# What is an $\infty$-category?

**Def:** A quasicategory $\mathcal{C}$ is a simplicial set ($\Delta^{op} \to \mathrm{Set}$) such that for $0<i<n$ we have horn fillings:

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & \mathcal{C} \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

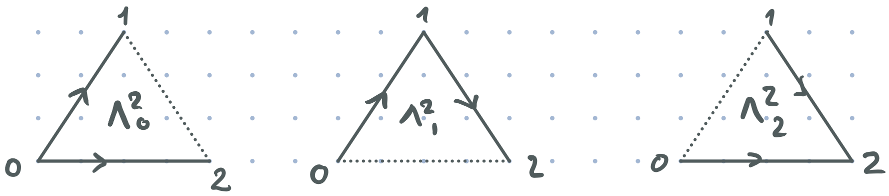
$\mathrm{Fun}(\mathcal{C},\mathcal{D}) \in s\mathrm{Set}$

$\mathrm{Fun}(\mathcal{C},\mathcal{D})_n := \mathrm{Hom}_{s\mathrm{Set}}(X \times \Delta^n, Y)$

$\mathcal{D}$ Kan $\Rightarrow$ $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ Kan

$\mathcal{D}$ $\infty$-cat $\Rightarrow$ $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ $\infty$-cat

where




Corresponds to 1-categories through the following:

**Thm:** The nerve functor $N : \mathrm{Cat}_1 \to s\mathrm{Set}$ is fully faithful with essential image given by those simplicial sets $X$ where the horn fillings are unique.

$$\begin{tikzcd} \mathrm{Cat}_1 \arrow[r, bend right, "N"'] \arrow[r, phantom, "\perp"] & \mathrm{Cat}_\infty \arrow[l, bend right, "\pi"'] \end{tikzcd}$$

$\pi\mathcal{C}$ has ob $= \mathcal{C}_0$, morp $=$ homot. classes of $\mathcal{C}_1$

One of the main differences between 1-categories and $\infty$-categories is that we work with a hom-space instead of set.

1 Categories are exactly the $\infty$-cats with discrete space of morphs

**Def:** $\mathcal{C} \in \mathrm{Cat}_\infty$, $a,b \in \mathcal{C}_0$. Then the hom space is the pullback

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(a,b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1,\mathcal{C}) =: \mathrm{Ar}(\mathcal{C}) \arrow[d, "{(s,t)}"] \\ \Delta^0 \arrow[r, "{(a,b)}"'] & \mathcal{C} \times \mathcal{C} \end{tikzcd}$$

morphisms

When we say 'space' what we mean is an '$\infty$-groupoid' / "anima" which means that $\pi\mathcal{C}$ is a groupoid

every morphisms is an eq.
$\rightsquigarrow$ eq. in $\infty$-cats if $[f]$ is iso in $\pi\mathcal{C}$

$\hookrightarrow$ But just think space!

**Thm:** $\infty$-groupoid iff Kan complex

horn filling for all $0 \le i \le n$

Ex: $\mathcal{D} \in \mathrm{Cat}_1 \rightsquigarrow \iota\mathcal{D} \subseteq \mathcal{D}$ non-full subcategory spanned by isomorphisms called the core. $\iota\mathcal{D}$ is obviously a groupoid. For $\mathcal{C} \in \mathrm{Cat}_\infty$ we define its core as the pullback

$$\begin{tikzcd} \iota\mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{C} \arrow[d] \\ N(\iota(\pi\mathcal{C})) \arrow[r] & N(\pi\mathcal{C}) \end{tikzcd}$$

Then $\iota\mathcal{C}$ is a space, and in fact the largest space contained in $\mathcal{C}$.

Other than as nerve of 1-categories our main source of $\infty$-categories is through the so called coherent nerve:

Construction: Let $\mathcal{C}$ be "simplicially enriched" category, i.e. $\mathrm{Map}_{\mathcal{C}}(a,b) \in \mathrm{sSet}$. The coherent nerve is the simplicial set given by

$$(N_\Delta \mathcal{C})_n := \mathrm{Hom}_{\mathrm{sCat}_1}(C(\Delta^n), \mathcal{C})$$

where $C(\Delta^n)$ is a "thickening"

$C(\Delta^0) = 0$ $\qquad\qquad$ $C(\Delta^1) = 0 \longrightarrow 1$

$$C(\Delta^2) = \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] \arrow[rr, phantom, bend left=20, "\Uparrow"] & & 2 \end{tikzcd}$$

The idea is that we this way encode the homotopies between maps.

Ex:
- If $\mathcal{C} \in \mathrm{sCat}$ s.t. $\forall x,y \in \mathcal{C}$ $\mathrm{Map}_{\mathcal{C}}(x,y)$ is Kan $\Rightarrow N_\Delta \mathcal{C} \in \mathrm{Cat}_\infty$
- $\mathrm{sSet} \in \mathrm{sCat}$:

$$\mathrm{Map}_{\mathrm{sSet}}(X,Y) := \mathrm{Hom}_{\mathrm{sSet}}(\Delta^\bullet \times X, Y) \in \mathrm{sSet}$$

so in particular is $\mathrm{Kan} \subseteq \mathrm{sSet}$, the full subcategory spanned by Kan complexes, simplicially enriched.

<u>Fact</u>: $X \in sSet$, $K \in Kan \Rightarrow Fun(X,K) \in Kan$

$$\mathcal{S} := N_\Delta Kan$$

the $\infty$-category of spaces

- $qCat \subseteq sSet$ full subcategory of quasicategories

$\rightsquigarrow$ $Cat_\infty := N_\Delta qCat$.

We will throughout need to assume $\mathcal{C}$ is stable which means

1) It admits a <u>zero object</u>: $0$ initial + terminal

2) Every morphism $g: X \to Y$ admits a

fibration

$$\begin{tikzcd} ker(g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "g"] \\ 0 \arrow[r] & Y \end{tikzcd}$$

cofibration

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d] \\ 0 \arrow[r] & coker(g) \end{tikzcd}$$

3) Pullback $\Leftrightarrow$ Pushout

Every morphism is the cokernel of its kernel
and the kernel of its cokernel

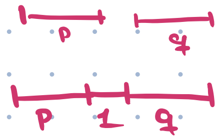
# Arrow categories

**Construction:** Twisted arrow category is the simplicial set

$$\mathrm{TwAr}(\mathcal{C})_n := \mathrm{Hom}_{\mathrm{sSet}}(\underbrace{(\Delta^n)^{op} * \Delta^n}, \mathcal{C})$$

Instead of thinking of $[n]^{op} * [n] = [2n+1]$ as $(0 < 1 < \cdots < 2n+1)$ we think of it as $(0_\ell < 1_\ell < \cdots < n_\ell < (n-1)_r < \cdots < 0_r)$ with $(-)_\ell$ and $(-)_r$ the "left" or "right" part

Join:




$[p] \sqcup [q]$

$\shortparallel$

$[p+1+q]$

of simplicial sets:

$$(X * Y)_n = \coprod_{[n] = [n_1] \sqcup [n_2]} X_{n_1} \times Y_{n_2}$$

**Ex:** $\boxed{\mathrm{TwAr}(\Delta^n)}$

Objects: Arrows in $\Delta^n$

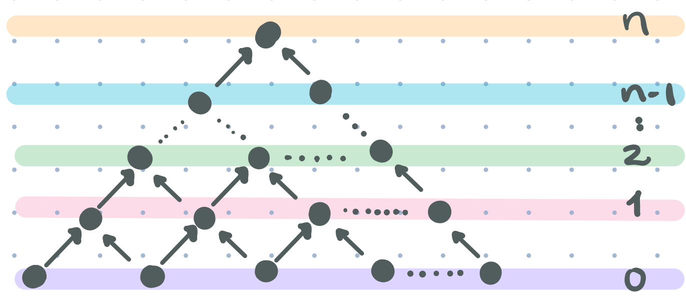
We organize these into an upwards pointing triangle with rows $0, 1, \ldots, n$ from the bottom to the top:

- Row $i$ = arrows that go up by $i$, i.e.

  $(0 \to i), (1 \to i+1), \ldots, (n-i \to n)$

- Row $0$ = All the identities

- Row $n$ = the one map $(0 \to n)$




The rest of the morphisms can then be taken as composites of these

**Def:** The *arrow category* $Ar(\mathcal{C}) := Fun(\Delta^1, \mathcal{C}) \in Cat_\infty$.

**Ex:** $\boxed{Ar(\Delta^n)}$ Objects in $\Delta^1$: $0 \to 1$

morphisms in $\Delta^1$:
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

$$\begin{tikzcd}
0 & 1 & 2 & 3 & & n-1 & n & \\
\bullet \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] & \cdots & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] & 0 \\
& \bullet \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] & \cdots & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] & 1 \\
& & \bullet \arrow[r] & \bullet \arrow[d] & \cdots & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] & 2 \\
& & & \bullet & \cdots & \bullet \arrow[r] & \bullet & 3 \\
& & & & & \vdots & \vdots & \\
& & & & & \bullet \arrow[r] & \bullet \arrow[d] & n-1 \\
& & & & & & \bullet & n
\end{tikzcd}$$

The idea is that we will use $TwAr(\mathcal{C})$ to define the $Q$-construction and $Ar(\mathcal{C})$ the $S_\bullet$-construction for a stable $\infty$-category. As throughout our entire reading course we will start with $S_\bullet$-construction.

**Prop:** $TwAr(\mathcal{C}) \simeq \iota Ar(\mathcal{C})$

## $K_0$ of a stable $\infty$-category

We first see that for a ring $R$, we don't need a projective module to define an object of $K_0(R)$ - A perfect module is sufficient, i.e. $M \in \mathrm{Mod}_R$ which admits a finite projective resolution

$$0 \to P_n \to P_{n-1} \to \dots \to P_0 \to M \to 0.$$

So for such $M$ we can put

$$[M] := \sum_{i=0}^{n} (-1)^i [P_i] \in K_0(R).$$

This is independent of choice of resolution

<u>Def:</u> $\mathcal{K}(R) := N_\Delta \mathrm{Ch}(R)$ the $\infty$-category of chai

$\mathcal{D}^{perf}(R) \subseteq \mathcal{D}(R)$ full sub $\infty$-category spanned by the finite complexes of finite projective $R$-modules

<u>Thm:</u> Sending a perfect complex $(P_n \to P_{n-1} \to \dots \to P_{-m})$ to it's "Euler characteristic" $\sum_{i=-m}^{n} (-1)^i [P_i]$ defines an isomorphism

$\pi_0 : \mathrm{sSet} \rightleftarrows \mathrm{Set} : \mathrm{const}$

$\pi_0 \sim$ set of connected components

$$(\pi_0 \, c \, \mathcal{D}^{perf}(R))/\sim_{ext} \xrightarrow{\sim} K_0(R)$$

where "$\sim_{ext}$" is the equivalence relation generated by

$B \sim_{ext} A \oplus C$ for all fibre sequences

$A \to B \to C$ in $\mathcal{D}^{perf}(R)$.

<u>Def:</u> For a stable $\infty$-category $\mathcal{C}$:

$$K_0(\mathcal{C}) := (\pi_0 c \mathcal{C})/\sim_{ext}$$

where again "$\sim_{ext}$" is the equivalence generated by $b \sim_{ext} a \oplus c$ for all fibre sequences $a \to b \to c$ in $\mathcal{C}$.

So we don't have to group-complete, instead we "split all fibre sequences".

# The $S_\bullet$-construction

<u>Construction:</u> Let $\mathcal{C}$ be a stable $\infty$-category. For all $[n] \in \Delta^{op}$ we let

$$S_n(\mathcal{C}) \subseteq \mathrm{Fun}(\mathrm{Ar}([n]), \mathcal{C})$$

be the full sub-$\infty$-category of those functors $F : \mathrm{Ar}([n]) \to \mathcal{C}$ satisfying

1) $F(i \leq i) = 0 \quad \forall i = 0, \ldots, n$

2) All 'squares' in $\mathrm{Ar}([n])$ go to a pushout/pullback square in $\mathcal{C}$

$$\begin{tikzcd}
0 & 1 & 2 & 3 & n-1 & n & \\
0 \arrow[r] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \bullet \arrow[r, dotted] \arrow[d] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \bullet \arrow[d] & 0 \\
 & 0 \arrow[r] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \bullet \arrow[r, dotted] \arrow[d] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \bullet \arrow[d] & 1 \\
 & & 0 \arrow[r] & \bullet \arrow[r, dotted] \arrow[d] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \bullet \arrow[d] & 2 \\
 & & & 0 \arrow[r, dotted] & \bullet \arrow[r] \arrow[d, dotted] & \bullet \arrow[d, dotted] & 3 \\
 & & & & 0 \arrow[r] & \bullet \arrow[d] & n-1 \\
 & & & & & 0 & n
\end{tikzcd}$$

It can be shown that

$$S_n(\mathcal{C}) \simeq \mathrm{Fun}([n-1], \mathcal{C})$$

which gives us it is stable again. Furthermore, we have that $\mathrm{Fun}(\mathrm{Ar}([n]), \mathcal{C})$ is functorial in both $[n]$ and $\mathcal{C}$, and one can check that the full subcategories $S_n(\mathcal{C})$ are preserved under the face and degeneracy maps in $\Delta^{op}$.

$\rightsquigarrow$ $S(\mathcal{C}) : \Delta^{op} \longrightarrow \underline{\mathrm{Cat}_\infty^{st}}$
stable $\infty$-categories

$\rightsquigarrow$ $S : \mathrm{Cat}_\infty^{st} \longrightarrow \underline{s\mathrm{Cat}_\infty^{st}}$

simplicial stable $\infty$-categories

Unraveling the definition: $S_0(\mathcal{C}) \simeq *$, $S_1(\mathcal{C}) \simeq \mathcal{C}$, $S_2(\mathcal{C}) \simeq \mathrm{Ar}(\mathcal{C})$ but we mainly think of a typical element of $S_2(\mathcal{C})$ as a diagram

$$\begin{tikzcd} 0 \arrow[r] & a \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & b \arrow[d] \\ & 0 \arrow[r] & a/b \arrow[d] \\ & & 0 \end{tikzcd}$$

rather than an arrow $a \to b$. We have the face maps

$$S_2(\mathcal{C}) \simeq \mathrm{Ar}(\mathcal{C}) \longrightarrow S_1(\mathcal{C}) \simeq \mathcal{C}$$

$$\begin{tikzcd} 0 \arrow[r] & a \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & b \arrow[d] \\ & 0 \arrow[r] & a/b \arrow[d] \\ & & 0 \end{tikzcd} \quad \begin{array}{l} \xmapsto{d_0} a/b \\ \xmapsto{d_1} b \\ \xmapsto{d_2} a \end{array}$$

Def: For a stable $\infty$-category $\mathcal{C}$ we define the algebraic K-theory space

$$k\mathcal{C} := \Omega|\iota S(\mathcal{C})|$$

$K\mathcal{C}$ is reserved for the spectrum version

Thm: There are canonical isomorphisms

$$K_0(\mathcal{C}) \simeq \pi_0 k(\mathcal{C}) \simeq \pi_1(|\iota \mathcal{C}|).$$

It can be shown that $k(\mathcal{C})$ admits the structure of a commutative groupoid structure over $\mathcal{S}$ functor $X: N(\mathrm{Fin}_*)^{op} \to \mathcal{S}$ s.t. $X \circ \mathrm{Cut}: \Delta^{op} \to \mathcal{S}$ is a Cartesian group

$\hookrightarrow$ i.e. it is an $E_\infty$-group since $\pi_0 k(\mathcal{C}) \simeq K_0(\mathcal{C})$ is an ordinary abelian group

$\rightsquigarrow$ We obtain a functor

$$\boxed{k: \mathrm{Cat}_\infty^{st} \longrightarrow \mathrm{CGrp}(\mathcal{S})}$$

To understand this better we need to compare it to an $\infty$-categorical version of the Q-construction.

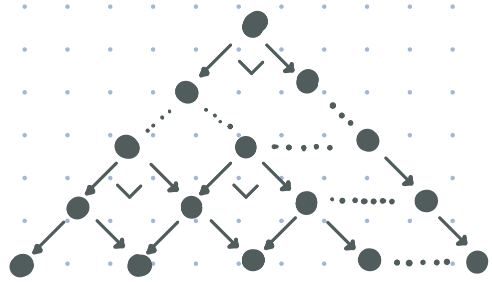
# Q - Construction

Note: We don't <u>need</u> to assume stable

**Construction:** Let $\mathcal{C}$ be any $\infty$-category and let

$$Q_n(\mathcal{C}) \subseteq \mathrm{Fun}(\mathrm{TwAr}([n])^{op}, \mathcal{C})$$

be the full subcategory spanned by those functors which takes every "square" in $\mathrm{TwAr}([n])$ to pullback squares in $\mathcal{C}$:



The $Q_n(\mathcal{C})$ assemble into a simplicial $\infty$-category

$$Q(\mathcal{C}) : \Delta^{op} \longrightarrow \mathrm{Cat}_\infty \qquad \text{Quillen Q-construction}$$

To see this we first note that

$$\mathrm{Fun}(\mathrm{TwAr}(-)^{op}, \mathcal{C}) : \Delta^{op} \longrightarrow \mathrm{Cat}_\infty$$

gives a functor, since

$$\mathrm{TwAr}(-) : \mathrm{sSet} \longrightarrow \mathrm{sSet}$$

is the right adjoint to

$$(-)^{op} * (-) : \mathrm{sSet} \longrightarrow \mathrm{sSet}.$$

One then has to prove that all the boundary and degeneracy maps preserves the full subcategory $Q_n(\mathcal{C})$.

This looks alot like a collection of 'spans', which explains the following

**Thm:** Let $\mathcal{C}$ be a stable $\infty$-category. The simplicial space

$$\iota Q(\mathcal{C}) : \Delta^{op} \longrightarrow \mathcal{S}$$

is a 'complete Segal space', which means we can 'straighten' it to a functor

$$\mathrm{Span}(\mathcal{C}) \longrightarrow \Delta^{op}$$ $\infty$-category of spans in $\mathcal{C}$

with fibers over $[n]$ being $Q_n(\mathcal{C})$.

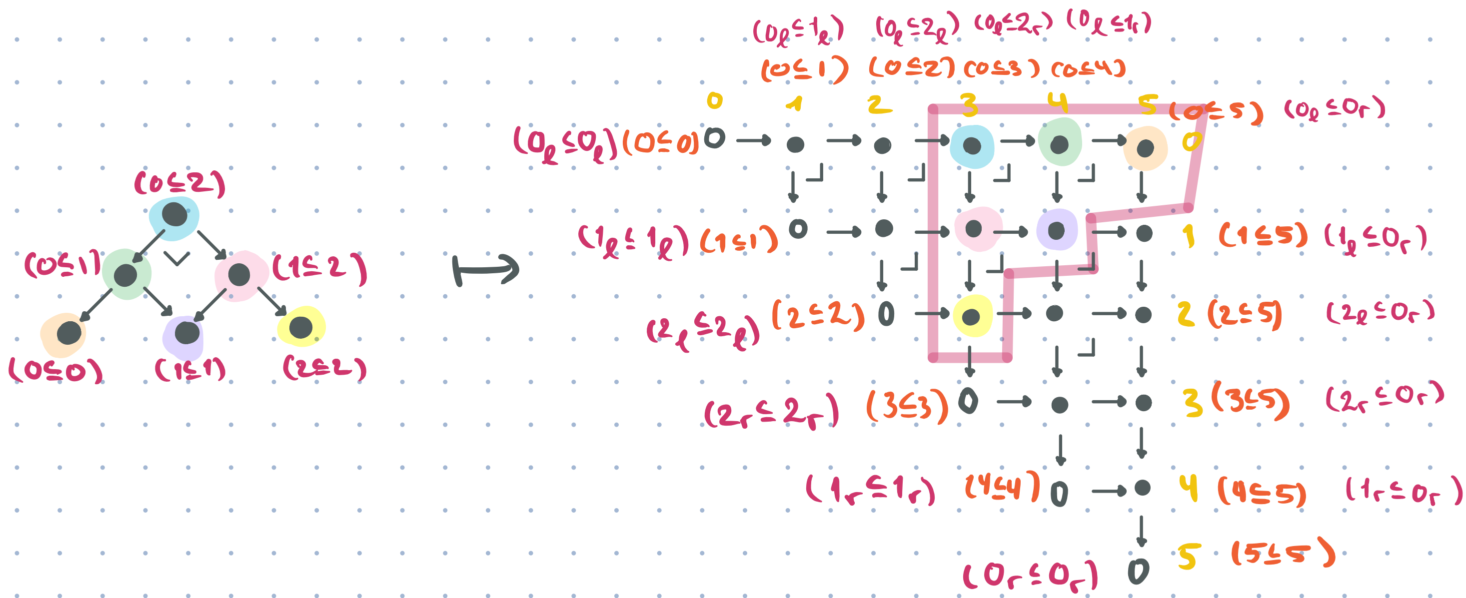
Fact: $|\iota Q\mathcal{C}| \simeq |\mathrm{Span}\,\mathcal{C}|$

Comparison theorem We have a functor

$$\mathrm{TwAr}([n]^{op}) \longrightarrow \mathrm{Ar}([n] * [n]^{op})$$
$$(i \le j) \longmapsto (i_\ell, j_r)$$

Case $n=2$:

$$\mathrm{TwAr}([2]^{op}) \longrightarrow \mathrm{Ar}([2]*[2]^{op}) \simeq \mathrm{Ar}([5])$$




We have a functor

$$\Delta^{op} \longrightarrow \Delta^{op} \rightsquigarrow (-)^{esd} : s\mathcal{S} \longrightarrow s\mathcal{S}$$ Edgewise subdivision
$$[n] \longmapsto [n]*[n]^{op}$$

Thm: The map $\mathrm{TwAr}([n])^{op} \to \mathrm{Ar}([n]*[n]^{op})$ restricts, by taking $\mathrm{Fun}(-,\mathcal{C})$, to a map

contravariant

$$S(\mathcal{C})^{esd} \xrightarrow{\sim} Q(\mathcal{C})$$

So we get

$$|\iota S\mathcal{C}| \simeq |\iota S\mathcal{C}^{esd}| \simeq |\iota Q(\mathcal{C})| \simeq |\mathrm{Span}(\mathcal{C})|$$

we get that

$$\boxed{k(\mathcal{C}) \simeq \Omega|\mathrm{Span}(\mathcal{C})|}$$

## Results

We have a functor

$$\iota\mathcal{C} \simeq \mathrm{Hom}_{\mathrm{Span}(\mathcal{C})}(0,0) \longrightarrow \mathrm{Hom}_{|\mathrm{Span}(\mathcal{C})|}(0,0) \simeq \Omega|\mathrm{Span}(\mathcal{C})| \simeq k(\mathcal{C})$$

Using $k(\mathcal{C})$ is an $\mathbb{E}_\infty$-group this map factors over an $\mathbb{E}_\infty$-group map

$$(\iota\mathcal{C})^{\infty\text{-grp}} \to k(\mathcal{C})$$

which through the inclusion $\mathrm{Proj}(R) \subseteq \iota\mathcal{D}^{\mathrm{perf}}(R)$ in particular gives a map

$$k(R) \simeq \mathrm{Proj}(R)^{\infty\text{-grp}} \to (\iota\mathcal{D}^{\mathrm{perf}}(R))^{\infty\text{-grp}} \longrightarrow k(\mathcal{D}^{\mathrm{perf}}(R))$$

which turns out to be an equivalence.

- Holds in higher generality:

Thm: If $\mathcal{C}$ is a stable $\infty$-category with an "exhausted weight structure", then

consists of 2 full subcategories $\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}$
- but not the same as a t-structure

$$\iota(\mathcal{C}^{\heartsuit})^{\infty\text{-grp}} \simeq k(\mathcal{C}).$$

'Resolution theorem'

## Universal Additive invariant

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Additive & factors over $\mathrm{CGrp}(\mathcal{S})$

Thm: $k : \mathrm{Cat}_\infty^{\mathrm{st}} \longrightarrow \mathcal{S}$ is the initial group-like functor which admits a natural transformation $\iota(-) \Rightarrow k : \mathrm{Cat}_\infty^{\mathrm{st}} \longrightarrow \mathrm{CMon}(\mathcal{S})$

More precisely:

The inclusion

$$\mathrm{Fun}^{\mathrm{grp}}(\mathrm{Cat}_\infty^{\mathrm{st}}, \mathrm{CGrp}(\mathcal{S})) \subseteq \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}_\infty^{\mathrm{st}}, \mathcal{S})$$

Additive: $F(0) =$ final object
& (split) Verdier squares $\mapsto$ Pullback

admits a left adjoint

$$(-)^{\mathrm{grp}} \simeq \Omega|\mathrm{Span}^{(-)}(-)|$$

and

$$k \simeq \iota^{\mathrm{grp}}.$$

Note: $\iota$ is additive but not group-like

Additivity theorem: $F : \mathrm{Cat}_\infty^{st} \longrightarrow \mathcal{S}$ grouplike $\Rightarrow F(\mathrm{Ar}(\mathcal{C})) \simeq F(\mathcal{C}) \times F(\mathcal{C})$

$\rightsquigarrow \quad k(\mathrm{Ar}(\mathcal{C})) \simeq k(\mathcal{C}) \times k(\mathcal{C})$