Recall: For $(\mathcal{C},\Omega)\in \mathrm{Cat}_\infty^p$, we define the L-groups:

- $L_0(\mathcal{C},\Omega) := \mathrm{coker}(\pi_0 \mathrm{Poinc}(\mathrm{Met}(\mathcal{C},\Omega)) \longrightarrow \pi_0(\mathrm{Poinc}(\mathcal{C},\Omega)))$
  $\simeq$ forms / cobordism
- $L_n(\mathcal{C},\Omega) := L_0(\mathcal{C},\Omega^{[-n]})$, $\Omega^{[-n]} := \Sigma^{-n}\Omega$

Today: L-theory space, additive functors, bordism invarience & L-theory spectrum.

§1 L-Theory space

Construction: For $[n]\in\Delta$:

$$\mathcal{T}_n := \underbrace{P_0([n])^{op}}_{\text{Functorial in } [n]\in\Delta}$$

opposite of category of non-empty posets (maps: incl.)

For $(\mathcal{C},\Omega)\in \mathrm{Cat}_\infty^p$ we want to equip $\mathrm{Fun}(\mathcal{T}_n,\mathcal{C})$ w. a Poincaré structure: (Poincaré by I.6.6.1)

$$\Omega^{\mathcal{T}_n} := \lim_{i\in\mathcal{T}_n} ev_i^*\Omega$$

$\lim_i (\mathrm{Fun}(\mathcal{T}_n,\mathcal{C})^{op} \xrightarrow{(ev_i)^{op}} \mathcal{C}^{op} \xrightarrow{\Omega} Sp) : \mathrm{Fun}(\mathcal{T}_n,\mathcal{C})^{op} \to Sp$ a weird is stable

where $ev_i : \mathrm{Fun}(\mathcal{T}_n,\mathcal{C}) \longrightarrow \mathcal{C}$ is evaluation at $i\in\mathcal{T}_n$. We then write

$$P_n(\mathcal{C},\Omega) := (\mathrm{Fun}(\mathcal{T}_n,\mathcal{C}), \Omega^{\mathcal{T}_n}) \in \mathrm{Cat}_\infty^p.$$

can be done more generally to obtain "cotensor"

$\rightsquigarrow$ P-construction

Reference: II. p.102

$$P(\mathcal{C},\Omega) : \Delta^{op} \longrightarrow \mathrm{Cat}_\infty^p$$
$$[n] \longmapsto P_n(\mathcal{C},\Omega)$$

Def: Let $\mathcal{E}$ be an ∞-category w. sifted colimits (e.g. Sp, $\mathcal{S}$) and $\mathcal{F}: \mathrm{Cat}_\infty^p \longrightarrow \mathcal{E}$ a functor.

$$\mathrm{ad}\,\mathcal{F} : \mathrm{Cat}_\infty^p \longrightarrow \mathcal{E}$$
$$(\mathcal{C},\Omega) \longmapsto |\mathcal{F}P(\mathcal{C},\Omega)|$$

Using functoriality of $P(-,-)$ we can extend this to a functor

$$\mathrm{ad} : \mathrm{Fun}(\mathrm{Cat}_\infty^p,\mathcal{E}) \longrightarrow \mathrm{Fun}(\mathrm{Cat}_\infty^p,\mathcal{E})$$
$$F \longmapsto \mathrm{ad}\,F = |FP(\mathcal{C},\Omega)|$$

Note: $\mathrm{Poinc}(\mathcal{C}, Ϙ)$ for $(\mathcal{C}, Ϙ) \in \mathrm{Cat}^p_\infty$ extends to a functor

$$\mathrm{Poinc}: \mathrm{Cat}^p_\infty \longrightarrow \mathcal{S}$$

Def: The L-theory space is the functor

$$\mathcal{L} := \mathrm{ad}(\mathrm{Poinc}): \mathrm{Cat}^p_\infty \longrightarrow \mathcal{S}$$
$$(\mathcal{C}, Ϙ) \longmapsto \mathrm{ad}(\mathrm{Poinc}(\mathcal{C}, Ϙ)) = |\mathrm{Poinc}\, P(\mathcal{C}, Ϙ)|$$

Rem: $\mathcal{L}(\mathcal{C}, Ϙ)$ (considered as simplicial space) is Kan $\forall (\mathcal{C}, Ϙ) \in \mathrm{Cat}^p_\infty$.

Claim: $L_0(\mathcal{C}, Ϙ) \cong \pi_0 \mathcal{L}(\mathcal{C}, Ϙ) \quad \forall (\mathcal{C}, Ϙ) \in \mathrm{Cat}^p_\infty$.

Pf: First we note that $P_0(\mathcal{C}, Ϙ) \simeq (\mathcal{C}, Ϙ)$ so we get a canonical map

$$\mathrm{Poinc}(\mathcal{C}, Ϙ) \longrightarrow \mathcal{L}(\mathcal{C}, Ϙ) = |\mathrm{Poinc}(P_\bullet(\mathcal{C}, Ϙ))|$$

u. 0-term $|\mathrm{Poinc}(\mathcal{C}, Ϙ)|$ and every term of a colimit have a map into the colimit.

We claim that the natural map

$$\pi_0 \mathrm{Poinc}(\mathcal{C}, Ϙ) \to \pi_0 \mathcal{L}(\mathcal{C}, Ϙ)$$

descends to an equivalence

$$\boxed{L_0(\mathcal{C}, Ϙ) \xrightarrow{\sim} \pi_0 \mathcal{L}(\mathcal{C}, Ϙ)}$$

Now in general for A, B monoids, $f, g: A \to B$ two maps.
Then

$$\mathrm{coeq}(f, g) \cong B/\sim,$$

where $\sim$ is the relation generated by

$$b \sim b' \iff \exists a \in A \text{ s.t. } b \overset{f}{\longleftarrow\!\shortmid} a \overset{g}{\shortmid\!\longrightarrow} b'$$

In our case, since $\mathcal{L}(\mathcal{C}, Ϙ)$ is Kan, we get

$$\pi_0 \mathcal{L}(\mathcal{C}, Ϙ) = \mathrm{coeq}(d_0, d_1: \pi_0 \mathcal{L}(\mathcal{C}, Ϙ)_1 \to \pi_0 \mathcal{L}(\mathcal{C}, Ϙ)_0)$$
$$\cong \pi_0 \mathcal{L}(\mathcal{C}, Ϙ)_0 / \sim$$
$$\cong \pi_0 \mathrm{Poinc}(\mathcal{C}, Ϙ) / \sim$$

where

$$(c,q) \sim (c',q')$$

iff

$$\exists\, x \in \pi_0 \mathrm{Poinc}(\mathrm{Met}(\mathcal{C},\mathfrak{Q})) \quad \text{s.t.} \quad d_0 x = (c,q), \quad d_1 x = (c',q')$$

which exactly means that (c,q) and (c',q') are cobordant. So since

$$L_0(\mathcal{C},\mathfrak{Q}) = {}^{\pi_0 \mathrm{Poinc}(\mathcal{C},\mathfrak{Q})}\text{forms} / \text{cobordism}$$

the result follows.

$\square$

§2 Additive functors ← David will talk more about this next time

Def: A sequence

Poincaré-Verdier inclusion, Poincaré-Verdier projection

$$(\mathcal{C}',Ϙ') \xrightarrow{(f',\eta')} (\mathcal{C},Ϙ) \xrightarrow{(f,\eta)} (\mathcal{C}'',Ϙ'')$$

in Cat_∞^p w. vanishing composition is called a Poincaré-Verdier sequence if it is both a fiber and a cofiber sequence in Cat_∞^p.

We say that it is split if the underlying functor

$$f':\mathcal{C}' \longrightarrow \mathcal{C},$$

or equivalently $f:\mathcal{C}\longrightarrow\mathcal{C}''$, admits both adjoints.

Def: (II.1.5.4) Let $\mathcal{E}$ be an ∞-category w. finite limits and $\mathcal{F}:\mathrm{Cat}_\infty^p\longrightarrow\mathcal{E}$ a functor. We say $\mathcal{F}$ is reduced if $\mathcal{F}(0)$ is terminal in $\mathcal{E}$. We further say a reduced functor $\mathcal{F}$ is additive if

[split Poincaré-Verdier sequences] $\longmapsto$ [Cartesian squares]

We denote the full subcategory spanned by such functors by

$$\mathrm{Fun}^{add}(\mathrm{Cat}_\infty^p,\mathcal{E})\subseteq\mathrm{Fun}(\mathrm{Cat}_\infty^p,\mathcal{E})$$

Prop: (II.4.4.2) Given a Poincaré-Verdier sequence

$$(\mathcal{C},Ϙ)\to(\mathcal{D},\Phi)\longrightarrow(\mathcal{E},\Psi)$$

the functor

$$\mathrm{Poinc}(\mathcal{P}(\mathcal{D},\Phi))\to\mathrm{Poinc}(\mathcal{P}(\mathcal{E},\Psi))$$

is a Kan fibration of simplicial spaces w. fibre $\mathrm{Poinc}(\mathcal{P}(\mathcal{C},Ϙ))$

Cor: The $\mathcal{L}$-theory space

$$\mathcal{L}:\mathrm{Cat}_\infty^p\longrightarrow\mathcal{S}$$

is additive.

Pf: First we need to show that $\mathcal{L}$ is reduced: Cat_∞^p is additive with 0-object

$$\Big(\overset{\text{clearly stable}}{\mathcal{C} = *},\ \begin{array}{c} Ϙ : *^{op} \to Sp \\ * \mapsto 0 \end{array}\Big),$$

and we see that

$$\mathcal{L}(*, Ϙ) = |\mathrm{Poinc}_{\bullet}(*, Ϙ)|$$

$$\begin{aligned} \mathcal{L}(*, Ϙ)_n &= \mathrm{Poinc}\,\rho_n(*, Ϙ) \\ &= \mathrm{Poinc}(*, Ϙ^{\Sigma n}) \\ &= \{(* \in *, q \in \Omega^\infty Ϙ(*)) + \text{some cond.}\} \\ &= \{*\} \end{aligned}$$

as desired.

To further see that $\mathcal{L}$ is indeed additive, we note that

$$\mathcal{L}(\mathcal{C}, Ϙ) = ad(\mathrm{Poinc}(\mathcal{C}, Ϙ)) = |\mathrm{Poinc}(\rho_\bullet(\mathcal{C}, Ϙ))|$$

so using II.4.4.2 together with the fact that geometric realization takes Kan fibrations w. fibre to Cartesian squares, the desired follows.

$\square$

§3 Bordism invariance

Let $(\mathcal{C},Ϙ), (\mathcal{D},\Phi) \in \mathrm{Cat}_\infty^h$, then we define the hermitian ∞-category **internal functor category**:

$$\mathrm{Fun}^{ex}((\mathcal{C},Ϙ),(\mathcal{D},\Phi)) := (\mathrm{Fun}^{ex}(\mathcal{C},\mathcal{D}), Ϙ_{\mathrm{Fun}})$$

but really nec. to define any

where the quadratic functor $Ϙ_{\mathrm{Fun}} : (\mathrm{Fun}^{ex}(\mathcal{C},\mathcal{D}))^{op} \longrightarrow \mathrm{Sp}$ is constructed by:

quadratic by I. 6.2.4

$$\begin{aligned} Ϙ_{\mathrm{Fun}} : \mathrm{Fun}^{ex}(\mathcal{C},\mathcal{D})^{op} &\longrightarrow \mathrm{Sp} \\ f &\longmapsto \mathrm{nat}(Ϙ, f^*\Phi) \end{aligned}$$

← spectrum of nat. trans

If $(\mathcal{C},Ϙ),(\mathcal{D},\Phi) \in \mathrm{Cat}_\infty^p$

then $\mathrm{Fun}^{ex}((\mathcal{C},Ϙ),(\mathcal{D},\Phi)) \in \mathrm{Cat}_\infty^p$.

$$\begin{tikzcd} \mathcal{C} \arrow[d, "Ϙ"'] \\ \mathrm{Sp} \end{tikzcd} \Longrightarrow \begin{tikzcd} \mathcal{C} \arrow[d, "f"] \\ \mathcal{D} \arrow[d, "\Phi"] \\ \mathrm{Sp} \end{tikzcd}$$

Def. (I. p.45) A **cobordism** between two poincare objects $(X,q),(X',q')$ in $(\mathcal{C},Ϙ) \in \mathrm{Cat}_\infty^p$, is a span of the form

$$X \xleftarrow{\alpha} \underset{\mathcal{C}}{\overset{}{W}} \xrightarrow{\beta} X'$$

($W \in \mathcal{C}$)

\+ a path $\eta : \alpha^* q \longrightarrow \beta^* q'$ in $\Omega^\infty Ϙ(W)$

s.t.

$$\mathrm{fib}(W \longrightarrow X) \simeq \Omega D_Ϙ(\mathrm{fib}(W \longrightarrow X'))$$

say more about this? "Poincaré-Lefschetz condition"

Def. (II.3.1.1) Let $(\mathcal{C},Ϙ),(\mathcal{D},\Phi) \in \mathrm{Cat}_\infty^p$ and

$$f,g : (\mathcal{C},Ϙ) \longrightarrow (\mathcal{D},\Phi)$$

two Poincaré functors. A **cobordism** between f and g is a cobordism in $\mathrm{Fun}^{ex}((\mathcal{C},Ϙ),(\mathcal{D},\Phi))$ between the Poincare objects corresponding to f and g.

Rem: (II. p.67) A cobordism f to g is a span

$$f \xleftarrow{\alpha} h \xrightarrow{\beta} g$$

↙so in particular $h : \mathcal{C} \to \mathcal{D}$

in $\mathrm{Fun}^{ex}(\mathcal{C},\mathcal{D})$, and α, β natural transformations.

There exists a Poincaré ∞-category $Q_1(\mathcal{C},Ϙ)$ which has spans of $\mathcal{C}$ as the underlying stable ∞-category

Using this we note that the data of a cobordism f to g equivalently can be encoded in a Poincaré functor

$$\phi : (\mathcal{C},Ϙ) \longrightarrow Q_1(\mathcal{D},\Phi)$$

s.t.

$$d_0\phi = f \quad , \quad d_1\phi = g.$$

Def: I.3.5.1 A Poincaré functor $(f,\eta):(\mathcal{C},Ϙ)\to(\mathcal{D},\Phi)$ is a bordism equivalence if there exists a Poincaré functor $(g,\theta):(\mathcal{D},\Phi)\to(\mathcal{C},Ϙ)$ s.t. the composites

$$(f,\eta)\circ(g,\theta) \quad , \quad (g,\theta)\circ(f,\eta)$$

are cobordant to the respective identities.

Def. I.3.5.3 Let $\mathcal{E}$ be an ∞-category w. finite products. Then we say that an additive functor $\mathcal{F}:\mathrm{Cat}^p_\infty \to \mathcal{E}$ is bordism invariant if it sends bordism equivalences of Poincaré ∞-categories to equivalences in $\mathcal{E}$.

We write $\mathrm{Fun}^{\mathrm{bord}}(\mathrm{Cat}^p_\infty,\mathcal{E}) \subseteq \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^p_\infty,\mathcal{E})$ for the full subcategory spanned by the bordism invariant functors

Rem: II.p.91 Bordism invariant functors vanishes on all metabolic Poincaré ∞-categories:

Let $F:\mathrm{Cat}^p_\infty \to \mathcal{E}$ be bordism invariant. Then we claim that

Holds since F in particular is reduced

$$F(\mathrm{Met}(\mathcal{C},Ϙ)) \simeq 0 = F(0),$$

i.e. want a bordism equivalence since F is bordism invariant

$$\mathrm{Met}(\mathcal{C},Ϙ) \rightleftarrows (0,Ϙ)$$

Mapped to 0 by F

Terminal and initial object so canonical map into and out of

Proof: ✓

$\circlearrowleft \sim \mathrm{id}$: This composite is the 0-map, so want a bordism between

$$\mathrm{Met}(\mathcal{C}, Ϙ) \underset{0}{\overset{\mathrm{id}}{\rightrightarrows}} \mathrm{Met}(\mathcal{C}, Ϙ).$$

It can be shown that this bordism is given by the diagram

$q \in \Omega^\infty Ϙ(d)$

$$\mathrm{id}\begin{pmatrix} c \\ \downarrow \\ d \end{pmatrix} = \begin{tikzcd} c \arrow[d] & c \arrow[l, equal] \arrow[r] & 0 \arrow[d] \\ d & \mathrm{fib}(f^*q_\#) \arrow[l, "\sim"] \arrow[r] & 0 \end{tikzcd} = \begin{pmatrix} c' \\ \downarrow \\ d' \end{pmatrix} 0$$

objects in $\mathrm{Met}(\mathcal{C}, Ϙ)$

} Span in $\mathrm{Met}(\mathcal{C}, Ϙ)$ $\rightsquigarrow$ i.e. a cob. from $\mathrm{id} \to 0$

$\square$

Won't need this part

For so called "group-like" additive functors, we have that the converse holds, i.e. that it is bordism invariant exactly when it disappears on metabolics

Lem: II.1.5.7 The forgetful functor

$$\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^p_\infty, \mathrm{Mon}_{E_\infty}(\mathcal{E})) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^p_\infty, \mathcal{E})$$

is an equivalence.

Def: II.1.5.8 An additive functor $F: \mathrm{Cat}^p_\infty \to \mathcal{E}$ is called grouplike if the natural lift of F to $\mathrm{Mon}_{E_\infty}(\mathcal{E})$ takes values in the full subcategory $\mathrm{Grp}_{E_\infty}(\mathcal{E})$.

$\mathrm{Mon}_{E_\infty}(\mathcal{E})$: commutative monoids

$\mathrm{Grp}_{E_\infty}(\mathcal{E})$: E_∞-groups (or group-like monoids)

Rem: II.1.5.9 If $\mathcal{E}$ is additive (e.g. Sp), then any additive functor $F: \mathrm{Cat}^p_\infty \to \mathcal{E}$ is group-like since both forgetful functors $\mathrm{Mon}_{E_\infty}(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}$, $\mathrm{Grp}_{E_\infty}(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}$ are equivalences.

$\mathrm{Grp}_{E_\infty}(\mathcal{E}) \subseteq \mathrm{Mon}_{E_\infty}(\mathcal{E})$ which admits inverses

Rem: If π_0 of the image of F is a group, then F is group-like

$\rightsquigarrow$ we know $\mathcal{L}: \mathrm{Cat}^p_\infty \to \mathcal{S}$ is additive and

$\pi_0 \mathcal{L}(\mathcal{C},\Omega) \simeq L_0(\mathcal{C},\Omega)$ is a group. Hence $\mathcal{L}$ is grouplike.

Prop: II.3.5.4 Let $F: \mathrm{Cat}^p_\infty \to \mathcal{E}$ be a group-like additive functor. Then F is bordism invariant iff. $F(\mathrm{Met}(\mathcal{C},\Omega)) \simeq *$ for all $(\mathcal{C},\Omega) \in \mathrm{Cat}^p_\infty$.

Prop: For any $(\mathcal{C},\Omega) \in \mathrm{Cat}^p_\infty$ we have

$$\pi_n \mathcal{L}(\mathcal{C},\Omega) \simeq L_n(\mathcal{C},\Omega) = \underbrace{L_0(\mathcal{C},\Omega^{[-n]})}_{\text{Forms/cobordism}}$$

Pf: See seperate document

Prop: II.3.5.5 $\mathcal{L}: \mathrm{Cat}^p_\infty \to \mathcal{S}$ is bordism invariant:

Pf: Consider

$$d_0, d_1 : Q_1(\mathcal{C},\Omega) \longrightarrow (\mathcal{C},\Omega)$$

$$x \leftarrow y \rightarrow z \quad \xmapsto{d_1} \quad x$$

$$x \leftarrow y \rightarrow z \quad \xmapsto{d_0} \quad z$$

$\pi_n \mathcal{L}(-)$

$$\pi_n \mathcal{L}(Q_1(\mathcal{C},\Omega)) \longrightarrow \pi_n \mathcal{L}(\mathcal{C},\Omega)$$

$$[x \leftarrow y \rightarrow z] \longmapsto [x] \sim [z]$$

so we get that d_0 and d_1 induce the same map on L-groups.

Now, let $f, g: \mathcal{C} \to \mathcal{D}$ be cobordant. Then we equivalently have a Poincare functor

$$H: (\mathcal{C},\Omega) \to Q_1(\mathcal{D},\Phi)$$

s.t.

$$d_0 H \simeq f \quad , \quad d_1 H \simeq g.$$

Then using that we by the above argument knows that

$$\pi_n \mathcal{L}(d_0) \simeq \pi_n \mathcal{L}(d_1)$$

we get

$$\begin{aligned} \pi_n \mathcal{L}(f) &\simeq \pi_n \mathcal{L}(d_0 H) \\ &\simeq \pi_n \mathcal{L}(d_1 H) \\ &\simeq \pi_n \mathcal{L}(g) \end{aligned}$$

hence any corbordant functors (between Poincare ∞-categories) induces equivalent maps on the

$\underbrace{\text{homotopy groups of the } \mathcal{L}\text{-spaces}}_{\text{which equivalently is the L-groups}}$

Now, assume that $(F,\eta): (\mathcal{C},\varphi) \to (\mathcal{D},\Phi)$ is bordism equivalent, with inverse given by

$$(G,\theta): (\mathcal{D},\Phi) \to (\mathcal{C},\varphi),$$

i.e. $(F,\eta)\circ(G,\theta) \overset{\text{cobordant}}{\sim} \mathrm{id}_{(\mathcal{D},\Phi)}$ and

$(G,\theta)\circ(F,\eta) \sim_{cob} \mathrm{id}_{(\mathcal{C},\varphi)}$

Hence by the above we get that both composites induces identity on L-groups $\pi_n\mathcal{L}$, so (F,η) induces an isomorphism w. inverse induced by (G,θ).

$\rightsquigarrow \pi_n\mathcal{L}(F,\eta)$ iso $\forall n$ $\Rightarrow \mathcal{L}(F,\eta)$ eq. $\square$

§4 L-Theory Spectrum

Prop: II.3.5.8 Assume $F: \mathrm{Cat}_\infty^p \to \mathcal{E}$ is bordism invariant. Then the natural

$$\Omega F(\mathcal{C}, Ϙ) \to F(\mathcal{C}, Ϙ^{[-1]})$$

arising from the metabolic Poincaré-Verdier sequence is an equivalence.

Pf: The metabolic Poincaré-Verdier sequence has the form

Split Poincaré-Verdier fibre sequence $(\mathcal{C}, Ϙ^{[-1]}) \to \mathrm{Met}(\mathcal{C}, Ϙ) \xrightarrow{\mathrm{met}} (\mathcal{C}, Ϙ)$

$$\downarrow$$

Fibre sequence since F is in particular additive $F(\mathcal{C}, Ϙ^{[-1]}) \to F(\mathrm{Met}(\mathcal{C}, Ϙ)) \to F(\mathcal{C}, Ϙ)$

By II.3.5.4 we know F is bordism invariant iff $F(\mathrm{Met}(\mathcal{C}, Ϙ)) \simeq * \ \forall (\mathcal{C}, Ϙ) \in \mathrm{Cat}_\infty^p$, so we get fibre sequence

$$\begin{tikzcd}
F(\mathcal{C}, Ϙ^{[-1]}) \arrow[r] \arrow[d] \arrow[dr, dashed, "\simeq"] & * \arrow[d] \\
* \arrow[r] & F(\mathcal{C}, Ϙ)
\end{tikzcd}$$

$\Omega F(\mathcal{C}, Ϙ)$

such a pullback is per definition the loopspace, i.e.

$$F(\mathcal{C}, Ϙ^{[-1]}) \xrightarrow{\sim} \Omega F(\mathcal{C}, Ϙ)$$ □

Since $\mathcal{L}: \mathrm{Cat}_\infty^p \to \mathcal{S}$ is bordism invariant we get

$$\Omega \mathcal{L}(\mathcal{C}, Ϙ) \simeq \mathcal{L}(\mathcal{C}, Ϙ^{[-1]})$$

$$\rightsquigarrow \pi_i \mathcal{L}(\mathcal{C}, Ϙ) \simeq \pi_0 \mathcal{L}(\mathcal{C}, Ϙ^{[-i]})$$

$$= L_0(\mathcal{C}, Ϙ^{[-i]})$$

$$= L_i(\mathcal{C}, Ϙ)$$

In general, II.3.5.8 gives us that the image of any bordism invariant functor can be delooped by shifting the Poincaré structure:

$$[\mathcal{L}(\mathcal{C},\mathfrak{Q}), \mathcal{L}(\mathcal{C},\mathfrak{Q}^{[1]}), \mathcal{L}(\mathcal{C},\mathfrak{Q}^{[2]}), \dots]$$

with structure maps given by II.3.5.8.

Def: We define the L-Theory spectrum $L: \mathrm{Cat}_\infty^p \to \mathrm{Sp}$ by

$$L(\mathcal{C},\mathfrak{Q}) = [\mathcal{L}(\mathcal{C},\mathfrak{Q}), \mathcal{L}(\mathcal{C},\mathfrak{Q}^{[1]}), \mathcal{L}(\mathcal{C},\mathfrak{Q}^{[2]}), \dots]$$ with the structure maps from above

Cor: II.4.4.5 By definition we have canonical equivalences

$$\Omega^{\infty-i} L(\mathcal{C},\mathfrak{Q}) \simeq \mathcal{L}(\mathcal{C},\mathfrak{Q}^{[i]}) \qquad \forall\, i\in\mathbb{Z}$$

hence we have isomorphisms:

$$\pi_i L(\mathcal{C},\mathfrak{Q}) \cong L_0(\mathcal{C},\mathfrak{Q}^{[-i]}) = L_i(\mathcal{C},\mathfrak{Q})$$

Pf:

$$\begin{aligned}\pi_i L(\mathcal{C},\mathfrak{Q}) &\simeq \pi_0 \Omega^{\infty+i} L(\mathcal{C},\mathfrak{Q}) \\ &\simeq \pi_0 \mathcal{L}(\mathcal{C},\mathfrak{Q}^{[-i]}) \\ &\simeq L_0(\mathcal{C},\mathfrak{Q}^{[-i]}) \\ &=: L_i(\mathcal{C},\mathfrak{Q})\end{aligned}$$

Cor: II.4.1.6 The functor $L: \mathrm{Cat}_\infty^p \to \mathrm{Sp}$ is bordism invariant.

Pf: Follows by the facts that

$$\pi_i L(\mathcal{C},\mathfrak{Q}) \cong \pi_0 \mathcal{L}(\mathcal{C},\mathfrak{Q}^{[i]}) \quad \forall i\in\mathbb{Z}$$

and that $\mathcal{L}$ is bordism invariant.

Thm: |Ranicki Periodicity| Let R be a discrete ring spectrum and M a discrete invertible module w. involution. Then

$$\Omega^2 L(\mathcal{D}^b(R), \mathfrak{Q}_M^{\geq n}) \simeq L(\mathcal{D}^b(R), \mathfrak{Q}_{-M}^{\geq n+1})$$

In particular we get that $L(\mathcal{D}^b(R), \mathfrak{Q}_M^{\geq n})$ is 4-periodic.

Pf: First recall the categorical equivalence

$$(\mathcal{D}^b(R), \Sigma^2\mathfrak{Q}_M^{\geq n}) \xleftarrow[\Sigma]{\simeq} (\mathcal{D}^b(R), \mathfrak{Q}_{-M}^{\geq n+1}).$$

By II.4.1.6 we know that $L: \mathrm{Cat}_\infty^p \to \mathrm{Sp}$ is bordism invariant, so by II.3.5.8 we get that suspending the spectrum is equivalent to suspending the quadratic functor, hence

$$\Omega^2 L(\mathcal{D}^b(R), Ϙ^{\geq n}_M) \simeq L(\mathcal{D}^b(R), \Sigma^2 Ϙ^{\geq n}_M)$$

So by categorical equivalence we get

$$\begin{aligned}\Omega^2 L(\mathcal{D}^b(R), Ϙ^{\geq n}_M) &\simeq L(\mathcal{D}^b(R), \Sigma^2 Ϙ^{\geq n}_M) \\ &\simeq L(\mathcal{D}^b(R), Ϙ^{\geq n+1}_{-M})\end{aligned}$$

as desired. $\square$