuse of notation we often identify $A \in \mathrm{Alg}(\mathcal{C})$ with it's image $\Theta(A)$ in $\mathcal{C}$. For each $n \geq 0$ a choice of ordering on $\{1, \dots, n\}$ determines an active morphism $\{a\}_{1 \leq i \leq n} \to a$ in $\mathrm{Assoc}^{\otimes}$, which induces a morphism $\Theta(A)^{\otimes n} \longrightarrow \Theta(A)$ in $\mathcal{C}$. In particular

- $n = 2$ and standard ordering of $\{1,2\}$: $m : \Theta(A) \otimes \Theta(A) \to \Theta(A)$
  - $\hookrightarrow$ Associative and unital up to homotopy
    - $\rightsquigarrow$ In particular, it endows $\Theta(A)$ with the structure of an associative algebra object in $h\mathcal{C}$.

LEFT & RIGHT MODULES

Classically: C monoidal category w. unit A, A associative algebra object of C. A <u>left A-module</u> in C is an object $M \in \mathcal{C}$ + action map $a: A \otimes M \to M$ s.t.

$$\begin{tikzcd} A\otimes A\otimes M \arrow[r, "m\otimes id"] \arrow[d, "id\otimes a"'] & A\otimes M \arrow[d, "a"] \\ A\otimes M \arrow[r, "a"] & M \end{tikzcd} \qquad \begin{tikzcd} 1\otimes M \arrow[r, "u\otimes id"] \arrow[dr] & A\otimes M \arrow[d, "a"] \\ & M \end{tikzcd}$$

where $\left.\begin{array}{l} m = \text{multiplication} \\ u = \text{unit} \end{array}\right\}$ in A

$\rightsquigarrow$ All left A-modules $= LMod_A(\mathcal{C})$

We wish to introduce a larger ∞-operad $\mathcal{LM}^{\otimes}$ which contains $Assoc^{\otimes}$. If $A \in Alg(\mathcal{C})$ we want a left A-module to be a map of ∞-operads $M: \mathcal{LM}^{\otimes} \to \mathcal{C}^{\otimes}$ s.t. $M|_{Assoc^{\otimes}} = A$.

<u>Def:</u> Define a colored operad $\underset{\sim}{LM}$ as:

- $ob LM = \{a, m\}$
- Let $\{X_i\}_{i\in I}$ be a finite collection of objects of LM. Then
  - $\hookrightarrow$ $Mul_{LM}(\{X_i\}, a) = \begin{cases} \text{all linear orderings of } I & \text{if all } X_i = a \\ \text{Empty} & \text{otw} \end{cases}$
  - $\hookrightarrow$ $Mul_{LM}(\{X_i\}, m) = \{$All linear orderings $\{i_1 < \dots < i_n\}$ on I s.t. $X_{i_n} = m$ & $X_{i_j} = a$ for $j < n$

  $a, a, \dots, a, m$

<u>Rem:</u> 4.2.1.2 $a \in LM \rightsquigarrow$ sub-colored operad of LM isomorphic to Assoc

<u>Rem:</u> 4.2.1.3 We first in this (1-categorical) case see how this can be used to understand modules.

Assume $\mathcal{C}$ symmetric monoidal, $F: LM \to \mathcal{C}$ map of colored operads

$\rightsquigarrow$ $F|_{Assoc}: Assoc \to \mathcal{C}$ $\rightsquigarrow$ Associative algebra object $F(a) = A \in \mathcal{C}$. Let $M = F(m) \in \mathcal{C}$.

Then the unique operation $\phi \in Mul_{LM}(\{a, m\}, m)$ determines $F(\phi): A \otimes M \to M$, which exhibits M as a left A-module.

$\rightsquigarrow$ So $\begin{array}{l} LM \to \mathcal{C} \\ a \mapsto \text{associative algebra} \\ m \mapsto \text{left module over that algebra} \end{array}$

<u>Notation:</u> Apply construction 2.1.1.7 to LM to obtain the category $LM^{\otimes}$ from LM. Unwinding this construction we see that

1) $ob LM^{\otimes} = \{(\langle n\rangle, S) \mid S \subset \langle n\rangle^{\circ}\}$

2) $(\langle n\rangle, S) \to (\langle n'\rangle, S') = \alpha: \langle n\rangle \to \langle n'\rangle$ in $Assoc^{\otimes}$ s.t.

   i) $S \cup \{*\} \overset{\alpha}{\mapsto} S' \cup \{*\}$

   ii) $s' \in S' \Rightarrow \alpha^{-1}\{s'\}$ contains exactly one element of S, and that object is maximal w.r.t. the linear ordering of $\alpha^{-1}\{s'\}$.

Rem:

$$\begin{array}{ccc} \mathbb{LM} & & \mathbb{LM}^{\otimes} \\ a & \longleftrightarrow & (\langle 1\rangle, \emptyset) \\ m & \longleftrightarrow & (\langle 1\rangle, \langle 1\rangle^{\circ}) \end{array}$$

Def: $\mathcal{LM}^{\otimes} = N(\mathbb{LM}^{\otimes})$. This is an ∞-operad via the forgetful map $\mathcal{LM}^{\otimes} \to \mathrm{Fin}_*$

Rem: The underlying ∞-category $\mathcal{LM}$ of $\mathcal{LM}^{\otimes}$ is isomorphic to the discrete simplicial set $\Delta^0 \amalg \Delta^0$ w. two vertices, corresponding to $a, m \in \mathbb{LM}$.

Rem: $\mathrm{Assoc} \hookrightarrow \mathbb{LM} \rightsquigarrow \mathrm{Assoc}^{\otimes} \hookrightarrow \mathcal{LM}^{\otimes}$, which is an isomorphism from $\mathrm{Assoc}^{\otimes}$ onto the full subcategory of $\mathcal{LM}^{\otimes}$ spanned by objects of the form $(\langle n\rangle, \emptyset)$.

Notation: Let $\mathcal{C}^{\otimes} \to \mathcal{LM}^{\otimes}$ be a fibration of ∞-operads. Write

$$\mathcal{C}^{\otimes}_a := \mathcal{C}^{\otimes} \times_{\mathcal{LM}^{\otimes}} \mathrm{Assoc}^{\otimes}, \quad \text{Underlying } \infty\text{-category } \mathcal{C}_a = \mathcal{C}^{\otimes} \times_{\mathcal{LM}^{\otimes}} \{a\}$$

$$\mathcal{C}^{\otimes}_m = \mathcal{C}^{\otimes} \times_{\mathcal{LM}^{\otimes}} \{m\}$$

Def: Let $\mathcal{C}^{\otimes} \to \mathrm{Assoc}^{\otimes}$ be a fibration of ∞-operads, $q: \mathcal{O}^{\otimes} \to \mathcal{LM}$ fibration of ∞-operads s.t. $\mathcal{O}^{\otimes}_a \simeq \mathcal{C}^{\otimes}$. Write $\mathcal{O}^{\otimes}_m = \mathcal{M}$ ($\in \mathrm{Cat}_{\infty}$) (Normally we say q exhibits $\mathcal{M}$ as weakly enriched over $\mathcal{C}^{\otimes}$)

- $\mathrm{LMod}(\mathcal{M}) := \mathrm{Alg}_{/\mathcal{LM}}(\mathcal{O})$ ∞-category of left module objects of $\mathcal{M}$ $\quad \begin{array}{c}\mathcal{LM} \to \mathcal{O} \\ \searrow \ \swarrow \\ \mathcal{LM}\end{array}$
- Composition w. $\mathrm{Assoc}^{\otimes} \hookrightarrow \mathcal{LM}^{\otimes}$ determines a categorical fibration

$$\mathrm{LMod}(\mathcal{M}) = \mathrm{Alg}_{/\mathcal{LM}}(\mathcal{O}) \xrightarrow{\circ\, \mathrm{incl}} \mathrm{Alg}_{/\mathrm{Assoc}}(\mathcal{C}) = \mathrm{Alg}(\mathcal{C})$$

$\rightsquigarrow$ $\mathrm{LMod}_A(\mathcal{M}) = \mathrm{LMod}(\mathcal{M}) \times_{\mathrm{Alg}(\mathcal{C})} \{A\}$ ∞-category of left A-modules objects of $\mathcal{M}$.

$\rightsquigarrow$ We think of $\mathrm{LMod}(\mathcal{O}_m)$ as given by pairs (A, M) where A is an associative algebra object of $\mathcal{O}_a$ and M is a left A-module in $\mathcal{O}_m$.

Ex: 4.2.1.16 $\mathcal{C}^{\otimes} \to \mathrm{Assoc}^{\otimes}$ fibration of ∞-operads, $\mathcal{O}^{\otimes} = \mathcal{C}^{\otimes} \times_{\mathrm{Assoc}^{\otimes}} \mathcal{LM}^{\otimes}$. Then "$\mathcal{O}^{\otimes}$ exhibits $\mathcal{C}$ as weakly enriched over $\mathcal{C}^{\otimes}$" $\rightsquigarrow$ Can consider $\mathrm{LMod}(\mathcal{C}) = \mathrm{Alg}_{\mathcal{LM}/\mathrm{Assoc}}(\mathcal{C})$ $\quad \begin{array}{c}\mathcal{LM}^{\otimes} \to \mathcal{C}^{\otimes} \\ \searrow \ \swarrow \\ \mathrm{Assoc}^{\otimes}\end{array}$

∞-cat

Ex: 4.2.1.18 Let $\mathcal{C}^{\otimes} \to \mathrm{Assoc}^{\otimes}$ be a monoidal ∞-category. Then

$$\mathrm{LMod}(\mathcal{C}) = \mathrm{Alg}_{\mathcal{LM}/\mathrm{Assoc}}(\mathcal{C}) \ni \begin{array}{c}\mathcal{LM}^{\otimes} \xrightarrow{F} \mathcal{C}^{\otimes} \\ \searrow \ \swarrow \\ \mathrm{Assoc}^{\otimes}\end{array}$$

$\rightsquigarrow$ $F|_{\mathrm{Assoc}} \in \mathrm{Alg}(\mathcal{C})$, which we identify with its underlying object $F(a) = A \in \mathcal{C}$.

$\rightsquigarrow$ Also have $F(m) = M \in \mathcal{C}$

The unique operation $\varphi \in \mathrm{Mul}_{\mathbb{LM}}(\{a, m\}, m)$ determines a map

$$a: A \otimes M \to M \quad \text{in } \mathcal{C}$$

which is well-defined up to homotopy.

Since F is defined on all of $\mathcal{LM}^{\otimes}$, we get that this action map is compatible with the associative multiplication on A, up to coherent homotopy. In particular, if

$m: A\otimes A \to A$ multiplication
on A
$u: 1 \longrightarrow A$ unit map

$$\rightsquigarrow \quad \begin{tikzcd} A\otimes A\otimes M \arrow[r, "m\otimes id"] \arrow[d, "id\otimes a"] & A\otimes M \arrow[d, "a"] \\ A\otimes M \arrow[r, "a"] & M \end{tikzcd} \qquad \begin{tikzcd} 1\otimes M \arrow[rr, "u\otimes id"] \arrow[dr] & & A\otimes M \arrow[dl, "a"] \\ & M & \end{tikzcd}$$

commutes up to homotopy.

<u>Rem:</u> If $\mathcal{C}^{\otimes} \xrightarrow{q} \mathcal{LM}^{\otimes}$ is a cocartesian fibration of ∞-ops, then the induced map $\mathcal{C}_a^{\otimes} \longrightarrow Assoc^{\otimes}$ is also a cocartesian fibration of ∞-operads.

We further see that by straightening q is classified by a map $\chi: \mathcal{LM}^{\otimes} \longrightarrow Cat_{\infty}$, which is an "$\mathcal{LM}$-monoid object of Cat_{∞}". We have

$$\chi \in Mon_{\mathcal{LM}}(Cat_{\infty}) \simeq Alg_{\mathcal{LM}}(Cat_{\infty}) = LMod(Cat_{\infty})$$

More informally: q can be thought of as giving an associative algebra $\mathcal{C}_a$ in Cat_{∞} together with a left module $\mathcal{C}_m$ over $\mathcal{C}_a$. In particular q determines an action map

$$\otimes : \mathcal{C}_a \times \mathcal{C}_m \to \mathcal{C}_m$$

which is well-defined up to homotopy - and compatible with the monoidal structure on $\mathcal{C}_a$