Paul 05.10.22 : Introduction

Motivation

X a topological space $\rightsquigarrow$ $\pi_0 X$, $x \in X$, $\pi_1(X,x)$

$$\pi_{\leq 1}(X,x) = \text{cat} \begin{cases} \text{ob} = \text{pts in } X \\ \text{morp} = \text{hmpty classes of rings in } X \end{cases}$$

$$\mathrm{Hom}_{\pi X}(x,x) = \pi_1(X,x)$$

Q: What sort of "gadget" contains the data of a homotopy theory? — ∞-categories

- All (small) categories w. functors and nat. trans form a 2Cat.
- Convenient way to do homotopy theory!

Def: A poset $\mathcal{P}$ is a set with a partial order

Every poset $\mathcal{P}$ determines a category:

$$\mathrm{ob}(\mathcal{P}) = \mathcal{P}$$

$$\mathrm{Hom}_{\mathcal{P}}(x,y) = \begin{cases} \emptyset & x \not\leq y \\ * & x \leq y \end{cases}$$

Ex: $[n] = \{0 \leq 1 \leq \dots \leq n\}$, $n \geq 0$

$$\Delta = \begin{cases} \mathrm{ob} = \{[n]\} \\ \mathrm{Hom}_{\Delta}([n],[m]) = \overset{\text{order preserving}}{\mathrm{Fun}}([n],[m]) \end{cases}$$

Distinguished maps:

$d_i : [n] \longrightarrow [n+1]$ injective and misses i

$s_j : [n] \longrightarrow [n-1]$ surjective and $|s_j^{-1}(i)| = 2$

Def: A simplicial set is a functor $\Delta^{op} \to \mathrm{Set}$

$X_0 := X([0])$ the 'objects' of X

$X_1 := X([1])$ the 'morphisms' of X.

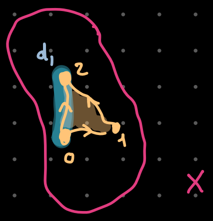
Ex: Δ^n the standard n-simplex $:= \mathrm{Hom}_{\Delta}(\cdot, [n])$

Prop: $X_n \cong \mathrm{Hom}_{sSet}(\Delta^n, X)$

Def: Topological n-simplex $\Delta^n_{top} = \{(x_0, \_, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0, \Sigma x_i = 1\}$

Def: Singular simplicial set of a space X is

$$\mathrm{Sing}_n(X) = \mathrm{Hom}_{Top}(\Delta^n_{top}, X)$$



Def: Geometric realization of a sSet X :

$$|X_\bullet| = \coprod X_m \times \Delta^n_{top} \Big/ \sim$$

$\sim$ glueing along faces & degeneracy maps

This is a topological space (CW-complex).

Prop: There exists an adjunction

$$|\cdot| : sSet \rightleftarrows Top : \mathrm{Sing}_\bullet$$

$\rightsquigarrow$ have units & counits

$|\mathrm{Sing}_\bullet(X)| \longrightarrow X$ weak htpy eq. - CW-approximation

CW Top

Def: $\mathcal{C}$ a cat, the nerve $N_\bullet\mathcal{C} : [n] = \mathrm{Fun}([n], \mathcal{C})$ $N : Cat \to sSet$ fully faithful

Def: The homotopy category hX of a simplicial set X :

$\mathrm{ob}\, hX = X_0$

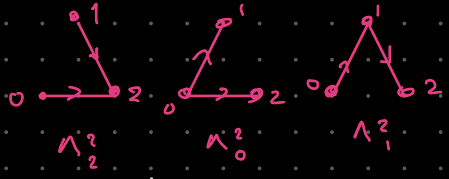
Hom_{hX} = free compositions of composable morphisms in $X_1 / \sim$

Punch line: $h \dashv N$, $\mathcal{C} \cong hN\mathcal{C}$

$\sim$ $g \cdot f \sim h$ $\exists\, \sigma$ s.t.
$d_0\sigma = g$
$d_2\sigma = f$
$s_0(x) = id_x$

Construction: $(\partial\Delta^n)_k$ = maps $[k] \to [n]$ that is not surj

Λ^n_j := The j^{th} horn = maps $[k] \to [n]$ s.t.
$\exists i \in [n] \setminus j$ not in the image



$(I^n)_k$:= Spine $\subseteq (\Delta^n)_k$: maps $[k] \to [n]$ whose image is

$\{j\}, \{j, j+1\}$.

Def.: A Kan complex is a simplicial set with the horn extension property for all horns:

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\forall f"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

$\forall\, n \geq 2,\ \forall\, 0 \leq i \leq n$

Ex.: $X \in Top \rightsquigarrow Sing_{\cdot}(X) \in Kan$ follows by the adjunction $|\cdot| \dashv Sing_{\cdot}$ and $|\Lambda^n_i| \hookrightarrow |\Delta^n|$ def. retracts.

Thm.: $N.\mathcal{C}$ is a Kan complex $\Leftrightarrow$ $\mathcal{C}$ is a groupoid
all morphisms are invertible

Trivia: $\mathcal{C}$ a category $\rightsquigarrow B\mathcal{C} = |N.\mathcal{C}|$ is the classifying space of $\mathcal{C}$

↳ Book reccommodation: "Classifying spaces & classifying topos" – MacLane

G group, model it as BG – obj. $*$, $Hom_{BG}(*,*) \cong G$

$|BG| =$ Classifying space of G (in the classical sense)

↳ What does this classify for general $\mathcal{C}$?

Def.: A simplicial set is a composer if it has the spine extension property, i.e.

$$\begin{tikzcd} I^n \arrow[r, "\forall f"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

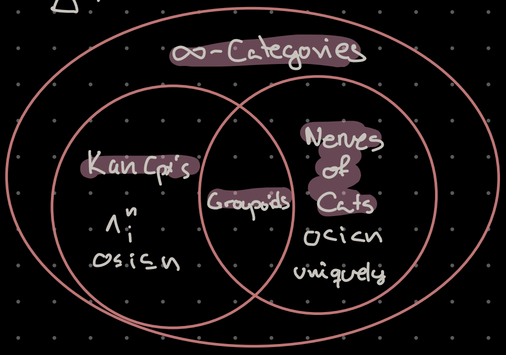
Weaker than ∞-cat requires

Prop.: In $N.\mathcal{C}$ there are unique inner horn extensions

Def.: An ∞-category is a simplicial set $\mathcal{C}$ with the horn extension property for all inner horns, i.e.

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\forall f"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

$\forall\, n \geq 2,\ \forall\, 0 < i < n$



Prop:

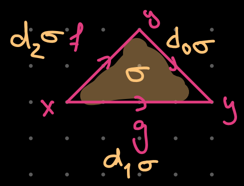
- $X \in \mathrm{Top}$; $\mathrm{Sing}.X \in \mathrm{Cat}_\infty$
- $\mathcal{C} \in \mathrm{Cat}$; $N.\mathcal{C} \in \mathrm{Cat}_\infty$
- $\mathcal{C} \in \mathrm{Cat}_2$; $N^{\text{Deskin}} \mathcal{C} \in \mathrm{Cat}_\infty$
- $\mathcal{C} \in \underbrace{\mathrm{Cat}_{\text{kan}}}_{\text{Kan-enriched}}$; $N^{hc} \mathcal{C} \in \mathrm{Cat}_\infty$

$\mathcal{C} \in \mathrm{Cat}_\Delta$ is only a composer

$N^{hc}\, \mathrm{Top} = \mathcal{S}$ ∞-cat. of spaces

Def: Say $f, g \in X_1$ are equivalent $f \sim g$ if there exists $\sigma \in X_2$ s.t.

$$d_2\sigma = f$$
$$d_1\sigma = g$$
$$d_0\sigma = id_y := s_0(y)$$



Prop: If X is a composer & has 2 and/or 3 inner horn extensions, then this is an equivalence then this is an equivalence relation.

Or if X is an ∞-category!

Def. πX = 0 simplices & 1 simplices $/\sim_{\text{equivalent}}$

Prop. $\pi X \cong hX$

Def. $X \in \mathrm{Cat}_\infty$, call $f \in X_1$ an equivalence if $f \in hX$ is an isomorphism

An ∞-groupoid is an ∞-cat where all 1-morphisms are all invertible

Thm: Kan complex $\Longleftrightarrow$ ∞-Groupoid

2: Daniel Marlowe - Small object argument

§1: Small object argument

Notation: $\Lambda_\bullet = \{\Lambda^n_j \hookrightarrow \Delta^n \mid n \geq 1\}$

Inner: $0 < j < n$

Left: $0 \leq j < n$

Right: $0 < j \leq n$.

Given $\mathcal{S} \subseteq \mathrm{ArSset}$ denote by

$\mathcal{X}_L(\mathcal{S}) = \{g \text{ w. LLP wrt. } s \in \mathcal{S}\}$

$\mathcal{X}_R(\mathcal{S}) = \{g \text{ w. RLP wrt. } s \in \mathcal{S}\}$

$\mathcal{X}(\mathcal{S}) := \mathcal{X}_L(\mathcal{X}_R(\mathcal{S}))$

Def. $S \subseteq \mathrm{Mor}(\mathrm{sSet})$ is saturated if it is closed under

i) Pushouts

(ii) arbitrary coproducts) If working w. not just $\mathbb{N}$-indexed colims, this will follow from iv.

iii) Retract

iv) $\mathbb{N}$-indexed colimits

$$\begin{tikzcd} X_0 \arrow[r, "\text{in } \mathcal{S}"] \arrow[rrr, bend right] & X_1 \arrow[r] & \cdots \arrow[r] & \operatorname{colim} X_n \end{tikzcd}$$

$\Rightarrow$ This in $\mathcal{S}$

Ex:
- All maps
- Cofibrations
- Monomorphisms

Def $\bar{\mathcal{S}}$ saturated closure

Lem: $\forall \mathcal{S} : \mathcal{X}_L(\mathcal{S})$ is saturated

Rem: $\bar{\mathcal{S}} \subset \mathcal{X}(\mathcal{S})$

Prop |Quillen| Given $\mathcal{S} = \{A_i \xrightarrow{f_i} B_i\}$ s.t.

$*$ A_i have $< \infty$ non-degenerate simplexes

then for any map of sSets $f : X \to Y$, $\exists$

$$\begin{tikzcd} x \arrow[r, "s\in\overline{\mathcal{S}}"] \arrow[dr, "f"'] & z \arrow[d, "p\in\chi_R(\mathcal{S})"] \\ & y \end{tikzcd}$$

Pf: Consider the family $\Theta_S = \left\{ \begin{tikzcd} A_i \arrow[r] \arrow[d, "s_i\in\mathcal{S}"'] & x \arrow[d, "f"] \\ B_i \arrow[r] & y \end{tikzcd} \right\}$. Look at

$$\begin{tikzcd} \coprod_{\Theta_S} A_i \arrow[r] \arrow[d, "\Sigma f_i"'] & x \arrow[d, "s_1\in\overline{\mathcal{S}}"] \arrow[ddr, bend left, "f"] & \\ \coprod_{\Theta_S} B_i \arrow[r] \arrow[drr, bend right] \arrow[ur, phantom, "\ulcorner"] & E^1(f) \arrow[dr, "\exists!"] & \\ & & y \end{tikzcd}$$

Iterate $E^1(f)\to y, \dots,$ produce a sequence

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "s\in\mathcal{S}"'] & E^1(f) \arrow[r] & \cdots \arrow[r] & y \\ & E^\omega(f) \arrow[urr, "p"'] & & \end{tikzcd}$$

⇝ Consider

$$\begin{tikzcd} A_i \arrow[r] \arrow[d, "f_i"'] & E^\omega(f) \arrow[d, "p"] \\ B_i \arrow[r] & y \end{tikzcd} \qquad \begin{tikzcd} \operatorname{colim}_k \operatorname{Hom}(A_i, E^k(f)) \arrow[d] \\ \operatorname{Hom}(A_i, E^\omega(f)) \end{tikzcd}$$

$\Rightarrow \exists N$ s.t. we get a factorization

$$\begin{tikzcd} A_i \arrow[r] \arrow[d, "f_i"'] & E^N(f) \arrow[r, "\iota_n"] \arrow[d] & E^\omega(f) \arrow[d, "p"] \\ B_i \arrow[r] & E^{N+1}(f) \arrow[r] \arrow[ur, "\iota_{n+1}"] & y \end{tikzcd} \Rightarrow p\in\chi_R(\mathcal{S})$$

$\sqcap$

Cor: For $\mathcal{S}$ satisfying $(*)$: $\overline{\mathcal{S}} = \chi(\mathcal{S})$.

Pf: $\overline{\mathcal{S}} \subseteq \chi(\mathcal{S})$ ✓

To see $\chi(\mathcal{S}) \subseteq \overline{\mathcal{S}}$: Let $f\in\chi(\mathcal{S})$:

$$\begin{tikzcd} x \arrow[r, "s\in\overline{\mathcal{S}}"] \arrow[d] & z \arrow[d, "p"] \\ y \arrow[r, equal] \arrow[ur, "\exists\sigma"] & y \end{tikzcd} \rightsquigarrow \begin{tikzcd} x \arrow[r, equal] \arrow[d] & x \arrow[r, equal] \arrow[d, "s"] & x \arrow[d, "f"] \\ y \arrow[r, "\sigma"] \arrow[rr, bend right, "id"'] & z \arrow[r, "p"] & y \end{tikzcd}$$

□

Now, given
$$\Lambda = \{\Lambda^n_j \hookrightarrow \Delta^n\}$$

We write
$$\mathcal{X}_R(\Lambda_i) = \{\text{inner fibrations}\}$$
$$\Lambda_L = \{\text{left fibration}\}$$
$$\Lambda_R = \{\text{Right fibration}\}$$
Cor 1.3.16

Def: Anodyne $= \mathcal{X}_L \mathcal{X}_R(\Lambda_{i,l,r})$

Rem: f monomorphism ⇝ f is anodyne iff it induces a weak eq. o realization.

Def: $\{\text{Trivial fibration}\} = \mathcal{X}_R(\{\partial\Delta^n \longrightarrow \Delta^n \mid n \geq 0\})$

$$\mathcal{X}_R(\partial\Delta) = \mathcal{X}_R(\overline{\partial\Delta}).$$

§2: Inner horns

So ∞-categories $= \mathcal{X}_R(\Lambda_i)$

Recall: $C \in \mathrm{Cat}_1$ is cartesian closed if $\forall a \in C$: $(-) \times a : C \to C$ admits a right adjoint, which we denote by $b \mapsto b^a$

For sSets; $(Y^X)_n = \mathrm{Hom}_{sSet}(X \times \Delta^n, Y)$ $\qquad a \times (-) \dashv (-)^a$

consider:
$$A \overset{i}{\hookrightarrow} B, \quad S \xrightarrow{g} T, \quad X \xrightarrow{f} Y$$

$$\begin{tikzcd}
X^B \arrow[rrd, bend left] \arrow[rd, "{\langle f, i \rangle}"] \arrow[rdd, bend right] & & \\
& X^A \times_{Y^A} Y^B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^A \arrow[d, "f_*"] \\
& Y^B \arrow[r, "i_*"'] & Y^A
\end{tikzcd}$$

Pullback exponential

$$\begin{tikzcd}
A \times S \arrow[r] \arrow[d] & A \times T \arrow[d] \arrow[rdd, bend left] & \\
B \times S \arrow[r] \arrow[rrd, bend right] & (A \times T) \coprod_{(A \times S)} (B \times S) \arrow[ul, phantom, "\ulcorner", very near start] \arrow[rd, "i \otimes g"] & \\
& & B \times T
\end{tikzcd}$$

Pushout product

$\rightsquigarrow$ $i \boxtimes (-) \dashv \langle -, i \rangle$

$$\begin{tikzcd} \mathrm{Ar}\,s\mathrm{Set} \arrow[r, bend left, "i \boxtimes -"] & \mathrm{Ar}\,s\mathrm{Set} \arrow[l, bend left, "\langle -, i \rangle"] \end{tikzcd}$$

Now, assume we are given the following diagram:

$$\begin{tikzcd} S \arrow[r] \arrow[d, "g"'] & X^B \arrow[d] \\ T \arrow[r] \arrow[ur] & X^A \times_{Y^A} Y^B \end{tikzcd} \cong \begin{tikzcd} S \arrow[r] \arrow[d] & X^B \arrow[d] \arrow[dr] & \\ T \arrow[ur] \arrow[r, dash] \arrow[dr] & \arrow[r] & Y^B \arrow[d] \\ & X^A \arrow[r] & Y^A \end{tikzcd}$$

these lifting problems are equivalent

T

$$\begin{tikzcd} A \times S \arrow[r] \arrow[d] & B \times S \arrow[d] \arrow[dr] & \\ A \times T \arrow[r] \arrow[urr] & B \times T \arrow[r] \arrow[ur] & Y \end{tikzcd} \cong \begin{tikzcd} (A \times T) \amalg_{(A \times S)} (B \times S) \arrow[r] \arrow[d, "i \boxtimes g"] & X \arrow[d] \\ B \times T \arrow[r] \arrow[ur] & Y \end{tikzcd}$$

Lem: 1.3.31 For i, g monomorphisms. Then $i \boxtimes g$ is respectively inner, L, R anodyne if either of i, or g resp., is.

Lem.: $\Lambda = \{ \Lambda^n_j \hookrightarrow \Delta^n \mid n \geq 1, 0 < j < n \}$

$\bar{\Lambda}$ = anodyne maps $= \overline{\Lambda^\circ}$

$\Lambda^\circ = \{ (K \hookrightarrow L) \boxtimes (\Lambda^n_i \to \Delta^n) \mid K \to L \text{ mono} \}$

Prop: $X \xrightarrow{f} Y$ is an fib. of sSets, i monomorphisms, then $\langle f, i \rangle$ is resp. inner /L /R is f respectively is.

Moreover, if i is respectively inner/L/R anodyne, then $\langle f, i \rangle$ is a trivial fibration.

Pf.:

$$\begin{tikzcd} S \arrow[r] \arrow[d, "g"'] & X^B \arrow[d, "{\langle f, i \rangle}"] \\ T \arrow[r] \arrow[ur, "\exists ?"] & X^A \times_{Y^A} Y^B \end{tikzcd} \Leftrightarrow \begin{tikzcd} (A \times T) \amalg_{(A \times S)} (B \times S) \arrow[r] \arrow[d, "i \boxtimes g"'] & X \arrow[d, "f"] \\ B \times T \arrow[r] \arrow[ur, "\exists ?"] & Y \end{tikzcd}$$

$\square$

Ex: $X \xrightarrow{f} Y = \Delta^0$, $\emptyset \xrightarrow{i} B$ $\rightsquigarrow$

∞-cat

$$\begin{tikzcd} X^B \arrow[d, "{\langle f, i \rangle}"] & X \arrow[d] \\ \Delta^0 & \Delta^0 \end{tikzcd}$$

Again ∞-cat

inner iff inner

Def: Given $\mathcal{C},\mathcal{D}\in\mathrm{Cat}_\infty$, set $\mathrm{Fun}(\mathcal{C},\mathcal{D}) := \mathcal{D}^{\mathcal{C}} = \underline{\mathrm{Hom}}(\mathcal{C},\mathcal{D})$ $\in\mathrm{Cat}_\infty$

Prop: $\mathcal{C}\in\mathrm{sSet}$ is an ∞-category iff $\mathcal{C}^{\Delta^2}\to\mathcal{C}^{\Lambda^2_1}$ is a trivial fibration

Proof: Uses that

$$\overline{\Lambda} = \overline{\{(K\hookrightarrow L)\boxtimes(\Lambda^2_1\hookrightarrow\Delta^2)\}}$$

Consider

$$\begin{tikzcd}
K \arrow[r] \arrow[d, hook, "g"'] & \mathcal{C}^{\Delta^2} \arrow[d, dashed] \\
L \arrow[r] \arrow[ur] & \mathcal{C}^{\Lambda^2_1}
\end{tikzcd}
\qquad
\begin{tikzcd}
K\times\Lambda^2 \coprod_{L\times\Lambda^2_1} L\times\Delta^2 \arrow[r] \arrow[d, hook, "L\boxtimes g"'] & \mathcal{C} \arrow[d] \\
L\times\Delta^2 \arrow[r] \arrow[ur] & \Delta^0
\end{tikzcd}$$

Tommy Yang: Localization & cocartesian fibrations

26.10.2022 Lecture 4

Def: Let $\mathcal{C}$ be an ∞-category. A functor $f: \mathcal{C} \to \overset{S \subset \mathcal{C}_1}{\mathcal{C}[S^{-1}]}$ is a localization if

1) f takes S to equivalences
2) $\mathrm{Fun}(\mathcal{C}[S^{-1}], \mathcal{D}) \simeq \boxed{\mathrm{Fun}^S}(\mathcal{C}, \mathcal{D})$ those functors which maps S to equivalences

Uniqueness almost follows by definition, so we'll only focus on existence.

Lem: $\Delta^1 \to \mathcal{J} = N(0 \rightleftarrows 1)$ is a localization

Pf: Note that elements in $\mathcal{J}_k$ are of the form

$$j_0 \to \cdots \to j_k \quad \text{w. } j_i = 0 \text{ or } 1$$

and non-degenerates of the form

$$0 \to 1 \to 0 \to 1 \to \cdots$$

$$1 \to 0 \to 1 \to 0 \to \cdots$$

$\underline{\mathrm{Fun}(\mathcal{J}, \mathcal{D}) \to \widetilde{\mathrm{Fun}}}(\Delta^1, \mathcal{D}) \subset \mathrm{Fun}(\Delta^1, \mathcal{D})$.

Construct a filtration on $\mathcal{J}$: $\mathcal{F}_k$ smallest sun-simplex that contains $\underbrace{0 \to 1 \to 0 \to 1 \to \cdots}_{k+1}$. Consider the diagram

$$\begin{tikzcd}
\Lambda^k_0 \arrow[r] \arrow[d] & \mathcal{F}_{k-1}(\mathcal{J}) \arrow[d] \\
\Delta^k \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathcal{F}_k(\mathcal{J})
\end{tikzcd}$$

and

$$V_k: \Delta^k \longrightarrow \mathcal{F}_k(\mathcal{J})$$

$$\Delta^k[n] \longrightarrow \mathcal{F}_k(\mathcal{J})[n]$$

$$f: [n] \to [k] \mapsto f \pmod 2$$

In particular: $f(j) = j+1 \mapsto \underbrace{1 \to 0 \to 1 \to \cdots}_{k}$

$\mathcal{F}_{k-1}(\mathcal{J}) \to \mathcal{F}_k(\mathcal{J})$ is anodyne

$\rightsquigarrow \mathcal{F}_1(\mathcal{J}) = \Delta^1 \longrightarrow \lim_k \mathcal{F}_k(\mathcal{J}) = \mathcal{J}$ is anodyne.

We wish to show that

$$\mathrm{Fun}(\mathcal{J},\mathcal{D}) \to \mathrm{Fun}^{\simeq}(\Delta^1,\mathcal{D})$$

is an equivalence for all $\mathcal{D}$. We firts note that this factors through

$$\mathrm{Fun}(\mathcal{J},\mathcal{D}) \to \underbrace{\mathrm{Fun}^{\Delta^1}(\mathcal{F}_k(\mathcal{J}),\mathcal{D}) \to \mathrm{Fun}^{\Delta^1}(\mathcal{F}_{k-1}(\mathcal{J}),\mathcal{D})}_{\text{Will show this is a trivial fibration}} \to \mathrm{Fun}^{\simeq}(\Delta^1,\mathcal{D})$$

$\rightsquigarrow$ show

$$\mathrm{Fun}(\mathcal{J},\mathcal{D}) \simeq \lim_k \mathrm{Fun}^{\Delta^1}(\mathcal{F}_k(\mathcal{J}),\mathcal{D}) \to \mathrm{Fun}^{\simeq}(\Delta^1,\mathcal{D}) \simeq \mathrm{Fun}^{\simeq}(\mathcal{F}_1(\mathcal{J}),\mathcal{D})$$

is a trivial fibration

$$\begin{tikzcd}
 & \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{Fun}^{\Delta^1}(F_k(\mathcal{J}),\mathcal{D}) \arrow[r] \arrow[d] & \mathrm{Fun}^{\Delta^1}(\Delta^k,\mathcal{D}) \arrow[d] \\
\Delta^0 \arrow[r, "x"] & \Delta^n \arrow[r] \arrow[ur] \arrow[urr] & \mathrm{Fun}^{\Delta^1}(\mathcal{F}_{k-1}(\mathcal{J}),\mathcal{D}) \arrow[r] & \mathrm{Fun}^{\Delta^1}(\Lambda^k_0,\mathcal{D})
\end{tikzcd}$$

wanna show $\rightarrow$ lift exists, enough to show $\rightarrow$. By adj:

$$\begin{tikzcd}
\Delta^1 \arrow[r] & \Lambda^k_0 \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^n,\mathcal{D}) \arrow[r, "ev"] \arrow[d] & \mathcal{D} \\
 & \Delta^k \arrow[r] \arrow[ur] & \mathrm{Fun}(\partial\Delta^n,\mathcal{D}) &
\end{tikzcd}$$

En equivalene

inner fibration

exists by Joyals

□

Lem: $\forall \mathcal{C} \in \mathrm{Cat}_\infty \ \exists$ localisation along $\mathcal{C}_1$.

Pf:

$$\begin{tikzcd}
\coprod_{ob(\mathcal{C})} \Delta^1 \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "f"] \\
\coprod_{ob\mathcal{C}} \mathcal{J} \arrow[r] & Y \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Don't know a priori that this is an ∞-category

Take an inner anodyne map $g: Y \xrightarrow{\simeq} X$

$\Rightarrow$ $g \circ f$ is anodyne.

Just some ∞-cat.

Claim: X is an ∞-groupoid

Pf: Recall $h(-) \dashv N(-) \rightsquigarrow h(-)$ preserves pushouts, so applying it to the above diagram:

$$\begin{tikzcd}
\coprod \{0 \to 1\} \arrow[r] \arrow[d] & h\mathcal{C} \arrow[d] \\
\coprod \{0 \rightleftarrows 1\} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & hY
\end{tikzcd}$$

$\rightsquigarrow$ hY is a groupoid

<u>Claim:</u> $\mathrm{Fun}(X,\mathcal{D}) \to \mathrm{Fun}^{\simeq}(\mathcal{C},\mathcal{D})$ is an equivalence.

<u>Pf:</u>

$$\begin{tikzcd}
\mathrm{Fun}(Y,\mathcal{D}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}^{\simeq}(\mathcal{C},\mathcal{D}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\mathcal{C},\mathcal{D}) \arrow[d] \\
\prod_{\mathcal{C}_1} \mathrm{Fun}(\mathcal{J},\mathcal{C}) \arrow[r] & \prod_{\mathcal{C}_1} \mathrm{Fun}^{\simeq}(\Delta^1,\mathcal{D}) \arrow[r] & \prod_{\mathcal{C}_1} \mathrm{Fun}(\Delta^1,\mathcal{D})
\end{tikzcd}$$

$\rightsquigarrow$ LHS is also a pullback

$\rightsquigarrow$ Left lower horizontal map is a trivial fibration by above lemma

$\Rightarrow$ Same holds for upper horizontal map

□

Hence we have showed that localization exists when considered along all morphisms, so now we wish to show existence w.r.t. any collection of morphisms.

<u>Prop:</u> For every $S \subset \mathcal{C}_1$ $\exists$ a localization along S.

<u>Pf.</u> $\forall S \subset \mathcal{C}_1$ $\exists$ a smallest sub ∞-category $\mathcal{C}_S$ containing S:

$$\begin{tikzcd}
\mathcal{C}_S \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\
X \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & W
\end{tikzcd}$$

Localization of $\mathcal{C}_S$ along all morphisms

Again, we don't know if this is an ∞-cat, so have to consider $\mathcal{D}$ instead

Take an inner anodyne $W \to \mathcal{D}$ $\Rightarrow$ $W \to \mathcal{D}$ weak cat. eq. in Joyal's model

some ∞-cat

$\rightsquigarrow$

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{D},\mathcal{E}) \arrow[r, "g^*"] & \mathrm{Fun}(W,\mathcal{E}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}^{S}(\mathcal{C},\mathcal{E}) \arrow[d] \\
 & \mathrm{Fun}(X,\mathcal{E}) \arrow[r] & \mathrm{Fun}^{S}(\mathcal{C}_S,\mathcal{E})
\end{tikzcd}$$

Trivial fibrations

$\rightsquigarrow$ g^* is an equivalence, thus $\mathrm{Fun}(\mathcal{D},\mathcal{E}) \simeq \mathrm{Fun}^{S}(\mathcal{C},\mathcal{E})$.

§ coCartesian fibrations

Motivation: For X a "nice" topological space, we have

$$\mathrm{Cov}(X) \simeq \mathrm{Fun}(\Pi_{\leq 1}(X), \mathrm{Set})$$

↖ covering spaces

Let $\mathcal{C} \in \mathrm{Cat}_1$, $F : \mathcal{C} \to \mathrm{Cat}_1$, then we have the so called Grothendieck construction

$$\int_{\mathcal{C}} F = \begin{cases} \text{ob}: & (A, X), \quad A \in \mathcal{C},\ X \in F(A) \\ \text{morp}: & (f, \varphi), \quad f: A \to B \in \mathrm{Mor}(\mathcal{C}) \\ & \varphi : f_!(x) \to Y \in \mathrm{mor}(F(B)) \\ & f_! : F(A) \to F(B) \quad \text{s.t. } F(f) = f_! \end{cases}$$

Observation: $\int_{\mathcal{C}} F \to \mathcal{C}$, $(A, X) \mapsto A$, can be charaterized by functo that has enough coCartesian morphisms.

Def: Given $\pi : \mathcal{D} \to \mathcal{C}$, a map $\phi : X \to Y$ is π-coCartesian if there exists the following (unique) lift:

$$\begin{tikzcd}
\left\{\begin{array}{c} X \xrightarrow{\psi} Z \\ \phi \downarrow \quad \\ Y \quad \end{array}\right\} \arrow[r] \arrow[d, mapsto] & \mathcal{D} \arrow[d, "\pi"] \\
\left\{\begin{array}{c} \pi(X) \xrightarrow{\pi(\psi)} \pi(Z) \\ \pi(\phi) \downarrow \quad \\ \pi(Y) \quad \end{array}\right\} \arrow[r] \arrow[ur, "\exists!"] & \mathcal{C}
\end{tikzcd}$$

$\pi : \mathcal{D} \to \mathcal{C}$ is further called coCartesian if $\forall f : A \to B$ in $\mathcal{C}$ and $X \in \mathcal{D}$ such that $\pi(x) \to A$ there exists a π-coCartesian morphism $\phi : X \to Y$ s.t. $\pi\phi = f$.

Grothendieck correspondence: $\int_{\mathcal{C}} : \mathrm{LFun}(\mathcal{C}, \mathrm{Cat}) \xrightarrow{\simeq} \mathrm{coCart}(\mathcal{C})$

Time to generalise this to ∞-categories: First idea is to take the nerve of the diagram used to define it in the 1-categorical case, which gives us

$$\begin{tikzcd}
\Lambda^2_0 \arrow[r] \arrow[d] & N(\mathcal{D}) \arrow[d] \\
\Delta^2 \arrow[r] & N(\mathcal{C})
\end{tikzcd}$$

Def: Let $p: X \to S$ be a functor of ssets. Then $f: \Delta^1 \to X$ a morphism is said to be p-coCartesian if $\forall\, n \geq 2$:

$$\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[r] \arrow[rr, bend left, "f"] & \Lambda^n_0 \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
& \Delta^n \arrow[r] \arrow[ur, "\exists"] & S
\end{tikzcd}$$

We further say $p: X \to S$ is coCartesian if p is an inner fibration, and there exists a lift

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
\Delta^1 \arrow[r] \arrow[ur, "\sigma"] & S
\end{tikzcd}$$

s.t. σ is p-coCartesian in X.

Ex: Right fibrations are Cartesian.

Next goal is to compare Cartesian fibrations with right fibrations.

Lem: A Cartesian fibration $p: \mathcal{E} \to \mathcal{C}$ is a right fibration $\Leftrightarrow$ Every morphism in $\mathcal{E}$ is p-Cartesian.

Prop: A Cartesian fibration $p: \mathcal{E} \to \mathcal{C}$ is a right fibration $\Leftrightarrow$ for all $x \in \mathcal{C}$, the fibres $\mathcal{E}_x$ are ∞-groupoids.

Pf: $\Rightarrow$: ✓

$\Leftarrow$: We will show that every morphism in $\mathcal{E}$ is p-Cartesian. Take any $f: \Delta^1 \to \mathcal{E}$, which takes $\overset{f: z \to y}{z}$ to y. We can then choose φ to be a p-Cartesian lift of $p(f)$ which has target y.

Consider the following diagram:

$$\begin{tikzcd}
\Delta^1 \arrow[r, hook, "\varphi"] & \Lambda^2_2 \arrow[r] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\
& \Delta^2 \arrow[r] \arrow[ur, "\tau"] & \mathcal{C}
\end{tikzcd}$$

$$\begin{tikzcd}
z \arrow[r, "f"] & y \\
z \arrow[ur, "\varphi"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
p(z) \arrow[r, "p(f)"] \arrow[d, -, "id"] & p(y) \\
p(z) \arrow[ur, "p(f)"'] &
\end{tikzcd}$$

The lift $\tau : \Delta^2 \to \mathcal{E}$ is given by

$$\begin{tikzcd}
z \arrow[r, "f"] \arrow[d, "\psi"'] & y \\
z \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

w. ψ a morphism in $\overset{\infty\text{-groupoid}}{\mathcal{E}_{p(z)}}$

$\Longrightarrow$ ψ is an eq.

$\overset{\text{2-out-of-3}}{\Longrightarrow}$ f is Cartesian.

□

Thomas Read: (Un)-Straightening

Lecture 5
02.11.22

$$\{\text{coCart fib over } \mathcal{C}\} \longleftrightarrow \{\text{functors } \mathcal{C} \to \mathrm{Cat}_\infty\}$$

Intuitively, how does this work? Let $p: \mathcal{E} \to \mathcal{C}$ be a coCartesian fibration.

Given $x \in \mathcal{C} \rightsquigarrow \mathcal{E}_x \in \mathrm{Cat}_\infty$. Given $f: x \to y$ we want $f_!: \mathcal{E}_x \to \mathcal{E}_y$.

Let $z \in \mathcal{E}_x$, take coCart lift of f

$$z \to f_! z$$

Given $\alpha: z \to z' \rightsquigarrow f_! \alpha$

$$\begin{tikzcd} \mathcal{E} & z \arrow[r, "f_!"'] \arrow[d, "\alpha"'] & f_! z \arrow[d, dashed, "f_! \alpha"] \\ & z' \arrow[r] & f_! z' \\ \mathcal{C} & x \arrow[r, "f"] \arrow[d, equal] & y \arrow[d, equal] \\ & x \arrow[r, "f"'] & y \end{tikzcd}$$

because coCart

(This is unique up to cont. choice)

In 1-Category land we would be done (the choices would be unique), but in ∞-category land this is more involved.

<u>Def:</u> $K \in \mathrm{sSet}$, $f: K \to \mathcal{C}$, $p: \mathcal{E} \to \mathcal{C}$ coCart. Define $\mathrm{Fun}_f^{cc}(K, \mathcal{E})$ as the full subcategory spanned by those functors $K \to \mathcal{E}$ which lifts f and sends all $\underset{\text{1-simplices}}{\underline{\text{morphisms}}}$ of K to coCartesian morphisms.

<u>Cor:</u> 3.2.18 Let $i: K \to L$ be left anodyne, $f: L \to \mathcal{C}$. Then

$$\mathrm{Fun}_f^{cc}(L, \mathcal{E}) \xrightarrow{i^*} \mathrm{Fun}_{fi}^{cc}(K, \mathcal{E})$$

is a trivial fibration.

Recall that $\{0\} \hookrightarrow \Delta^n$ is left anodyne, so we get:

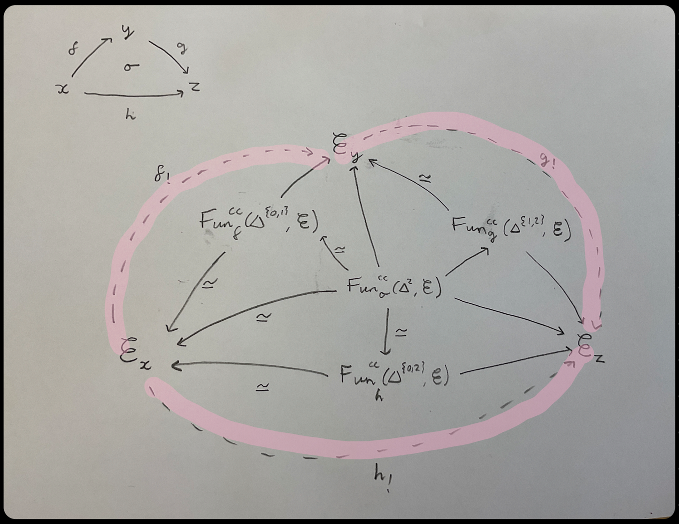
$$\mathrm{Fun}_f^{cc}(\Delta^1, \mathcal{E}) \underset{\overset{\curvearrowleft}{s}}{\longrightarrow} \mathrm{Fun}_{\underset{\text{domain of } f}{x}}^{cc}(\{0\}, \mathcal{E}) \simeq \mathcal{E}_x$$

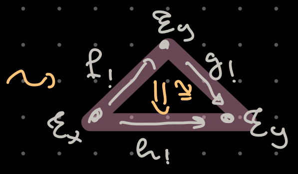
is a trivial fibration, so it admits a section:

$$\boxed{f_!: \mathcal{E}_x \xrightarrow[s]{} \mathrm{Fun}_f^{cc}(\Delta^1, \mathcal{E}) \longrightarrow \mathcal{E}_y.}$$

An actual functor!

What does $\mathcal{C} \to \mathrm{Cat}_\infty$ do on 2-simplices then?





This might get ugly, but defining functors between ∞-categories are hard!

If A is a 1 category, then we have a functor $A \to \underbrace{\mathrm{Cat}^1_\infty}_{\text{1-cat of } \infty\text{-cats}}$

$\rightsquigarrow N(A) \to N(\mathrm{Cat}^1_\infty) \to N(\mathrm{Cat}^1_\infty)[\mathrm{Eq}^{-1}] \simeq \mathrm{Cat}_\infty$

Given $W \subseteq A \rightsquigarrow N(A)[W^{-1}] \longrightarrow \mathrm{Cat}_\infty$.

By choosing a clever choice of W, A we can get $N(A)[W^{-1}] \simeq \mathcal{C}$.

Clever choice of A: $A = \Delta^{op}/\mathcal{C}$ category of simplices, defined by

- objects $= (n, \sigma : \Delta^n \to \mathcal{C})$
- morphism $(n, \sigma) \to (m, \tau) = \alpha : [m] \longrightarrow [n]$ s.t.

$$\begin{tikzcd} \Delta^m \arrow[rr, "\alpha_*"] \arrow[dr, "\tau"'] & \arrow[d, phantom, "\circlearrowright"] & \Delta^n \arrow[dl, "\sigma"] \\ & \mathcal{C} & \end{tikzcd}$$

We see that $N(\Delta^{op}/\mathcal{C})$ is "baycentric subdivision":

$\sigma : \Delta^2 \to \mathcal{C}$

$* = \sigma \to y \to z$

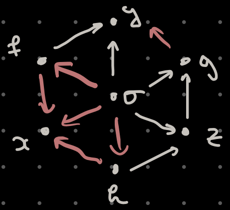
Define $W_\mathcal{C}$ to be the set of maps of the form

$(n, \sigma) \to (m, \tau)$, with $\alpha : [m] \to [n]$,

such that $\alpha(0) = 0$.

So the red maps

are in $W_{\mathcal{C}}$:



Want: $N(\Delta^{op}/\mathcal{C})[W_{\mathcal{C}}^{-1}] \to \mathcal{C}$ Joyal equivalence

Have canonical map

initial vertex map $IV: N(\Delta^{op}/\mathcal{C}) \to \mathcal{C}$

k-simplex $[n_0] \xrightarrow{\alpha_0} [n_1] \to \cdots \xrightarrow{\alpha_n} [n_k]$ $\mapsto$ $\Delta^{\{0,\alpha_k(0),\alpha_k(\alpha_{k-1}(0)),\dots,\alpha_n(\cdots(\alpha_1(0))\}}$

$\sigma: \Delta^{n_k} \to \mathcal{C}$ $\hookrightarrow \Delta^{n_k} \xrightarrow{\sigma} \mathcal{C}$

$w \in W_{\mathcal{C}} \Rightarrow IV(w)$ degenerate, so IV induces $N(\Delta^{op}/\mathcal{C})[W_{\mathcal{C}}^{-1}] \to \mathcal{C}$.

First generalize to $X \in sSet$, $W \subseteq X_1$

Define $L(X,W)$ as the pushout

$$\begin{tikzcd} \coprod \Delta^1 \arrow[r] \arrow[d] & X \arrow[d] \\ \coprod_{f \in W_X} \boxed{J} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & L(X,W) \end{tikzcd}$$

from def of localization

$\rightsquigarrow$ This satisfies that for $X = \mathcal{D}$ an ∞-category:

$L(\mathcal{D},W) \simeq \mathcal{D}[W^{-1}]$ so it is indeed a generalization

So want to show that

induced by IV

$L(N(\Delta^{op}/X), W_X) \to X$

to be a weak categorical equivalence.

Fact: $X \mapsto L(N(\Delta^{op}/X), W_X)$ preserves monomorphisms and colimits.

$\Rightarrow$ Suffices to consider the case $X = \Delta^n$:

$IV: N(\Delta^{op}/\Delta^n) \to \Delta^n = N([n])$

induced by

$$\begin{aligned} \widetilde{Iv} : \Delta^{op}/\Delta^n &\longrightarrow [n] \\ (m, \sigma : \Delta^m \to \Delta^n) &\mapsto \text{label of } \sigma(0) \end{aligned}$$

Have

$$\begin{aligned} i : [n] &\longrightarrow \Delta^{op} \\ k &\longmapsto \Delta^{\{k,\dots,n\}} \hookrightarrow \Delta^n \end{aligned}$$

via morphisms in W_{Δ^n}

$$\widetilde{Iv}\, i \simeq id \quad , \quad i\, \widetilde{Iv} \Rightarrow id$$

So when taking the nerve we get a Joyal equivalence ⚠

$\rightsquigarrow$ $p : \mathcal{E} \to \mathcal{C}$ cocartesian. Have

$$\begin{aligned} \Delta^{op}/\mathcal{C} &\longrightarrow Cat^1_\infty \\ (n, \sigma : \Delta^n \to \mathcal{C}) &\mapsto Fun^{cc}_\sigma(\Delta^n, \mathcal{E}) \end{aligned}$$

Given $(n,\sigma) \to (m,\tau)$ in $W_\mathcal{C}$

$$\begin{tikzcd} Fun^{cc}_\sigma(\Delta^n, \mathcal{E}) \arrow[rr] \arrow[dr, "\sim"'] & ? & Fun^{cc}_\tau(\Delta^m, \mathcal{E}) \arrow[dl, "\sim"] \\ & Fun^{cc}_{\{c\}}(\Delta^0, \mathcal{E}) & \end{tikzcd}$$

By 2-out-of-3 this is again an eq.

So we get the straightening of $p : \mathcal{E} \to \mathcal{C}$

$$\mathcal{C} \xrightarrow[\text{Joyal eq}]{} N(\Delta^{op}/\mathcal{C})[W_\mathcal{C}^{-1}] \longrightarrow Cat_\infty$$

Thm: Equivalence

$$coCart(\mathcal{C}) \underset{\xrightarrow{\text{straightning}}}{\overset{\xleftarrow{\text{Unstraightning}}}{\simeq}} Fun(\mathcal{C}, Cat_\infty)$$

Special case:

- $LFib(\mathcal{C}) \simeq Fun(\mathcal{C}, \mathcal{S})$
- X a Kan complex : $\mathcal{S}/X \simeq Fun(X, \mathcal{S})$

A comment about unstraightning:

$$\mathrm{id}: \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty \quad \longleftrightarrow \quad p: (\mathrm{Cat}_\infty)_{*//} \longrightarrow \mathrm{Cat}_\infty$$

universal cocart. fib

objects: pairs $(\mathcal{C}, x \in \mathcal{C}_0)$

mor $(\mathcal{C}, x) \to (\mathcal{D}, y)$ is $F: \mathcal{C} \to \mathcal{D}$, $\alpha: y \to Fx$

$$\rightsquigarrow \qquad \begin{tikzcd} F^*(\mathrm{Cat}_\infty)_{*//} \arrow[r] \arrow[d, "F^*p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Cat}_{*//} \arrow[d, "p"] \\ \mathcal{C} \arrow[r, "F"'] & \mathrm{Cat}_\infty \end{tikzcd}$$

Hannah MacDermott: Adjoint functor "theorem"

Talk 8 – 23.11.22

Cofinal functors

Def: $K, L, X \in \text{sSet}$, $f : K \to L$, $p : X \to L$ inner fibration. We define the

$$\begin{tikzcd}
\mathrm{Fun}_X(K,L) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(K,X) \arrow[d, "p_*"] \\
\Delta^0 \arrow[r, "f"'] & \mathrm{Fun}(K,L)
\end{tikzcd}$$

$$\mathrm{Fun}_L(L,X) = \mathrm{Fun}_{\mathrm{id}_L}(L,X)$$

$f : K \to L$ is cofinal if for all right fibrations $p : X \to L$, the map

$$\mathrm{Fun}_L(L,X) \to \mathrm{Fun}_f(K,X)$$

is a Joyal equivalence.

Thm: 4.4.8 f is cofinal iff for any ∞-category $\mathcal{C}$ and $p : L \to \mathcal{C}$, the induced map

$$\mathcal{C}^{p/} \to \mathcal{C}^{pf/}$$

is a Joyal equivalence.

Cor: 4.4.9 Joyal equivalences are cofinal

Cor: 4.4.10 p admits a colimit iff pf does, and f preserves this colimit.

Thm: 4.4.20 $f : \mathcal{C} \to \mathcal{D}$ map of simplicial sets, $\mathcal{D}$ an ∞-category. Then

$$f \text{ is cofinal} \Leftrightarrow \mathcal{C}_{d/} \text{ is weakly } \underset{|\mathcal{C}_{d/}| \simeq *}{\underline{\text{contractible}}} \text{ for every object in } \mathcal{D}$$

Prop: 5.2.6 $\mathcal{C}, \mathcal{D}$ ∞-categories, $F : \mathcal{C} \to \mathcal{D}$. F admits a right adjoint iff $\mathcal{C}/d$ admits a terminal object for each $d \in \mathcal{D}$.

Pf: Choose $d \rightsquigarrow (Gd, f : FGd \to d)$ terminal object in $\mathcal{C}_{/d}$. By 5.1.10, sufficient to prove that

$$\mathrm{map}_{\mathcal{C}}(c, G d) \longrightarrow \mathrm{map}_{\mathcal{D}}(F, FGd) \longrightarrow \mathrm{map}_{\mathcal{D}}(F, d)$$

is an equivalence. Consider the following diagram

$$\begin{tikzcd}
\mathrm{map}_{\mathcal{C}/d}((c,\alpha),(Gd,f)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] \arrow[rr, bend left, "\sim"] & \mathrm{map}_{\mathcal{D}/d}(c,f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\textcircled{\lrcorner}\ 3.3.18"] & \Delta^0 \arrow[d, "\alpha"] \\
\mathrm{map}_{\mathcal{C}}(c, G d) \arrow[r] \arrow[rr, bend right, "\sim"'] & \mathrm{map}_{\mathcal{D}}(F, FGd) \arrow[r, "f_*"] & \mathrm{map}_{\mathcal{D}}(F, d)
\end{tikzcd}$$

$\sim$ on components lying over the component of α

$\rightsquigarrow$ True for all $\alpha : Fc \to d$.

Joyal equivalent to a small ∞-cat.

$\square$

Thm: 5.2.2 $\mathcal{C}$ locally small cocomplete ∞-category. If there exists $\mathcal{C}_0 \subseteq \mathcal{C}$ full essentially small colimit dense, then

For any $c \in \mathcal{C}$, have a diagram s.t. $c = \mathrm{colim}(K \to \mathcal{C}_0 \to \mathcal{C})$

locally small

$$F : \mathcal{C} \to \mathcal{D} \text{ admits a right adjoint} \iff F \text{ preserves colimits.}$$

Def: Let S be a small set of objects in $\mathcal{C}$. Then S is weakly terminal if for every object $c \in \mathcal{C}$, there exists an object $s \in S$ s.t. $\mathrm{map}_{\mathcal{C}}(c, s)$ is not empty.

An object t is said to be weakly terminal if the set $\{t\}$ is weakly terminal.

Lem: 5.2.10 $\mathcal{C}$ cocomplete, and there exists $\mathcal{C}_0 \subseteq \mathcal{C}$ essentially small colimit-dense full subcategory. Then $\mathcal{C}$ has a weakly terminal object.

Pf: Consider $\mathcal{C}' \simeq \mathcal{C}$, $t = \mathrm{colim}_{\mathcal{C}}(\mathcal{C}' \to \mathcal{C}) = \mathrm{colim}_{\mathcal{C}}(\mathcal{C}_0 \to \mathcal{C})$. Write c in $\mathcal{C}$ as $\mathrm{colim}_{\mathcal{C}}(K \to \mathcal{C}_0 \to \mathcal{C})$. Obtain a map $c \to t$.

Prop: 5.2.11 $\mathcal{C}$ locally small cocomplete, $S \subseteq \mathcal{C}$ weakly terminal and assume there exists a full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ spanned by S. Then $\mathcal{C}_0 \to \mathcal{C}$ is cofinal

Pf: Sufficient to show that $(\mathcal{C}_0)_{x/}$ is weakly contractible for all x in $\mathcal{C}$

$\rightsquigarrow$ Sufficient to show that any functor

$$K \longrightarrow (\mathcal{C}_0)_{x/}$$

factors through $K * \Delta^0$ which is weakly cont.

$\mathcal{C}$ cocomplete $\Rightarrow$ $\mathcal{C}_{x/}$ cocomplete 4.3.35

Take colimit cone

$$\begin{tikzcd} K \arrow[r] \arrow[d] & (\mathcal{C}_0)_{x/} \arrow[r] & \mathcal{C}_{x/} \\ K * \Delta^0 \arrow[urr, "\mu"'] & & \end{tikzcd}$$

$\mu(\infty): x \to y$, pick $s \in S$ $y \to s$. Choosing a composite $x \to y \to s$ gives a 2-simplex $\Delta^2 \to \mathcal{C}$, adjoint to a map $\sigma: \Delta^1 \to \mathcal{C}^{x/}$

$$\begin{tikzcd} K * \Delta^0 \sqcup_{\Delta^0} \Delta^1 \arrow[r, "{\mu, \sigma}"] \arrow[d] & \mathcal{C}_{x/} \\ K * \Delta^1 \arrow[ur, "\mu'"'] & \end{tikzcd}$$

1.4.22

Lift exists because the vertical map is inner anodyne and $\mathcal{C}_{x/}$ is an ∞-category. Restricting μ' to $K * \Delta^{\{1\}}$ gives a map $\mu'': K * \Delta^0 \to \mathcal{C}_{x/}$. $(\mathcal{C}_0)_{x/}$ is full in $\mathcal{C}_{x/}$, so sufficient to show that we can find $K * \Delta^0 \to \mathcal{C}_{x/}$ which sends objects to $(\mathcal{C}_0)_{x/}$.

$$\left.\begin{array}{l} \mu'' \text{ agrees w. } \mu \text{ on } K \\ \mu''(\infty) = \mu'(1) = \mu(1) = \sigma(1) = s \end{array}\right\} \text{Both in } \mathcal{C}_0$$

$\square$

Cor: $\mathcal{C}$ locally small cocomplete, $S \subseteq \mathcal{C}$ weakly terminal. Then $\mathcal{C}$ admits a terminal object.
5.2.12

Pf: Again take $\mathcal{C}_0$ to be the full subcategory spanned by S. $\mathcal{C}_0$ is small so $\mathcal{C}_0 \to \mathcal{C}$ has a colimit. $\mathcal{C}_0 \to \mathcal{C}$ is cofinal

$$\rightsquigarrow \operatorname{colim}(\mathcal{C}_0 \to \mathcal{C} \xrightarrow{id} \mathcal{C}) = \operatorname{colim}(\mathcal{C} \xrightarrow{id} \mathcal{C})$$

lemma 4.3.15

A colimit of the identity contains a terminal object.

$\square$

Proof of adjoint functor theorem. Enough to show that every $\mathcal{C}_{/d}$ has a terminal object

$\rightsquigarrow$ Enough to show that $(\mathcal{C}_0)_{/d} \subseteq \mathcal{C}_{/d}$ essentially ✓ small, colimit dense ✓ full subcategory and $\mathcal{C}_{/d}$ is locally small and cocomplete 3.3.18, 5.1.30

Need $\{$equivalence classes in $(\mathcal{C}_0)_{/d}\}$ to be small $\sim$ but $\{$equivalence classes in $\mathcal{C}\}$ and $\mathrm{map}(Fc, d)$ are all small.

Sufficient to prove that $(\mathcal{C}_0)_{/d} \subseteq \mathcal{C}_{/d}$ is colimit-dense. Take $(c, \alpha : Fc \to d)$ in $\mathcal{C}_{/d}$, and write

$$c = \operatorname{colim}(K \to \mathcal{C}_0 \to \mathcal{C})$$

Get a cone

$$K * \Delta^0 \to \mathcal{C}.$$

Want to lift this to $\mathcal{C}_{/d}$ $\rightsquigarrow$ get a diagram

$$K \to \mathcal{C}_{/c} \xrightarrow{F} \mathcal{D}_{/Fc} \xrightarrow{\alpha} \mathcal{D}_{/d}$$

$$\begin{tikzcd}
K \arrow[rr] \arrow[dd] \arrow[dr] & & \mathcal{D}_{/d} \arrow[dd] \\
& (\mathcal{C}_0)_{/d} \arrow[ur] \arrow[dl] & \\
\mathcal{C}_0 \arrow[rr] & & \mathcal{D}
\end{tikzcd}$$

4.3.34

$\operatorname{colim}(K \to \mathcal{C}_{0/d})$ is (c, α)

MOTIVATION

There is useful analoges between homotopy theory and chain complexes (homological algebra) with values in an abelian category

- "homotopies" between chain maps
- "contractible" chain complexes

chain cpx's

The analoge of the homotopy category of topological spaces is the 'derived category' of an abelian category, and analouge of stable homotopy theory is the homotopy category of spectra - Both which are triangulated categories, which gives a good setting for doing homological algebra. Working with triangulated categories can be very useful in practice but their theory can be very lacking, e.g:

- The category of functors between two triangulated categories doesn't inherit a triangulated structure
- Being triangulated is extra structure instead of a property
- Often can't use 'descent' ("glueing arguments)

Many of these problems can be traced back to the fact that we identify things without remembering <u>why</u> they're identified. Using ∞-categories let's us keep track of all this extra information.

We will therefor introduce the two ∞-categorical analouges of these, see which properties these admit and use this as motivation for our definition.

think left/right modules over a ring R

<u>Ex</u>: Let $\mathcal{A}$ be a Groethendieck abelian category, i.e. locally presentable abelian category in which the small filtered colimit of a collection of SES's is again a SES.

Let $Ch(\mathcal{A})$ = 1-Cat. of chain complexes in $\mathcal{A}$

$$\text{Model structure on } Ch(\mathcal{A}) = \begin{cases} \text{cofib's} = \text{levelwise monomorphisms} \\ \text{weak eq.s} = \text{quasi-isomorphism} \end{cases}$$

We then define the <u>(unbounded) derived ∞-category</u> of $\mathcal{A}$ to be

$$\mathcal{D}(A) := N_{dg}(Ch(A)^{\circ}),$$

the differential graded nerve of the bifibrant objects w.r.t. this model structure

↪ The localization of the category of chain complexes + the class of quasi-isomorphisms

Ex: First recall the ∞-category of spaces $N^{ch}(Kan) = \mathcal{S}$. We wish to consider the "pointed" spaces:

$$\mathcal{S}_* \subseteq Fun(\Delta^1, \mathcal{S})$$

spanned by those functors which takes 0 to a "terminal" object of $\mathcal{S}$.

We define the ∞-category of spectra as

$$Sp := \lim(\cdots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*)$$

↪ Ω is the loop functor which we will define later

↪ Think of it as a collection of spaces $\{X_n\}_{n\in\mathbb{Z}}$ + $X_n \overset{\text{htpy}}{\simeq} \Omega X_{n+1}$.

PROPERTIES THEY SHARE:

1) Pointed = Admits a zero-object

= An object which is both initial & terminal

- In Sp: spectrum which is a point at each level

  = Those whose homotopy groups all vanish

- In $\mathcal{D}(A)$: Acyclic complexes

  = Those whose homology all vanish

Rem: Initial and terminal object is only unique up to contractible space of choices. We see this already at 1-categorical level: E.g. any one-point set is terminal in the category of sets, but $\{0\}$ and $\{1\}$ are two different (yet isomorphic) terminal objects

Rem: In a pointed ∞-category, we get a 'zero morphism' between any two objects by composing the initial/terminal map

$$x \to 0 \to y$$

2) Concept of (co)fibres or (co)kernels

- In $\mathcal{D}(A)$: Given a chain map $f: C \to D$ between chain complexes

  $\rightsquigarrow$ Associated SES

  $$0 \to C \to cyl(f) \to cone(f) \to 0$$

  where $cyl(f)$ is a chain complex quasi-isomorphic to D.

  ↪ We can often say interesting things about f by studying $cone(f)$, e.g. there is a LES relating $H_*(C)$, $H_*(D)$ and $H(cone(f))$, so $cone(f)$ is acyclic iff f is a quasi-isomorphism.

  ↪ We can view $cone(f)$ as a 'cokernel' or 'cofibre' of f.

- In Sp: If we have a map $f: X \to Y$ between spectra, we get a fibre sequence

  $$fib(f) \to X \xrightarrow{f} Y$$

  which comes with an associated LES on homotopy groups. There is also a notion of cofibre sequences, and it turns out that fibre sequences are exactly the same as cofibre sequences, i.e.

  $$cofib(fib(f) \to X) \simeq Y.$$

Let's make this notion more precise:

Def: A triangle in a pointed ∞-category is a diagram $\Delta^1 \times \Delta^1 \to \mathcal{C}$ of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

Rem: More explicitly: A triangle consists of

i) Two 1-simplices $X \xrightarrow{f} Y \xrightarrow{g} Z$

ii) A 2-simplex

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

witnessing a composite h of f and g.

iii) A 2-simplex

$$\begin{tikzcd} X \arrow[d] \arrow[dr, "h"] & \\ 0 \arrow[r] & Z \end{tikzcd}$$

witnessing a null-homotopy of h.

Rem: A $\Delta^1 \times \Delta^1 \to \mathcal{C}$ diagram $\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$ determines a unique! triangle by precomposing with the map $\Delta^1 \times \Delta^1 \to \Delta^1 \times \Delta^1$ that flips the two factors.

Def: Let $\mathcal{C}$ be pointed and $p : \Delta^1 \times \Delta^1 \to \mathcal{C}$ the triangle

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z \end{tikzcd}.$$

We say p is a <u>fibre sequence</u> if p is a pullback diagram (we say that p is a fibre of g). Dually, we say that p is a <u>cofibre sequence</u> if p is a pushout diagram (p is a cofibre of f).

Now, lets finally define stable:

Def: An ∞-category $\mathcal{C}$ is <u>stable</u> if

i) It is pointed

ii) Every morphism admits a fibre and a cofibre sequence

iii) A triangle is a fibre sequence iff it is a cofibre sequence.

Rem: So being stable is a _property_, not extra structure.

↳ Think of given a set X, then it doesn't make sense to ask if it is a group without also specifying $m: X\times X \to X$ group multiplication. On the other hand, it does make sense to ask if a group G is abelian.

↳ Why does this difference matter? Often easier to formulate cleaner abstract arguments when working w. properties.

Prop: |HA.1.4.2.27| Let $\mathcal{C}$ be pointed. Then the following is equivalent;

i) $\mathcal{C}$ is stable

ii) $\mathcal{C}$ has finite colimits and the suspension functor $\Sigma: \mathcal{C} \to \mathcal{C}$, determined by the existence of the pushout square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \Sigma X, \end{tikzcd}$$

is an equivalence

iii) $\mathcal{C}$ has finite limits, and the loop functor $\Omega: \mathcal{C} \to \mathcal{C}$, determined by the pullback square

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & 0 \arrow[d] \\ 0 \arrow[r] & X, \end{tikzcd}$$

is an equivalence

Pf: We will show i)<=>(ii), the case of (iii) is dual. It is clear that (i) => (ii), so assume $\mathcal{C}$ admits finite colimits, and that Σ is an equivalence.

Claim 1: Every cofiber sequence is a fiber sequence

Pf: Let $p: \Delta^1 \times \Delta^1 \to \mathcal{C}$ denote the cofiber sequence determined by the square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "P"] \\ 0 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & C, \end{tikzcd} \qquad (*)$$

and

$$P_0: \Delta^1 \amalg_{\Delta\{1\}} \Delta^1 \to \mathcal{C}$$

the restriction of p to the lower and right edges To show the square is Cartesian, we need to show that the induced map

$$\mathcal{C}_{/p} \longrightarrow \mathcal{C}_{/p_0}$$

is an equivalence (trivial Kan fibration). Sufficient to show that for each $X \in \mathcal{C}$

$$\mathcal{C}_{/p} \times_{\mathcal{C}} \{X\} \longrightarrow \mathcal{C}_{/p_0} \times_{\mathcal{C}} \{X\}$$

is a weak equivalence of Kan complexes. We see that $\mathcal{C}_{/p_0} \times_{\mathcal{C}} \{X\}$ classifies diagrams of the form

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, "p"] & 0 \arrow[d] \\
0 \arrow[r] & C \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \Sigma A
\end{tikzcd}$$

by taking the cofiber

Pushout

(Note that the cofiber of p is ΣA by $(*)$).

$\rightsquigarrow$ The outer diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, "p"] & 0 \arrow[d] \arrow[dd, bend left] \\
0 \arrow[r] \arrow[drr, bend right] & C \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \Sigma A \\
 & & \Sigma X \arrow[u, "\exists!"']
\end{tikzcd}$$

$$\Sigma X \longrightarrow \Sigma A$$

$$\begin{tikzcd}
\rightsquigarrow \quad \Sigma X \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
 & \Sigma A \arrow[r] \arrow[d, "f"] & 0 \arrow[d] \\
 & \Sigma B \arrow[r] & \Sigma C ,
\end{tikzcd}$$

(essentially uniquely extended!)

Let $q : \Delta^1 \times \Delta^1 \to \mathcal{C}$ be the map corresponding to

$$\begin{tikzcd}
\Sigma A \arrow[r] \arrow[d, "f"] & 0 \arrow[d] \\
\Sigma B \arrow[r] & \Sigma C
\end{tikzcd}$$

and $q_0 = q|_{\Delta^1 \amalg_{\Delta^{\{1\}}} \Delta^1}$ be the restriction to the right and bottom edges. By the above we see that we get (a contractible choice of) maps of the form

$$\mathcal{C}_{/p_0} \times_{\mathcal{C}} \{X\} \longrightarrow \mathcal{C}_{/q} \times_{\mathcal{C}} \{\Sigma X\}$$

Hence we obtain a sequence of maps of the form

$$\begin{tikzcd}
\mathcal{C}_{/p} \times_{\mathcal{C}} \{X\} \arrow[r] \arrow[rr, bend right] & \mathcal{C}_{/p_0} \times_{\mathcal{C}} \{X\} \arrow[r] \arrow[rr, bend right] & \mathcal{C}_{/q} \times_{\mathcal{C}} \{\Sigma X\} \arrow[r] & \mathcal{C}_{/q_0} \times_{\mathcal{C}} \{\Sigma X\}
\end{tikzcd}$$

The interesting comps

These two compositions can be shown to agree with the maps induced by $\Sigma \rightsquigarrow$ weak equivalences!

2-out-of-6 $\Rightarrow$ Each map is a weak equivalence!

Claim 2: Every map in $\mathcal{C}$ has a fiber

Pf: Let $p: B \to C$ be a map and $A \in \mathcal{C}$ s.t. p admits a cofiber seq. of the form

$$\begin{tikzcd}
B \arrow[r] \arrow[d, "p"'] & 0 \arrow[d] \\
C \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \Sigma A
\end{tikzcd}$$

Exists since ΣA is an eq.

From claim 1 we know this is also a fiber sequence:

$$\begin{tikzcd}
A \arrow[r, "\exists!"] \arrow[d] \arrow[rr, bend left] \arrow[dr, phantom, "\lrcorner"] & B \arrow[r] \arrow[d, "p"] \arrow[dr, phantom, "\lrcorner"] & 0 \arrow[d] \\
0 \arrow[r] \arrow[rr, bend right] & C \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \Sigma A
\end{tikzcd}$$

$\rightsquigarrow$ Left hand square is again a pullback ✓

Claim 3: Every fiber sequence is a cofiber sequence

Pf: Consider $\mathcal{C}^{op}$. By claim II it has all cofibers. Since $\Sigma_{\mathcal{C}}$ is an equivalence, so is $\Sigma_{\mathcal{C}^{op}}$. Applying claim I to $\mathcal{C}^{op}$ we get that all fiber sequences in $\mathcal{C}$ is a cofiber sequence.

□

Cor: $\mathcal{C}$ stable $\Rightarrow$ $\mathcal{C}^{op}$ stable

Cor: If $\mathcal{C}$ is a stable, then Σ, Ω are inverse equivalences.

Pf: Let $X \in \mathcal{C}$, then we have a cofiber sequence

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \Sigma X
\end{tikzcd}
\quad \overset{\text{Also fib seq}}{\rightsquigarrow} \quad
\begin{tikzcd}
X \simeq \Omega(\Sigma X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & 0 \arrow[d] \\
0 \arrow[r] & \Sigma X
\end{tikzcd}$$

□

Cor: Ω^∞ = unique map : $Sp = \operatorname{colim}(\mathcal{S}_* \xrightarrow{\Omega} \cdots \xrightarrow{\Omega} \mathcal{S}_*) \to \mathcal{S}_*$

$\Sigma^\infty : \mathcal{S}_* \to Sp$

$\rightsquigarrow$ Adjunction $\begin{tikzcd} Sp \arrow[r, bend right, "\Omega^\infty"'] \arrow[r, phantom, "\perp"] & \mathcal{S} \arrow[l, bend right, "\Sigma^\infty"'] \end{tikzcd}$

~~**Def:** A functor $f:\mathcal{C}\to\mathcal{D}$ between stable ∞-categories are *exact* if it preserves the zero object (i.e. reduced) and (co)fiber sequences~~

Construction of mapping spectrum: We first note that

$$\mathrm{Fun}^{\mathrm{lex}\,\leftarrow\text{ preserves all limits}}(\mathcal{C},\mathrm{Sp})\xrightarrow[\Omega^\infty\circ(-)]{\sim}\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C},\mathcal{S}_*)$$

Using that $\mathrm{Map}_{\mathcal{C}}(X,-):\mathcal{C}\to\mathcal{S}$ preserves all limits, we get that it corresponds to

$$\mathrm{map}_{\mathcal{C}}(X,-):\mathcal{C}\to\mathrm{Sp}$$

n^{th} space of $\mathrm{map}_{\mathcal{C}}(X,Y)$ is $\mathrm{Map}_{\mathcal{C}}(X,\Sigma^n Y)$

s.t.

$$\Omega^\infty\,\mathrm{map}_{\mathcal{C}}(X,-)\simeq\mathrm{Map}_{\mathcal{C}}(X,-).$$

All of this is natural in $X\in\mathcal{C}$, so can build the mapping spectrum

$$\mathrm{map}_{\mathcal{C}}(-,-):\mathcal{C}^{op}\times\mathcal{C}\to\mathrm{Sp}.$$

Any stable ∞-category is spectrally enriched.

TRIANGULATED STRUCTURE

As mentioned in the beginning, another approach to capturing the "stable" properties of the derived category and the category of spectra, is using triangulated categories. As it turns out, the homotopy category of any stable ∞-category can be equipped with a triangulated structure (actually 2).

First recall the following definitions:

Def: A 1-Category is additive if

1) It is Ab-enriched (Each hom-set carries the structure of an abelian grap, and composition is bilinear)
2) Admits finite coproducts

Def: A triangulated category is an additive 1-category C with an equivalence $\Sigma: C \to C$ together with a class of distinguished triangles, each which is a diagram of the form

$$X \to Y \to Z \to \Sigma X$$

satisfying the following:

TR1: i) $\forall X \in C$, $X \xrightarrow{id} X \to 0 \to \Sigma X$ is a distinguished triangle

ii) $\forall u: X \to Y$ $\exists Z \in C$ + $X \xrightarrow{u} Y \to Z \to \Sigma X$ distinguished

iii) Class of distinguished triangles are closed under iso.

TR2: $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ distinguished iff

$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$ distinguished

TR3:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & Z \arrow[r] \arrow[d, dashed, "\exists"] \arrow[dr, phantom, "\circlearrowleft"] & \Sigma X \arrow[d, "\Sigma u"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'
\end{tikzcd}$$

Distinguished

TR4: "Octahedral axiom"

The main point of writing out this definition is to apprechiate how compact the definition of stable ∞-category is!

Thm: $\mathcal{C}$ stable $\Rightarrow$ $h\mathcal{C}$ admits a triangulated structure

We won't go into to many details, but we will describe why $h\mathcal{C}$ is additive and the distinguished triangles.

$h\mathcal{C}$ is additive: We define translation functor

$$[n]: \mathcal{C} \longrightarrow \mathcal{C}$$

as n^{th}-fold suspension (for $n \geq 0$, $(-n)^{th}$-fold for $n \leq 0$). Using that limits commutes w. $\mathrm{Map}_{\mathcal{C}}(X,-)$ (and Ω is a limit), we get

$$\mathrm{Map}_{\mathcal{C}}(X,Y) \simeq \mathrm{Map}_{\mathcal{C}}(X, \Omega^n \Sigma^n Y) \simeq \Omega^n \mathrm{Map}_{\mathcal{C}}(X, Y[n]) \quad \text{of sSets}$$

$\rightsquigarrow$ $\mathrm{Map}_{\mathcal{C}}(X,Y)$ is an ∞-loop space, so $\pi_0 \mathrm{Map}_{\mathcal{C}}(X,Y) \simeq \mathrm{Hom}_{h\mathcal{C}}(X,Y)$ is an abelian group under loop sum

To see that $h\mathcal{C}$ admits finite coproducts, we see that $\mathcal{C}$ admits such:

Note that

1) $X \simeq \mathrm{cofib}(X[-1] \xrightarrow{u} 0)$

2) $Y \simeq \mathrm{cofib}(0 \xrightarrow{v} Y)$

3) $u \amalg v = (X[-1] \xrightarrow{0} Y)$ in $\mathrm{Fun}(\Delta^1, \mathcal{C})$

4) cofibers preserves colimits

$\Rightarrow$ $X \amalg Y = \mathrm{cofib}(X[-1] \xrightarrow{0} Y)$

Def: Let $\mathcal{C}$ be a stable ∞-category. We say that a diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

in $h\mathcal{C}$ is a distinguished triangle, if there exists a diagram $\Delta^1 \times \Delta^2 \to \mathcal{C}$ of the form

$$\begin{tikzcd}
X \arrow[r, "\tilde{f}"] \arrow[d] & Y \arrow[r] \arrow[d, "\tilde{g}"] \arrow[dl, phantom, "\ulcorner"] & 0 \arrow[d] \arrow[dl, phantom, "\ulcorner"] \\
0 \arrow[r] & Z \arrow[r, "\tilde{h}"'] & W
\end{tikzcd}
\quad \text{s.t.} \quad
\begin{tikzcd}
Z \arrow[r, "\tilde{h}"] \arrow[dr, "h"'] & W \arrow[d, "\simeq"] \arrow[dl, phantom, "2"] \\
 & \Sigma X
\end{tikzcd}$$

Rem: 1) Diagram 1 implies that $w \simeq \Sigma X$, which is the vertical map in the triangle

2) We made a choice of having triangles of the form $\Delta^1 \times \Delta^2 \to \mathcal{C}$ rather than of the form $\Delta^2 \times \Delta^1 \to \mathcal{C}$. Either choice works, but the two choices yields two different triangulated structures on $h\mathcal{C}$.

T-STRUCTURES

As mentioned in the beginning, a reason to consider stable ∞-categories is that it is a form of analouge of chain complexes and gives us a framework to do (homological) algebra. An example of this is that given a filtered chain complex

$$\dots \subseteq F_{p-1} \subseteq F_p \subseteq \dots \subseteq C$$

we get a spectral sequence

$$E^1_{p,q} = H_{p+q}(F_p/F_{p-1}) \Rightarrow H_{p+q}(C)$$

whose convergence is conditional on finiteness properties of the filtration. This spectral sequence arises from the SES of chain complexes

$$0 \to F_{p-1} \to F_p \to F_p/F_{p-1} \to 0.$$

This can be generalised to stable ∞-categories by adding a so called "t-structure", which in particular associates an abelian category.

Notation: $\mathcal{C}[n] = \Sigma^n \mathcal{C}$, $\mathcal{C}[-n] = \Omega^n \mathcal{C}$

Def: Let $\mathcal{C}$ be a stable ∞-category. A t-structure on $\mathcal{C}$ consists of a pair $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ of full subcategories of $\mathcal{C}$ s.t.

1) $\mathcal{C}_{\geq 0}[1] \subseteq \mathcal{C}_{\geq 0}$ (closed under suspension) ($\mathcal{C}_{\geq 0}$ = connective objects

$\mathcal{C}_{\leq 0}[-1] \subseteq \mathcal{C}_{\leq 0}$ (closed under loop] $\mathcal{C}_{\leq 0}$ = coconnective objects)

2) $\forall X, Y \in \mathcal{C}$: $\mathrm{Map}_{\mathcal{C}}(X, Y[-1]) \simeq *$

3) $\forall X \in \mathcal{C}$ $\exists$ fiber sequence

$$\begin{tikzcd}
\tau_{\geq 0} X \arrow[r] \arrow[d] & X \arrow[d] \\
0 \arrow[r] & \tau_{\leq -1} X
\end{tikzcd}$$

w. $\tau_{\geq 0} X \in \mathcal{C}_{\geq 0}$

$\tau_{\leq -1} X[1] \in \mathcal{C}_{\leq 0}$.

Write

$$\mathcal{C}_{\geq n} := \mathcal{C}_{\geq 0}[n], \quad \mathcal{C}_{\leq n} := \mathcal{C}_{\leq 0}[n]$$

Ex: ① $\mathcal{C} = Ch(R\text{-Mod}) \quad (\mathcal{D}(R))$

$\mathcal{C}_{\geq 0} = \{X \mid H_n X = 0,\ n<0\}$

$\mathcal{C}_{\leq 0} = \{X \mid H_n X = 0,\ n>0\}$

② $\mathcal{C} = Sp$

$\mathcal{C}_{\geq 0} = \{X \mid \pi_n X = 0,\ n<0\}$

$\mathcal{C}_{\leq 0} = \{X \mid \pi_n X = 0,\ n>0\}$

Postnikov t-structure

Def: The inclusions of $\mathcal{C}_{\geq n}, \mathcal{C}_{\leq n} \hookrightarrow \mathcal{C}$ admits adjoints:

coconnective cover

$$\begin{tikzcd}
\mathcal{C}_{\geq n} \arrow[r, bend left, "incl"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "\tau_{\geq n}"] \arrow[r, bend left, "\tau_{\leq n}"] \arrow[r, phantom, "\perp"] & \mathcal{C}_{\leq n} \arrow[l, bend left, "incl"]
\end{tikzcd}$$

connective cover

$\rightsquigarrow$ Fits into a commutative diagram (in sSets)

$$\begin{tikzcd}
\mathcal{C}_{\geq n} \arrow[r, bend right, "incl"'] \arrow[r, phantom, "\top"] \arrow[d, "\tau_{\leq m}"] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{C} \arrow[l, bend right, "\tau_{\geq n}"'] \arrow[d, "\tau_{\leq m}"] \\
\mathcal{C}_{\geq n} \cap \mathcal{C}_{\leq m} \arrow[r, bend left, "incl"] \arrow[r, phantom, "\top"] & \mathcal{C}_{\leq m} \arrow[l, bend left, "\tau_{\leq m}"]
\end{tikzcd}$$

$\rightsquigarrow$ $\theta : \tau_{\leq m} \circ \tau_{\geq n} \overset{\sim}{\Rightarrow} \tau_{\geq n} \circ \tau_{\leq m}$ of functors $\mathcal{C} \to \mathcal{C}_{\geq n} \cap \mathcal{C}_{\leq m}$.

Def: The heart of $\mathcal{C}$ is $\mathcal{C}^\heartsuit := \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0} \subseteq \mathcal{C}$

Rem: $\pi_n Map_{\mathcal{C}^\heartsuit}(X,Y) \cong \pi_n Map_{\mathcal{C}}(X,Y) \cong \pi_0 Map_{\mathcal{C}}\underbrace{(X, \Omega^n Y) = 0}_{\text{since}\ \Omega^n Y \in \mathcal{C}_{\leq -n}}$ for $n > 0$

$\rightsquigarrow$ $\mathcal{C}^\heartsuit \simeq N(h(\mathcal{C}^\heartsuit)) \simeq N((h\mathcal{C})^\heartsuit)$

Can be used to show it is abelian!

Ex: 1) $\mathcal{D}(R)^\heartsuit = R\text{-Mod}$

2) $Sp^\heartsuit = Ab$

Def: $\pi_0 := \tau_{\leq 0} \circ \tau_{\geq 0} \cong \tau_{\geq 0} \circ \tau_{\leq 0} : \mathcal{C} \to \mathcal{C}^\heartsuit$

$\pi_n : \mathcal{C} \xrightarrow{\Omega^n} \mathcal{C} \xrightarrow{\pi_0} \mathcal{C}^\heartsuit$

Ex: 1) $\mathcal{C} = \mathcal{D}(R) \Rightarrow \pi_n$ is the homology functor

2) $\mathcal{C} = Sp \Rightarrow \pi_n$ is the "normal" homotopy groups on spectra

Now, lets turn our focus to the spectral sequence promised as motivation for t-structures.

Def: A filtered object of $\mathcal{C} \in \mathrm{Cat}_\infty$ is a functor $F: (\mathbb{Z}, \leq) \longrightarrow \mathcal{C}$. We define the p^{th} graded piece of F as

$$gr_p(F) := \mathrm{cofib}(F(p-1) \to F(p)).$$

From this we obtain an exact couple

$$\begin{tikzcd}
\pi_{p+q}(F(p)) \arrow[r, "{(1,-1)}"] \arrow[dr, "{(-1,0)}"'] & \pi_{p+q}(F(p)) \arrow[d, "{(0,0)}"] \\
 & \pi_{p+q}(gr_p(F))
\end{tikzcd}$$

Recall that π_i lands in the abelian 1-category $\mathcal{C}^\heartsuit$, where we can do all our normal homological algebra. So the exact couple gives us a spectral sequence

$$E^1_{p,q} = \pi_{p+q}(gr_p(F))$$

in $\mathcal{C}^\heartsuit$ with Serre differentials, i.e.

$$d^r: E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}.$$

Under nice circumstances, this converges to $\pi_{p+q}(\mathrm{colim}\, F)$ - e.g. if $\mathcal{C}$ has sequential colimits (i.e. colimits for any diagram $(\mathbb{Z}_{\geq 0}, \leq) \to \mathcal{C}$), $\mathcal{C}_{\leq 0}$ is closed under sequential colimits and $F(p) \simeq 0$ $\forall p << 0$.

Ex: $\mathcal{C} = \mathcal{D}(R)$ $\rightsquigarrow$ This SS corresponds to the classical spectral sequence on filtered colimits.

INTRODUCTION TO HIGHER ALGEBRA

§1: SYMMETRIC MONOIDAL → OPERADS → ∞-LAND

The basic algebraic structure we want to generalise is

commutative monoid: Set M + multiplication $M \times M \to M$, unit $1 \in M$ s.t.

$$1x = x,\quad xy = yx,\quad x(yz) = (xy)z \quad \forall x,y,z \in M$$

⇝ For categories this is a <u>symmetric monoidal category</u>: Category C + $1 \in C$ unit + $\otimes : C \times C \to C$

When working with categories its unnatural to ask for $X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$, instead we want this structure to be given by extra data in form isomorphisms

$$\alpha_X : 1 \otimes X \simeq X \qquad \beta_{X,Y} : X \otimes Y \simeq Y \otimes X$$

Just called monoidal if we do not assume this

$$\gamma_{X,Y,Z} : X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z$$

+ coherence data!

It is clear that this approach can not be used to generalize to ∞-categories. Lets spell out how to get an equivalent description of symmetric monoidal: The idea is that instead of giving the bifunctor $\otimes : C \times C \to C$, we instead for each n-tuple $c_1, \_, c_n$ and $d \in C$ we specify the collection of maps

$$c_1 \otimes \_ \otimes c_n \to d \quad + \quad \text{composition data (and coherence)}$$

This is done by colored operads (what I called multicategories in an earlier talk):

<u>Def:</u> A <u>coloured operad</u> $\mathcal{O}$ consists of

- A set of objects $\mathrm{ob}\,\mathcal{O}$ (sometimes called the colours)
- $\forall X_0, \_, X_n, Y$ in $\mathcal{O}$: A set of multimorphisms $\mathrm{Mul}_{\mathcal{O}}(X_0, \_, X_n; Y)$
- Composition: Given

  $$(X_1, \_, X_{i_1}) \to Y_1, \_, (X_{i_{n-1}}, \_, X_{i_n}) \to Y_n$$

  can compose this with $(Y_1, \_, Y_n) \to Z$ to obtain

  $$(X_1, \_, X_{i_n}) \to Z$$

- Unit: $\mathrm{id}_X : (X) \to X$

S.t. the composition law is unital and associative

<u>Rem:</u> Every colored operad $\mathcal{O}$ has an underlying category by setting:

- objects = $\mathrm{ob}\,\mathcal{O}$
- $\mathrm{Hom}(X,Y) = \mathrm{Mul}_{\mathcal{O}}(\{X\}, Y)$.

$\rightsquigarrow$ So can view a coloured operad as a category + extra data in form of the collection of multimorphisms.

<u>Rem:</u> Given a (symmetric) monoidal category $(C, \otimes)$ we can obtain a coloured operad $C^{\otimes}$ w. underlying category C, by assigning

$$\mathrm{Mul}_C(X_0, \_, X_n; Y) = \mathrm{Hom}_C(X_0 \otimes \_ \otimes X_n; Y).$$

We can recover the symmetric monoidal structure of $C^{\otimes}$ (up to canonical isomorphism) by Yoneda's Lemma. E.g. the tensor product $X \otimes Y$ is characterized by the fact that it corepresents the functor $Z \mapsto \mathrm{Mul}_C(X, Y; Z)$.

$\rightsquigarrow$ Can consider symmetric monoidal categories as a special case of coloured operads.

To identify which assumptions it is on a colored operad that makes it into a symmetric monoidal.

First recall:

<u>Def:</u> $\mathrm{Fin}_* = \{\langle n\rangle\}$ pointed sets

$f: \langle n\rangle \to \langle m\rangle$ is <u>inert</u> if it takes some point to the basepoint and injective (isomorphic) on the rest.

$\rho_i: \langle n\rangle \to \langle 1\rangle$ takes $\begin{matrix} j \mapsto 0 & \text{if } j \neq i \\ i \mapsto i & \end{matrix}$, f is <u>active</u> if $f^{-1}(0) = 0$

<u>Construction:</u>
2.1.1.7

Let O be a colored operad $\rightsquigarrow$ $O^{\otimes}$ category:

- Objects = sequences of objects of O $X_0, \_, X_n$
- $\{X_i\}_{1 \leq i \leq m} \to \{Y_j\}_{1 \leq j \leq n}$ is given by a map $\alpha: \langle m\rangle \to \langle n\rangle$ in Fin_* together with a collection of multimorphisms
$$\{\varphi_j \in \mathrm{Mul}_O(\{X_i\}_{i \in \alpha^{-1}\{j\}}; Y_j)\}_{0 \leq j \leq n}$$
in O.
- Composition of morphisms in $O^{\otimes}$ is determined by composition laws on Fin_* and on O.

By construction, $O^{\otimes}$ comes equipped with a forgetful functor

$$\pi: O^{\otimes} \to \mathrm{Fin}_*$$
$$(x_0, \_, x_n) \mapsto \langle n\rangle.$$

Using π we can reconstruct the operad structure:

- Write $O^{\otimes}_{\langle n\rangle} = \pi^{-1}\{\langle n\rangle\}$

- $p^i: \langle n\rangle \to \langle 1\rangle \rightsquigarrow p^i_!: \mathcal{O}^\otimes_{\langle n\rangle} \to \mathcal{O}^\otimes_{\langle 1\rangle} \simeq \mathcal{O}$ ← underlying category of $\mathcal{O}$

  which induces equivalences: $\mathcal{O}^\otimes_{\langle n\rangle} \simeq \mathcal{O}^{\times n}$.

  $(x_0, -, x_n) \mapsto \overline{X}$.

- $\mathrm{Mul}_{\mathcal{O}}(x_0,-,x_n; Y) \leftrightarrow \{ f: \overline{X} \to Y$ in $\mathcal{O}^\otimes$ s.t. $\pi(f): \langle n\rangle \to \langle 1\rangle$ satisfies $\pi(f)^{-1}(0) = \{0\}\}$

  ↳ $\pi: \mathcal{O}^\otimes \to \mathrm{Fin}_*$ determines $\mathrm{Mul}_{\mathcal{O}}(x_0,-,x_n; Y)$ in the colored operad $\mathcal{O}$

- Can show that composition law for morphisms in $\mathcal{O}$ can be recovered from the one in $\mathcal{O}^\otimes$.

$\rightsquigarrow$ Can think of a coloured operad as an ordinary category $\mathcal{O}^\otimes$ together with a forgetful functor $\pi: \mathcal{O}^\otimes \to \mathrm{Fin}_*$ s.t. $\mathcal{O}^\otimes_{\langle n\rangle} \simeq (\mathcal{O}^\otimes_{\langle 1\rangle})^{\times n}$.

It turns out that if π is further an "op-fibration" then this is exactly the symmetric monoidal categories.

So this is what we generalise to ∞-land:

Def: An <u>∞-operad</u> is a functor of ∞-categories $p: \mathcal{O}^\otimes \to N(\mathrm{Fin}_*)$ s.t.

1) coCart. lift of every inert $f: \langle n\rangle \to \langle m\rangle$

   ↳ In particular it induces $f_!: \mathcal{O}^\otimes_{\langle n\rangle} \to \mathcal{O}^\otimes_{\langle m\rangle}$.

2) For each $n \geq 0$, the functors $\{p^i_!: \mathcal{O}^\otimes_{\langle n\rangle} \to \mathcal{O}\}_{1 \leq i \leq n}$ determines an equivalence of categories $\mathcal{O}^{\times n} \simeq \mathcal{O}^\otimes_{\langle n\rangle}$.

\+ ...

So we in particular get that $(x_0, -, x_n) \in \mathcal{O}^{\times n}$ corresponds to an object of $\mathcal{O}^\otimes_{\langle n\rangle}$

Def: A <u>symmetric monoidal ∞-category</u> is an ∞-operad $\mathcal{C}^\otimes \to \mathrm{Fin}_*$ which is also coCartesian fibration.

Let's understand why this gives the desired structure:

Note that: $\langle n\rangle \to \langle m\rangle$ in $\mathrm{Fin}_* \rightsquigarrow \mathcal{C}^\otimes_{\langle n\rangle} \longrightarrow \mathcal{C}^\otimes_{\langle m\rangle}$

so the active morphisms $\langle 0\rangle \to \langle 1\rangle$, $\langle 2\rangle \to \langle 1\rangle$

$$\rightsquigarrow \underbrace{\Delta^0 \to \mathcal{C}}_{\substack{\text{An object} \\ \text{we denote by } 1}}, \quad \underbrace{\mathcal{C} \times \mathcal{C} \to \mathcal{C}}_{\text{our tensor}}$$

The reason we went through all this work introducing ∞-operads is because they are exactly what we need to get nice algebraic structures, and symmetric monoidal categories is just one of many 'monoidal'-ish categories.

In general:

Def: We say that $p: \mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ exhibits $\mathcal{C}$ as $\mathcal{O}$-monoidal ∞-category if

1) p is coCartesian
2) $\mathcal{O}^{\otimes} \xrightarrow{q} \mathrm{Fin}_*$ is an ∞-operad and $\mathcal{C}^{\otimes} \xrightarrow{p} \mathcal{O}^{\otimes} \xrightarrow{q} \mathrm{Fin}_*$ exhibits $\mathcal{C}^{\otimes}$ as an ∞-operad.

Ex: Symmetric monoidal $\leftrightarrow$ Fin_*-monoidal.

Rem: If $\mathcal{C}$ is $\mathcal{O}$-monoidal by $p: \mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$, then any

$$X \in \mathcal{O}^{\otimes}_{\langle n \rangle} \longleftrightarrow \{X_0, \_, X_n\} \in \mathcal{O}$$

Given

$$f \in \mathrm{Mul}_{\mathcal{O}}(X_0, \_, X_n; Y) \rightsquigarrow \prod_{0 \le i \le n} \mathcal{C}_{X_i} \simeq \underset{\substack{\uparrow \\ \text{fiber over } X}}{\mathcal{C}_X} \xrightarrow{\otimes f} \mathcal{C}_Y$$

It can be shown that one of the key elements is that p in particular is a morphism of ∞-operads:

ALGEBRA OBJECTS

Def: A morphism of ∞-operads:

$$\begin{tikzcd} p\text{-coCart} \arrow[rr, mapsto] \arrow[d, mapsto, squiggly] & & q\text{-coCart} \\ \text{inert} & \mathcal{O}^{\otimes} \arrow[r, "f"] \arrow[d, "p"'] & \mathcal{O}'^{\otimes} \arrow[dl, "q"] \\ & \mathrm{Fin}_* & \end{tikzcd} \rightsquigarrow \mathrm{Alg}_{\mathcal{O}}(\mathcal{O}') \subseteq \mathrm{Fun}_{\mathrm{Fin}_*}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes})$$

Def: Assume $p: \mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ be a fibration of ∞-operads. (i.e. morphism + fibration on underlying)
and given $\alpha: \mathcal{O}'^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \subseteq \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})$$

$$\begin{tikzcd} \mathcal{O}'^{\otimes} \arrow[r] \arrow[dr, "\alpha"'] & \mathcal{C}^{\otimes} \arrow[d, "p"] \\ & \mathcal{O}^{\otimes} \end{tikzcd}$$

Equivalently:

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) = \text{fib. of } \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}) \xrightarrow{\circ p} \mathrm{Alg}_{\mathcal{O}'}(\mathcal{O}) \quad \text{at } \alpha$$

$$\mathcal{O}'^{\otimes} \to \mathcal{C}^{\otimes} \longmapsto \mathcal{O}'^{\otimes} \to \mathcal{C}^{\otimes} \xrightarrow{p} \mathcal{O}^{\otimes}$$

Case $\mathcal{O}^{\otimes} = \mathcal{O}'^{\otimes}$, $\alpha = id$: $\mathrm{Alg}_{\mathcal{O}/\mathcal{O}}(\mathcal{C}) =: \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$

$$\begin{tikzcd} \mathcal{O}^{\otimes} \arrow[r] \arrow[d, equal, "id"'] & \mathcal{C}^{\otimes} \arrow[dl, "p"] \\ \mathcal{O} & \end{tikzcd}$$

Case $\mathcal{O}'^{\otimes} = \mathcal{O}^{\otimes} = Fin_*$:

$$\mathrm{CAlg}(\mathcal{C}) := \mathrm{Alg}_{\mathcal{O}/\mathcal{O}}(\mathcal{C}) = \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$$

$$\begin{tikzcd} Fin_* \arrow[r] \arrow[d, equal] & \mathcal{C}^{\otimes} \arrow[dl] \\ Fin_* & \end{tikzcd}$$

It's specific cases which gives us algebras!

Base case is associative algebra

Classically: An associative algebra object in a monoidal category C is an object $A \in \mathcal{C}$ + unit map $e: 1 \to A$ and multiplication $m: A \times A \to A$ s.t.

$$\begin{tikzcd} 1 \otimes A \arrow[r, "e \otimes id"] \arrow[dr, squiggly] & A \otimes A \arrow[d, "m"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} A \otimes 1 \arrow[r] \arrow[dr] & A \otimes A \arrow[d, "m"] \\ & A \end{tikzcd}$$

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "m \otimes id"] \arrow[d, "id \otimes m"] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"'] & A \end{tikzcd}$$

We will apply the formalisme of ∞-operads to introduce the notion of monoidal ∞-category, and to each monoidal ∞-category associate another ∞-category $\mathrm{Alg}(\mathcal{C})$ of associative algebra objects of $\mathcal{C}$.

Def: Colored operad Assoc (associative algebra)

- $\mathrm{ob}\,\mathrm{Assoc} = \{a\}$
- $\mathrm{Mul}_{\mathrm{Assoc}}(\{a\}_{i \in I}, a) =$ set of linear orderings on I.

  any finite set

+ composition..

$\rightsquigarrow$ Obtain a category $\mathrm{Assoc}^{\otimes}$ by applying construction 2.1.1.7

Unwinding the def of $\mathrm{Assoc}^{\otimes}$:

- $\mathrm{ob}\,\mathrm{Assoc}^{\otimes} = \mathrm{ob}\,Fin_*$
- $\langle m \rangle \to \langle n \rangle$ in Fin_* $\rightsquigarrow$ $\langle m \rangle \to \langle n \rangle$ in $\mathrm{Assoc}^{\otimes}$ consists of a pair $(\alpha, \{\leq_i\}_{1 \leq i \leq n})$ where $\alpha: \langle m \rangle \to \langle n \rangle$ is a map in Fin_* and $\leq_i$ is a linear ordering on the inverse image $f^{-1}\{i\} \subseteq \langle m \rangle$ for $1 \leq i \leq n$

$\rightsquigarrow$ $\mathrm{Assoc}^{\otimes} = N(\mathrm{Assoc}^{\otimes}) \in \mathrm{Cat}_{\infty}$.

Notation: We write $\mathrm{Assoc} := \mathrm{Assoc}^{\otimes} \times_{Fin_*} \{\langle 1 \rangle\}$

Note that as a simplicial set, Assoc is isomorphic to the 0-simplex Δ^0; But we use the notation Assoc to emphasize the role of the simplicial set as the underlying ∞-category for the ∞-operad $\mathrm{Assoc}^{\otimes}$.

<u>Def:</u> A <u>monoidal ∞-category</u> is a coCartesian fibration of ∞-operads $\mathcal{C}^{\otimes} \to \mathrm{Assoc}^{\otimes}$.

<u>Def:</u> $\mathcal{C}^{\otimes} \in \mathrm{Op}^{\infty}$ equipped w. a fibration $q: \mathcal{C}^{\otimes} \to \mathrm{Assoc}^{\otimes}$. Then the <u>$\infty$-category of associative algebra objects of $\mathcal{C}$</u> is $\mathrm{Alg}(\mathcal{C}) := \mathrm{Alg}_{/\mathrm{Assoc}}(\mathcal{C})$ (∞-operad sections of q)

$$\begin{tikzcd}
\mathrm{Assoc}^{\otimes} \arrow[r, "A"] \arrow[dr, equal, "id"'] & \mathcal{C}^{\otimes} \arrow[d, "q"] \\
 & \mathrm{Assoc}^{\otimes}
\end{tikzcd}$$

Let's understand what this means: Let $\mathcal{C}^{\otimes} \to \mathrm{Assoc}^{\otimes}$ be a monoidal ∞-category. Then

Rem 4.1.1.12-13

- $\mathcal{C}^{\otimes}_{\langle n \rangle} \simeq \mathcal{C}^{\otimes} \times_{\mathrm{Assoc}^{\otimes}} \{\langle n \rangle\} \simeq \mathcal{C}^n$, and for every linear ordering on $\{1, \dots, n\}$, the corresponding map $\langle n \rangle \to \langle 1 \rangle$ in $\mathrm{Assoc}^{\otimes}$ induces a functor $\mathcal{C}^n \to \mathcal{C}$. In particular
  - $n = 0 \rightsquigarrow$ unit object $1 \in \mathcal{C}$
  - $n = 2$, standard ordering on $\{1,2\} \rightsquigarrow \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
- Evaluating on $a \in \mathrm{Assoc}^{\otimes}$ determines a forgetful functor

$$\Theta : \mathrm{Alg}(\mathcal{C}) \to \mathcal{C}$$

By abuse of notation we often identify $A \in \mathrm{Alg}(\mathcal{C})$ with it's image $\Theta(A)$ in $\mathcal{C}$. For each $n \geq 0$ a choice of ordering on $\{1, \_, n\}$ determines an active morphism $\{a\}_{1 \leq i \leq n} \to a$ in $\mathrm{Assoc}^{\otimes}$, which induces a morphism $\Theta(A)^{\otimes n} \longrightarrow \Theta(A)$ in $\mathcal{C}$. In particular

- $n = 2$ and standard ordering of $\{1,2\}$: $m : \Theta(A) \otimes \Theta(A) \to \Theta(A)$
  - ↳ Associative and unital up to homotopy
    - $\rightsquigarrow$ In particular, it endows $\Theta(A)$ with the structure of an associative algebra object in $h\mathcal{C}$.

<u>LEFT & RIGHT MODULES</u>

Classically: C monoidal category w. unit A, A associative algebra object of C. A <u>left A-module</u> in C is an object $M \in \mathcal{C}$ + action map $a: A \otimes M \to M$ s.t.

$$\begin{tikzcd}
A\otimes A\otimes M \arrow[r, "m\otimes id"] \arrow[d, "id\otimes a"] & A\otimes M \arrow[d, "a"] \\
A\otimes M \arrow[r, "a"] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
1\otimes M \arrow[rr, "u\otimes id"] \arrow[dr] & & A\otimes M \arrow[dl, "a"] \\
& M &
\end{tikzcd}$$

where m = multiplication, u = unit $\Big\}$ in A

$\rightsquigarrow$ All left A-modules $= \mathrm{LMod}_A(\mathcal{C})$

We wish to introduce a larger ∞-operad $\mathcal{LM}^{\otimes}$ which contains $\mathrm{Assoc}^{\otimes}$. If $A \in \mathrm{Alg}(\mathcal{C})$ we want a left A-module to be a map of ∞-operads $M: \mathcal{LM}^{\otimes} \to \mathcal{C}^{\otimes}$ s.t. $M|_{\mathrm{Assoc}^{\otimes}} = A$.

Def: Define a colored operad $\underline{LM}$ as:

- $\mathrm{ob}\,LM = \{a, m\}$
- Let $\{X_i\}_{i\in I}$ be a finite collection of objects of LM. Then
  - $\hookrightarrow$ $\mathrm{Mul}_{LM}(\{X_i\}, a) = \begin{cases} \text{all linear orderings of } I & \text{if all } X_i = a \\ \text{Empty} & \text{otw} \end{cases}$
  - $\hookrightarrow$ $\mathrm{Mul}_{LM}(\{X_i\}, m) = \{\text{All linear orderings } \{i_1 < \dots < i_n\} \text{ on } I \text{ s.t. } X_{i_n} = m \;\&\; X_{i_j} = a \text{ for } j < n\}$

$a, a, \dots, a, m$

Rem: 4.2.1.2 $a \in LM \rightsquigarrow$ sub-colored operad of LM isomorphic to Assoc

Rem: 4.2.1.3 We first in this (1-categorical) case see how this can be used to understand modules.

Assume $\mathcal{C}$ symmetric monoidal, $F: LM \to \mathcal{C}$ map of colored operads

$\rightsquigarrow$ $F|_{\mathrm{Assoc}}: \mathrm{Assoc} \to \mathcal{C}$ $\rightsquigarrow$ Associative algebra object $F(a) = A \in \mathcal{C}$. Let $M = F(m) \in \mathcal{C}$.

Then the unique operation $\phi \in \mathrm{Mul}_{LM}(\{a, m\}, m)$ determines $F(\phi): A \otimes M \to M$, which exhibits M as a left A-module.

$\rightsquigarrow$ So $LM \to \mathcal{C}$
$a \mapsto$ associative algebra
$m \mapsto$ left module over that algebra

Notation: Apply construction 2.1.1.7 to LM to obtain the category $LM^{\otimes}$ from LM. Unwinding this construction we see that

1) $\mathrm{ob}\,LM^{\otimes} = \{(\langle n\rangle, S) \mid S \subset \langle n\rangle^{\circ}\}$

2) $(\langle n\rangle, S) \to (\langle n'\rangle, S') = \alpha: \langle n\rangle \to \langle n'\rangle$ in $\mathrm{Assoc}^{\otimes}$ s.t.

   i) $S \cup \{*\} \overset{\alpha}{\mapsto} S' \cup \{*\}$

   ii) $s' \in S' \Rightarrow \alpha^{-1}\{s'\}$ contains exactly one element of S, and that object is maximal w.r.t. the linear ordering of $\alpha^{-1}\{s'\}$.

Rem:

$$\begin{array}{ccc} \mathrm{LM} & & \mathrm{LM}^\otimes \\ a & \longleftrightarrow & (\langle 1\rangle, \emptyset) \\ m & \longleftrightarrow & (\langle 1\rangle, \langle 1\rangle^\circ) \end{array}$$

Def: $\mathcal{LM}^\otimes = N(\mathrm{LM}^\otimes)$. This is an ∞-operad via the forgetful map $\mathcal{LM}^\otimes \to \mathrm{Fin}_*$

Rem: The underlying ∞-category $\mathcal{LM}$ of $\mathcal{LM}^\otimes$ is isomorphic to the discrete simplicial set $\Delta^0 \amalg \Delta^0$ w. two vertices, corresponding to $a, m \in \mathrm{LM}$.

Rem: $\mathrm{Assoc} \hookrightarrow \mathrm{LM} \rightsquigarrow \mathrm{Assoc}^\otimes \hookrightarrow \mathcal{LM}^\otimes$, which is an isomorphism from $\mathrm{Assoc}^\otimes$ onto the full subcategory of $\mathcal{LM}^\otimes$ spanned by objects of the form $(\langle n\rangle, \emptyset)$.

Notation: Let $\mathcal{C}^\otimes \to \mathcal{LM}^\otimes$ be a fibration of ∞-operads. Write

$$\mathcal{C}^\otimes_a := \mathcal{C}^\otimes \times_{\mathcal{LM}^\otimes} \mathrm{Assoc}^\otimes, \quad \text{Underlying ∞-category } \mathcal{C}_a = \mathcal{C}^\otimes \times_{\mathcal{LM}^\otimes} \{a\}$$

$$\mathcal{C}^\otimes_m = \mathcal{C}^\otimes \times_{\mathcal{LM}^\otimes} \{m\}$$

Def: Let $\mathcal{C}^\otimes \to \mathrm{Assoc}^\otimes$ be a fibration of ∞-operads, $q: \mathcal{O}^\otimes \to \mathcal{LM}$ fibration of ∞-operads s.t. $\mathcal{O}^\otimes_a \simeq \mathcal{C}^\otimes$. Write $\mathcal{O}^\otimes_m = \mathcal{M}$ ($\in \mathrm{Cat}_\infty$) (Normally we say q exhibits $\mathcal{M}$ as weakly enriched over $\mathcal{C}^\otimes$)

- $\mathrm{LMod}(\mathcal{M}) := \mathrm{Alg}_{/\mathcal{LM}}(\mathcal{O})$ ∞-category of left module objects of $\mathcal{M}$ $$\begin{tikzcd} \mathcal{LM} \arrow[r] \arrow[rd] & \mathcal{O} \arrow[d] \\ & \mathcal{LM} \end{tikzcd}$$
- Composition w. $\mathrm{Assoc}^\otimes \hookrightarrow \mathcal{LM}^\otimes$ determines a categorical fibration

$$\mathrm{LMod}(\mathcal{M}) = \mathrm{Alg}_{/\mathcal{LM}}(\mathcal{O}) \xrightarrow{\circ \mathrm{incl}} \mathrm{Alg}_{/\mathrm{Assoc}}(\mathcal{C}) = \mathrm{Alg}(\mathcal{C})$$

$$\rightsquigarrow \mathrm{LMod}_A(\mathcal{M}) = \mathrm{LMod}(\mathcal{M}) \times_{\mathrm{Alg}(\mathcal{C})} \{A\} \quad \text{∞-category of left A-modules objects of } \mathcal{M}.$$

$\rightsquigarrow$ We think of $\mathrm{LMod}(\mathcal{O}_m)$ as given by pairs (A, M) where A is an associative algebra object of $\mathcal{O}_a$ and M is a left A-module in $\mathcal{O}_m$.

Ex: 4.2.1.16 $\mathcal{C}^\otimes \to \mathrm{Assoc}^\otimes$ fibration of ∞-operads, $\mathcal{O}^\otimes = \mathcal{C}^\otimes \times_{\mathrm{Assoc}^\otimes} \mathcal{LM}^\otimes$. Then "$\mathcal{O}^\otimes$ exhibits $\mathcal{C}$ as ∞-cat weakly enriched over $\mathcal{C}^\otimes$ $\rightsquigarrow$ Can consider $\mathrm{LMod}(\mathcal{C}) = \mathrm{Alg}_{\mathcal{LM}/\mathrm{Assoc}}(\mathcal{C})$

$$\begin{tikzcd} \mathcal{LM}^\otimes \arrow[r] \arrow[rd] & \mathcal{C}^\otimes \arrow[d] \\ & \mathrm{Assoc}^\otimes \end{tikzcd}$$

Ex: 4.2.1.18 Let $\mathcal{C}^\otimes \to \mathrm{Assoc}^\otimes$ be a monoidal ∞-category. Then

$$\mathrm{LMod}(\mathcal{C}) = \mathrm{Alg}_{\mathcal{LM}/\mathrm{Assoc}}(\mathcal{C}) = \begin{tikzcd} \mathcal{LM}^\otimes \arrow[r, "F"] \arrow[rd, dotted] & \mathcal{C}^\otimes \arrow[d, dashed] \\ & \mathrm{Assoc}^\otimes \end{tikzcd}$$

$\rightsquigarrow$ $F|_{\mathrm{Assoc}} \in \mathrm{Alg}(\mathcal{C})$, which we identify with its underlying object $F(a) = A \in \mathcal{C}$.

$\rightsquigarrow$ Also have $F(m) = M \in \mathcal{C}$

The unique operation $\varphi \in \mathrm{Mul}_{\mathrm{LM}}(\{a, m\}, m)$ determines a map

$$a: A \otimes M \to M \quad \text{in } \mathcal{C}$$

which is well-defined up to homotopy.

Since F is defined on all of $\mathcal{LM}^\otimes$, we get that this action map is compatible with the associative multiplication on A, up to coherent homotopy. In particular, if

$m: A\otimes A \to A$ multiplication

on A

$u: 1 \longrightarrow A$ unit map

$$\rightsquigarrow \quad \begin{tikzcd} A\otimes A\otimes M \arrow[r, "m\otimes id"] \arrow[d, "id\otimes a"] & A\otimes M \arrow[d, "a"] \\ A\otimes M \arrow[r, "a"] & M \end{tikzcd} \qquad \begin{tikzcd} 1\otimes M \arrow[r, "u\otimes id"] \arrow[dr] & A\otimes M \arrow[d, "a"] \\ & M \end{tikzcd}$$

commutes up to homotopy.

Rem: If $\mathcal{C}^\otimes \xrightarrow{q} \mathcal{LM}^\otimes$ is a cocartesian fibration of ∞-ops, then the induced map $\mathcal{C}_a^\otimes \to Assoc^\otimes$ is also a cocartesian fibration of ∞-operads.

We further see that by straightening q is classified by a map $\chi: \mathcal{LM}^\otimes \longrightarrow Cat_\infty$, which is an "$\mathcal{LM}$-monoid object of Cat_∞". We have

$$\chi \in Mon_{\mathcal{LM}}(Cat_\infty) \simeq Alg_{\mathcal{LM}}(Cat_\infty) = LMod(Cat_\infty)$$

More informally: q can be thought of as giving an associative algebra $\mathcal{C}_a$ in Cat_∞ together with a left module $\mathcal{C}_m$ over $\mathcal{C}_a$. In particular q determines an action map

$$\otimes: \mathcal{C}_a \times \mathcal{C}_m \to \mathcal{C}_m$$

which is well-defined up to homotopy - and compatible with the monoidal structure on $\mathcal{C}_a$