THR of Poincaré ∞-Categories

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§0: Motivation

Algebraic K-theory is hard!

§0: Motivation for algebraic K-theory

K-theory is connected to <u>many</u> <u>areas</u>! Geometry, topology, algebra, number theory

Algebraic
Arithmetic Objects ⟶ K-groups
Geometric

contains detailed information regarding the original object

Cool stuff using K-theory

- Whitehead torsion & group ⇝ s-cobordism theorem
- Motivic cohomology & specifically Chow groups
- Groethendieck-Riemann-Rock theorem
- Atiyah-Singer index theorem 2nd proof ~'62
- Early example of an extraordinary cohomology theory '59 Atiyah-Hirzebruch

History time! K_0 - '57 Groethendieck

Introduced while Groethendieck formulated the Groethendieck-Riemann-Roch theorem

R a unital ring

$$\left\{ \begin{array}{c} \text{finitely generated} \\ \text{projective } R\text{-modules} \end{array} \right\} \Big/ \cong$$ is a commutative monoid

$\langle P\rangle + \langle Q\rangle = \langle P \oplus Q\rangle$

unit $\langle 0\rangle$

Group complete $\longrightarrow$ $K_0(R) = (\cdots)^{grp} = \{\langle P\rangle - \langle Q\rangle\}$ abelian group

↑ Formal differences

Idea: Forces exact sequences to split

- Universal way of assigning an invariant in a way that is compatible with exact sequences.

History time! K_1 ~ '40s

Whitehead
Bass-Schanuel

$$K_1(R) = GL(R)/[GL(R),GL(R)]$$

$GL_n(R)$ = Group of invertible $n \times n$-matrices with entries in R

$GL_n(R) \subset GL_{n+1}(R) \subset \cdots \subset GL(R) := \bigcup_{n=0}^{\infty} GL_n(R)$

⊂ Commutater subgroup

Similar group was earlier introduced by Whitehead:

- Poincaré: attempted to define the Betti numbers of a manifold via triangulations
- Triangulations stable under subdivision $\overset{\text{Whitehead}}{\rightsquigarrow}$ simple homotopy equivalence
  $\rightsquigarrow$ torsion

  simple homotopy equivalence is a finer invariant than homotopye equivalence
- Torsion (homotopy equivalence) $\in \underbrace{\text{Whitehead group}}$

Quotient of $K_1(\mathbb{Z}\pi)$

Integral group ring of π_1(target)

History time! Higher K-groups - Quillen

Q-construction '72 Need categorical input!

Still has it's roots in Grothendieck's K_0, but produces a category instead of an abelian group

Exact category, $\mathcal{C}$ $\rightsquigarrow$ $Q\mathcal{C}$ $= \begin{cases} \text{ob } Q\mathcal{C} = \text{ob}\,\mathcal{C} \\ \text{morphisms} = \text{short exact sequences} \end{cases}$ category

satisfies properties similar to, but weaker than, properties satisfied by Mod_R, $\text{Vect}(\mathbb{R})$

↙ Fixes indexing

$K(\mathcal{C}) := \Omega B Q\mathcal{C}$ space

Recovers K_0 and K_1!

K-theory is hard

$$> K_n(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & , \ n=0 \\ \mathbb{Z}/2 & , \ n=1,2 \\ \mathbb{Z}/48 & , \ n=3 \\ 0 & , \ n=4 \\ \mathbb{Z} \oplus \left(\begin{array}{l}\text{3-torsion} \\ \text{finite} \\ \text{group}\end{array}\right) & , \ n=5 \\ ? & , \ n>5 \end{cases}$$

> R a $\underbrace{\text{number ring}}_{\text{for example } \mathbb{Z}[\sqrt{2}]} \rightsquigarrow K_0(R) \cong \mathbb{Z} \oplus C(R)$

$\uparrow$

Ideal class group

~ Difficult object studied in number theory

Tool: Topological Hochschild homology THH

- Another invariant - still hard, but a bit nicer
- on commutative ring objects: 'Tensor it with the circle'

This can tell us something about K-theory through the **trace map**

$$tr : K(R) \longrightarrow THH(R)$$

Dundas-McCarthy '96: Not an equivalence, but the difference is 'locally constant'

$$\begin{tikzcd}
 & & \mathrm{THH}(R) & & \\
 & & K(R) \arrow[u, "tr"'] \arrow[dl, bend right, "\text{Generalize to stable } \infty\text{-categories}"'] \arrow[dr, bend left, "\text{Add ``complex conjugation''}"] & & \\
\mathrm{THH}(\mathcal{C}) & K(\mathcal{C}) \arrow[l, "tr"'] \arrow[dr, bend right] & & GW(R) \arrow[r, "tr"] \arrow[dl, bend left] & \mathrm{THR}(R) \\
 & & \boxed{?} & &
\end{tikzcd}$$

Poincaré ∞-categories $(\mathcal{C}, \Qoppa)$

$$GW(\mathcal{C}, \Qoppa) \xrightarrow{tr} \mathrm{THR}(\mathcal{C}, \Qoppa)$$

§ Extending to GW(R)

"Enhanced spaces"
$\{X_n\}_{n\geq 0} + \sigma : \Sigma X_i \xrightarrow{\sim} X_{i+1}$

Vector bundle structure	Structure on K-groups	K-group	K-theory spectrum
Complex vector bundle $E \longrightarrow X$	Abelian groups $[E]$ with direct sum	$K^0(X)$ = Groethendieck group of complex v.b.	Complex K-theory KU
Complex Conjugation $E^* \longrightarrow X$	C_2-action on abelian groups $[E] \longmapsto [E^*]$	$t : K^0(X) \longrightarrow K^0(X)$ $[E] \longmapsto [E^*]$ $t \in C_2,\ t^2 = 1$	$C_2 \circlearrowright KU$ $\rightsquigarrow KR \in Sp^{C_2}$ $\hookrightarrow = \mathrm{Fun}(BC_2, Sp)$
↱ Duality, $D^2 = \mathrm{Id}$ $$\begin{tikzcd} E \arrow[r, "D"] \arrow[d] & E^* \arrow[d] \\ X \arrow[r, "\tau"'] & X \end{tikzcd}$$ τ ← C_2-action	C_2-Mackey functors	Atiyah's real K-theory $$\begin{tikzcd} {[E,D]} \arrow[r, mapsto] & {[E]} \\ KR^0(X) \arrow[r, bend left, "res"] & K^0(X) \arrow[l, bend left, "tran"] \arrow[loop right, "t"] \\ {[E \oplus E^*]} & {[E]} \arrow[l, mapsto] \end{tikzcd}$$ ↰ hyperbolic duality	Genuine C_2-Spectrum KR

Equivariant notions

$$\begin{tikzcd}
E \arrow[r] & X \arrow[r, squiggly] & KU
\end{tikzcd}$$

$$\begin{tikzcd}
E \arrow[r, "D"] \arrow[d] & E^* \arrow[d] \arrow[r, squiggly] & KR \text{ a genuine } C_2\text{-spectrum} \\
X \arrow[r, "\tau"'] & X &
\end{tikzcd}$$

$A \circlearrowleft C_2$: Abelian group with C_2-action

Orbits — Norm map — Fixed points — 0th Tate cohomology

$$\begin{tikzcd}
A_{C_2} := A/\langle ta - a\rangle \arrow[r, "Nm"] & A^{C_2} := \{a \mid ta = a\} \arrow[r] & \operatorname{coker}(Nm) \\
a \arrow[r, mapsto] & a + ta &
\end{tikzcd}$$

$X \circlearrowleft C_2$: Spectrum with C_2-action

Homotopy Orbits — Homotopy fixed points — Tate construction

$$\begin{tikzcd}
X_{hC_2} \arrow[r, "Nm"] & X^{hC_2} \arrow[r] & X^{tC_2}
\end{tikzcd}$$

$\operatorname{colim}_{BC_2} X$ $\qquad$ $\lim_{BC_2} X$

Example: $A \circlearrowleft C_2 \rightsquigarrow HA \circlearrowleft C_2$

- $\pi_n(HA_{hC_2}) = H_n(C_2, A)$ Group homology
- $\pi_n(HA^{hC_2}) = H^{-n}(C_2, A)$ Group cohomology
- $\pi_n(HA^{tC_2}) = \widehat{H}^{-n}(C_2, A)$ Tate cohomology

$\xrightarrow{n=0}$

- $H_0(C_2, A) \cong A_{C_2}$
- $H^0(C_2, A) \cong A^{C_2}$
- $\widehat{H}^0(C_2, A) \cong \operatorname{coker}(Nm)$

For KR:

- Underlyi spectrum : KU
- $KR^{tC_2} \simeq$ $*$
- $KR^{hC_2} \simeq KR_{hC_2} \simeq KO$ ← Real K-theory ~ Uses real vector bundles

Duality, $D^2 = Id$

$$\begin{tikzcd} E \arrow[r, "D"] \arrow[d] & E^* \arrow[d] \\ X \arrow[r, "I"'] & X \end{tikzcd}$$

C_2-action

1. Complex vector bundle
2. Add complex conjugation
3. Add duality on E and C_2-action on X

⤳ Genuine C_2-spectrum KR

> Underlying spectrum: KU
> ∼ Complex K-theory

> $KR_{hC_2} \simeq KR^{hC_2} \simeq KO$
> ∼ Real K-theory

For a ring

> R a ring ⤳ K(R)

> Add anti-involution $(-)^* : R \to R^{op}$

> Symmetric bilinear form (P, q)

- P : finitely generated projective R-module
- $q : P \otimes P \to R$: $R \otimes R$-linear such that $q(x \otimes y) \simeq q(y \otimes x)^*$

> Assume (P, q) is unimodular :

$$q_\# : P \xrightarrow{\sim} DP := hom_R(P, R)$$
$$x \longmapsto (y \mapsto q(x \otimes y))$$

§ Extending to GW(R)

Forms a monoid

$$\begin{tikzcd}
\left(\left\{\begin{array}{l}\text{finitely generated}\\ \text{projective } R\text{-modules}\end{array}\right\}/\cong,\ \oplus\right) \arrow[d, squiggly, "(-)^{grp}"] & & \left(\left\{\begin{array}{l}\text{Unimodular symmetric}\\ \text{bilinear forms on } R\end{array}\right\}/\cong,\ \oplus\right) \arrow[ll, "\text{forget}"'] \arrow[d, squiggly, "(-)^{grp}"'] & & \\
K_0(R) \arrow[rr] & & GW_0(R) \arrow[r, "\text{coker}"] & L_0(R) \arrow[r] & 0
\end{tikzcd}$$

Groethendieck-Witt group

$$P \longmapsto \left(P\oplus P^*,\ P\oplus P^* \xrightarrow[\begin{pmatrix}0 & \mathrm{id}_{P^*}\\ \mathrm{can} & 0\end{pmatrix}]{\sim} (P\oplus P^*)^* \cong P^*\oplus P^{**}\right)$$

L-theory

$$\text{"forms/}\underbrace{\text{cobordism}}\text{"}$$

There exist $L \subseteq P$ s.t. $q|_L = 0$ and

$$0 \to L \hookrightarrow P \to L^* \to 0$$

is exact

$\rightsquigarrow P \cong L \oplus L^*$

This sequence does not extend on the left

$\rightsquigarrow$ Find a "derived functor" which does

$\rightsquigarrow$ In homotopy theory: Fiber sequence of spaces/spectra which on homotopy groups recovers the original sequence

$$K_0(R)_{C_2} \xrightarrow{hyp} GW_0(R) \longrightarrow L_0(R) \longrightarrow 0$$

Theorem: (Schlichting) If 2 is invertible, then there exists a fiber sequence

$$K(R)_{C_2} \longrightarrow GW(R) \longrightarrow L(R)$$

which extends to the desired long exact sequence of groups

$$\begin{tikzcd}
\cdots \arrow[r] & K_1(R)_{C_2} \arrow[r] & GW_1(R) \arrow[r] & L_1(R) \arrow[dll, rounded corners] \\
& K_0(R)_{C_2} \arrow[r] & GW_0(R) \arrow[r] & L_0(R)
\end{tikzcd}$$

Quillen: Categorical input necessary

Waldhausen & Thomason: Need a higher categorical input

$$R \looparrowright K.(R) \qquad \text{Group}$$

Perfect Derived category

Chain complexes of finitely generated Projective R-modules [quasi-isomorphism^{-1}]

$$Q^P(R) \looparrowright K(R) \qquad \text{space/spectrum}$$

§2: Extend to $\mathcal{Q}^p(R)$

Good Idea: Forms ~ "Extra structure" on $\mathcal{Q}^p(R)$ which describes the "space of forms" for any $X \in \mathcal{Q}^p(R)$

$\rightsquigarrow$ This can be axiomatized!

#9: Calmes-Dotto-Harpaz-Hepestreit-Land-Moi-Nardin-Nikolaus-Steimle

Note: Throughout we will work over $\mathcal{Q}^p(R)$ but this could be done for any stable ∞-category $\mathcal{C}$.

"Extra Structure" on $Q^p(R)$ which describes the "space of forms" for any $X \in Q^p(R)$

Def: A Poincaré (∞-) category is a pair $(R, \Qoppa)$ where

$$\Qoppa : Q^p(R)^{op} \longrightarrow Sp$$

is perfectly quadratic, i.e.

- Reduced preserves the 0-object
- 2-excisive weaker notion of 'pushout square $\mapsto$ Pullback square'
- Induces a perfect duality : The unit of the adjunction

$$\begin{tikzcd} Q^p(R) \arrow[r, bend left, "D_{\Qoppa}"] \arrow[r, phantom, "\perp"] & Q^p(R)^{op} \arrow[l, bend left, "D_{\Qoppa}^{op}"] \end{tikzcd}$$

is an equivalence $ev : id_{\mathcal{C}} \overset{\sim}{\Rightarrow} D_{\Qoppa}^2$

Ex: Let R be an associative ring with involution $(-)^* : R^{op} \longrightarrow R$

Symmetric forms

$$\Qoppa^s : Q^p(R)^{op} \longrightarrow Sp$$
$$X \longmapsto map_{R \otimes R}(X \underset{R}{\otimes} X, R)^{hC_2}$$

space of forms a pair (X, q)

- $X \in Q^p(R)$
- $q \in \Omega^\infty \Qoppa^s(X) \simeq Map_{R \otimes R}(X \underset{R}{\otimes} X, R)^{hC_2}$

$\rightsquigarrow$ $q : X \otimes X \longrightarrow R$

$q(x \otimes y) \simeq q(y \otimes x)^*$

Theorem (#9) For $(R, Ϙ)$ a Poincaré ∞-category we have the following fibre sequence in Sp:

$$K(R)_{hC_2} \longrightarrow GW(R, Ϙ) \longrightarrow L(R, Ϙ)$$

Note: The "2 invertible" assumption is gone !

We furthermore have a C_2-spectrum $\mathbb{K}R(R, Ϙ)$ with

- Underlying spectrum: $K(R)$
- $\mathbb{K}R(R, Ϙ)^{C_2} \simeq GW(R, Ϙ)$

  fixed points working well with limits
- $\Phi^{C_2} \mathbb{K}R(R, Ϙ) \simeq L(R, Ϙ)$

  fixed points working well with colimits

Tate Square

$$\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & X^{C_2} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Phi^{C_2} X := (\Sigma^\infty \tilde{E}P_{C_2} \otimes X)^{C_2} \arrow[d] \\
X \arrow[r, "Nm"] & X^{hC_2} \arrow[r] & X_{hC_2}
\end{tikzcd}$$

§3 THR of a 'nice' ring R

"Tensor with the circle"

cyclic bar construction

$\mathrm{THH}(R) := R \otimes_{R\otimes R} R \simeq |N^{cy}_\bullet(R)|$

$\simeq \Big| [n] \mapsto$  $\Big| \circlearrowleft S^1$

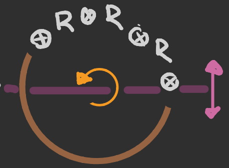
Dennis trace map

$tr : K(R) \longrightarrow \mathrm{THH}(R)$

↖ Bökstedt-Hsiang-Madsen: This is a genuine S^1-spectrum

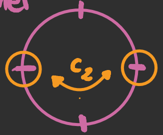
Idea: Refine THH to a (genuine) C_2-spectrum, using the reflection of the circle

Let R be a ring with anti-involution

$\mathrm{THR}(R) = R \otimes_{NR} R \simeq \Big| [n] \mapsto$  $\Big| \circlearrowleft C_2$

Hill-Hopkins-Ravenel norm

$NR = \underset{\curvearrowleft}{R \otimes R}$



Trace map of C_2-spectra

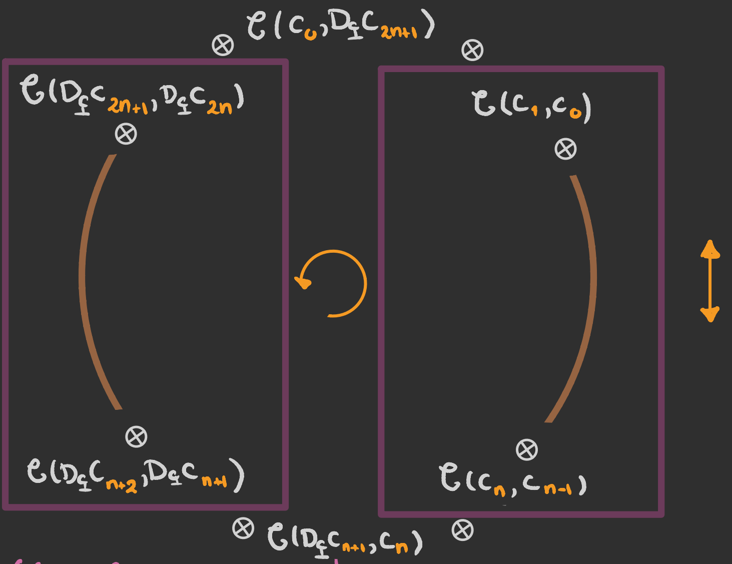
$tr : \mathrm{KR}(R) \longrightarrow \mathrm{THR}(R)$

Theorem (Dotto-Moi-Patchkovia-Reeh) $\Phi^{C_2}\mathrm{THR}(R) \simeq \Phi^{C_2}R \otimes_R \Phi^{C_2}R$

Of a Poincaré ∞-Category $(\mathcal{C}, Ϙ)$

$\mathrm{THR}(\mathcal{C}, Ϙ)$ Shoud be of the form

$[n] \longmapsto \operatorname{colim}_{c_0, \ldots, c_{2n+1} \in \mathcal{C}}$

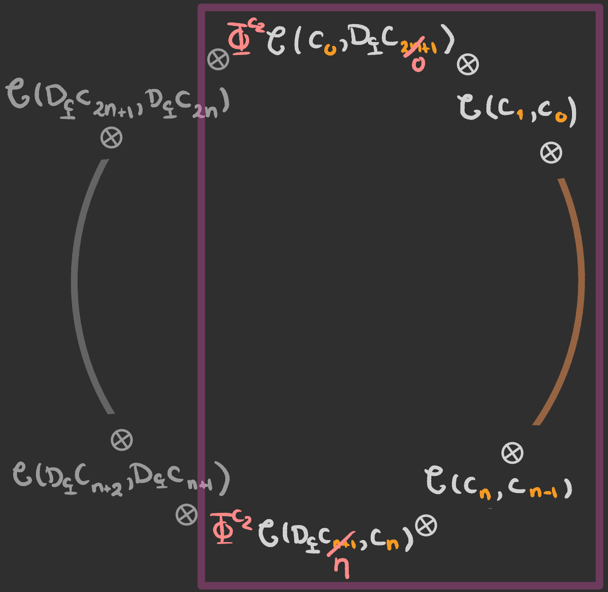


$(c_0, \ldots, c_{2n+1})$ summand $\leftrightarrow$ $(c_{2n+1}, \ldots, c_0)$-summand

Of a Poincaré ∞-Category $(\mathcal{C}, \Qoppa)$

Recall: $\Phi^{C_2}\mathrm{THR}(R) \simeq \Phi^{C_2}R \otimes_R \Phi^{C_2}R$

$$[n] \longmapsto \operatorname*{colim}_{c_0,\dots,c_{\underset{n}{\cancel{2n+1}}}\in\mathcal{C}}$$



$$\boxed{\Phi^{C_2}\mathrm{THR}(\mathcal{C},\Qoppa) \simeq \left| [n] \longmapsto \operatorname*{colim}_{c_0,\dots,c_n\in\mathcal{C}} \Phi^{C_2}\mathcal{C}(c_0, D_\Qoppa c_0) \otimes \mathcal{C}(c_1, c_0) \otimes \cdots \otimes \mathcal{C}(c_n, c_{n-1}) \otimes \Phi^{C_2}\mathcal{C}(D_\Qoppa c_n, c_n) \right|}$$

Theorem: | R. | There exists a functor which is objectwise on the above form

$$\Phi^{C_2}\mathrm{THR} : \mathrm{Cat}_\infty^{p} \longrightarrow \mathrm{Sp}$$

$$\Phi^{C_2}\mathrm{THR}(\mathcal{C}, Ϙ) \simeq \left| [n] \mapsto \operatorname*{colim}_{c_0, \dots, c_n \in \mathcal{C}} \Phi^{C_2}\mathcal{C}(c_0, D_Ϙ c_0) \otimes \mathcal{C}(c_1, c_0) \otimes \cdots \otimes \mathcal{C}(c_n, c_{n-1}) \otimes \Phi^{C_2}\mathcal{C}(D_Ϙ c_n, c_n) \right|$$

Morita invariance | R. |

> $\Phi^{C_2}\mathrm{THR}(R, Ϙ^{gs}) \simeq \Phi^{C_2}R \otimes_R \Phi^{C_2}R$

> $\Phi^{C_2}\mathrm{THR}(R, Ϙ^{s}) \simeq R^{tC_2} \otimes_R \Phi^{C_2}R$

Equivalent to the geometric fixed points of the Borel completion

Thank
you for
listening