THR of Poincaré ∞-Categories

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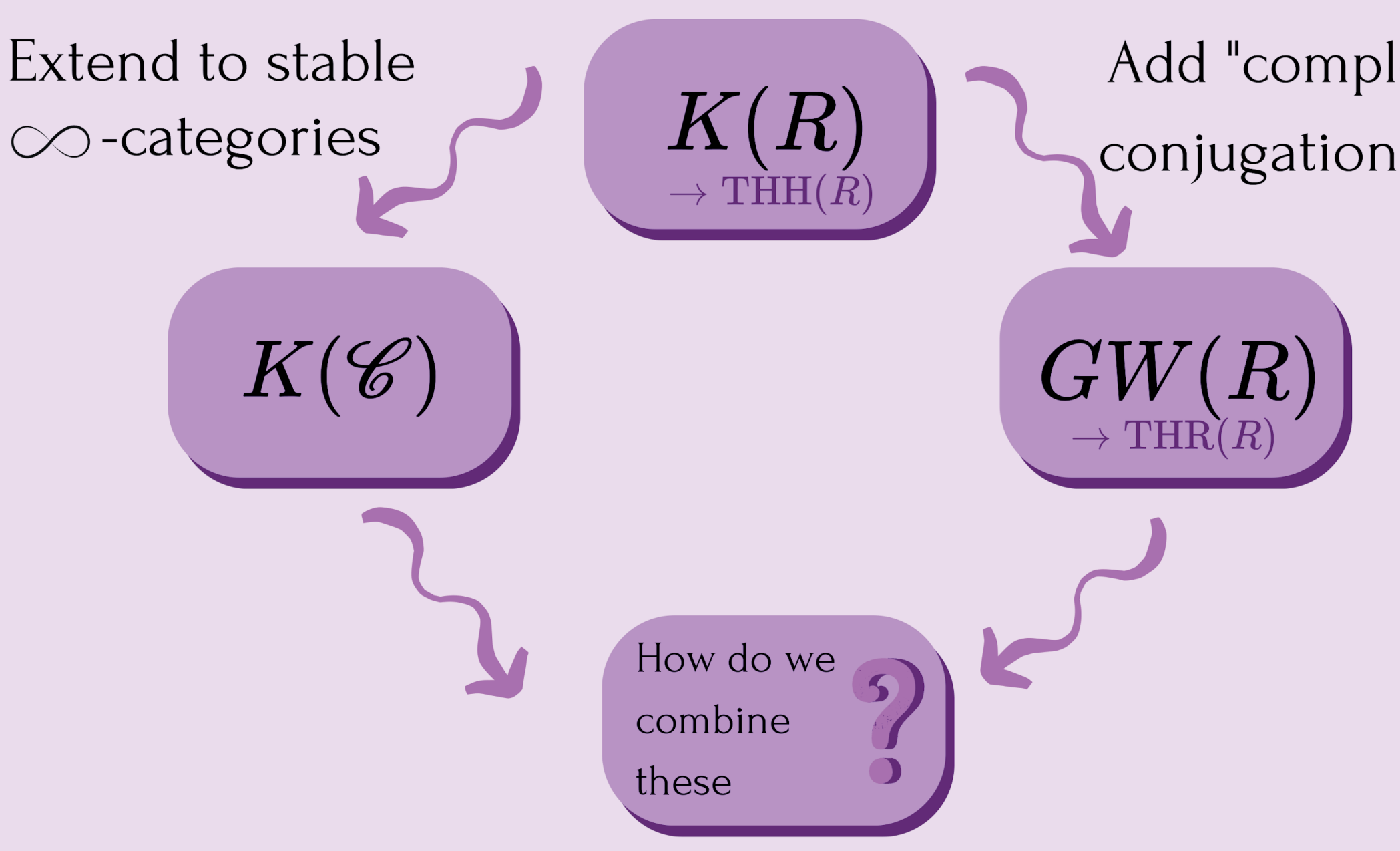
MOTIVATION Algebraic K-theory ~ Hard!

Algebraic K-theory was originally studied as an invariant of rings, but even the simplest cases has been notoriously hard to calculate. Because of this another invariant called Topological Hochschild Homology (THH) was introduced - Still not easy, but a bit more pleasant, and due to the so called trace map, which locally has a constant difference, we can use this as a tool to understand Algebraic K-theory

$$K(R) \xrightarrow{tr} THH(R)$$

The notion of K-theory has since then been generalized in several ways, but two main ways stand out: Adding extra structure to R that looks like complex conjugation, or consider stable ∞-categories instead of R.



Groethendick-Witt spectrum of Poincaré ∞-category $\mathcal{GW}(\mathscr{C}, \Qoppa)$

EXTENDING TO $GW(R)$

Warmup: Complex vector bundles

$$\begin{tikzcd} E \arrow[r, "\textcircled{3}\ q"] \arrow[d, "\textcircled{1}"'] & E^* \arrow[dl, "\textcircled{2}"] \\ X \arrow[loop below, "\textcircled{3}\ C_2"] & \end{tikzcd}$$

① Complex vector bundle

⇝ Complex K-theory KU

② Add complex conjugation

⇝ C_2-action on KU

③ Add C_2-action on X, and q with

$$q^*q = c : E \cong E^{**}$$

The natural isomorphism

⇝ Genuine C_2-spectrum

$$KR \text{ with} \begin{cases} \text{Underlying spectrum } KU \\ KR^{tC_2} \simeq * \\ KR_{hC_2} \simeq KR^{hC_2} \simeq KO \end{cases}$$

Forms on rings

Definition

Let R be a ring with anti-involution $(-)^* : R^{op} \to R$

- A symmetric bilinear form on R is a pair (P, q)
  - P: finitely generated projective R-Module
  - $q : P \otimes P \to R$ an $R \otimes R$-linear map with $q(x \otimes y) = q(y \otimes x)^*$
- A form (P, q) is unimodular if

$$q_\# : P \xrightarrow{\sim} \hom_R(P, R)$$
$$x \mapsto (y \mapsto q(x \otimes y))$$

How to study rings with forms

$$\begin{tikzcd} \left\{\begin{array}{l}\text{Finitely generated}\\ \text{projective R-modules}\end{array}\right\} \arrow[d, squiggly] & \left\{\begin{array}{l}\text{Unimodular symmetric}\\ \text{bilinear forms on R}\end{array}\right\} \arrow[d, squiggly] \\ K_0(R) \arrow[r, "\text{Hyp}"] & GW_0(R) \arrow[d, "\text{Cokernel}"] \\ & L_0(R) \end{tikzcd}$$

$$P \longmapsto (P \oplus P^*, \text{ hyperbolic form})$$

L-theory group ~"Forms/cobordism" $L_0(R)$

This can be promoted to a fiber sequence of spectra!

Theorem

There exists a fiber sequence in spectra

$$K(R)_{hC_2} \to GW(R) \to L(R)$$

How do we generalize this to ∞-categories?

Q A

The good idea
Forms = "Extra structure" on $\mathcal{D}^p(R)$ assigning to each object a space of forms

EXTENDING TO STABLE ∞ -CATEGORIES

Definition

A Hermitian ∞-category is a pair $(\mathscr{C}, \Qoppa)$ consisting of

- $\mathscr{C}$: stable ∞-category
- $\Qoppa : \mathscr{C}^{op} \to \mathcal{S}p$ quadratic functor:
  - $\Qoppa$ is reduced
  - $B_\Qoppa(x, y) = \mathrm{fib}(\Qoppa(x \oplus y) \to \Qoppa(x) \oplus \Qoppa(y))$ is bilinear
  - $\Lambda_\Qoppa(x) = \mathrm{cofib}(B_\Qoppa(x, x)_{hC_2} \to \Qoppa(x))$ is linear

2-excisive

Let $(\mathscr{C}, \Qoppa)$ be a Hermitian ∞-category. If $B_\Qoppa(-,-)$ is representable in each variable we get an adjunction

$$\begin{tikzcd} \mathscr{C}^{op} \arrow[r, shift left, "D_\Qoppa"] \arrow[r, phantom, "\perp"] & \mathscr{C} \arrow[l, shift left, "D_\Qoppa^{op}"] \end{tikzcd}$$

- $\Qoppa$ is perfect if the unit $\mathrm{ev} : \mathrm{id}_\mathscr{C} \Rightarrow D_\Qoppa D_\Qoppa^{op}$ is an equivalence
- A Hermitian ∞-category $(\mathscr{C}, \Qoppa)$ is called Poincaré if $\Qoppa$ is perfect quadratic

Classification of quadratic functors

$$\begin{tikzcd} \Qoppa(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Lambda_\Qoppa(x) \arrow[d] \\ B_\Qoppa(x,x)^{hC_2} \arrow[r] & B_\Qoppa(x,x)^{tC_2} \end{tikzcd}$$

⇝ C_2-spectrum $\mathrm{map}_\mathscr{C}(x, D_\Qoppa(x)) \simeq B_\Qoppa(x, x)$ with

- $\mathrm{map}_\mathscr{C}(x, D_\Qoppa(x))^{C_2} \simeq \Qoppa(x)$
- $\Phi^{C_2}\mathrm{map}_\mathscr{C}(x, D_\Qoppa(x)) \simeq \Lambda_\Qoppa(x)$

Example: Symmetric forms

Let R be a ring with anti-involution $(-)^* : R^{op} \to R$

$$\Qoppa^s : \mathcal{D}^p(R)^{op} \to \mathrm{Sp}$$
$$P \mapsto \mathrm{map}_{R \otimes R}(P \otimes P, R)^{hC_2}$$

⇝ $(\mathcal{D}^p(R), \Qoppa^s)$ is a Poincaré ∞-category

Space of forms:

$$q \in \Omega^\infty \Qoppa^s(P) \simeq \mathrm{Map}_{R \otimes R}(P \otimes P, R)^{hC_2}$$
$$q : P \otimes P \to R \text{ with } q(x \otimes y) \simeq q(y \otimes x)^*$$

Poincaré ∞-category ⇝ $\mathcal{GW}(\mathscr{C}, \Qoppa)$ and $\mathcal{L}(\mathscr{C}, \Qoppa)$

Theorem

For every Poincaré ∞-category $(\mathscr{C}, \Qoppa)$ there is a fiber sequence of spectra

$$K(\mathscr{C})_{hC_2} \to \mathcal{GW}(\mathscr{C}, \Qoppa) \to \mathcal{L}(\mathscr{C}, \Qoppa)$$

There is a genuine C_2-spectrum

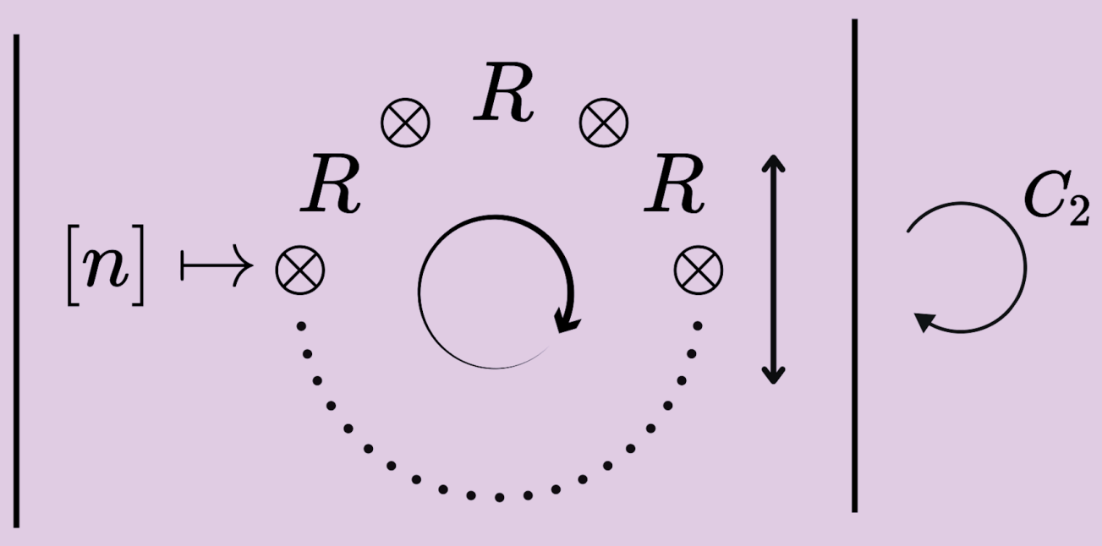
$$K\mathrm{R}(\mathscr{C}, \Qoppa) \text{ with} \begin{cases} \text{Underlying spectrum } K(\mathscr{C}) \\ K\mathrm{R}(\mathscr{C}, \Qoppa)^{C_2} \simeq \mathcal{GW}(\mathscr{C}, \Qoppa) \\ \Phi^{C_2} K\mathrm{R}(\mathscr{C}, \Qoppa) \simeq \mathcal{L}(\mathscr{C}, \Qoppa) \end{cases}$$

EXTENDING TOPOLOGICAL HOCHSCHILD HOMOLOGY

Classically

Let R be a ring with anti-involution

$$\mathrm{THR}(R) := R \otimes_{NR} R \simeq$$



Note: This works for any associative ring spectrum with anti-involution

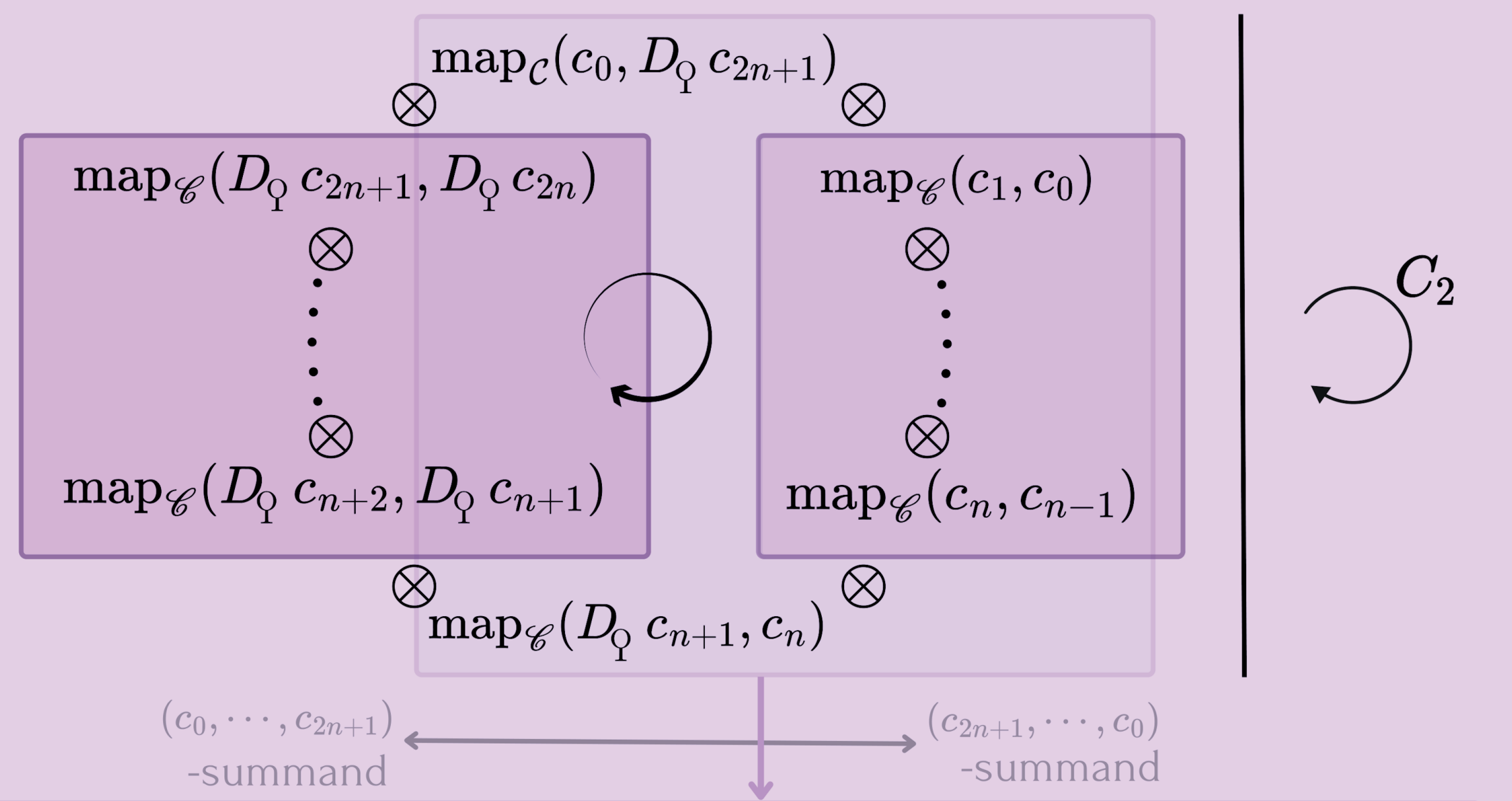
Property: $\Phi^{C_2}\mathrm{THR}(R) \simeq \Phi^{C_2} R \otimes_R \Phi^{C_2} R$

For Poincaré ∞-categories

Trying to resemble the picture above, a point-set model for $\mathrm{THR}(\mathscr{C}, \Qoppa)$ would be expected to be of the following form:

Note: This is indeed how it is for spectral categories with strict duality, but we do not have this point-set description for Poincaré ∞-categories

$$\mathrm{THR}(\mathscr{C}, \Qoppa) \sim \Big| [n] \mapsto \operatorname*{colim}_{c_0, \dots, c_{2n+1} \in \mathscr{C}^\simeq}$$



$$\Phi^{C_2}\mathrm{THR}(\mathscr{C}, \Qoppa) \sim \Big| [n] \mapsto \operatorname*{colim}_{c_0, \cdots, c_n \in \mathscr{C}^\simeq} \overbrace{\Phi^{C_2}\mathrm{map}_\mathscr{C}(c_0, D_\Qoppa\, c_0)}^{\Lambda_\Qoppa(c_0)} \otimes \mathrm{map}_\mathscr{C}(c_1, c_0) \otimes \cdots \otimes \mathrm{map}_\mathscr{C}(c_n, c_{n-1}) \otimes \overbrace{\Phi^{C_2}\mathrm{map}_\mathscr{C}(D_\Qoppa\, c_n, c_n)}^{\Lambda_\Qoppa(D_\Qoppa\, c_n)} \Big|$$

Theorem [R.]

There exists a functor

$$\Phi^{C_2}\mathrm{THR} : \mathrm{Cat}^p_\infty \longrightarrow \mathcal{S}p$$

which on objects is given by the above formula, and such that

$$\Phi^{C_2}\mathrm{THR}(\mathcal{D}^p(\underbrace{R}_{\text{Discrete ring}}), \overbrace{\Qoppa^{gs}}^{\text{Genuine symmetric}}) \simeq \Phi^{C_2} R \otimes_R \Phi^{C_2} R$$

In the case of symmetric forms:
$$\Phi^{C_2}\mathrm{THR}(\mathcal{D}^p(R), \Qoppa^s) \simeq \underbrace{R^{tC_2}}_{\text{equivalent to the geometric fixed points of the Borel completion}} \otimes_R \Phi^{C_2} R$$

For Poincaré ∞-categories: Calmés-Dotto-Harpaz-Hebestreit-Land-Moi-Nardin-Nikolaus-Steimle "Hermitian K-theory for stable ∞-categories" I, II & III [arXiv:2009.07223, arXiv:2009.07224, arXiv:2009.07225]
For spectral categories with duality: Dotto-Moi-Patchkoria-Reeh "Real topological Hochschild homology" [arXiv:1711.10226]