The goal is to prove that we have a quasifibration

$$B(\mathcal{S}^{-1}\mathcal{S}) \to B(\mathcal{S}^{-1}(\mathcal{S}^{-1}\mathcal{X})) \to B(\langle \mathcal{S}, \mathcal{X} \rangle)$$

and using this, show that if $B(\langle \mathcal{S}, \mathcal{X} \rangle)$ is contractible, then

$$B(\mathcal{S}^{-1}\mathcal{S}) \simeq B(\mathcal{S}^{-1}\mathcal{X}).$$

<u>Recall:</u> Given a monoidal category $\mathcal{S}$ acting ($+ : \mathcal{S} \times \mathcal{X} \to \mathcal{X}$) on $\mathcal{X}$ we define the <u>orbit category</u> $\langle \mathcal{S}, \mathcal{X} \rangle$:

- $\mathrm{ob}\langle \mathcal{S}, \mathcal{X} \rangle \simeq \mathrm{ob}\,\mathcal{X}$
- Morphism $F \to G$ is an isomorphism class $[A, \varphi]$ with
  - ↳ $A \in \mathrm{ob}\,\mathcal{S}$
  - ↳ $\varphi : A + F \to G$ morphism in $\mathcal{X}$
  - ↳ $(A, \varphi) \sim (A', \varphi')$ if there exists $a : A \to A'$ in $\mathcal{S}$ s.t.

$$\begin{tikzcd} A+F \arrow[rr, "a+\mathrm{id}_F"] \arrow[dr, "\varphi"'] & & A'+F \arrow[dl, "\varphi'"] \\ & G & \end{tikzcd} \circlearrowleft .$$

We then further defined the <u>localisation</u> $\mathcal{S}^{-1}\mathcal{X}$ to be $\langle \mathcal{S}, \mathcal{S} \times \mathcal{X} \rangle$ where the action of $\mathcal{S}$ on $\mathcal{S} \times \mathcal{X}$ is given by

$$\mathcal{S} \times \mathcal{S} \times \mathcal{X} \longrightarrow \mathcal{S} \times \mathcal{X}$$
$$(a, b, c) \longmapsto (a \Box b, a + c)$$

monoidal structure ← action of $\mathcal{S}$ on $\mathcal{X}$.

As last time we will need some assumptions:

<u>Assumptions:</u>

- $\mathcal{S} = i\mathcal{S}$
- $\forall B \in \mathcal{S}$:

$$\mathcal{S}(A, A') \longrightarrow \mathcal{S}(A+B, A'+B)$$
$$a \longmapsto a + \mathrm{id}_B$$

  is injective

- All morphisms in $\mathcal{X}$ are monomorphisms
- $\forall F \in \mathcal{X}$

$$\begin{aligned} \mathcal{S}(A,A') &\longrightarrow \mathcal{X}(A+F, A'+F) \\ a &\longmapsto a + \mathrm{id}_F \end{aligned}$$

is injective.

<u>Construction:</u> Recall that we last time showed that

$$\begin{aligned} p: \mathcal{S}^{-1}\mathcal{X} &\longrightarrow \langle \mathcal{S}, \mathcal{S} \rangle \\ (B,F) &\longmapsto B \end{aligned}$$

is cofibered. For $B \in \mathrm{ob}\,\mathcal{S}$ we get that $p^{-1}(B)$ is the collection of pairs (B,F) in $\mathcal{S}^{-1}\mathcal{X}$, hence for all $F \in \mathcal{X}$ $\rightsquigarrow$ so we identify $p^{-1}(B)$ with $\mathcal{X}$. Let

$$\varphi: B_0 \to B_1$$

be a morphism in $\langle \mathcal{S}, \mathcal{S} \rangle$ represented by $A + B_0 \to B_1$. Then the cobase change functor $\varphi_*: p^{-1}(B_0) \to p^{-1}(B_1)$ becomes the translation

$$\begin{aligned} T_A: \mathcal{X} &\longrightarrow \mathcal{X} \\ F &\longmapsto A+F. \end{aligned}$$

We say the action of $\mathcal{S}$ on $\mathcal{X}$ is <u>invertible</u> if for each $A \in \mathrm{ob}\,\mathcal{S}$, T_A induces a homotopy equivalence on classifying spaces.

<u>Ex:</u> The action of $\mathcal{S}$ on $\mathcal{S}^{-1}\mathcal{X}$ given by

$$\begin{aligned} +: \mathcal{S} \times \mathcal{S}^{-1}\mathcal{X} &\longrightarrow \mathcal{S}^{-1}\mathcal{X} \\ (A,(B,F)) &\longmapsto (B, A+F) \end{aligned}$$

is invertible: To see this compare the two functors

$$\begin{aligned} T_A: \mathcal{S}^{-1}\mathcal{X} &\longrightarrow \mathcal{S}^{-1}\mathcal{X} \\ (B,F) &\longmapsto (B, A+F) \end{aligned} \qquad \begin{aligned} S_A: \mathcal{S}^{-1}\mathcal{X} &\longrightarrow \mathcal{S}^{-1}\mathcal{X} \\ (B,F) &\longmapsto (A+B, F). \end{aligned}$$

We see that they commute,

$$T_A \circ S_A = S_A \circ T_A$$

and for each $A \in \mathcal{S}$ we have a natural transformation

$$\mathrm{id} \Rightarrow S_A \circ T_A$$

given by the morphism

$$[A; \mathrm{id}; \mathrm{id}] : (B, F) \longrightarrow (A+B, A+F)$$

in $\mathcal{S}^{-1}\mathcal{X}$. Then follows that it induces a homotopy equivalence on classifying spaces (2.4).

Cor: 11.17 The functor

$$T : \mathcal{X} \longrightarrow \mathcal{S}^{-1}\mathcal{X}$$
$$F \longmapsto (0, F)$$

induces a homotopy equivalence on classifying spaces iff $\mathcal{S}$ acts invertibly on $\mathcal{X}$.

Pf: "$\Leftarrow$" If $\mathcal{S}$ acts invertibly on $\mathcal{X}$ then by assumption we know that the cobase change functor

$$\varphi_* : p^{-1}(B_0) \to p^{-1}(B_1)$$

identified with the translation functor $T_A : \mathcal{X} \to \mathcal{X}$ induces a homotopy equivalence on classifying spaces. So it satisfies the assumptions of Quillen's theorem B for cofibered functors, hence we get that

$$B(p^{-1}(B_0)) \to B(\mathcal{S}^{-1}\mathcal{X}) \xrightarrow{Bp} B(\langle \mathcal{S}, \mathcal{S} \rangle)$$

is a quasifibration, with $B(p^{-1}(B_0)) \simeq B(\mathcal{X})$. Since $\langle \mathcal{S}, \mathcal{S} \rangle$ admits the initial object 0 from the monoidal structure on $\mathcal{S}$, we get that $B(\langle \mathcal{S}, \mathcal{S} \rangle) \simeq * \rightsquigarrow B(\mathcal{S}^{-1}\mathcal{X}) \overset{\text{hmpty}}{\simeq} B(p^{-1}(0)) \simeq B(\mathcal{X})$, hence T induces a homotopy equivalence.

"$\Rightarrow$" If $T : \mathcal{X} \to \mathcal{S}^{-1}\mathcal{X}$ induces a homotopy equivalence, then the action of $\mathcal{S}$ on $\mathcal{X}$ is invertible since by the above exampl we know this is the case for $\mathcal{S}^{-1}\mathcal{X}$;

$$\begin{tikzcd} \mathcal{X} \arrow[r, "T", "\simeq"'] \arrow[d, "T_A"'] & \mathcal{S}^{-1}\mathcal{X} \arrow[d, "T_A", "\simeq"'] \\ \mathcal{X} \arrow[r, "\simeq", "T"'] & \mathcal{S}^{-1}\mathcal{X} \end{tikzcd}$$

$\square$

We now turn our focus to how $\mathcal{S}$ acts on $\mathcal{S}^{-1}\mathcal{X}$ so we can understand how the localisction of

$$q : \mathcal{S}^{-1}\mathcal{X} \longrightarrow \langle \mathcal{S}, \mathcal{X} \rangle$$
$$(B,F) \longmapsto F$$

looks like. We let $\mathcal{S}$ act on $\mathcal{S}^{-1}\mathcal{X}$ by acting on the first component, i.e.

$$+ : \mathcal{S} \times \underbrace{\mathcal{S}^{-1}\mathcal{X}}_{=\langle \mathcal{S}, \mathcal{S}\times\mathcal{X}\rangle} \longrightarrow \mathcal{S}^{-1}\mathcal{X}$$
$$(A, (B,F)) \longmapsto (A+B, F)$$

Using that this action is fiberwise with respect to q, we can extend q to a functor

$$\hat{q} : \mathcal{S}^{-1}(\mathcal{S}^{-1}\mathcal{X}) \longrightarrow \langle \mathcal{S}, \mathcal{X} \rangle$$
$$(A, (B,F)) \longmapsto q(B,F) = F.$$

To describe what $\hat{q}$ does on morphisms we first need to understand how morphisms in $\mathcal{S}^{-1}(\mathcal{S}^{-1}\mathcal{X})$ looks like. Note that

$$\mathcal{S}^{-1}(\mathcal{S}^{-1}\mathcal{X}) = \langle \mathcal{S}, \langle \mathcal{S}, \mathcal{S}\times\mathcal{X}\rangle\rangle$$

so let

$$\varphi : (A,(B,F)) \to (A',(B',F'))$$

be a morphism in $\mathcal{S}^{-1}(\mathcal{S}^{-1}\mathcal{X})$. Then φ is an equivalence class of a 5-tuple $[V, W, \varphi_1, \varphi_2, \varphi_3]$ with

- $V, W \in \operatorname{ob}\mathcal{S}$
- $\left.\begin{array}{l} \varphi_1 : V+A \to A' \\ \varphi_2 : W+V+B \to B' \end{array}\right\}$ In $\mathcal{S}$
- $\varphi_3 : W+F \to F'$ In $\mathcal{X}$

So

$$\hat{q}([V,W,\varphi_1,\varphi_2,\varphi_3]) = [W,\varphi_3]$$

Thm: The localization

$$\hat{q} : \mathcal{S}^{-1}(\mathcal{S}^{-1}\mathcal{X}) \to \langle \mathcal{S},\mathcal{X}\rangle$$

of the functor q, induces a quasifibration

$$B(\mathcal{S}^{-1}\mathcal{S}) \to B(\mathcal{S}^{-1}(\mathcal{S}^{-1}\mathcal{X})) \xrightarrow{B(\hat{q})} B(\langle \mathcal{S},\mathcal{X}\rangle).$$

Pf: We wish to apply Quillen's theorem B for cofibered functors, hence we need to show

① $\hat{q}$ is cofibered

② Every cofiber lift of $\hat{q}$ induces a homotopy equivalence

③ $\hat{q}^{-1}(F) \cong \mathcal{S}^{-1}\mathcal{S}$ for all $F \in \mathcal{X}$

Pf 1: Given a morphism

$$[W,\varphi_3] : \hat{q}(A,(B,F)) \to F'$$

we have universal lift

$$\widehat{[W,\varphi_3]} = [0,W,id_A, id_{W+B},\varphi_3] : (A,(B,F)) \to (A,(W+B,F'))$$

which is just the universal lift of

$$q(B,F) = F \longrightarrow F' \quad \text{represented by } \varphi_2 : W+F \to F'$$

$$(B,F) \longrightarrow (W+B,F') \text{ represented by}$$

$$(A+B,A+F) \xrightarrow{(id,\varphi_2)} (A+B,F').$$

Hence $\hat{q}$ is cofibered.

Pf of 2: ($\hat{q}^{-1}(F) \cong \mathcal{S}^{-1}\mathcal{S}$) The objects of $\hat{q}^{-1}(F)$ is of the form $(A,(B,F))$ with $A,B \in \operatorname{ob}\mathcal{S}$. A morphism

$$\varphi : (A,(B,F)) \to (A',(B',F))$$

in $\hat{q}^{-1}(F)$ is represented by

$$(V,0,\varphi_1,\varphi_2,id_F)$$

which corresponds directly to

$$(V,\varphi_1,\varphi_2)\colon (A,B) \longrightarrow (A',B')$$

in $\mathcal{S}^{-1}\mathcal{S}$.

<u>Pf of 3:</u> Assume we are given a morphism $F \to F'$ in $\langle \mathcal{S},\mathcal{X}\rangle$ represented by $\varphi_3\colon W+F \to F'$. We then have that the cofiber lift

$$\hat{\varphi}_3\colon \hat{q}^{-1}(F) \longrightarrow \hat{q}^{-1}(F')$$

is given by

$$\begin{aligned} Tw\colon \mathcal{S}^{-1}\mathcal{S} &\longrightarrow \mathcal{S}^{-1}\mathcal{S} \\ (A,B) &\longmapsto (A, W+B) \end{aligned}$$

which we know from our earlier example (11.16) induces a homotopy equivalence with homotopy inverse induced by

$$S_W\colon (A,B) \mapsto (W+A,B).$$ $\square$

<u>Thm: 11.19</u> If the category $\langle \mathcal{S},\mathcal{X}\rangle$ has contractible classifying space, then the functor

$$\begin{aligned} i_F\colon \mathcal{S}^{-1}\mathcal{S} &\longrightarrow \mathcal{S}^{-1}\mathcal{X} \\ (A,B) &\longmapsto (A, B+F) \end{aligned}$$

induces a homotopy equivalence on classifying spaces for all $F \in \mathcal{X}$.

<u>Pf:</u> Consider the following (non-commutative!) square

$$\begin{tikzcd} & (A,B) \arrow[r, mapsto] & (A,(B,F)) & \\ (B,A) & \mathcal{S}^{-1}\mathcal{S} \arrow[r, "j_F"] & \mathcal{S}^{-1}(\mathcal{S}^{-1}\mathcal{X}) & (0,(B,G)) \\ (A,B) \arrow[u, mapsto] & \mathcal{S}^{-1}\mathcal{S} \arrow[r, "i_F"'] \arrow[u, "\gamma"] & \mathcal{S}^{-1}\mathcal{X} \arrow[u, "T"'] & (B,G) \arrow[u, mapsto] \\ & (A,B) \arrow[r, mapsto] & (A,B+F) & \end{tikzcd}$$

<u>Claim 1:</u> $B(j_F \circ \gamma) \simeq B(T \circ i_F)$

<u>Pf:</u> To prove this we first note that

$$T \circ i_F(A,B) = (0, (A, B+F))$$

$$j_F \circ \gamma(A,B) = (B, (A,F))$$

We introduce a third functor

$$k_F : \mathcal{S}^{-1}\mathcal{S} \longrightarrow \mathcal{S}^{-1}(\mathcal{S}^{-1}\mathcal{X})$$

$$(A,B) \longmapsto (B, (B+A, B+F)),$$

and see that we have natural transformations

$$(0, B, id, id, id) : j_F \circ \gamma \Rightarrow k_F \Leftarrow T \circ i_F : (B, 0, id, id, id).$$

which implies that $B(j_F \circ \gamma) \simeq B(T \circ i_F)$.

<u>Claim 2:</u> j_F induces a homotopy equivalence on classifying spaces

<u>Pf:</u> We have assumed $B(\langle \mathcal{S}, \mathcal{X} \rangle) \simeq *$ so by 11.18 we have a quasifibration

$$B(\mathcal{S}^{-1}\mathcal{S}) \xrightarrow{B(j_F)} B(\mathcal{S}^{-1}(\mathcal{S}^{-1}\mathcal{X})) \longrightarrow *$$

<u>Claim 3:</u> γ induces a homotopy equivalence on classifying spaces

<u>Pf:</u> $\gamma \circ \gamma = id$ so in particular $id \Rightarrow \gamma \circ \gamma$

<u>Claim 4:</u> T induces a homotopy equivalence on classifying spaces

<u>Pf:</u> From example 11.16 we get that $\mathcal{S}$ acts invertibly on $\mathcal{S}^{-1}\mathcal{X}$, so the desired follows by 11.17

So it follows that $B(i_F)$ is a homotopy equivalence.