§1: Motivation

Homotopy groups of spheres = HARD!

$\sim$ $\pi_n S^m = 0 \quad n<m$

$\pi_{i+n} S^n = \mathrm{Map}(S^{i+n}, S^n)$ only depends on n for large n, so it stabilizes

$\sim$ Try to understand just the stable part:

stable homotopy groups of spheres: $\pi_i^s = \lim_n \pi_{i+n} S^n$

The sphere spectrum captures these:

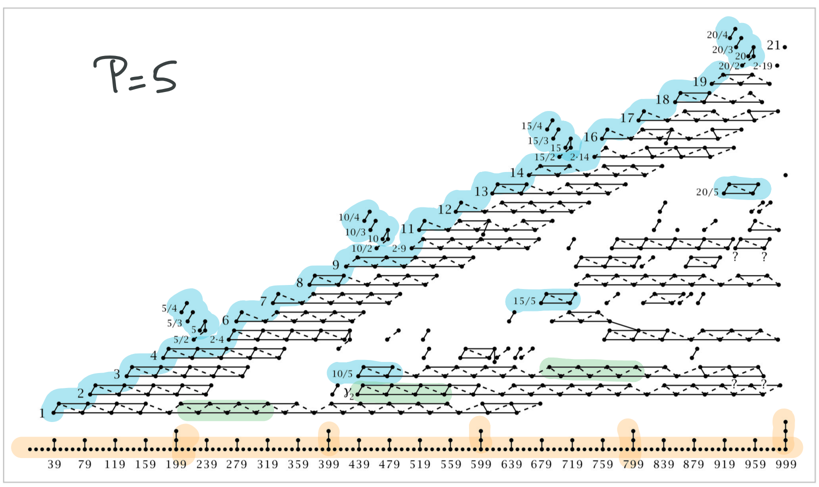
$$\pi_i \mathbb{S} = \pi_i^s = \lim_n \pi_{i+n} S^n$$

<u>Serre:</u> $\pi_n \mathbb{S} = \begin{cases} \mathbb{Z} & n=0 \\ \text{finite ab. groups} & n>0 \end{cases}$

Maybe enough to calculate one torsion at a time?

$\pi_\bullet \mathbb{S}$ — $\pi\mathbb{S}_2$, $\pi\mathbb{S}_3$, ..., $\pi\mathbb{S}_p$, ...

where $\pi\mathbb{S}_p$ describes the p-primary part



$\pi_\bullet \mathbb{S}_{(5)}$

$\bullet$ $\mathbb{Z}_5$

$\mathbb{Z}_{5^2}$

$\mathbb{Z}_{5^3}$

1st chromatic Layer - closely related to K-theory

- Period between each dot $= 2(p-1) = 8$

$\sim v_1$-periodic $=$ image of J

2nd chromatic Layer -

- v_2-periodic $\sim 2(p^2-1) = 48$

3rd Chromatic Layer -

- v_3-periodic $\sim$ $2(p^3-1)=248$

$\sim$ chromatic filtration:

$$\begin{tikzcd}
 & \pi_* S \arrow[dl, dash] \arrow[d, dash] \arrow[drr, dash] & & \\
\pi_* S_2 \arrow[d, dash] & \pi_* S_3 \arrow[d, dash] & \cdots & \pi_* S_p \arrow[d, dash] & \cdots \\
\vdots \arrow[d, dash] & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] \\
\pi_* S_{2,n} \arrow[d, dash] & \pi_* S_{3,n} \arrow[d, dash] & & \pi_* S_{p,n} \arrow[d, dash] \\
\vdots \arrow[d, dash] & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] \\
\pi_* S_{2,2} \arrow[d, dash] & \pi_* S_{3,2} \arrow[d, dash] & & \pi_* S_{p,2} \arrow[d, dash] \\
\pi_* S_{2,1} & \pi_* S_{3,1} & & \pi_* S_{p,1}
\end{tikzcd}$$

Have convergence theorems!

<u>Idea</u>: Fix p, construct spectra $K(0), K(1), K(2), \ldots$ called Morava K-theories

Define

$$L_n X := L_{K(0)\vee\cdots\vee K(n)} X$$

It's a result by Hopkins-Ravenel that

$$S_p \simeq \operatorname{holim}_n \underbrace{L_n S^0}_{n^{\text{th}}\text{ chromatic layer}}$$

So want to study S_p through this filtration.

§ Spectra

Def: An (Ω-) Spectrum E:

- Sequence of based spaces E_n, $n \geq 0$
- Weak equivalences
$$\omega_n : E_n \xrightarrow{\sim} \Omega E_{n+1}$$

Morphisms: $f: E \to F = \{f_n : E_n \to F_n\}$ s.t.

$$\begin{tikzcd} E_n \arrow[r, "f_n"] \arrow[d, "\sim"'] \arrow[dr, phantom, "\circlearrowleft"] & F_n \arrow[d, "\sim"] \\ \Omega E_{n+1} \arrow[r, "\Omega f_{n+1}"'] & \Omega F_{n+1} \end{tikzcd}$$

$\rightsquigarrow$ Sp

so not assuming w.eq

Def: Spectification of $E = \{E_n : n \geq 0\}$ together w. inclusions $\omega_n : E_n \hookrightarrow \Omega E_{n+1}$

$$\mathbb{L}E_n = \varinjlim_k \Omega^k E_{n+k} \quad , \quad \mathbb{L}\omega_n = \varinjlim_k \Omega^k \omega_{n+k}$$

is a spectrum ($Sp(E)$)

E.g.: $Sp(E)_0 = \mathbb{L}E_0 = \mathrm{colim}(E_0 \to \Omega E_1 \hookrightarrow \Omega^2 E_2 \hookrightarrow \cdots)$

The idea of a spectra comes from cohomology theories due to Brown representability theorem:

If E is a cohomology theory, then there exists a spectrum (E_n, ω_n) s.t.
$$\tilde{E}^n(X) \cong \underbrace{[X, E_n]}_{\text{Based hmpty classes of maps}} \sim E_n \text{ represents the (reduced) cohom. thy.}$$

Ex

- Complex K-theory: Bott periodicity gives

$U = \lim U(n)$
$U(n) = n\times n$ unitary matrices
$BU = \lim(BU(n) \hookrightarrow BU(n+1) \hookrightarrow \cdots)$

$$\Omega U = \mathbb{Z} \times BU \quad , \quad \Omega(\mathbb{Z} \times BU) \simeq U$$
$\leftarrow$ unitary group

Complex K-theory $K^*(-)$ is represented by
$$K = \{\mathbb{Z} \times BU, U, \mathbb{Z} \times BU, U, \ldots\}$$

- Suspension spectra: "Think of based spaces as living in spectra"

$X \in Top_*$. Then $\Sigma^\infty X$ is the spectification of

$(\Sigma^\infty X)_n = \Sigma^n X$, $\omega_n : \Sigma^n X \to \Omega \Sigma^{n+1} X$

$\Sigma X = X \wedge S^1$
$\Omega X = \mathrm{Maps}(S^1, X)$

The adjoint to the identity
$\sigma_n : \Sigma \Sigma^n X \xrightarrow{=} \Sigma^{n+1} X$

$\rightsquigarrow$ Sphere spectrum $\mathbb{S} = \Sigma^\infty S^0$

Construction Complex Cobordism MU is the spectrification of

$$\{\overset{0}{MU(1)}, \overset{1}{\Sigma MU(1)}, \overset{2}{MU(2)}, \overset{3}{\Sigma MU(2)}, \dots\}$$

- $MU(n)$ = Thom space of the canonical complex vector bundle

one-pt. compactification
(all points at ∞ are sent to one point)

$\gamma_n \longrightarrow BU(n)$ $(\simeq Gr_n(\mathbb{C}^\infty))$
$P \longmapsto [P]$ n-plane inside $\mathbb{C}^\infty$
pt. = planes
fibers = vectors in that plane

- Consider the following pullback:

Tautological v.b.
$\gamma_n := \{(V,x) \in BU(n) \times \mathbb{C}^\infty \mid x \in V \subseteq \mathbb{C}^\infty\}$

$$\begin{tikzcd}
(\gamma_n \oplus \overset{\mathbb{R}^2}{\overset{\shortparallel}{\mathbb{C}}} \cong) \; i^*\gamma_{n+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \gamma_{n+1} \arrow[d] \\
BU(n) \arrow[r, "i"] & BU(n+1)
\end{tikzcd}$$

trivial complex line bundle

$\rightsquigarrow$ Induces a map on Thom spaces

$$\Sigma^2 MU(n) \cong Th(i^*\gamma_{n+1}) \xrightarrow{\sigma_{2n+1}} Th(\gamma_{n+1}) = MU(n+1)$$

since adding a trivial bundle means you suspend w. the dim.

- $\omega_{2n+1} : \Sigma MU(2n+1) \to MU(2n)$ is adjoint to σ_{2n+1} / to get the loop, adjoin a factor
- $\omega_{2n} : MU(n) \to \Omega \Sigma MU(n)$ is adjoint to the identity

$$\boxed{\begin{aligned} &\omega_n : E_n \to \Omega E_{n+1} \\ &\omega_{2n+1} : \Sigma MU(2n+1) \longrightarrow \Omega MU(2n+2) \\ &\quad \Sigma^2 MU(2n+1) \longrightarrow MU(2n+2) \end{aligned}}$$

§3 : Stable homotopy Category

<u>Def:</u> $\mathcal{C}$ a category, $\mathcal{W} \subseteq \mathcal{C}$

- $\mathrm{iso}\,\mathcal{C} \subseteq \mathcal{W}$
- 2-out-of-3 of $\{f, g, g\circ f\}$ in $\mathcal{W}$, then so is the third

The <u>homotopy category</u> of $\mathcal{C}$ (if it exists!) is

- Category $\mathrm{Ho}(\mathcal{C},\mathcal{W})$
- $\iota : \mathcal{C} \to \mathrm{Ho}(\mathcal{C},\mathcal{W})$ which maps all maps in $\mathcal{W}$ to isomorphisms

  and it's the initial such map, i.e.:

$$\mathcal{W} \longmapsto \text{isos}$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\iota"'] \arrow[dr, phantom, "\Downarrow\cong"] & \mathcal{D} \\ \mathrm{Ho}(\mathcal{C},\mathcal{W}) \arrow[ur, "\exists!"'] & \end{tikzcd}$$

<u>Ex:</u> <u>Stable Homotopy Category</u>: $SH = \mathrm{Ho}(Sp, \mathcal{W}_0)$ where

$\mathcal{W}_0$ = weak-equivalences (= π_*-isos)

$$\boxed{\pi_r E = \varinjlim_n \pi_{r+n} E_n}$$

Several nice properties:

- $[X,Y] := SH(X,Y) \in Ab$
- Finite products and coproducts are equivalent

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner \ \ulcorner"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

- Closed symmetric monoidal with unit S^0:

$$\underset{(\otimes)}{- \wedge -} : SH \times SH \longrightarrow SH \quad , \quad F(-,-) : SH^{op} \times SH \to SH$$

$$F(X \wedge Y, Z) \cong F(X, F(Y,Z))$$

$(\Sigma^\infty X \wedge E)_n$
$=$ "$X \wedge E_n$ + spectrify"

- Triangulated

<u>Note</u>: There exists other models for spectra

We know that having a cohomology theory we get a spectrum, but we can also go the other way.

<u>Def:</u> $E \in Sp$, <u>E-homology</u> $\tilde{E}_* : SH \to Ab$ given by

$$\tilde{E}_n : SH \longrightarrow Ab$$
$$X \longmapsto \pi_n(E \wedge X)$$

<u>E-cohomology</u> $\tilde{E}^* : SH^{op} \to Ab$ given by

$$\tilde{E}^n : SH^{op} \longrightarrow Ab$$
$$X \longmapsto \pi_{-n} F(X,E) = [X, \Sigma^n E]$$

in SH

functions

Unreduced versions:

$$E_n := \tilde{E}_n(\$) = \pi_n E = \tilde{E}^{-n}(\$) =: E^{-n}$$

Ex: For $E = \$$, then the stable homotopy groups are the E-homology:

$$\tilde{\$}_*(X) \cong \pi_*(X) \quad (\text{For } X \text{ a space: } \pi_r^S X = \operatorname{colim}_{n+r} \pi_n X)$$

Weak equivalences in Sp are $\tilde{\$}_*$-isos and SH is obtained from Sp by inverting these.

Q: what about inverting other maps?

§4: Bousfield Localization

<u>Def:</u> • $f: X \to Y$ in SH is an <u>E-equivalence</u> if
$$\tilde{E}_*(f): \tilde{E}_*(X) \xrightarrow{\cong} \tilde{E}_*(Y)$$

• $W_E \subseteq Sp$ full subcategory of E-equivalences:
"E-local" $SH_E := Ho(Sp, W_E)$

• X is <u>E-acyclic</u> if $\tilde{E}_*(X) = 0$

• Y is <u>E-local</u> if $[X, Y] = 0 \quad \forall X$ E-acyclic

} "orthogonal to eachother in SH"

Write $Sp_E \subseteq Sp$ full subcat. of E-local spectra.

<u>Fact:</u> • $SH_E \cong Ho(Sp_E, \text{weak. eq. in } Sp_E)$

• $f: X \to Y$ in Sp_E is a weak eq. $\Leftrightarrow$ it is an E-equivalence

<u>Construction:</u> $\boxed{\text{Bousfield localization}}$ Let $E \in Sp$, $X \in SH$

• An <u>E-localization</u> is an E-equivalence $\eta: X \xrightarrow{E_*\text{-iso}} \underbrace{L_E X}_{\in Sp_E}$

It's a result by Bousfield that these exists and are unique in SH, so we get a functor:
$$L_E: SH \longrightarrow SH_E \quad + \quad \eta: \overset{\text{identity}}{1_{SH}} \Rightarrow L_E$$
(This is left adj. to the inclusion)

<u>Bousfield localization</u> w.r.t. E

It has the following universal property: (so initial)

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta"'] & Y \in Sp_E \\ L_E X \arrow[ur, dashed, "\exists! f_E \in SH"'] & \end{tikzcd}$$

<u>Fact:</u> There is a distinguished triangle
$$C_E X \to X \xrightarrow{\eta_X} L_E X \longrightarrow \Sigma C_E X$$
(so sits in a cofiber seq. $C_E X \to X \to L_E X$

where $C_E X$ is the terminal E_*-acyclic spectrum with a map to X.

We use this for p-localization and -completion

<u>Ex:</u> Let $E = \mathbb{S}_{(p)}$ <u>p-local sphere</u>, which is the spectrum representing the homology theory
$$X \mapsto \pi_* X \otimes_{\mathbb{Z}} \overset{p\text{ local}}{\mathbb{Z}_{(p)}}.$$

$\rightsquigarrow$ <u>p-local spectra</u> $=$ E-local spectra $\rightsquigarrow X_{(p)} := L_{\mathbb{S}_{(p)}} X$

<u>Fact:</u> For any spectrum X: $\pi_* L_{\mathbb{S}_{(p)}} X \cong \pi_* X \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$

$\rightsquigarrow$ In particular: A spectrum is p-local iff it's homotopy groups are

<u>Ex:</u> Let $E = \mathbb{S}/p$ be the mod-p sphere.

$\rightsquigarrow$ <u>p-complete</u> = E-local $\rightsquigarrow X_p := L_{\mathbb{S}/p} X$

inverse limit $= \varprojlim_i \mathbb{Z}/p^i$

<u>Fact:</u> $L_{\mathbb{S}/p} X \simeq \varprojlim X/p^n \rightsquigarrow \pi_* L_{\mathbb{S}/p} X \simeq \pi_* X \otimes \mathbb{Z}_p$

so there is a connection between the homotopy groups of p-local and p-complete spectra.

The Eilenberg-MacLane space

<u>Ex:</u> $S^1 = K(\mathbb{Z}, 1) \in Sp$

$\rightsquigarrow S^1_p = K(\mathbb{Z}_p, 1)$

$S^1_{(p)} = K(\mathbb{Z}_{(p)}, 1)$

§2: Complex oriented Cohomology theories

Paul Pantea 18.01.23

G group, BG = classifying space of G

↳ Classifies principal G-bundles

$BG = |NBG|$

* one object category, $\mathrm{Hom}_{BG}(*,*) \cong G$

$EG := |EG_\cdot|$: Contractible free G-space

$EG_n := G \times G \times \dots \times G$ simplicial space

This has a G-action, and from this POU.

$$BG = EG/G$$

We obtain the universal G-bundle

$$EG \longrightarrow BG,$$

i.e. $\forall$ principal G-bundles $V \to X$ arises as the pullback of $EG \to BG$

$$\begin{tikzcd} f^*\pi = V \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & EG \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & BG \end{tikzcd} \qquad [X, BG] \cong \{\text{principal } G\text{-bundle over } X\}/\text{iso}$$

<u>Statement</u>:
- $G = O(n) \rightsquigarrow$ principal G-bundle = real n-dimensional vector bundle ($\simeq GL(n)$)
- $G = U(n) \rightsquigarrow$ principal G-bundle = complex n-dimensional vector bundle

<u>Why care</u>: M manifold $\rightsquigarrow$ TM is a real vector bundle

In particular, if we consider $G = U(1) = S^1$ then

$$[X, BU(1)] \cong [X, \mathbb{CP}^\infty] \cong [X, K(\mathbb{Z},2)] \cong H^2(X;\mathbb{Z})$$

Once you have picked the isomorphisms $\Rightarrow$ complex line bundle $V \to X$ over X

$$\rightsquigarrow c_1(V) \in H^2(X,\mathbb{Z}) \quad \text{first chern class}$$

When we have chosen such for the universal bundle $H^*(BU(1);\mathbb{Z}) \cong \mathbb{Z}[t]$, $|t| = 2$

it is easy to generalise using the pullback

$$\begin{tikzcd} V \arrow[r] \arrow[d] & \mathcal{O}(1) = \gamma_1 \arrow[d] \\ X \arrow[r, "f"'] & BU(1) \end{tikzcd} \qquad \begin{aligned} f^* H^2(BU(1);\mathbb{Z}) &\longrightarrow H^2(X;\mathbb{Z}) \\ t &\longmapsto f^*(t) = c_1(V) \end{aligned}$$

What about higher dimensional?

e.g. $s_1 = t_1 + \cdots + t_n$, $s_2 = t_1t_2 + t_1t_3 + \cdots + t_{n-1}t_n$, $s_n = s_1 \cdots s_n$

Prop: $H^*(BU(n);\mathbb{Z}) \cong \mathbb{Z}[s_1,\dots,s_n]$ where s_i = fundemental symmetric polynomial in $\underbrace{t_1,\dots,t_n}_{c_1(\pi_i^* \gamma_n)}$ where

$$\begin{array}{c} \gamma_1 \\ \downarrow \\ BU(1) \end{array} \quad \leftarrow \text{universal bundle}$$

$$\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty \xrightarrow[\pi_i]{} \mathbb{C}P^\infty$$

Def: $c_i(\gamma_n) := s_i$ and for $V \to X$ an n-dimensional complex vector bundle, classified by $X \xrightarrow{f} BU(n)$, we extend this to

$$c_i(V) = f^*(s_i) \qquad i^{\text{th}} \text{ Chern class}$$

The inclusion $BU(n) \hookrightarrow BU(n+1)$ makes Chern classes compatible.

We write

$$c(V) := c_1(V) + c_2(V) + \cdots + c_n(V) + \cdots$$

Prop:
- $c(V \oplus W) = c(V) \cdot c(V)$
- $c(\mathcal{O}(1)) = 1 + t$

Now, consider the multiplication map

$$\mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{m} \mathbb{C}P^\infty$$

show

$$\boxed{c(\mathcal{L}^1 \otimes \mathcal{L}^2) = c(\mathcal{L}^1) + c(\mathcal{L}^2)} \qquad \mathcal{L} = \text{line bundle}$$

So why did this work? It boils down to the fact that

$$H^*(\mathbb{C}P^\infty;\mathbb{Z}) = \mathbb{Z}[[t]]$$
$$H^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[[t_1,t_2]]$$

Def: Complex oriented cohomology theory E is a multiplicative cohomology theory such that

$$\tilde{E}^2(\mathbb{C}P^\infty) \longrightarrow \tilde{E}^2(\mathbb{C}P^1) \cong \tilde{E}^2(S^2) \cong E^0(pt) \ni 1$$

is surjective.

A complex orientation is such a lift $x \in \tilde{E}^2(\mathbb{C}P^\infty)$.

Atiyah-Hirzebruch spectral sequence tells us that

$$E_2^{p,q} = H^p(\mathbb{C}P^\infty; E^q(pt)) \Longrightarrow E^{p+q}(\mathbb{C}P^\infty)$$

Lurie says that a cohomology theory is complex oriented iff AHSS collapses at the E_2-page for $\mathbb{CP}^\infty$.

Prop: If E is an even cohomology theory $\Rightarrow$ AHSS collapses

$$\Rightarrow E^*(\mathbb{CP}^\infty) \cong E^*(pt.)[[t]]$$

Ex: • Complex K-theory is even: $KU^*(\mathbb{CP}^\infty) \cong \mathbb{Z}[x^{\pm 1}][[t]]$

• MU is also even: $MU^* \cong \mathbb{Z}[x_1, x_2, \ldots]$, $|x_i| = 2i$

Which to consider this on the tensor of line bundles:

$$c^E(\mathcal{L}^1 \otimes \mathcal{L}^2) = F_E(c(\mathcal{L}^1), c(\mathcal{L}^2))$$

$\rightsquigarrow$ Formal group laws

§4 Landweber exact functor theorem

Thomas Read
01.02.23

Complex oriented cohomology theory E

$\Downarrow$

1^{st} E-Chern class for line bundles

$\Downarrow$

Formal group laws

Today - how do we go the other direction?

?

$\Uparrow$

Ring hom. $MU_* \cong L \to R$

$\Uparrow$

FGL over R

The idea for our approach comes from the following theorem:

Thm: [Conner-Floyd]

$$KU_*(X) \cong MU_*(X) \otimes_{MU_*} KU_*$$

Generalize this:

$$E_*(X) := MU_*(X) \otimes_{MU_*} R$$

Q: Is this a homology theory?

Since MU_* is already such, most axioms holds trivially. "All" we are missing to check is exactness:

$A \to B \to C$ cofiber sequence of spaces

Want: $MU_*(A) \otimes_{MU_*} R \to \cdots$ exact would be nice if flat... But not so interesting

We have that

$$0 \to MU_*(A) \to MU_*(B) \to MU_*(C) \to 0$$

is an exact sequence of $(MU_*, MU_*(MU))$-comodules

The Landweber exact functor theorem gives us a condition for when

$$(-) \otimes_{MU_*} R : \text{comod}(MU_*MU) \to Ab$$

exact.

Note: Might be a cohomology theory without this being exact - so does not give us a complete description.

Recall: Given an R-module M, a sequence $r_1, \dots, r_d \in R$ is regular for M if

$$M/(r_1,\dots,r_{k-1}) \xrightarrow{\cdot r_k} M/(r_1,\dots,r_{k+1})$$

"nice filtration" of the module

is injective for $1 \leq k \leq d$

Thm: | Landweber Exact functor theorem | For p prime, let $v_i \in MU_*$ be the coefficient of x^{p^i} in $[p](x)$ for the universal formal group law.

↳ multiplication by p map

Further let M be an MU_*-module. Then

$$(-) \otimes_{MU_*} M : \mathrm{Comod}(MU_* MU) \to Ab$$

is exact iff $\forall$ primes p, $p, v_1, v_2, \dots$ is regular for M.

Cor: Let F be a FGL over R corresponding to

$$\theta : MU_* \to R.$$

Further let $v_i \in R$ be the coefficients of x^{p^i} in $[p]_F(x)$. Then $(-) \otimes_{MU_*} R$ is exact iff $\forall\, p$, $p, v_1, v_2, \dots$ is regular for M.

Variation in the litterature: Defining v_i using the Lazard ring in the following way:

$$MU_* \cong \mathbb{Z}[t_1, t_2, \dots] \ , \ v_i \equiv -t_{p^i-1} \mod p$$

So when our FGL satisfies this, we get a cohomology theory, and one can show that by going back from cohomology theories to FGLs, we will recover the original FGL.

Ex: | $F_{H\mathbb{Z}}(x,y) = x+y$ |

$[p](x) = px \rightsquigarrow v_i = 0$ for $i \geq 1$

$\rightsquigarrow \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z}$ injective ✓

$\mathbb{Z}/p \xrightarrow{\cdot 0} \mathbb{Z}/p$ not injective ✗ $\rightsquigarrow$ Not Landweber exact

Ex: | $F_{H\mathbb{Q}}(x,y) = x+y$ | "Fixes" the above:

$\mathbb{Q} \xrightarrow{\cdot p} \mathbb{Q}$ injective ✓

$0 = \mathbb{Q}/p\mathbb{Q} \xrightarrow{\cdot p} \mathbb{Q}/p\mathbb{Q} = 0$ injective ✓ $\rightsquigarrow$ Landweber exact.

Ex: $\boxed{F_{KU}(x,y) = x+y+\beta xy}$ $KU_* = \mathbb{Z}[\beta^{\pm 1}]$, $|\beta| = 2$

$v_1 = \beta^{p-1}$, $v_i = 0$ for $i>1$.

$\mathbb{Z}[\beta^{\pm 1}] \xrightarrow{\cdot p} \mathbb{Z}[\beta^{\pm 1}]$ injective ✓

$\rightsquigarrow$ Landweber exact

$\mathbb{Z}[\beta^{\pm 1}]/p \xrightarrow{\cdot \beta^{p^i}} \mathbb{Z}[\beta^{\pm 1}]/p$ injective ✓

Rest will just become $0 \xrightarrow{\cdot 0} 0$ injective

Construction: $BP_* = (MU_*)_{(p)} / (t_i \mid i \neq p^n - 1)$

$\cong \underline{\mathbb{Z}_{(p)}[v_1, v_2, \dots]} \in \text{comod}(MU_* MU)$

v_i instead of t_i, since the image of the generators of MU_* is $t_{p^i-1} \simeq v_i$.

$\sim$ These are the v_i's we care about in the Landweber exact functor theorem

For $q \neq p$, q is a unit

$BP_* \xrightarrow{\cdot p} BP_*$ injective

$\rightsquigarrow$ Landweber exact

$\mathbb{Z}_{(p)}[v_n, v_{n-1}, \dots]/p \xrightarrow{\cdot v_n} \mathbb{Z}_{(p)}[v_n, \dots]/p$ injective

So this gives rise to the spectrum BP.

Fact: Plays the role of MU in the setting of p-typical: For R an $\mathbb{Z}_{(p)}$-algebra, p-typical FGLs correspond to $BP_* \longrightarrow R$.

Construction: Johnson-Wilson theory $\forall n, p$

$E(n)_* := BP_*[v_n^{-1}]/(v_{n+1}, v_{n+2}, \dots)$ using that this is Landweber exact

$\rightsquigarrow$ $K(n)_* = E(n)_* / (p, v_1, \dots, v_{n-1}) \cong \mathbb{F}_p[v_n^{\pm 1}]$ $\underline{\underline{\text{not}}}$ exact.

Fix a perfect field k of characteristic p and FGL $f(x,y) \in k[\![x,y]\!]$ of height n over k.

Wish to understand which FGL's are in some sense "close" to f.

extension of k "infinitesimal thickening"

<u>Def:</u> A deformation of f over a complete local ring A equipped with a surjection $\varphi: A \longrightarrow A/m_A$, where m_A is a maximal ideal of A, is a pair (f_A, i):

$$f_A \in FGL(A), \quad i: k \longrightarrow A/m_A$$

such that

$$i^* f = \varphi^* f_A$$

$$\begin{tikzcd} f \ FGL(k) \arrow[dr, "i^*"'] & & FGL(A) \ f_A \arrow[dl, "\varphi^*"] \\ & FGL(A/m_A) & \end{tikzcd}$$

An $*$- isomorphism of deformations:

only defined with same $i: k \to A/m_A$

$$(f_A, i) \sim (f_A', i) = \alpha: f_A \xrightarrow{\sim} f_A' \text{ s.t. } \varphi_* \alpha = id$$

$$\begin{tikzcd} f_A \arrow[r, "\sim", "\alpha"'] & f_A' \\ i^* f = \varphi^* f_A \arrow[r, "\sim", "\varphi^* \alpha"'] & \varphi^* f_A' \end{tikzcd} \quad \text{in } FGL(A/m_A)$$

they only differ by an invertible power series $\alpha(t) \in A[\![t]\!]$ s.t. $\alpha(t) = t \mod m_A$

$Def_f(A)$ = Deformations of f over A and $*$-isomorphisms.

This defines a groupoid. Every morphism is invertible and can be described as

$$Def(A) \simeq \coprod_{j: k \to A/m_A} Def(A)_j$$

full subcategory of (f_A, i) w. $i = j$.

This extends to a functor for fixed k, f

$$\text{Inf. thicks of } k \ \{(A, m_A, \varphi)\} \longrightarrow \text{Groupoids}$$

$$A \longmapsto Def(A)$$

and we wish to understand this one better.

<u>Ex:</u> Let's consider a specific deformation of f:

$W(k)$ = ring of Witt vectors of k $\quad R = W(k)[\![v_1, \ldots, v_{n-1}]\!]$ Lubin-Tate ring

Have a canonical map

$$\varphi: R \longrightarrow R/m_R \cong k, \quad m_R = (p, v_1, \ldots, v_n)$$

$f \in FGL(k)$ of height n is classified by a map characterisation by height

$$\alpha_0: L_{(p)} \longrightarrow k, \quad L_{(p)} \simeq \mathbb{Z}_{(p)}[t_1, t_2, \ldots] \quad |t_i| = -2i$$

Lazard ring

Assume that $\qquad t_{p^i-1} = \underline{v_i}$ $\qquad$ Recall from last time: v_i the coefficient of x^{p^i} in $[p](x)$

for $1 \le i \le n-1$. Since f has height n we get

$$t_{p^i-1} \to 0 \in k$$

Next, let $\alpha : L_{(p)} \to R$ be any homomorphism which lifts α_0 and maps $t_{p^i-1} \mapsto v_i$ for $0 < i < n$ notes says m?

$f \longmapsto \bar{f}$

$FGL(k) \longrightarrow Def(R)$

$$\begin{tikzcd} & & v_i \\ & & R \arrow[d, "\varphi"] \\ t_{p^i-1} \arrow[uurr, bend left] \arrow[ddrr, bend right] & L_{(p)} \arrow[ur, "\alpha"] \arrow[r, "\alpha_0"'] & k \\ & & 0 \end{tikzcd}$$

$\rightsquigarrow$ α classifies a formal group law $E(k,f) \in FGL(R)$ s.t. $\varphi * E(k,f) \cong f$ Lubin-Tate FGL

$\rightsquigarrow E(k,f) \in Def(R)$ $\qquad$ since $k \cong R/m_R$ we just have $i = id$

<u>Thm:</u> |Lubin-Tate| $E(k,f) \in FGL(R)$, $R = W(k)[[u_1, \dots, u_n]]$ is a "universal deformation of f": For all (A, m_A, φ_A), $E(k,f)$ gives a bijection

$E(k,f)$ represents $Def_f(A)$ $\qquad Hom_{/k}(R, A) \xrightarrow{\sim} Def_f(A) = \{(f_A, i_A) \text{ s.t. } i_A^* f = \varphi_A^* f_A\}$

$h: R \to A \longmapsto h^* E(k,f) \cong_* G$

$$\begin{tikzcd} R \arrow[r, "h"] \arrow[dr, "\varphi"'] & A \arrow[d, "j"] \\ & k \end{tikzcd}$$

$\rightsquigarrow$ $L_{(p)} \xrightarrow{\alpha} R \xrightarrow{h} A$ determines a formal group law $\bar{h} \in FGL(A)$ s.t. $j^* \bar{h} = f$ in $FGL(k)$ due to the following diagram:

Intuition when $A/m_A \cong k$

$$\begin{tikzcd} & R \arrow[r, "h"] \arrow[d, "\varphi"] & A \arrow[dl, "j"] \\ L_{(p)} \arrow[ur, "\alpha"] \arrow[r, "\alpha_0"'] & k & \end{tikzcd}$$

<u>Cor:</u> • $\pi_0 Def_f(A)_i \cong m_A^{\times(n-1)}$ $\quad$ corresponds to choices for the coefficients x^{p^k}, $k = 1, \dots, n-1$ in $[p]_G(x)$

• $\pi_1 Def_f(A)_i = \{1\}$ $\rightsquigarrow \exists! \ast$-iso $(f_A, i) \cong (f'_A, i)$.

<u>Key parts of proof:</u>

1) $A \mapsto Def(A)$ <u>formally smooth</u>, i.e. $A \to A'$ surjective $\Rightarrow Def(A) \to Def(A')$ surjective

$\hookrightarrow$ Because any FGL over A extends to a FGL over A', since L is polynomial

2) $A \to B \leftarrow C$ surjective maps $\Rightarrow \mathrm{Def}(A \times_B C) \to \mathrm{Def}(A) \times_{\mathrm{Def}(B)} \mathrm{Def}(C)$ bijection

↳ Argument for this uses Spec.

Proof is then done by induction on the length of A.

Rem: $E(k, f)$ Landweber exact:

- $v_0 = p, v_1, \dots, v_n$ regular per construction
- v_n has invertible image in $R/(v_0, v_1, \dots, v_{n-1}) \cong k$ by the assumption that f has height n. Depends not only on n, but also k and f

$\rightsquigarrow$ Cohomology theory E_n Morava E-theory satisfying Periodic version of $L_{K(n)}E_n$

$$(E_n)_* := \pi_* E_n \cong W(k)[[u_1, \dots, u_{n-1}]][\beta^{\pm 1}], \quad |\beta| = 2$$

$$(E_n)_0 \cong W(k)[[u_1, \dots, u_{n-1}]]$$

Motivation: L_{E_n} behaves like restricting to the open substack $M_{FG}^{\leq n} \subseteq M_{FG} \times \mathrm{Spec}\, \mathbb{Z}_{(p)}$

A connection to the Johnson-Wilson spectrum: $L_{E_n} \cong L_{E(n)}$ — Smashing (preserves direct sum)

A way to try and understand this thing is by the "Morava stabilizer group" which acts on $(E_n)_*$

automorphisms of f over k

$$G_n := \mathrm{Aut}(k, f) \quad \text{Morava stabilizer group}$$

$\rightsquigarrow$ Produce automorphisms of the universal deformation by naturality

$\rightsquigarrow$ G_n acts on $(E_n)_0$ which extends $(E_n)_*$

Ex: $n = 1$:

Write K for the complex K-theory spectrum. This has a canonical complex orientation which determines a FGL

$$F_K(x, y) = x + y + \beta xy, \quad \beta \in \pi_2 K \text{ Bott element}$$

Fix p and write K_p for the p-adic completion of K. Then

$$\pi_0 K_p \cong \mathbb{Z}_p$$

we get that

$$F_{\mathbb{Z}_p}(x, y) = x + y + xy \quad \in FGL(\mathbb{Z}_p)$$

is a deformation of

$$F_{\mathbb{F}_p}(x, y) = x + y + xy \quad \in FGL(\mathbb{F}_p) \quad \rightsquigarrow F_{\mathbb{Z}_p} \in \mathrm{Def}_{F_{\mathbb{F}_p}}(\mathbb{Z}_p)$$

$\rightsquigarrow$ Fix $F_{\mathbb{F}_p}$ as above. Then $E_1 \cong K_p$, $(E_1)_* \cong \mathbb{Z}_p[\beta^{\pm 1}]$

In this case

$$\mathbb{G}_1 \cong \mathbb{Z}_p^\times \circlearrowright (E_1)_* \quad \text{the opposite of the Adams operations}$$

For every p-adic unit λ, we let ψ^λ denote the corresponding map in $\mathbb{G}_1$

$\rightsquigarrow$ Gives $\psi^\lambda : K_p \to K_p$, and these are what is given by the Adams ops.

Construction of Morava K-theory:

$$\pi_* MU_{(p)} \cong L_{(p)} \cong \mathbb{Z}_{(p)}[t_1, \ldots, t_n]$$

where we may assume $v_i = t_{p^i - 1}$ for $i > 0$ and by convention we set $t_0 = p \in \pi_0 MU_{(p)}$

Write

$$M(k) := \operatorname{cofib}(\Sigma^{2k} MU_{(p)} \xrightarrow{\cdot t_k} MU_{(p)})$$

Prop: $M(k)$ homotopy associative algebra over $MU_{(p)}$

Fix prime p and $n > 0$: Morava K-theory

$$K(n) := MU_{(p)}[v_n^{-1}] \wedge_{MU_{(p)}} \left(\bigotimes_{k \neq p^n - 1} M(k) \right)$$

which by the above has the structure of a homotopy associative $MU_{(p)}$-algebra.

If $p \neq 2$ its homotopy commutative!

One can calculate that

$$\pi_* K(n) \cong (\pi_* MU_{(p)})[v_n^{-1}] / (t_0, t_1, \ldots, t_{p^n - 2}, t_{p^n}, \ldots)$$

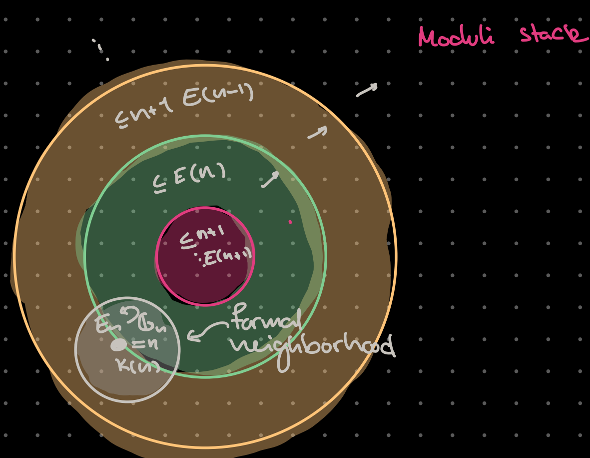
$$\cong \mathbb{F}_p[v_n^{\pm 1}] \quad , \quad |v_n| = 2(p^n - 1)$$

We get a map of ring spectra

$$MU_{(p)} \to K(n) \quad \rightsquigarrow \text{complex orientation on } K(n)$$

$$\rightsquigarrow \text{a FGL}(\pi_* K(n)) \text{ with height } n$$

One can show that this construction of $K(n)$ is independent of all the choices we made.



General motivation/ Intuition

Height $\varphi: MU_* \to R_*$ $\quad FGL(R_*)$

- $\varphi(v_n)$ unit in R_*
  $\rightsquigarrow$ F height $\leq n$ at p
- If further $\varphi(v_i) = 0$ for $0 \leq i < n$
  $\rightsquigarrow$ F height $= n$

$v_n \in MU_{2(p^n-1)}$ coefficient of x^p in

$$[p]_{F_{MU}}(x) = \underbrace{x \underset{F}{+} \cdots \underset{F}{+} x}_{p\text{-times}} = px + \cdots + v_1 x^p + \cdots + v_n x^{p^n} + \cdots$$

$$v_n = \begin{cases} p & n=0 \\ x_{p^n-1} & \bmod (p, x_1, \ldots, x_{p^n-2}) \end{cases}$$

§6: Morava K-theory

Paul Pantae - 22.02.23

<u>Recall:</u> k perfect field of char p, $f \in FGL_n(k)$, A an infinitesimal thickening of k, $A \to k$. Then a deformation of f over A is a $g \in FGL(A)$ s.t.

$$\begin{tikzcd} & A \arrow[d] \\ L_{(p)} \arrow[ur, "g"] \arrow[r, "f"'] \arrow[ur, phantom, "\cong", shift right=3ex] & k \end{tikzcd}$$

We have that $R = W(k)[[u_1, \dots, u_{n-1}]] \to k$ is the universal deformation in the sense that $FG(A) \simeq Def(A) \cong Hom(R, A)$.

$\rightsquigarrow$ The Lubin-Tate formal group law

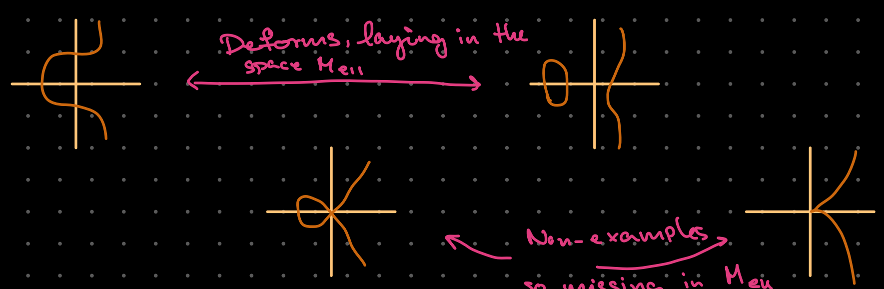
$\rightsquigarrow$ Landweber exact $E(n)$ p-local ring spectrum

Formal groups = Group object in formal schemes

$\rightsquigarrow$ M_{FG} = Moduli stack of formal groups

<u>Recall:</u> Elliptic curve $y^2 = x^3 + ax + b$ when 2, 3 are invertible in R, $\Delta \neq 0$

Over $\mathbb{R}$:



Consider $F: Sch/\mathbb{C} \longrightarrow Set$
└ site of schemes over spec $\mathbb{C}$

$$F(x) \cong Hom(x, M_{ell})$$

Issue: M_{ell} can't be a <u>scheme</u> — to 'small' because elliptic curves have non-trivial automorphisms

$\rightsquigarrow$ M_{ell} will be a <u>stack</u>

<u>Thm:</u> [Goerss-Hopkins-Miller] The structure sheaf of M_{ell} can be lifted to a sheaf of $\mathbb{E}_\infty$-ring spectra $\mathcal{O}^{top}$, and the spectrum of global sections of this $\mathcal{O}^{top}$ is by definition Tmf = the spectrum of topological modular forms

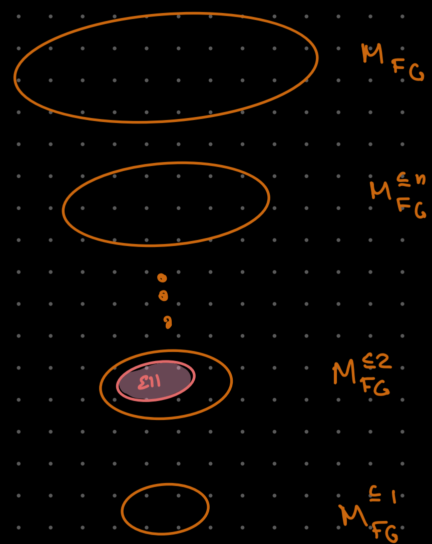
story of "elliptic cohomology"

This is a good approximation of height 2.

Elliptic curves $\longmapsto$ formal groups of height ≤ 2

$E \longmapsto \hat{E}$ complete

tmf allows you to understand $E(2)$-local spectra.



Slogan "Localizing spectra at $E(n)$ is like restricting $\mathcal{M}_{FG}^{\leq n} \subseteq \mathcal{M}_{FG}$"

Want to look at $\mathcal{M}_{FG}^{n}$ - those of exactly height n

$$\Sigma^{2k} MU_{(p)} \xrightarrow{\cdot t_k} MU_{(p)} \longrightarrow M(k)$$

$MU_{(p)}$ is a ring spectrum $\rightsquigarrow$ $M(k)$ is an algebra over $MU_{(p)}$

$$K(n) := MU_{(p)}[v_n^{-1}] \otimes_{MU_{(p)}} \Big(\bigotimes_{k \neq p^n-1} M(k) \Big)$$ Morava K-theory

$L_{K(n)}$ is like restricting to $\mathcal{M}_{FG}^{n}$.

Thm: The following is a homotopy pullback square

$$\begin{tikzcd} L_{E(n)} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & L_{E(n-1)} X \arrow[d] \\ L_{K(n)} X \arrow[r] & L_{E(n-1)} L_{K(n)} X \end{tikzcd}$$

Fracture square or the Hasse square

comes from Hasse-Minkowski thm

$$\begin{tikzcd} \mathbb{Z}_{(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{Q} \arrow[d] \\ \hat{\mathbb{Z}}_p \arrow[r] & \mathbb{Q}_p \cong \mathbb{Q} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}_p \end{tikzcd}$$

Thm: | Chromatic convergence |

$$X \xrightarrow{\sim} \operatorname{holim} \left(L_{E(n)} \to L_{E(n-1)} \to \cdots \right)$$

Prop: $\pi_* K(n) \cong \mathbb{F}_p[v_n^{\pm 1}]$

Def. A graded field is an evenly graded ring s.t. either

$$R \cong R_0 \cong k \quad , \quad R \cong k[\beta^{\pm 1}], \ |\beta| = 2k, k > 0.$$

Def. A ring spectrum E is a field if $\pi_* E$ is a graded field

Ex: $H\mathbb{Q}$, $H\mathbb{F}_p$, $K(n)$

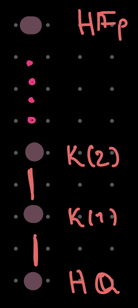
Thm: | Uniqueness | $K(n)$ is the unique spectrum having height n fgl and which is a field.

Prop: E field $\Rightarrow$ E-module is w.e.q. to $\bigoplus_a \Sigma^{k_a} E$

$L_{H\mathbb{F}_p}$ = p-completion of spectra

$L_{H\mathbb{Z}_p}$ = p-localization of spectra

Balmer Spectrum $\mathrm{Spc}(Sp^{fin}_{(p)})$



Prop: $L_{E(n)}$ is smashing, i.e. it preserves direct sum

$\Longrightarrow$ $L_{E(n)} X \simeq L_{E(n)} \mathbb{S} \wedge X$

$L_{K(n)}$ is <u>not</u> smashing

We have

$E(0)$-local SHT = $H\mathbb{Q}$-local SHT $\qquad \mathcal{D}(H\mathbb{Q}) \simeq \mathcal{D}(\mathbb{Q})$

$E(1) \simeq KU_p^\wedge$

§7: The known big theorems & the telescope conjecture

08.03.23

Where are we?

- Elements v_n: $v_n \in MU_{(p^n-1)}$ is the coefficient of x^{p} in
$$[p]_{F_{MU}}(x) = px + \dots + v_1 x^p + \dots + v_n x^{p^n} + \dots$$
- Height: A FGL classified by $\varphi: MU_* \to R_*$ has height $\leq n$ if $\varphi(v_n)$ is a unit and $\varphi(v_i) = 0$ for $0 \leq i < n$.
- Landweber Exact Functor theorem $\rightsquigarrow$ Morava E-theory
$$E(n)_* \simeq \mathbb{Z}_{(p)}[v_1, \dots, v_n, v_n^{\pm 1}] \qquad \text{height } n$$
$$W(k)[[u_1, \dots, u_{n-1}]][\beta^{\pm 1}]$$
- Morava K-theory $K(n)$ for $0 \leq n < \infty$
  - $K(n)_* \cong \mathbb{F}_p[v_n^{\pm 1}]$ height n
  - $K(n)_* K(m) = 0$ if $m \neq n$
  - X finite & $K(n)_* X = 0 \Rightarrow K(n-1)_* X = 0$.

Want to understand localizations with these things.

Need wedge of spectra

$$E \vee F = \overset{\text{spectrification}}{\mathbb{L}}\left(E_n \vee F_n,\ E_n \vee F_n \xrightarrow{w_n^E \vee w_n^F} \Omega E_{n+1} \vee \Omega F_{n+1} \longrightarrow \Omega(E_{n+1} \vee F_{n+1})\right)$$

↳ Both product and coproduct in SH

Chromatic fracture square

Write $L_n X := L_{K(0) \vee \dots \vee K(n)} X$

Intuition:

- L_n = inverting v_n
- $L_{K(n)}$ = inverting v_n and completing at $(p, v_1, \dots, v_{n-1})$

Thm: $L_{E(n)} \cong L_n \cong L_{v_n^{-1} MU_{(p)}}$

There are clearly natural transformations $L_n \to L_{n-1}$ so we get
$$\dots \to L_{E(n)} \to L_{E(n-1)} \to \dots$$

Def chromatic tower of $X \in Sp$
$$\dots \to L_{E(n)} X \to L_{E(n-1)} X \to \dots$$

Monochromatic layers = Fibers of the maps in this tower

The natural transformation $\eta_X : X \to L_{E(n)}X$ gives a map

$$X \to \operatorname{holim}_n (L_{E(n)}X)$$

If this is an equivalence, we say chromatically complete

<u>Thm:</u> |Chromatic convergence - Barthel| X connective spectrum w. finite projective dimension is chromatically complete

In particular

- S^0 p-locally is chromatically complete
- p-local finite spectra are chromatically complete

<u>Thm:</u> |Smash product theorem| $L_nX \simeq L_{E(n)}X \simeq L_{E(n)}(S^0) \wedge X \simeq (L_n S^0) \wedge X$ smashing

<u>Thm:</u> |Localization theorem| $BP \wedge L_{E(n)}X \simeq X \wedge L_{E(n)}BP$ can compute $BP_*(L_nX)$ in terms of BP_*X

$\Rightarrow$ If $v_{n-1}^{-1}BP(X) \simeq 0$ then $BP \wedge L_nX \simeq X \wedge v_n^{-1}BP \Rightarrow BP_*L_nX \simeq v_n^{-1}BP_*X$

Want to understand these maps $L_{E(n)} \to L_{E(n-1)}$.

<u>Thm</u> |Hasse square| natural map $L_{E(n)} \simeq L_n \to L_{K(n)}$

Chromatic fracture square

$$\begin{tikzcd} L_{E(n)}X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L_{K(n)} \arrow[d] \\ L_{E(n-1)}X \arrow[r] & L_{E(n-1)}L_{K(n)}(X) \end{tikzcd}$$

$L_{E(n-1)}$ on $X \to L_{K(n)}(X)$

<u>Chromatic splitting conjecture:</u> This glueing process is as simple as possible without being trivial

Consider the following diagram

$$\begin{tikzcd} L_nX \arrow[r] \arrow[d] & L_{K(n)}X \arrow[d] \arrow[dl, "\alpha_n"'] \\ L_{n-1}X \arrow[r, shift left] & L_{n-1}L_{K(n)}X \arrow[l, shift left, "\gamma_n"] \end{tikzcd}$$

Turns out that there exists such an α_n making the top triangle commutes exactly if there exists a map γ_n splitting $L_{n-1}X \to L_{n-1}L_{K(n)}X$

<u>Weak CSC:</u> X p-completion of a finite spectrum $\Rightarrow$ γ_n exists for all n

This would imply that taking the limit of

$$L_{K(n-1)}X_p \xrightarrow{\alpha_{n-1}} L_nX_p \to L_{K(n)}X_p$$

gives an equivalence

$X \xrightarrow{\simeq}_p \lim_n L_{K(n)} X$ From chromatic convergence theorem by cofinality

"Finite spectrum X can be recovered from its monochromatic pieces $L_{K(n)}X$"

Another consequence: $f: X \to Y$ map between (finite) spectra and $L_{K(n)}f: L_{K(n)}X \to L_{K(n)}Y$ is null $\Rightarrow$ f is null

General version is known for

$n=1, p\geq 2$: Adams-Bousfield-Baird-Ravenel
$n=2, p>5$: Hopkins based on Shimomura-Yabe
$n=2, p=3$: Goerss-Henn-Mahowald
$n=2, p=2$: Beaudry-Goerss-Henn
$n>2, p\geq 2$: Wide open

Brutal computational – No tactics that can be generalised

There are two different approaches to consider a "filtration" of the chromatic tower. The first one:

Algebraic chromatic filtration of a p-local spectrum X is for $n\geq 1$

$$C^a_n(X) := \ker(\pi_* X \to \pi_* L_{n-1} X) \qquad C^a_0(X) := \pi_* X$$

The other filtration will be a bit harder to construct, and relies on another localization.

Geometric chromatic filtration

Def A full subcategory $\mathcal{T}$ of the (homotopy) category of p-local spectra is thick if

- $0 \in \mathcal{T}$
- Closed under fibers and cofibers
- Closed under retracts

Def: A p-local finite spectrum X is of type n if

$$K(i)_* X \cong \begin{cases} \neq 0 & i=n \\ =0 & i<n \end{cases}$$

Ex: • $S^0_{(p)}$ type 0 since $K(0)_*(S^0_{(p)}) \neq 0$
• $S\mathbb{Z}/p$ type 1 $K(0)_* S\mathbb{Z}/p = 0$, $K(1)_* S\mathbb{Z}/p \neq 0$

Write

$$\mathcal{P}_n = \{\text{finite } p\text{-local spectra of type } \geq n\}$$

i.e. those s.t. $K(m)_* X \cong 0$, $m<n$
$\hookrightarrow$ since finite, $K(m)_* X = 0 \Rightarrow K(m-1)_* X = 0$, so enough to consider $n-1$

Note: Every such finite p-local spectrum is of type n for some n, and it can be shown that for all $n \geq 0$ there exists one of type n so all these $\mathcal{P}_{\geq n}$'s are different

Prop: $\mathcal{P}_n$ is a thick subcategory Actually "thick prime tensor ideals of $SH^c_{(p)}$"

The LES of $K(m)$-homology gives us that a cofiber sequence

$$X' \to X \to X''$$ satisfies 2-out-of-3 w.r.t. $\mathcal{P}_{\geq n}$

A retract of a type n spectrum is again type n.

Thm: | Thick subcategory theorem - Ravenel/Mitchell/Hopkins-Smith |

Put $\mathcal{P}_0 =$ Category of p-local finite spectra $SH^c_{(p)}$. Then

$$\mathcal{P}_0 \supsetneq \mathcal{P}_1 \supsetneq \cdots \supsetneq \mathcal{P}_n \supsetneq \mathcal{P}_{n+1} \supsetneq \cdots \supsetneq *$$

If $\mathcal{C}$ is a thick subcategory, then $\mathcal{C} \simeq \mathcal{P}_n$ for some $n \geq 0$.

So $\mathcal{P}_n$ are all of the thick subcategories $\sim$ The thick subcategories are the kernels of $K(n)_*$

Cor: Let X be of type n, then $L_n X \simeq L_{K(n)} X$

Pf Follows by the chromatic fracture square

Being of 'type n' can equivalently be described as existence of some specific maps.

First we consider how to construct spectra of a specific type:

$n=0$: $H_*(X;\mathbb{Q}) \neq 0$ $\sim$ take e.g. $\mathbb{S}_{(p)}$

$n=1$: Define X to be the mod p more spectrum which is defined by the cofiber

$$S \xrightarrow{\cdot p} S \longrightarrow X$$

This has no rational homology. Furthermore, since multiplication by p annihilates $K(1)_* S \simeq \mathbb{F}_p[v_1^{\pm 1}]$, the map $K(1)_* S \to K(1)_* X$ is injective so in particular $K(1)_* X \neq 0$ $\rightsquigarrow$ X type 1

$n > 1$ is much harder! We wish to proceed inductively.

Assume X is of type n. Then we wish to construct a self-map

$$f : \Sigma^k X \to X$$

So we can form the cofiber sequence

$$\Sigma^k X \to X \to X/f$$

By looking at LES on $K(m)_*$ such that X/f is of type $n+1$.

Turns out this is exactly the case when

- f induces an isomorphism $K(n)_*X \to K(n)_*X$ K(n)-homology of X/f vanish
- f does not induce an isomorphism $K(n-1)_*X \to K(n-1)_*X$ K(n-1)-homology does not vanish

This motivates the following definition:

Def: A v_n-self map on a p-local finite spectrum X, is a map $f: \Sigma^k X \to X$ s.t.

- f induces an isomorphism $K(n)_*X \to K(n)_*X$
- For $m \neq n$, the induced map $K(m)_*X \to K(m)_*X$ is nilpotent.

This is equivalent to saying

$$K(m)_* f = \begin{cases} 0 & n \neq m \\ v_n^{N} & n = m \end{cases}$$

For a suitable power

Can be done more generally (Nilpotence II, Hopkins-Smith)

Ex: If X has type $> n$, then $K(n)_*X$ vanishes, so the zero map $0: X \to X$ is a v_n-self map.

Thm: | Periodicity theorem |

Follows by showing $\mathcal{J} = \{$p-local finite spectra w. v_n-self map$\}$ is thick, followed by thick subcategory theorem.

- A spectrum X has type n iff it admits a v_n-self map
- Furthermore, if f, g both are v_n-self maps, then $\exists i, j \geq 0$ s.t.

$$f^i = g^j$$

Essentially unique!

$\rightsquigarrow$ Want to think of these as periodic operators induces is on $K(n)_*$-hom and iterating will give us the same back at some point

So, if we have a type n spectrum and a v_n-self map we can construct a spectrum of type $n+1$:

Ex:

- $S \xrightarrow{p \,=\, v_0^1} S \to S\mathbb{Z}/p$ type 1 $\sim$ sometimes denoted $M(1)$
- p odd,

$$\alpha: \Sigma^{2(p-1)} M(1) \to M(1) \qquad \text{Adams map}$$

satisfies $K(1)_*(\alpha) = v_1^1$. The cofiber has type 2 and we write $M(1,1)$

In general: We inductively define a type $n+1$ spectrum as follows.

- cokernel of a v_0-self-map f_0 satisfying $K(1)_*(f_0) = v_0^{i_0}$

  $\rightsquigarrow$ $M(i_0)$ type 1

- cokernel of a v_1-self-map $f_1 : \Sigma^{2(p-1)i_1} M(i_0) \to M(i_0)$ s.t.
  $K(1)_*(f_1) = v_1^{i_1}$
  $\rightsquigarrow$ $M(i_0, i_1)$ type 2
-
- ⋮

$M(i_0, i_1, \ldots, i_n)$ is the type $n+1$ spectrum defined as the cofiber of a v_n-self map

$$f_n : \Sigma^{2(p^n-1)i_n} M(i_0, \ldots, i_{n-1}) \longrightarrow M(i_0, \ldots, i_{n-1})$$

satisfying

$$K(n)_*(f_n) = v_n^{i_n}.$$

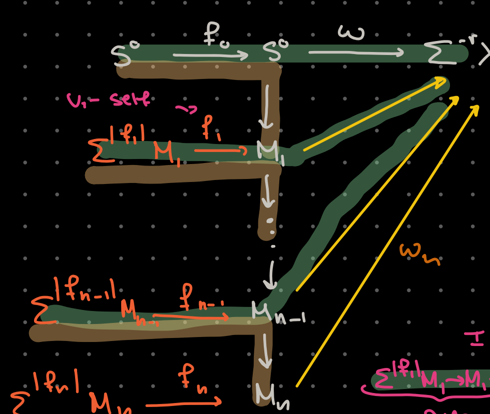
There is a lot of choices when constructing these $M(i_0, \ldots, i_n)$!

"Periodic families":

Construction: Write $M_n := M(i_0, \ldots, i_{n-1})$ type n $K(n)_* f_n = v_n^{i_n}$

Let $w \in \pi_r X$ $S^r \xrightarrow{w} X \rightsquigarrow S^0 \xrightarrow{w} \Sigma^{-r} X$

- w is v_{n-1}-torsion if there exists a diagram



If w is p-torsion $\rightsquigarrow f_0 = p^{i_0}$ (v_0)

so if the comp $w \circ f_0 \simeq 0$, we get that it extends, since $S^0 \xrightarrow{f_0} S^0 \longrightarrow M_1$ cofibre seq

If we can find a power of f_1 s.t. $\underbrace{\Sigma^{|f_1|} M_1 \to M_1 \to \Sigma^{-r} X}_{\simeq 0}$, then we can again extend

So we are assuming we can continue this process until a type n spectrum M_n.

- w is v_n-periodic if for any v_n-self map f_n of M_n, $w_n \circ f_n \neq 0$ so we can't continue the constr.

Def Geometric Chromatic Filtration

$C_0^g(X) = \pi_* X$

$C_n^g(X) = v_{n-1}$-torsion elements $n \geq 1$

Decreasing filtration: $C_0^g(X) \supseteq C_1^g(X) \supseteq C_2^g(X) \supseteq \cdots$

We now have two filtrations - when are they the same? Telescope conjecture

Telescope conjecture

Recall that by the periodicity theorem tells us that a v_n-self map $f: \Sigma^k X \to X$, for X a type n spectrum, is essentially unique, so the following colimit is independent of f.

Telescope of f $\quad X[f^{-1}] := \mathrm{colim}(X \xrightarrow{\Sigma^{-k} f} \Sigma^{-k} X \xrightarrow{\Sigma^{-2k} f} \Sigma^{-2k} X \to \cdots)$

<u>Def:</u> For $M_n = M(i_0, \ldots, i_{n-1})$ w. v_n self map f_n, write $\mathrm{Tel}(n) := M_n[f_n^{-1}]$.

Telescopic localization

sometime people write 'f' for finite - It's a finite localisation w. $\ker(L_n^t)$ generated by any (finite) $(n+1)$-type spectra

$$L_n^t X := L_{\mathrm{Tel}(0) \vee \cdots \vee \mathrm{Tel}(n)} X$$

$\llcorner$ p-local spectrum

<u>Prop:</u> If X is of type $\geq n$ and f is a v_n-self-map of X, then

$$L_n^t X \simeq X[f^{-1}].$$

<u>Prop:</u> L_n^t is a finite smashing localisation

This explains the name: It is the colimit of the telescope of a map

Using this we can redefine the geometric chromatic filtration

$$C_n^g X = \begin{cases} \pi_* X & n = 0 \\ \ker(\pi_* X \to \pi_* L_{n-1}^t X) & n \geq 1 \end{cases}$$

p-local spectrum

This is very similar to the algebraic one now!

$$C_n^a X = \begin{cases} \pi_* X \\ \ker(\pi_* X \to L_{n-1} X) \end{cases}$$

There exist a natural transformation:

$$L_n^t X \to L_n X$$

which is known to be an equivalence if

- X is $E(m)$-local for some $m \geq 0$
- X is an MU-module spectrum localization theorem

<u>Telescope conjecture:</u> For every spectrum X, Ravenel made this conjecture

$$L_n^t X \xrightarrow{\sim} L_n X.$$

and the conjecture that it is false

Known to be true for $n = 0$, $p \geq 2$ - Bousfield (tautology $\mathrm{Tel}(0) = S\mathbb{Q} = H\mathbb{Q} = K(0)$)
$n = 1$, $p \geq 2$ $\quad$ >2 Miller
$=2$ Mahowal

$\rightsquigarrow$ completely open for $n \geq 1$, $P \geq 2$. But attempts to disprove

Prop: For $n \geq 1$ the following is equivalent.

- $L_{n-1}^{t} \simeq L_{n-1} \Rightarrow L_n^t \simeq L_n$
- There exists a type n spectrum X w. $X[f^{-1}] \not\simeq L_n X$

Using the thick subcategory theorem

so one example or counter example is enough to settle the passage from $n-1$ to n.

Periodic families

Let $w \in \pi_r X$ be v_n-periodic, and $M = M_n$ as above w. v_n-self map s.t. (cw-spectrum)

$$\Sigma^d M \xrightarrow{f_n} M \xrightarrow{w_n} \Sigma^{-r} X \quad \text{non-zero}$$

Let M^r = r-skelton of M and cofiber sequences

$$M^{r-1} \longrightarrow M^r \longrightarrow M^r_r \; , \quad M^{r-1} \longrightarrow M \longrightarrow M_r = M_r^{\dim M}$$

take r-skeleton and quotient out w. (r-1)-skeleton

Then there exists an r s.t. we can form the following diagram

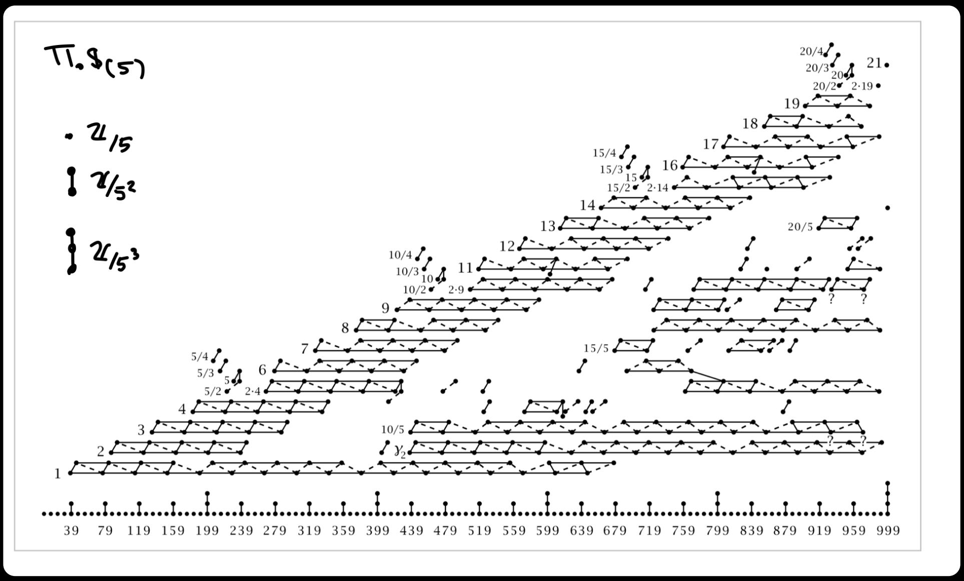
$$\begin{array}{ccccc} \Sigma^d M & \xrightarrow{f_n} & M & \xrightarrow{w_n} & \Sigma^{-r} X \\ \downarrow & & & & \\ \Sigma^d M_r & & & & \end{array}$$

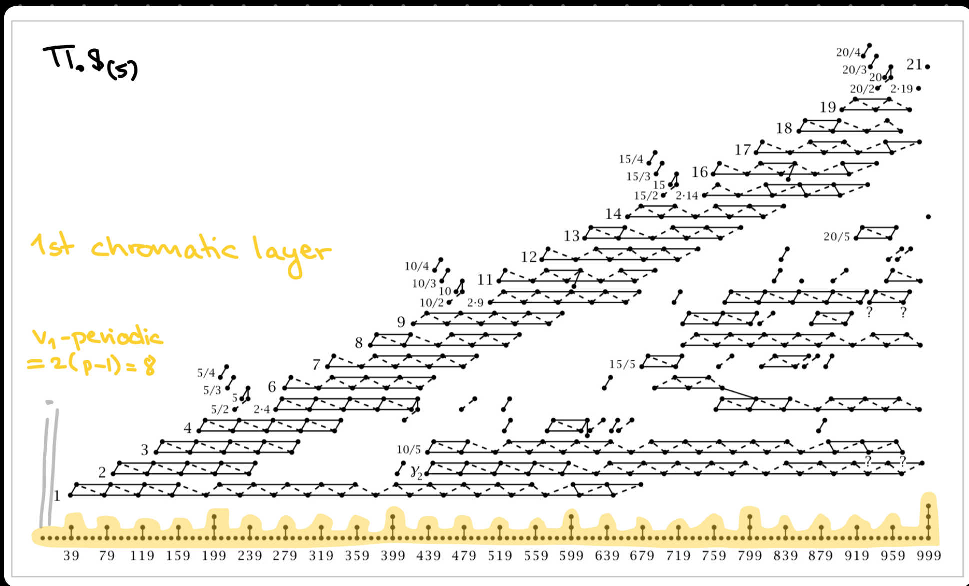
$S^k \simeq \Sigma^d M^r_r \xrightarrow{incl} \Sigma^d M_r$ $\quad \exists g$ s.t. $g \circ i$ non-trivial

$\rightsquigarrow$ i.e. $w_n \circ f_n$ is non-trivial on "some cell of M" - A cell that detects it

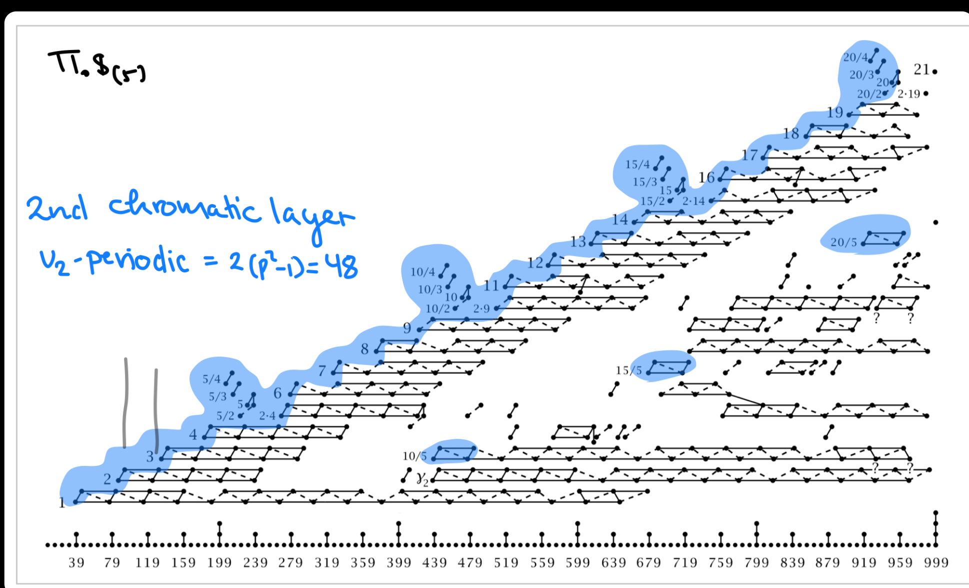
Such elements $g \circ i \in \pi_{k+r} X$ are part of the v_n-periodic family of w.

Thinking of f_n as "multiplication by v_n"

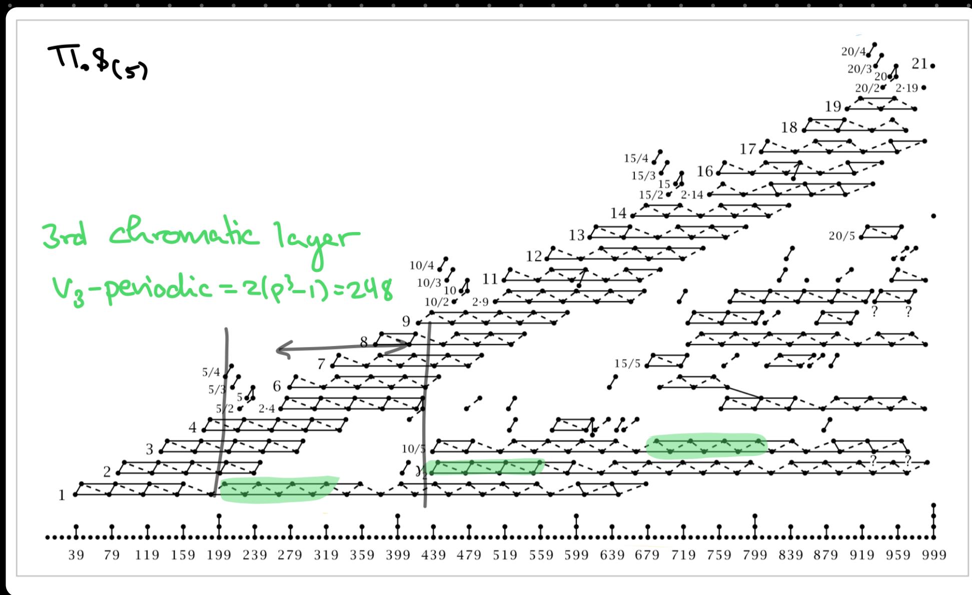


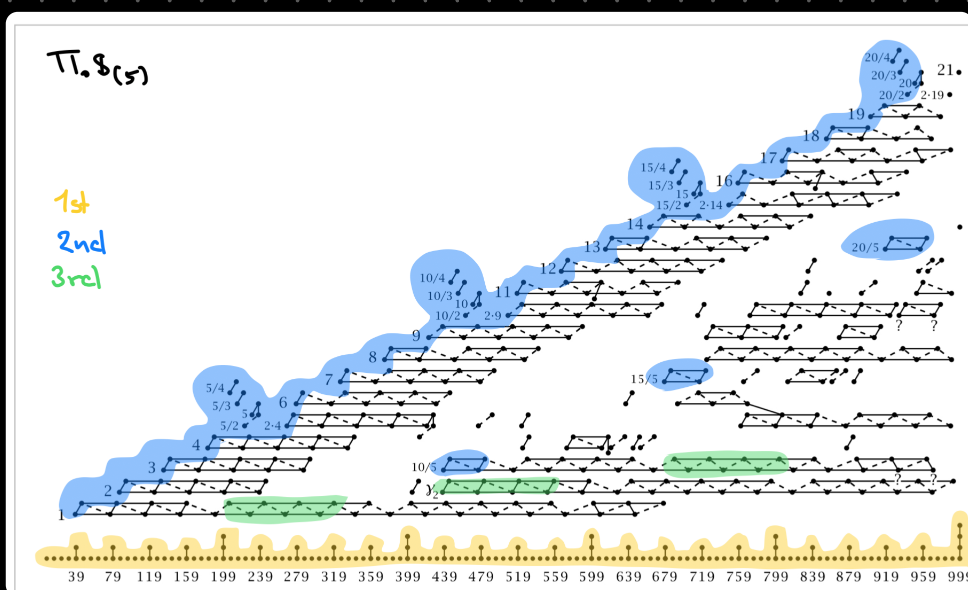


$\pi_* S_{(5)}$
1st chromatic layer
v_1-periodic
$= 2(p-1) = 8$
39 79 119 159 199 239 279 319 359 399 439 479 519 559 599 639 679 719 759 799 839 879 919 959 999



$\pi_* S_{(5)}$
2nd chromatic layer
v_2-periodic $= 2(p^2-1) = 48$
39 79 119 159 199 239 279 319 359 399 439 479 519 559 599 639 679 719 759 799 839 879 919 959 999


π.S(5)
3rd chromatic layer
V3-periodic = 2(p³-1)=248
39 79 119 159 199 239 279 319 359 399 439 479 519 559 599 639 679 719 759 799 839 879 919 959 999


π.S(5)
1st
2nd
3rd
39 79 119 159 199 239 279 319 359 399 439 479 519 559 599 639 679 719 759 799 839 879 919 959 999

§8: The Red-Blue shift Conjecture

Talk 8
15.03.23

First: Finish what we discussed last time - how does these v_n-self maps describe periodic families?

so $K(m)_* X = 0$, $m < n$

Recall:
- Type n = finite p-local spectrum X s.t. $K(n)_* X \neq 0$, $K(n-1)_* X = 0$
- v_n-self map on finite p-local spextrum X: $f : \Sigma^k X \to X$ s.t.

$$K(m)_* f = \begin{cases} 0 & n \neq m \\ v_n^{N} & n = m \end{cases}$$

For some suitable power N

Motivation for this definition was that if X is type n and f a v_n-self map, then the cokernel of f is of type $n+1$.

$$\Sigma^k \xrightarrow{f} X \longrightarrow X/f$$

In general: We inductively define a type $n+1$ spectrum as follows

- cokernel of a v_0-self-map f_0 satisfying $K(1)_*(f_0) = v_0^{i_0}$
  $\rightsquigarrow$ $M(i_0)$ type 1
- cokernel of a v_1-self-map $f_1 : \Sigma^{2(p-1)i_1} M(i_0) \to M(i_0)$ s.t.
  $K(1)_*(f_1) = v_1^{i_1}$
  $\rightsquigarrow$ $M(i_0, i_1)$ type 2
-
-
-

$M(i_0, i_1, \dots, i_n)$ is the type $n+1$ spectrum defined as the cofiber of a v_n-self map

$$f_n : \Sigma^{2(p^n-1)i_n} M(i_0, \dots, i_{n-1}) \longrightarrow M(i_0, \dots, i_{n-1})$$

satisfying

$$K(n)_*(f_n) = v_n^{i_n}.$$

There is a lot of choices when constructing these $M(i_0, \dots, i_n)$!

"Periodic families":

Construction: Write $M_n := M(i_0, \dots, i_{n-1})$ type n $K(n)_* f_n = v_n^{i_n}$

Let $\omega \in \pi_r X$ $S^r \xrightarrow{\omega} X$ $\rightsquigarrow$ $S^0 \xrightarrow{\omega} \Sigma^{-r} X$

- ω is v_{n-1}-torsion if there exists a diagram

$$\begin{tikzcd}
 & S^0 \arrow[r, "f_0"] \arrow[d] & S^0 \arrow[r, "\omega"] & \Sigma^{-r}X \\
\Sigma^{|f_1|}M_1 \arrow[r, "f_1"] & M_1 \arrow[urr] \arrow[d] & & \\
 & \vdots \arrow[d] & & \\
\Sigma^{|f_{n-1}|}M_{n-1} \arrow[r, "f_{n-1}"] & M_{n-1} \arrow[uuurr] \arrow[d] & & \\
\Sigma^{|f_n|}M_n \arrow[r, "f_n"] & M_n \arrow[uuuurr, "\omega_n"'] & &
\end{tikzcd}$$

v_1-self $\rightsquigarrow f_1$

If ω is p-torsion

$\rightsquigarrow f_0 = \underbrace{p^{i_0}}_{v_0}$

so if the comp $\omega \circ f_0 \simeq 0$, we get that it extends, since

$S^0 \xrightarrow{f_0} S^0 \longrightarrow M_1$ cofibre seq

If we can find a power of f_1 s.t. $\underbrace{\Sigma^{|f_1|}M_1 \to M_1 \to \Sigma^{-r}X}_{\simeq 0}$, then we can again extend

So we are assuming we can continue this process until a type n spectrum M_n.

- ω is v_n-periodic if for any v_n-self map f_n of M_n, $\omega_n \circ f_n \neq 0$ so we can't continue the constr.

Periodic families

Let $\omega \in \pi_r X$ be v_n-periodic, and $M = M_n$ (cw-spectrum) as above w. v_n-self map s.t.

$$\Sigma^d M \xrightarrow{f_n} M \xrightarrow{\omega_n} \Sigma^{-r} X \quad \text{non-zero}$$

Let M^r = r-skelton of M and cofiber sequences

$$M^{r-1} \longrightarrow M^r \longrightarrow M^r_r \quad , \quad M^{r-1} \longrightarrow M \longrightarrow M_r = M_r^{\dim M}$$

take r-skeleton and quotient out w-(r-1)-skeleton

Then there exists an r s.t. we can form the following diagram

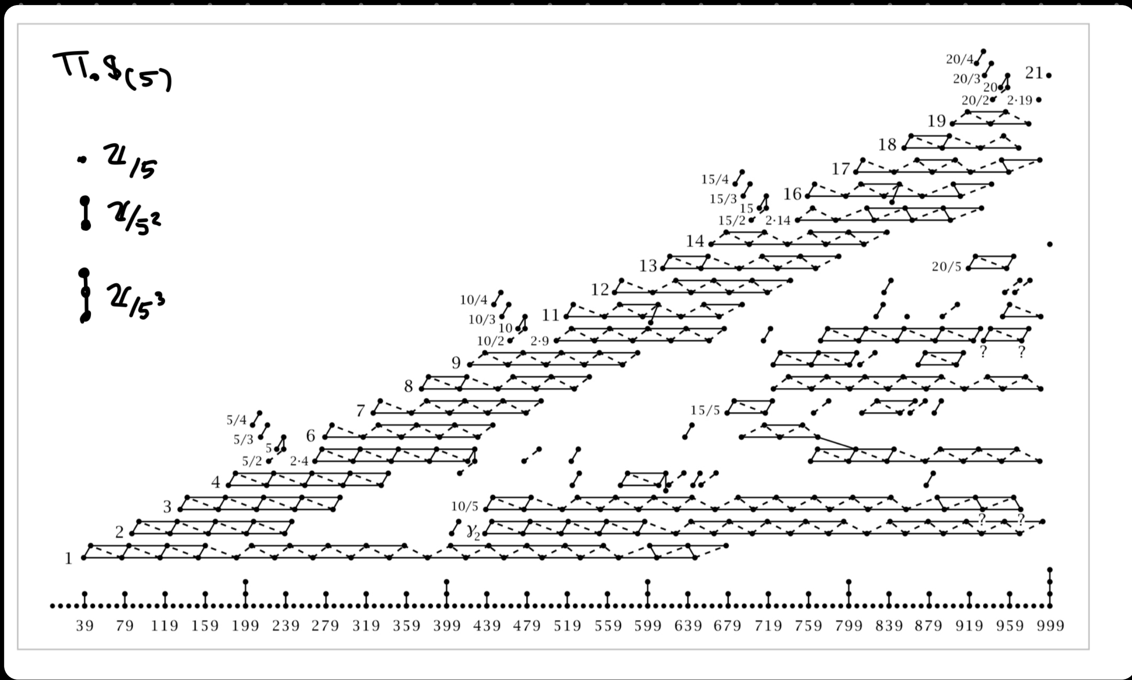
$$\begin{tikzcd}
 & & \Sigma^d M \arrow[r, "f_n"] \arrow[d] & M \arrow[r, "\omega_n"] & \Sigma^{-r}X \\
S^k \cong & \Sigma^d M^r_r \arrow[r, "\mathrm{incl}", "i"'] & \Sigma^d M_r \arrow[urr, "g"'] & &
\end{tikzcd}$$

$\exists g$ s.t. $g \circ i$ non-trivial

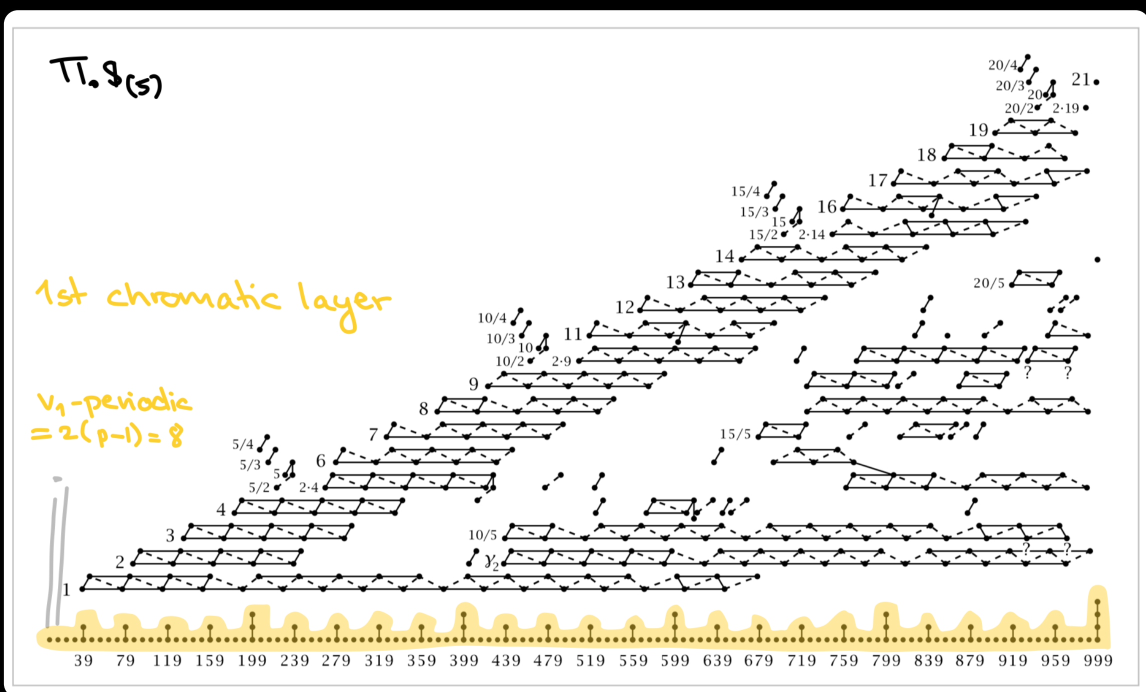
$\rightsquigarrow$ i.e. $\omega_n \circ f_n$ is non-trivial on "some cell of M" - A cell that detects it

Such elements $g \circ i \in \pi_{k+r} X$ are part of the v_n-periodic family of ω.

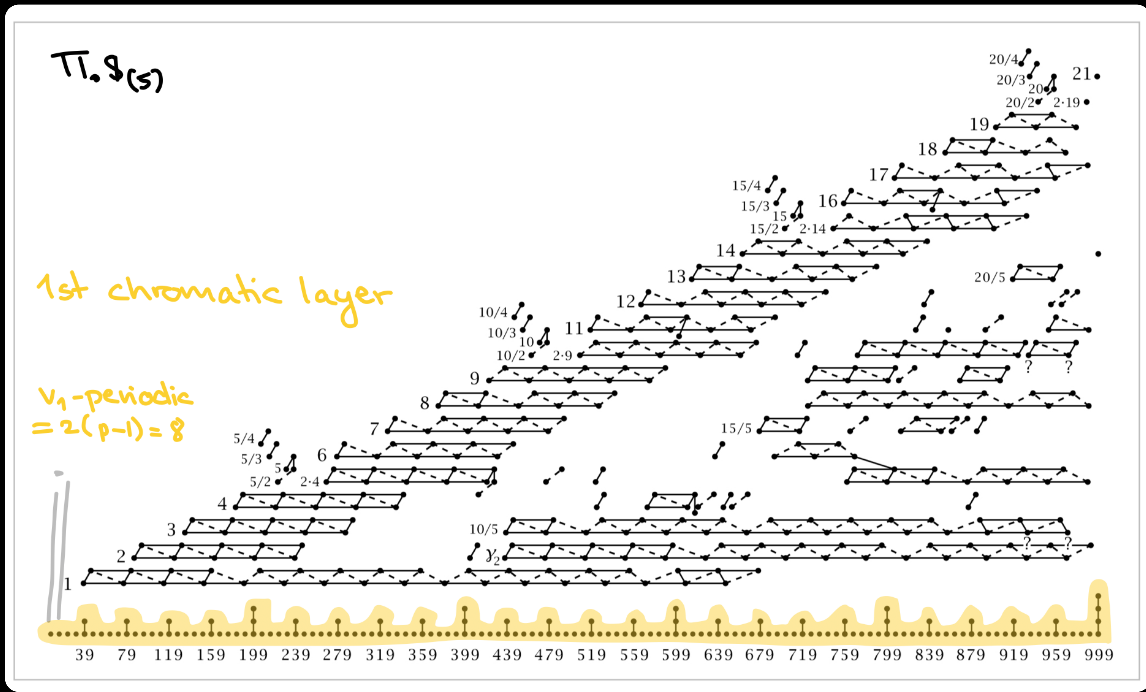
Thinking of f_n as "multiplication by v_n"



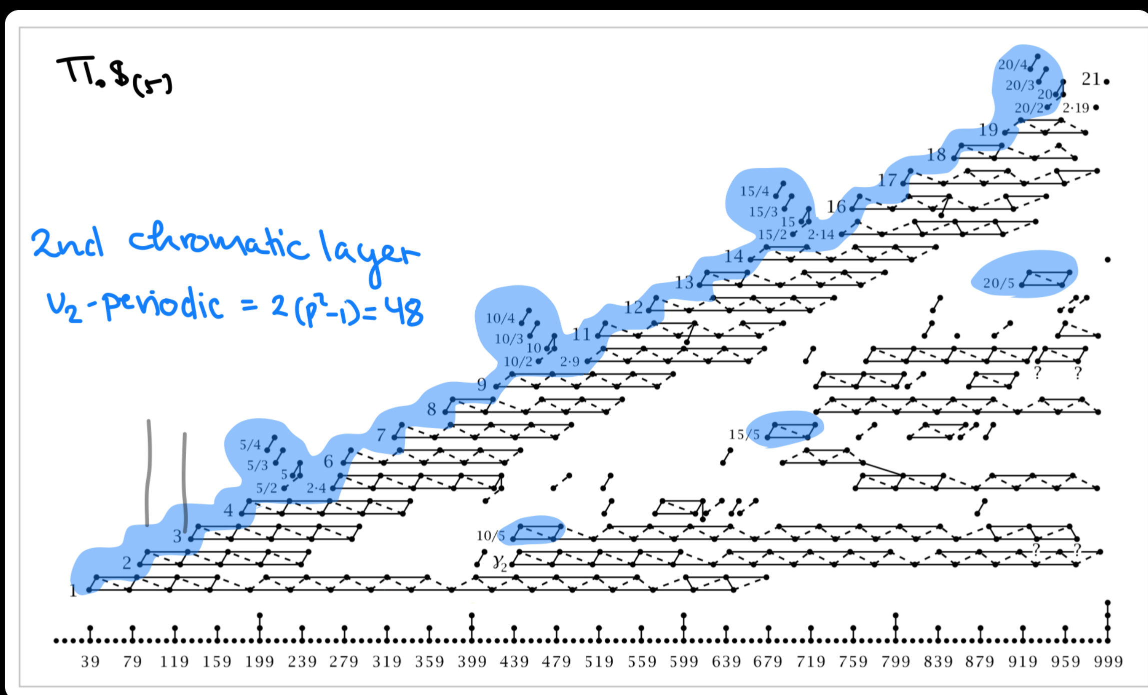
π_* S_(5)
Z/5
Z/5²
Z/5³
1 2 3 4 5/2 5/3 5/4 5 2·4 6 7 8 9 10/2 10/3 10/4 10 2·9 11 12 13 14 15/2 15/3 15/4 15 2·14 16 17 18 19 20/2 20/3 20/4 20 2·19 21
10/5 15/5 20/5
39 79 119 159 199 239 279 319 359 399 439 479 519 559 599 639 679 719 759 799 839 879 919 959 999



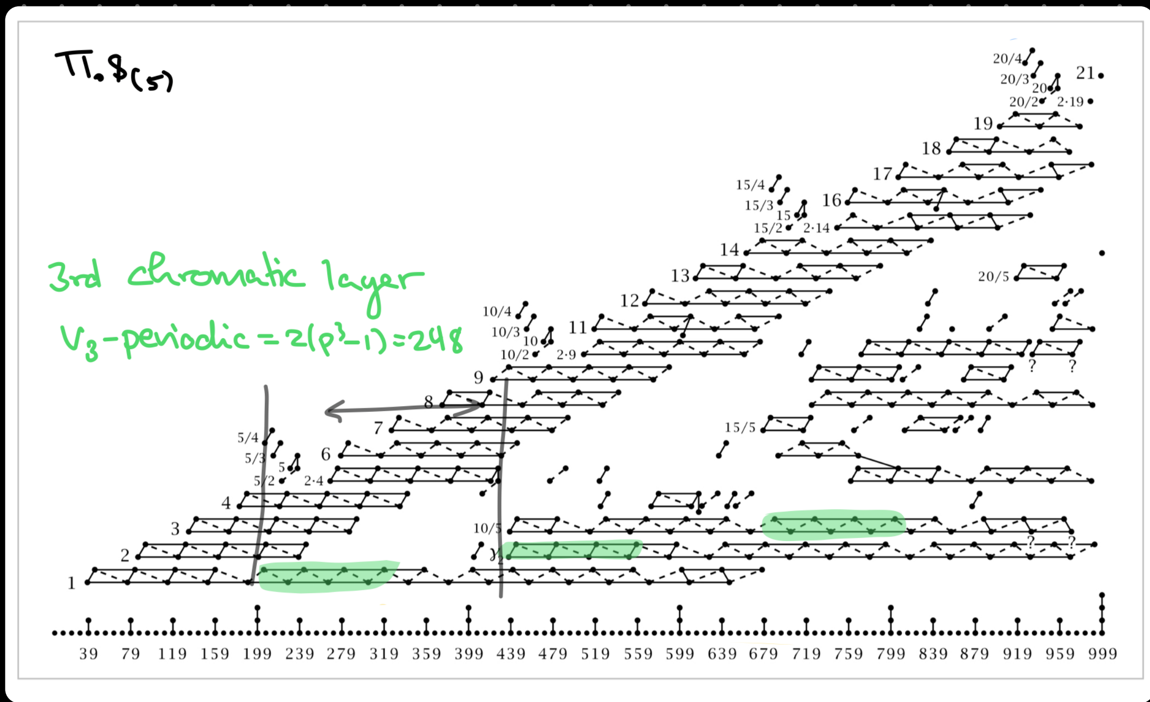
π_* S_(5)
1st chromatic layer
v_1-periodic = 2(p-1) = 8
1 2 3 4 5/2 5/3 5/4 5 2·4 6 7 8 9 10/2 10/3 10/4 10 2·9 11 12 13 14 15/2 15/3 15/4 15 2·14 16 17 18 19 20/2 20/3 20/4 20 2·19 21
10/5 15/5 20/5
39 79 119 159 199 239 279 319 359 399 439 479 519 559 599 639 679 719 759 799 839 879 919 959 999

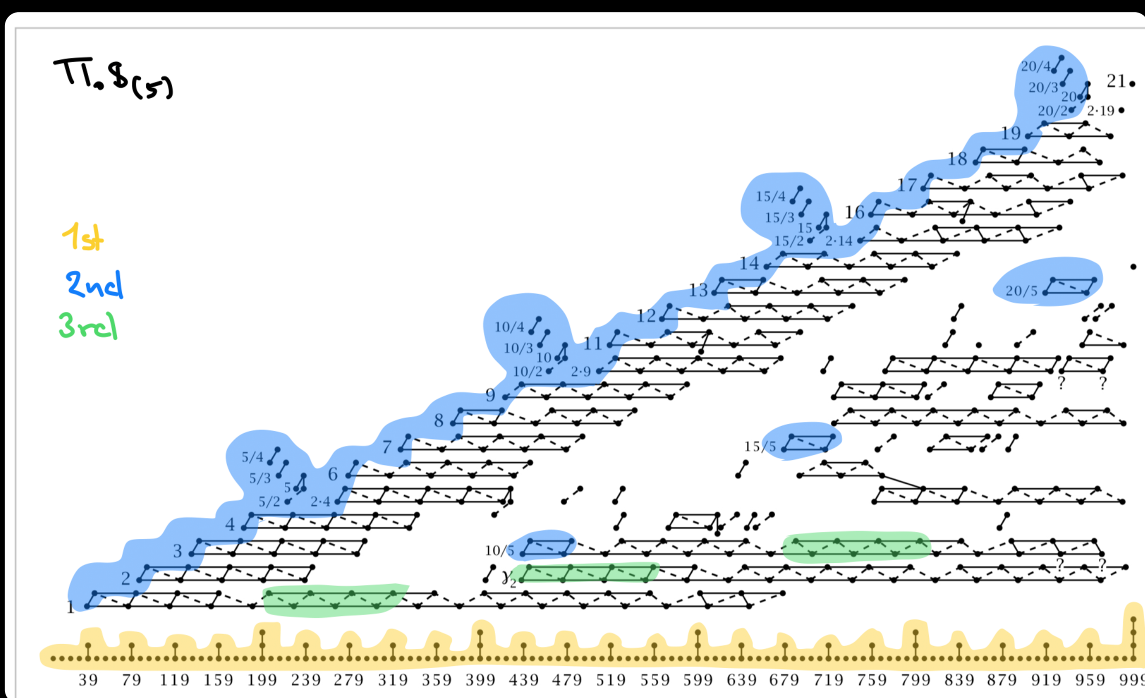


$\pi_*\mathbb{S}_{(5)}$
1st chromatic layer
v_1-periodic $= 2(p-1) = 8$
39 79 119 159 199 239 279 319 359 399 439 479 519 559 599 639 679 719 759 799 839 879 919 959 999

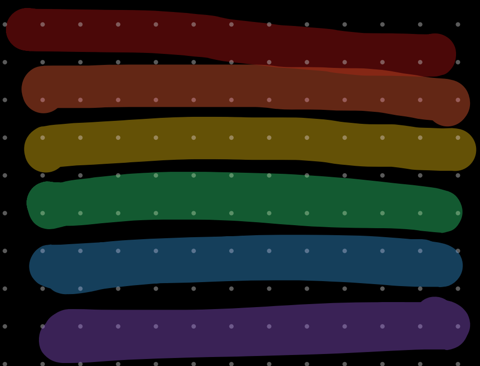


$\pi_*\mathbb{S}_{(5)}$
2nd chromatic layer
v_2-periodic $= 2(p^2-1) = 48$
39 79 119 159 199 239 279 319 359 399 439 479 519 559 599 639 679 719 759 799 839 879 919 959 999


π*S(5)
3rd chromatic layer
v3-periodic = 2(p³−1) = 248
39 79 119 159 199 239 279 319 359 399 439 479 519 559 599 639 679 719 759 799 839 879 919 959 999


π*S(5)
1st
2nd
3rd
39 79 119 159 199 239 279 319 359 399 439 479 519 559 599 639 679 719 759 799 839 879 919 959 999

Red-Blue Shift Conjecture



"Increasing wavelengths"

Red shift — Increases chromatic information

Blue shift — Decreases chromatic information

"Decreases wavelength"

Conjecture: Algebraic K-theory exhibits Red shif

Conjecture: Tate construction exhibits blue shift

RED SHIFT

Idea: R structured ring spectrum related to a FGL, height n

$$\Downarrow$$

$K(R)$ related to a FGL height $n+1$

In terms of periodic families: Homotopy of R is v_n-periodic, but not v_{n+1}-periodic, then often $K(R)$ is v_{n+1}-periodic, but not v_{n+2}.

Thm: |Nullstellensatz '22| The red shift conjecture holds for all (non-zero) commutative ring spectra

To discuss this in more detail, recall that the periodicity theorem tells us that

$$X \text{ type } n \iff X \text{ admits a } v_n\text{-self map}$$

Furthermore, we introduced

$f: \Sigma^k X \to X$ v_n-self map on X type n

Telescope of f $\quad X[f^{-1}] := \operatorname{colim}(X \xrightarrow{\Sigma f} \Sigma^{-k} X \xrightarrow{\Sigma^{-2k} f} \Sigma^{-2k} X \to \cdots)$

Def: For $M_n = M(i_0, \ldots, i_{n-1})$ w. v_n self map f_n, write $Tel(n) := M_n[f_n^{-1}]$.

Telescopic localization

sometime people write 'f' for finite - It's a finite localisation w. $\ker(L_n^f)$ generated by any (finite) $(n+1)$-type spectra

$$L_n^f X := L_{Tel(0) \vee \cdots \vee Tel(n)} X$$

↳ p-local spectrum

Prop: If X is of type $\geq n$ and f is a v_n-self-map of X, then

$$L_n^f X \simeq X[f^{-1}].$$

We further know that if X is $E(m)$-local, for some m then

$$L_n^t X \simeq L_n X \simeq L_{E(n)} X$$

To ease notation today: $T(n) := Tel(n)$. We wish to study $T(n)$-local $\mathbb{E}_\infty$-algebras

Can be shown that $\quad L_{T(n)} R \simeq 0 \iff L_{K(n)} R \simeq 0$

In the case of ring spectra we make the following definition

Def. The **height** of $0 \neq R \in CAlg(Sp)$ is

$$\text{height}(R) := \max\{n \geq -1 \mid T(n) \otimes R \neq 0\}$$

where we set $\quad T(-1) := \mathbb{S}$.

The motivation for this comes from the following theorem.

Thm: | Hahn '16 | $R \in CAlg(Sp)$, $n \geq 0$:

$$R \otimes T(n) = 0 \implies R \otimes T(n+1) = 0.$$

$T(n)$-acyclic $\implies$ $T(n+1)$-acyclic

R height $n \implies L_{T(h)} R$ vanishes for all $h > n$

So the idea is, that if the higher chromatic information for some such R is zero, then the chromatic information is truncated at that level, i.e. there is no "even higher" information either.

So could equivalently:

$$\text{height}(R) = \max\{k \mid L_{T(k)} R \neq 0\}$$

Thm: | Redshift for $\mathbb{E}_\infty$ | Let $0 \neq R \in CAlg(Sp)$ s.t. height$(R) = n$. Then height$(K(R)) = n+1$. so $K(R)$ has a bit more fancy structure

Building blocks to prove that the jump is exactly one:

① No crazy jumps

It was proved in 2020 by Clausen-Mathew-Naumann-Noel that no crazy crazy jumps in height can occour:

Thm: $R \in CAlg(Sp)$: height$(R) = n \implies$ height$(K(R)) \leq n+1$

We can restate this as follows: $\boxed{L_{T(n)} R \simeq 0 \implies L_{T(n+1)} K(R) = 0}$

② Maps of $\mathbb{E}_\infty$-ring spectra & height

Lem: $\text{height}(R) = n$ iff $\exists A \in \text{CAlg}(Sp)$, $\text{height}(A) = n$ together with a map of $\mathbb{E}_\infty$-rings $A \to R$.

So to figure out the height of R, we can try to compare it with something we know have height n.

Intuition about why this holds:

"$\Rightarrow$" use the identity

"$\Leftarrow$" The zero-ring have no non-zero modules, so if we have such a map $A \to R$ from A non-zero, then R is non-zero.

$\sim$ Same holds $T(n)$-locally

Gives us information about the height of R, since it is defined by non-vanishing and vanishing of $T(n)$-locally, by using our known $\mathbb{E}_\infty$-map.

③ Height of K-theory of Lubin-Tate theory

Recall that

Landweber Exact functor theorem $\rightsquigarrow$ Morava E-theory

$$(E_n)_* \simeq \mathbb{Z}_{(p)}[v_1, \dots, v_n, v_n^{\pm 1}]$$
$$W(k)[\![v_1, \dots, v_{n-1}]\!][\beta^{\pm 1}]$$

height n Lubin-Tate theory.

We have $E_n \in \text{CAlg}(Sp)$, and we want $\text{height}(K(n)) = n+1$. This is done by constructing an $\mathbb{E}_\infty$-map

$$K(E_n) \to A$$

where $\text{height}(A) = n+1$.

- $\text{height}(E_{n+1}) = n+1$
- Blue shift: $\text{height}(E_{n+1}^{tC_p}) = n$ — trivial action
- $L_{T(n+1)} K(E_{n+1}^{tC_p}) \neq 0 \Rightarrow K(E_{n+1}^{tC_p}) = n+1$

Red blue shift "inverses"

Dream: Construct a map $E_n \to E_{n+1}^{tC_p}$ which would induce a map on K-theory, but sadly this does not work...

Instead: Construct a map "up to a sequence of Etale extensions"

$$f: E_n \to \underbrace{L_{K(n)} (E_{n+1}^{tC_p})^{Sh}_p}$$

K-theory of this has height n+1.

4 Nullstellensatz

Def: Let $\mathcal{C}$ be a presentable ∞-category. A non-terminal object $C \in \mathcal{C}$ is nullstellensatzian if every compact object in $\mathcal{C}_{C/}$ has a map to the initial object C.

∞-categorical analogue of "finitely generated"

Hilbert's Nullstellensatz Let L be an algebraically closed field, $\mathcal{I}$ some ideal of the polynomial ring $L[X_1, \dots, X_n]$. Then for all common roots of polynomials in L, there exists an L-algebra map

$$L[X_1, \dots, X_n]/\mathcal{I} \longrightarrow L$$

Note that $L[X_1, \dots, X_n]/\mathcal{I}$ is a finitely generated commutative algebra over L

$\rightsquigarrow$ L is "nullstellensatzian"

Rem: Nullstellensatzian objects in $\mathrm{CAlg}(\mathrm{Ab})$ are exactly the algebraically closed fields L, since all of the 'compact objects' finitely generated in $\mathrm{CAlg}_{L/}$, have a map $A \to L$ by Hilbert's nullstellensatz

nullstellensatzian objects ~ those that behave like algebraically closed fields in the category of rings

Chromatic Nullstellensatz Let $0 \neq R \in \mathrm{CAlg}(Sp_{T(n)})$. Then R is nullstellensatzian iff there exists some algebraically closed field L, such that

$$R \simeq E_n(L).$$

Object in $\mathrm{CAlg}(Sp_{T(n)})$ ~ $\mathbb{E}_\infty$-ring R + $L_{T(n)} R \neq 0$ ($T(n)$-local)

Hence: Nullstellensatz $T(n)$-local $\mathbb{E}_\infty$-rings are exactly the Lubin-Tate theories over algebraically closed fields

"$E_n(L)$ = 'algebraically closed field objects' in $Sp_{T(n)}$"

An important result they use to prove this, is the existence of maps

$$A \longrightarrow E_n(L)$$

More generally: $R \in \mathrm{CAlg}(Sp_{T(n)})$
$\Rightarrow \exists$ perfect algebra A of Krull dimension 0 and a nilpotence detecting map $R \to E_n(A)$

for any $T(n)$-local $\mathbb{E}_\infty$-ring A.

What have we gained? For any $\mathbb{E}_\infty$-ring spectrum R we have

- R is $T(n)$-acyclic

or

- There is a map of $\mathbb{E}_\infty$-ring spectra $R \to E_n(L)$

The map

$$R \longrightarrow E_n(L) \quad \rightsquigarrow \quad K(R) \longrightarrow K(E_n(L))$$

We know redshift holds for Lubin-Tate spectra, so

$$\text{height}(K(E_n(L))) = \text{height}(E_n(L)) + 1 = n+1$$

So we conclude

$$\boxed{\text{height}(K(R)) = n+1}$$