Talk 1: Introduction

12.01.22

§1 Motivation: K_0 and K_1

The roots of algebraic K-theory is two abelian groups $K_0(R)$ and $K_1(R)$ defined for any unital ring R.

Let R be a unital ring, we let $\mathcal{P}_R$ denote the category w.

$$\mathrm{ob}\mathcal{P}_R = \text{finitely generated (right) } R\text{-modules}$$

$$\text{morphisms} = \text{homomorphisms of } R\text{-modules}$$

and let $I(R)$ denote the set of isomorphism classes is $\mathrm{ob}\mathcal{P}_R$.

For $P \in \mathcal{P}_R$ we write $\langle P \rangle$ for the isomorphism class in $I(R)$ it defines.

A set with a binary operation which is associative & unit

$I(R)$ is a commutative monoid with addition given by

$$\langle P \rangle + \langle Q \rangle = \langle P \oplus Q \rangle$$

and unit $\langle 0 \rangle$.

So we can take the "group completion" of this to obtain an abelian group (originally called Groethendieck group):

$$K_0(R) = \{ \underbrace{\langle P \rangle - \langle Q \rangle}_{\text{formal differences}} \mid P, Q \in \mathrm{ob}\mathcal{P}_R \}$$

"Klasse" Groethendieck 1957

with addition

$$(\langle P_0 \rangle - \langle Q_0 \rangle) + (\langle P_1 \rangle - \langle Q_1 \rangle) = \langle P_0 \oplus P_1 \rangle - \langle Q_0 \oplus Q_1 \rangle$$

Alternative: $K_0(R) = I(R) \times I(R) /\sim$

$$(\langle P \rangle, \langle Q \rangle) \sim (\langle P \oplus N \rangle, \langle Q \oplus N \rangle)$$

some $N \in I(R)$

<u>Note:</u> $K_0(R)$ only depends on R through $\mathcal{P}_R$

$K_1(R)$ was defined by Whitehead in 1940's:

$GL_n(R)$ = group of invertible $n \times n$ matrices w. entries in R

then we have inclusions

$$GL_n(R) \subset GL_{n+1}(R) \subset \cdots \qquad GL(R) := \bigcup_{n=0}^{\infty} GL_n(R)$$

$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

$$K_1(R) := GL(R) / \underbrace{[GL(R), GL(R)]}$$

commutator subgroup of $GL(R)$ generated by its commutators

$$[g,h] = ghg^{-1}h^{-1}$$

By the universal property of $[G,G]$ we get the following universal property of $K_1(R)$:

$G/[G,G]$ is abelian group, and every homomorphism from G into an abelian group factors through $G/[G,G]$

Every homomorphism from $GL(R)$ into an abelian group, must factor through the natural quotient $GL(R) \to K_1(R)$.

It was showed by Whitehead 1950 that

$$[GL(R), GL(R)] \cong E(R),$$

where

$$E(R) = \bigcup_{n=0}^{\infty} E_n(R)$$

— subgroup of $GL_n(R)$ generated by all elementary matrices $\underbrace{e_{i,j}(r)}$ w. $1 \leq i,j \leq n$, $r \in R$

matrix w. 1 in diagonal, r in i-j and 0 otw.

is the group of unital matrices, hence

$$K_1(R) \cong GL(R) / E(R)$$

Both K_0 and K_1 are interesting but very hard to explicitly desribe/compute.

— Higher algebraic K-theory —

Around 1970, Quillen defined the higher algebraic K-groups of a ring R as the homotopy groups of a a certain topological space K(R):

$$K_n(R) := \pi_n K(R),$$

satisfying that

$$\pi_0 K(R) \cong K_0(R), \quad \pi_1 K(R) \cong K_1(R).$$

He did this first through the "+-construction"

$$K(R) = BGl(R)$$

but this only defined $K_n(R)$ for $n \geq 1$

— can be fixed by writing $K(R) := K_0(R) \times BGl(R)^+$

He later defined it through the "Q-construction", which holds in greater generality, agrees with the +-construction in the relevant cases: $K(\mathcal{C}) := \Omega BQ\mathcal{C}$

Later Waldhausen extended Quillen's K-space from rings to ring spectra throug his S-dot construction:

$$K(\mathcal{C}) := \Omega |wS_\bullet \mathcal{C}|$$

We will define K-theory space K(R), firstly for a ring R, using Waldhausen's constructions instead of Quillen's — But we will later introduce this aswell and show that they agree.

A necessary preliminarie to understand Waldhausen's S-dot construction is simplicial & -bisimplicial sets.

§2 Simplicial sets

<u>Def:</u>

- $\Delta :=$ category w.

$$\mathrm{ob}\,\Delta = \{[n] \mid \forall n \geq 0\} \quad \{0 \to 1 \to 2 \to \cdots \to n\}$$

$$\Delta([n],[m]) = \text{weakly increasing maps}$$

$$\theta : [n] \longrightarrow [m]$$

$$\theta(i) \leq \theta(j) \quad \forall i \leq j$$

- A simplicial object in a category $\mathcal{C}$ is a functor

$$X : \Delta^{op} \longrightarrow \mathcal{C}.$$

For $\mathcal{C} = \mathrm{Set}$ it's called simplicial set

sSet

- This forms a category

$$s\mathcal{C} := \mathrm{Fun}(\Delta^{op}, \mathcal{C}).$$

Lets unravel the definition. $X \in s\mathcal{C}$ consists of

$$X[n] \in \mathcal{C} \quad \forall n \in \mathbb{N}$$

and morphisms in $\mathcal{C}$:

$$\alpha^* : X[k] \longrightarrow X[n] \quad \forall\ \alpha : [n] \to [k] \text{ in } \Delta$$

The indices changes direction since it's contravariant

s.t.
- $(\alpha \circ \beta)^* = \beta^* \circ \alpha^*$
- $(id)^* = id$

A morphism of simplicial sets $f : X \longrightarrow Y$ consists of maps $f[n] : X[n] \to Y[n]$ in $\mathcal{C}$ s.t.

$$\begin{tikzcd} X[n] \arrow[r, "f{[n]}"] & Y[n] \\ X[k] \arrow[u, "\alpha^*"] \arrow[r, "f{[k]}"'] & Y[k] \arrow[u, "\alpha^*"'] \end{tikzcd} \qquad \forall\ \alpha : [n] \longrightarrow [k].$$

There are special maps in Δ that "generates" this category:

$s^i : [n+1] \longrightarrow [n] \quad \forall i \in \{0, \_, n\}$ (co)degeneracies

unique surjective map in Δ s.t. $|(s^i)^{-1}(i)| = 2$.

$$s^i(j) = \begin{cases} j & j \leq i \\ j-1 & j > i \end{cases}$$

$$[0, \_, i, i+1, \_, n+1] \mapsto [0, \_, i-1, i, i, i+1, \_, n]$$

and

$d^i : [n] \longrightarrow [n+1] \quad \forall i \in \{0, \dots, n+1\}$ (co) faces

unique injection in Δ s.t. $i \notin \mathrm{Im}(d^i)$.

$$d^i(j) = \begin{cases} j & j < i \\ j+1 & j \geq i \end{cases}$$

$$[0, \_, i, \_, n] \longmapsto [0, \_, i-1, i+1, \_, n+1]$$

It turns out that every morphism in Δ factors (non-uniquely) as a composite of d^i and s^i's

$\rightsquigarrow$ A simplicial object is equivalent to objects X_n $\forall n \in \mathbb{N}$ together with maps

degeneracy $s_i := (s^i)^* : X[n] \to X[n+1] \quad \forall i \in \{0, \_, n\}$

face $d_i := (d^i)^* : X[n+1] \longrightarrow X[n] \quad \forall i \in \{0, \_, n+1\}$

subject to the "simplicial relations".

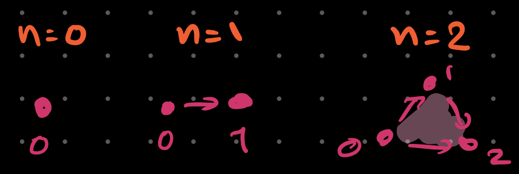
Def. n-simplex $\Delta^n := \mathrm{Hom}_\Delta(-, [n]) : \Delta^{op} \longrightarrow \mathrm{Set}$

$$\Delta^n[k] = \mathrm{Hom}_\Delta([k], [n])$$

We get that

$$\alpha : [n] \longrightarrow [m]$$

$$\rightsquigarrow \quad \alpha_* : \Delta^n \longrightarrow \Delta^m$$



$$(f:[k]\to[n]) \longmapsto \alpha_* f = \alpha\circ f : [k]\to[m]$$

Assembles to a functor

$$\Delta^\bullet : \Delta \longrightarrow sSet$$

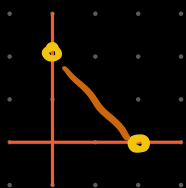
$$\begin{tikzcd}
{[n]} \arrow[r, mapsto] \arrow[d, "\alpha"] & \Delta^n \arrow[d, "\alpha_*"] \\
{[m]} \arrow[r, mapsto] & \Delta^m
\end{tikzcd}$$

<u>Def.</u> Topological n-simplex

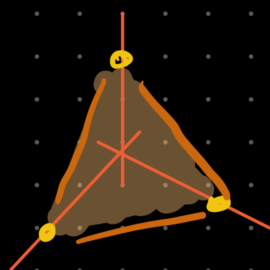
$$\Delta^n_{top} := \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; 0 \leq t_i \leq 1,\ \textstyle\sum_{i=0}^n t_i = 1 \right\}$$

convex hull

n=1



n=2



⇝ Assembles into a cosimplicial space

$$\Delta^\bullet_{top} : \Delta \longrightarrow Top$$

$$\begin{tikzcd}
{[n]} \arrow[r, mapsto] \arrow[d, "\alpha"'] & \Delta^n_{top} \arrow[d, "\alpha_*"] & (t_0, \ldots, t_n) \arrow[d, mapsto] \\
{[k]} \arrow[r, mapsto] & \Delta^k_{top} & (u_0, \ldots, u_k)
\end{tikzcd}$$

$$u_i = \sum_{j \in \alpha^{-1}(i)} t_j$$

$$\Sigma_\emptyset = 0$$

Def: The **nerve** $N\mathcal{C}$ of a category $\mathcal{C}$ is the simplicial set w.

$$N\mathcal{C}[m] := \mathrm{Hom}_{cat}([m], \mathcal{C}) \quad + \text{ simplicial operation w. precomposition}$$

So can think of $N\mathcal{C}[m]$ as a composable string

$$c_0 \longrightarrow c_1 \longrightarrow \cdots \longrightarrow c_m$$

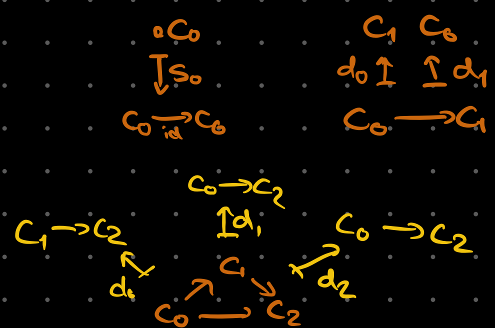
- $N\mathcal{C}[0]$ = objects of $\mathcal{C}$
- $N\mathcal{C}[1]$ = morphisms of $\mathcal{C}$

This gives a (fully faithful) functor

$$N : \mathrm{Cat} \longrightarrow \mathrm{sSet}$$

Ex: $N([m]) \cong \Delta^m$

$$\begin{array}{c} N\mathcal{C}[0] = \mathrm{ob}\,\mathcal{C} \\ \text{target} = d_0 \uparrow \quad \downarrow s_0 = id \quad \uparrow d_1 = \text{source} \\ N\mathcal{C}[1] = \mathrm{Hom}\,\mathcal{C} \\ d_0 = pr \uparrow \quad \uparrow d_1 = comp \uparrow \quad d_2 = pr \\ N\mathcal{C}[2] = \{ c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \} \end{array}$$



Def: The **geometric realization** of $X : \Delta^{op} \to \mathrm{Set}$ is the topological space

$$|X| := \Big(\underbrace{\coprod_{n \in \mathbb{N}} \underbrace{X_n \times \Delta^n_{top}}_{\text{Product topology}}}_{\text{Disjoint union}} \Big) \Big/ \begin{array}{l} \alpha : [n] \longrightarrow [k] \\ x \in X_k, \; t \in \Delta^n_{top} \\ (\alpha^* x, t) \sim (x, \alpha_* t) \end{array}$$

This defines a functor

$$|{-}| : \mathrm{sSet} \longrightarrow \mathrm{Top}$$

$$\begin{array}{ccccc} X & \longmapsto & |X| & & [x, t] \\ \downarrow f & & \downarrow |f| & & \downarrow \\ Y & \longmapsto & |Y| & & [f(x), t] \end{array}$$

The idea is to replace each $x \in X_n$ with Δ^n_{top} and then glue them together

Def: The **classifying space** of a category $\mathcal{C}$ is the geometric realization of its nerve

$$B\mathcal{C} := |N\mathcal{C}|$$

⤳ Is a functor

$$B : \mathrm{Cat} \longrightarrow \mathrm{Top}$$

§3 Bisimplicial sets

We can extend this to **Bisimplicial sets**

$$X[-,-] : \Delta^{op} \times \Delta^{op} \longrightarrow \mathrm{Set}$$

- Introduce bisimplicial sets
  - p.64 (including $X_L[-], X_R[-] : \Delta^{op} \to k\text{-top}$, so define k-top (5.3))
  - State Realization theorem (6.12)
  - Def 6.13 (Good simplicial space)
  - Thm 6.14

So given

$$\theta : [m] \longrightarrow [m'] \,, \quad \eta : [n] \longrightarrow [n'] \quad \text{in } \Delta$$

we get a commutative diagram

$$\begin{tikzcd} X[m,n] \arrow[r, "\theta^*"] \arrow[d, "\eta^*"] & X[m,n'] \arrow[d, "\eta^*"] \\ X[m',n] \arrow[r, "\theta^*"] & X[m',n'] \end{tikzcd}$$

to such a bisimplicial set there is 3 simplicial sets:

variable in left $X_L[n][-] := X[-,n] : \Delta^{op} \longrightarrow \mathrm{Set}$

Variable in right $X_R[m][-] := X[m,-] : \Delta^{op} \longrightarrow \mathrm{Set}$

Diagonal $\delta X[-] : \Delta^{op} \xrightarrow{\text{diag}} \Delta^{op} \times \Delta^{op} \xrightarrow{X[-,-]} \mathrm{Set}$

We will need the simplicial spaces we obtain by taking geometric realization of $X_L[n][-]$ and $X_R[m][-]$, for which we need the following:

Def. i) $U \subset X \in \mathrm{Top}$ is **k-open** or compactly open if for every continuous map $f : K \to X$ w. K compact Hausdorff, $f^{-1}(U) \subset K$ is open in K.

ii) X is a **k-space** if k-open subsets are open

iii) $k\text{-top} \subset \mathrm{Top}$ full subcat. of k-spaces

Note that compact Hausdorff spaces are k-spaces, and it can be shown that $|X| \in$ k-top.

$$X_L[-] : \Delta^{op} \longrightarrow \text{k-top}, \quad X_L[n] := |X_L[n][-]|$$
$$X_R[-] : \Delta^{op} \longrightarrow \text{k-top}, \quad X_R[m] := |X_R[m][-]|$$

Lem: 6.8 Natural homeomorphisms
$$|X_L[-]| \cong |\delta X[-]| \cong |X_R[-]|$$

Therefor one can write $|X[-,-]|$ for any of these homeomorphic spaces.

Lem: 6.12 |Realization lemma| Let
$$f[-,-] : X[-,-] \longrightarrow Y[-,-]$$
be a morphism of bisimplicial sets and assume that for any $m \geq 0$, the map
$$|f[m,-]| : |X[m,-]| \longrightarrow |Y[m,-]|$$
is a homotopy equivalence. Then the induced map on the realization on the diagonals
$$|\delta f[-]| : |\delta X[-]| \longrightarrow |\delta Y[-]|$$
is a homotopy equivalence

Equivalently $|f_R[-]| : |X_R[-]| \to |Y_R[-]|$ is a homotopy eq.

The same holds w.r.t. the other variable.

One would assume that if $f[n]$ is a homotopy equivalence for all $[n]$, then the same would hold for $|f[-]|$ - but this is not true in general.

Def: A simplicial space $X[-]$ is good if the inclusion
$$s_i(X[n-1]) \longrightarrow X[n]$$
is a closed cofibration for all i and n

Ex: $X_L[-]$ and $X_R[-]$ are both good simplicial spaces

Recall: A continuous inclusion $f: A \to B$ is a cofibration provided that given maps

$$g: B \to C, \quad G: A \times I \to C$$

with $G(a,0) = g(f(a))$, there exists $F: B \times I \to C$ s.t.

$$F(f(a), t) = G(a,t), \quad F(b,0) = g(b)$$

$\forall (a,t) \in A \times I, b \in B$. It's called a closed cofibration if it maps closed subspaces to closed subspaces.

Thm 6.14: |Segal, May| Let $f[-]: X[-] \to Y[-]$ be a map of good simplicial spaces. If each $f[n]$ is a homotopy equivalence, then $|f[-]|: |X[-]| \to |Y[-]|$ is a homotopy equivalence.