§8: The Red-Blue shift Conjecture

Talk 8
15.03.23

First: Finish what we discussed last time - how does these v_n-self maps describe periodic families?

Recall:
- Type n = finite p-local spectrum X s.t. $K(n)_*X \neq 0$, $K(n-1)_*X = 0$ (so $K(m)_*X = 0$, $m<n$)
- v_n-self map on finite p-local spextrum X: $f: \Sigma^k X \to X$ s.t.

$$K(m)_* f = \begin{cases} 0 & n \neq m \\ v_n^{N} & n = m \end{cases}$$

For some suitable power N

Motivation for this definition was that if X is type n and f a v_n-self map, then the cokernel of f is of type $n+1$.

$$\Sigma^k \xrightarrow{f} X \longrightarrow X/f$$

In general: We inductively define a type $n+1$ spectrum as follows

- cokernel of a v_0-self-map f_0 satisfying $K(1)_*(f_0) = v_0^{i_0}$
  $\rightsquigarrow$ $M(i_0)$ type 1
- cokernel of a v_1-self-map $f_1: \Sigma^{2(p-1)i_1} M(i_0) \to M(i_0)$ s.t. $K(1)_*(f_1) = v_1^{i_1}$
  $\rightsquigarrow$ $M(i_0, i_1)$ type 2
- ⋮

$M(i_0, i_1, \ldots, i_n)$ is the type $n+1$ spectrum defined as the cofiber of a v_n-self map

$$f_n: \Sigma^{2(p^n-1)i_n} M(i_0, \ldots, i_{n-1}) \longrightarrow M(i_0, \ldots, i_{n-1})$$

satisfying

$$K(n)_*(f_n) = v_n^{i_n}.$$

There is a lot of choices when constructing these $M(i_0, \ldots, i_n)$!

"Periodic families":

Construction: Write $M_n := M(i_0, \ldots, i_{n-1})$ type n $K(n)_* f_n = v_n^{i_n}$

Let $w \in \pi_r X$ $\quad S^r \xrightarrow{w} X \rightsquigarrow S^0 \xrightarrow{w} \Sigma^{-r} X$

- ω is v_{n-1}-torsion if there exists a diagram

v_1-self $\rightsquigarrow f_1$

$$\begin{tikzcd}
S^0 \arrow[r, "f_0"] & S^0 \arrow[r, "\omega"] \arrow[d] & \Sigma^{-r}X \\
\Sigma^{|f_1|}M_1 \arrow[r, "f_1"] & M_1 \arrow[d, dotted] \arrow[ur] & \\
\Sigma^{|f_{n-1}|}M_{n-1} \arrow[r, "f_{n-1}"] & M_{n-1} \arrow[d] \arrow[uur] & \\
\Sigma^{|f_n|}M_n \arrow[r, "f_n"] & M_n \arrow[uuur, "\omega_n"'] &
\end{tikzcd}$$

If ω is p-torsion

$\rightsquigarrow f_0 = \underbrace{p}_{v_0}^{i_0}$

so if the comp $\omega \circ f_0 \simeq 0$, we get that it extends, since

$S^0 \xrightarrow{f_0} S^0 \longrightarrow M_1$ cofibre seq

If we can find a power of f_1 s.t. $\underbrace{\Sigma^{|f_1|}M_1 \to M_1 \to \Sigma^{-r}X}_{\simeq 0}$, then we can again extend

So we are assuming we can continue this process until a type n spectrum M_n.

- ω is v_n-periodic if for any v_n-self map f_n of M_n, $\omega_n \circ f_n \neq 0$ so we can't continue the constr.

Periodic families

Let $\omega \in \pi_r X$ be v_n-periodic, and $M = M_n$ as above w. v_n-self map s.t. (cw-spectrum)

$$\Sigma^d M \xrightarrow{f_n} M \xrightarrow{\omega_n} \Sigma^{-r}X \qquad \text{non-zero}$$

Let M^r = r-skeleton of M and cofiber sequences

$$M^{r-1} \longrightarrow M^r \longrightarrow M^r_r \;, \quad M^{r-1} \longrightarrow M \longrightarrow M_r = M_r^{\dim M}$$

take r-skeleton and quotient out w-(r-1)-skeleton

Then there exists an r s.t. we can form the following diagram

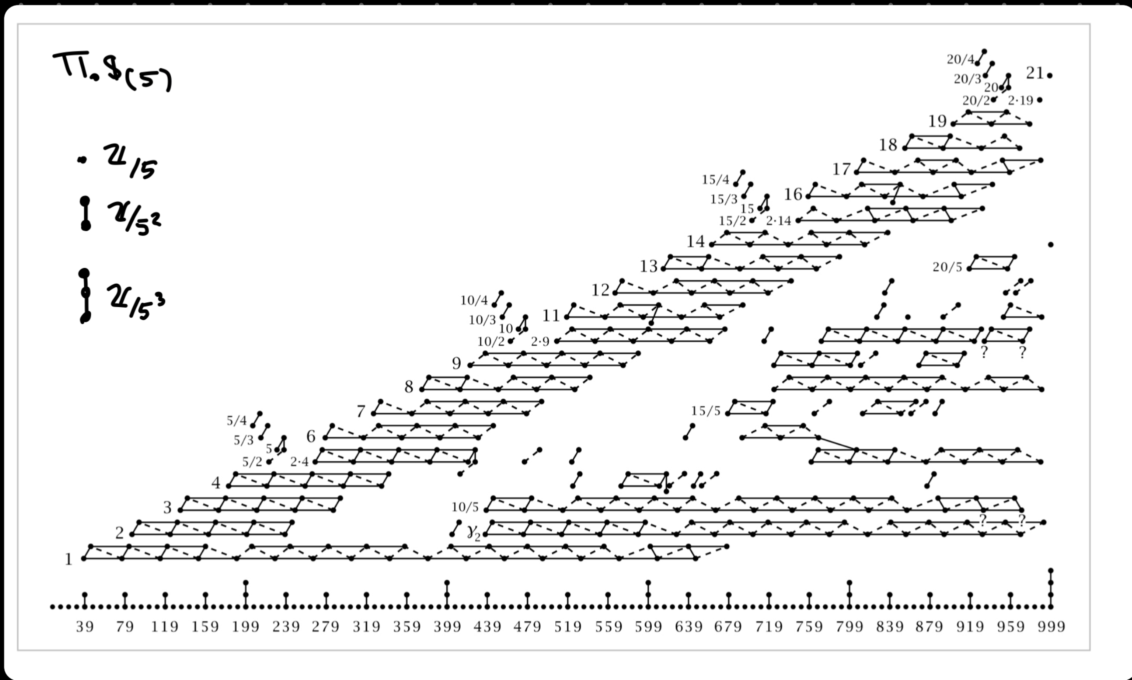
$$\begin{tikzcd}
 & \Sigma^d M \arrow[r, "f_n"] \arrow[d] & M \arrow[r, "\omega_n"] & \Sigma^{-r}X \\
S^k \cong \Sigma^d M^r_r \arrow[r, "\mathrm{incl}", "i"'] & \Sigma^d M_r \arrow[urr] & &
\end{tikzcd}$$

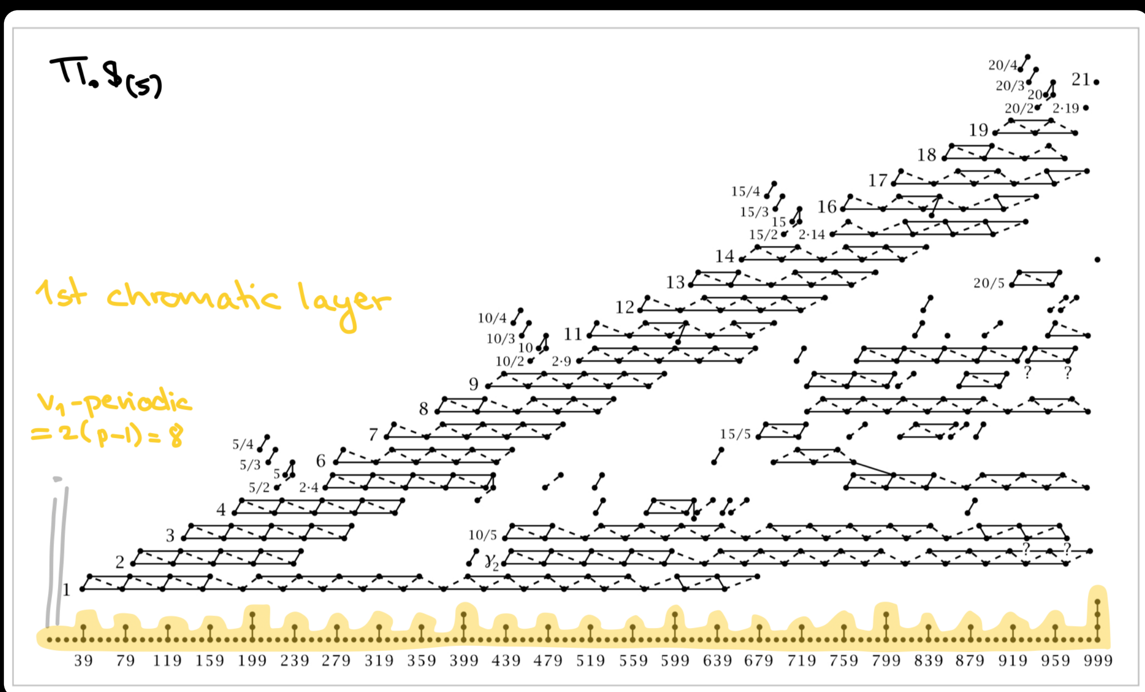
$\exists g$ s.t. $g \circ i$ non-trivial

$\rightsquigarrow$ i.e. $\omega_n \circ f_n$ is non-trivial on "some cell of M" - A cell that detects i

Such elements $g \circ i \in \pi_{k+r} X$ are part of the v_n-periodic family of ω.

Thinking of f_n as "multiplication by v_n"


$\pi_* S_{(5)}$
$\mathbb{Z}/5$
$\mathbb{Z}/5^2$
$\mathbb{Z}/5^3$
39 79 119 159 199 239 279 319 359 399 439 479 519 559 599 639 679 719 759 799 839 879 919 959 999


$\pi_* S_{(5)}$
1st chromatic layer
v_1-periodic $= 2(p-1) = 8$
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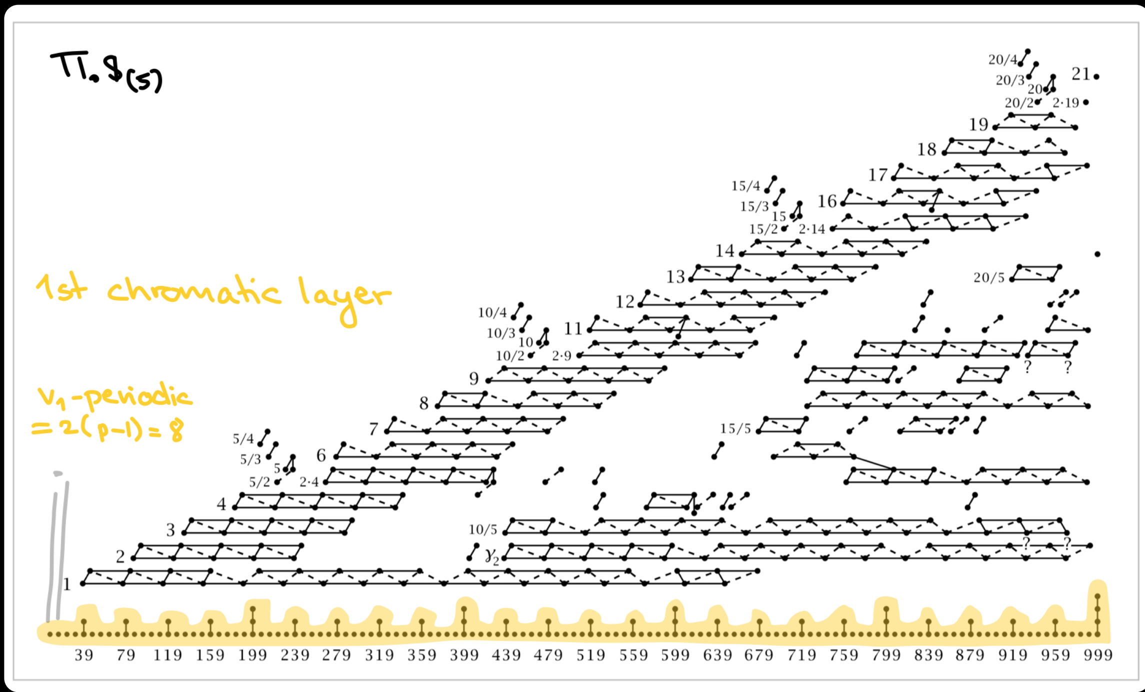
$\pi_* \mathbb{S}_{(5)}$
1st chromatic layer
v_1-periodic $= 2(p-1) = 8$
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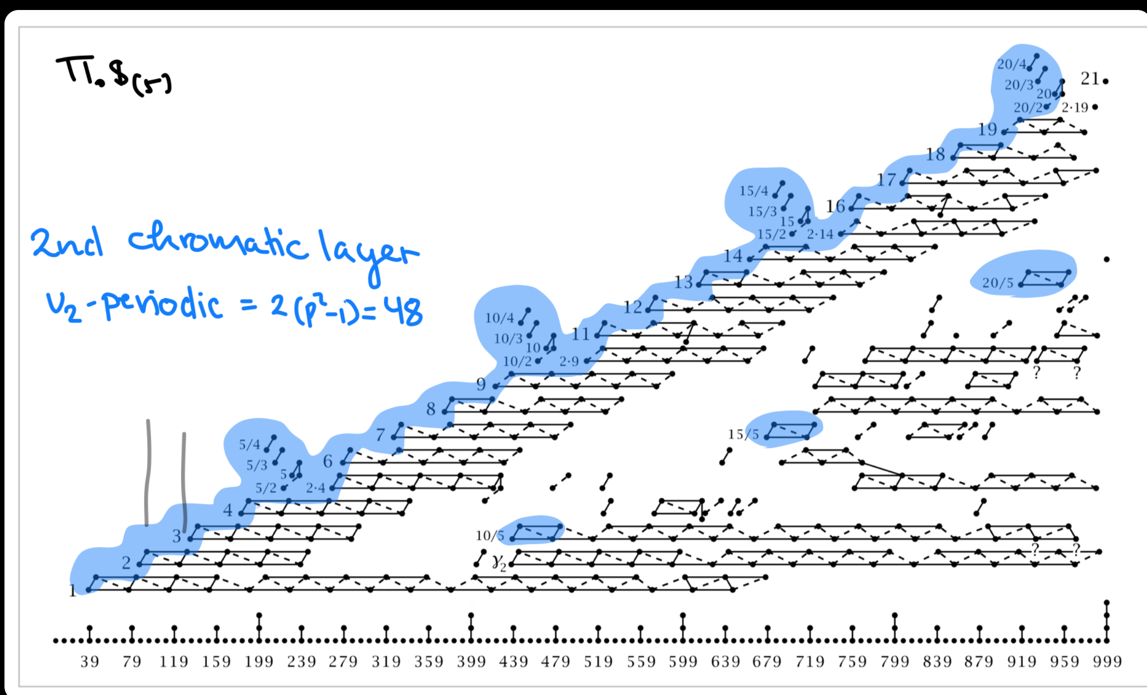
$\pi_* \mathbb{S}_{(5)}$
2nd chromatic layer
v_2-periodic $= 2(p^2-1) = 48$
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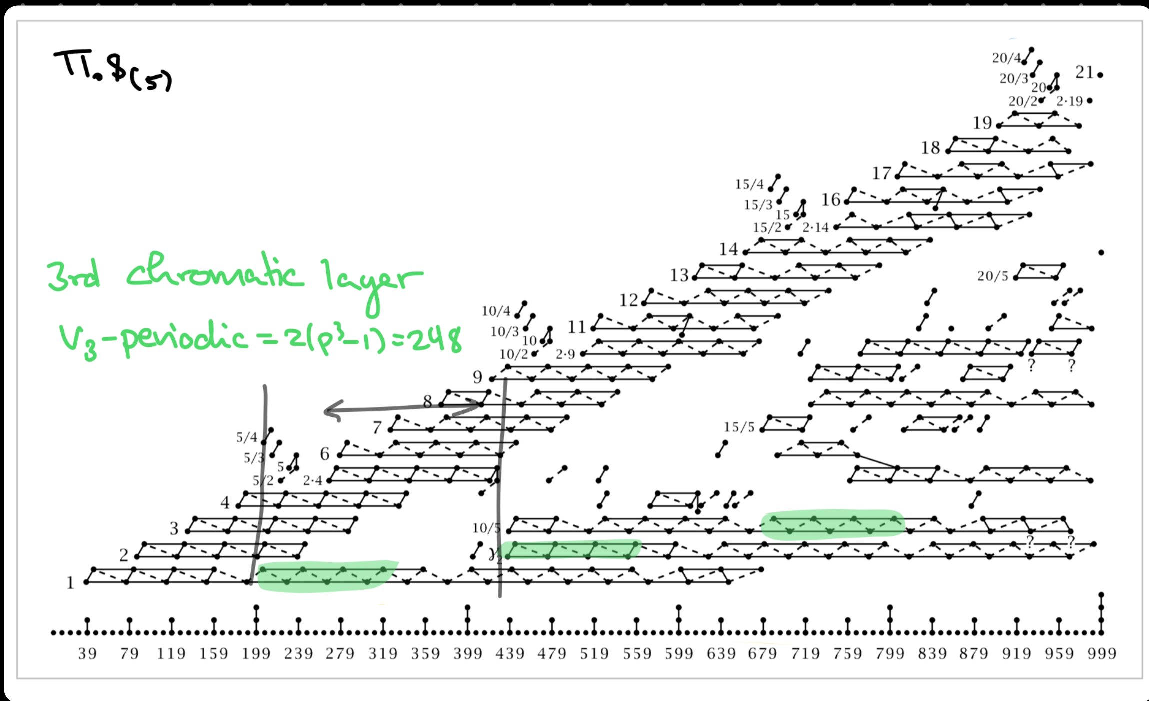
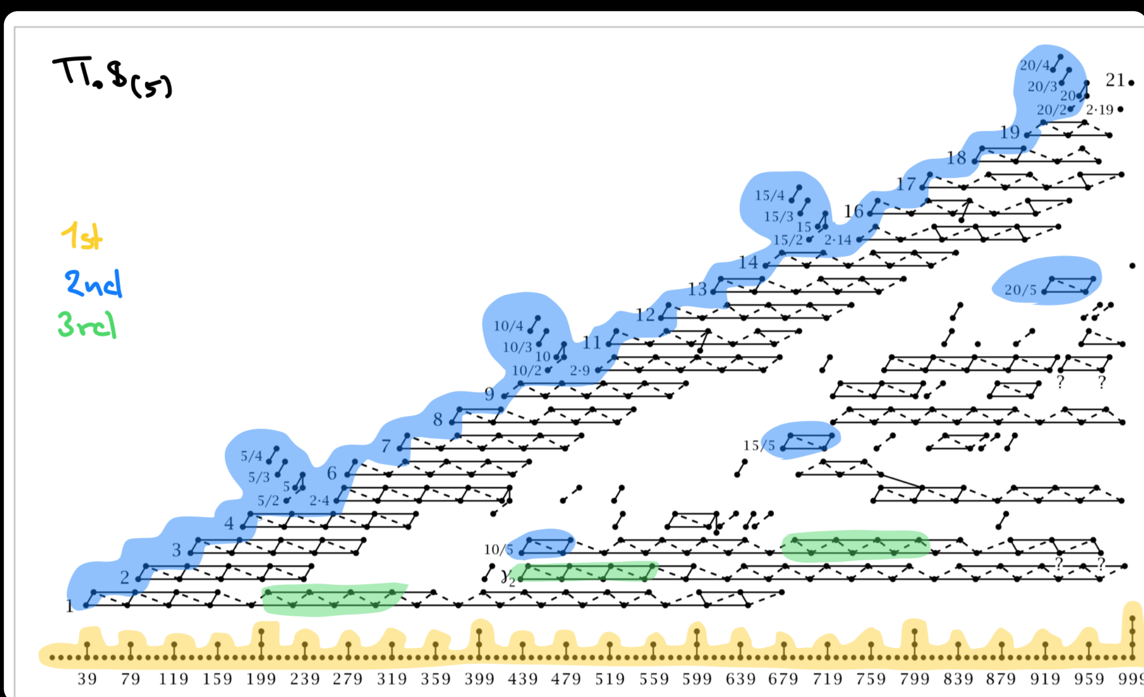
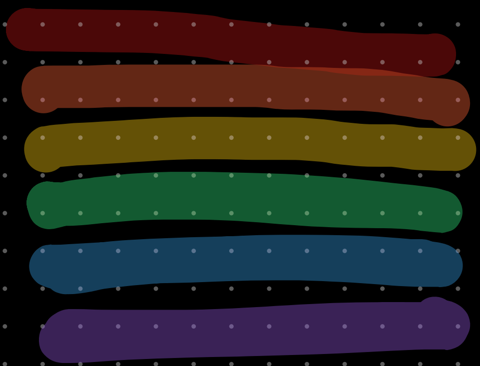

$\pi_* S_{(5)}$
3rd chromatic layer
v_3-periodic $= 2(p^3-1) = 248$
39 79 119 159 199 239 279 319 359 399 439 479 519 559 599 639 679 719 759 799 839 879 919 959 999


$\pi_* S_{(5)}$
1st
2nd
3rd
39 79 119 159 199 239 279 319 359 399 439 479 519 559 599 639 679 719 759 799 839 879 919 959 999

Red-Blue Shift Conjecture



"Increasing wavelengths"

Red shift — Increases chromatic information

Blue shift — Decreases chromatic information

"Decreases wavelength"

Conjecture: Algebraic K-theory exhibits Red shif

Conjecture: Tate construction exhibits blue shift

RED SHIFT

Idea: R structured ring spectrum related to a FGL, height n

$\Downarrow$

K(R) related to a FGL height n+1

In terms of periodic families: Homotopy of R is v_n-periodic, but not v_{n+1}-periodic, then often K(R) is v_{n+1}-periodic, but not v_{n+2}.

Thm: |Nullstellensatz '22| The red shift conjecture holds for all (non-zero) commutative ring spectra

To discuss this in more detail, recall that the periodicity theorem tells us that

$$X \text{ type } n \iff X \text{ admits a } v_n\text{-self map}$$

Furthermore, we introduced

$f: \Sigma^k X \to X$ v_n-self map on X type n

Telescope of f $\quad X[f^{-1}] := \operatorname{colim}\left(X \xrightarrow{\Sigma f} \Sigma^{-k} X \xrightarrow{\Sigma^{-2k} f} \Sigma^{-2k} X \to \cdots\right)$

Def: For $M_n = M(i_0, \dots, i_{n-1})$ w. v_n-self map f_n, write $\mathrm{Tel}(n) := M_n[f_n^{-1}]$.

Telescopic localization

sometimes people write 'f' for finite - It's a finite localisation w. $\ker(L_n^f)$ generated by any (finite) (n+1)-type spectra

$$L_n^f X := L_{\mathrm{Tel}(0) \vee \dots \vee \mathrm{Tel}(n)} X$$

p-local spectrum

Prop: If X is of type $\geq n$ and f is a v_n-self-map of X, then

$$L_n^f X \simeq X[f^{-1}].$$

We further know that if X is $E(m)$-local, for some m then

$$L_n^t X \simeq L_n X \simeq L_{E(n)} X$$

To ease notation today: $T(n) := \mathrm{Tel}(n)$

We wish to study $T(n)$-local $\mathbb{E}_\infty$-algebras

Can be shown that $\quad L_{T(n)} R \simeq 0 \iff L_{K(n)} R \simeq 0$

In the case of ring spectra we make the following definition

Def. The height of $0 \neq R \in \mathrm{CAlg}(Sp)$ is

$$\mathrm{height}(R) := \max\{n \geq -1 \mid T(n) \otimes R \neq 0\}$$

where we set $\quad T(-1) := \mathbb{S}$.

The motivation for this comes from the following theorem:

Thm: | Hahn '16 | $R \in \mathrm{CAlg}(Sp)$, $n \geq 0$:

$$R \otimes T(n) = 0 \implies R \otimes T(n+1) = 0.$$

$T(n)$-acyclic $\implies$ $T(n+1)$-acyclic

R height $n \implies L_{T(k)} R$ vanishes for all $k > n$

So the idea is, that if the higher chromatic information for some such R is zero, then the chromatic information is truncated at that level, i.e. there is no "even higher" information either.

So could equivalently:

$$\mathrm{height}(R) = \max\{k \mid L_{T(k)} R \neq 0\}$$

Thm: | Redshift for $\mathbb{E}_\infty$ | Let $0 \neq R \in \mathrm{CAlg}(Sp)$ s.t. $\mathrm{height}(R) = n$. Then $\mathrm{height}(K(R)) = n+1$.

so $K(R)$ has a bit more fancy structure

Building blocks to prove that the jump is exactly one:

1. No crazy jumps

   It was proved in 2020 by Clausen–Mathew–Naumann–Noel that no crazy crazy jumps in height can occour:

   Thm: $R \in \mathrm{CAlg}(Sp)$: $\mathrm{height}(R) = n \implies \mathrm{height}(K(R)) \leq n+1$

   We can restate this as follows: $\boxed{L_{T(n)} R \simeq 0 \implies L_{T(n+1)} K(R) = 0}$

② Maps of $\mathbb{E}_\infty$-ring spectra & height

Lem: height(R) = n iff $\exists A \in \mathrm{CAlg}(\mathrm{Sp})$, height(A) = n together with a map of $\mathbb{E}_\infty$-rings $A \to R$.

So to figure out the height of R, we can try to compare it with something we know have height n.

Intuition about why this holds:

"$\Rightarrow$" use the identity

"$\Leftarrow$" The zero-ring have no non-zero modules, so if we have such a map $A \to R$ from A non-zero, then R is non-zero.

$\rightsquigarrow$ Same holds $T(n)$-locally

Gives us information about the height of R, since it is defined by non-vanishing and vanishing of $T(n)$-locally, by using our known $\mathbb{E}_\infty$-map.

③ Height of K-theory of Lubin-Tate theory

Recall that

Landweber Exact functor theorem $\rightsquigarrow$ Morava E-theory

$$(E_n)_* \simeq \mathbb{Z}_{(p)}[v_1, \dots, v_n, v_n^{\pm 1}]$$

$$W(k)[\![v_1, \dots, v_{n-1}]\!][\beta^{\pm 1}]$$

height n Lubin-Tate theory.

We have $E_n \in \mathrm{CAlg}(\mathrm{Sp})$, and we want height$(K(E_n)) = n+1$. This is done by constructing an $\mathbb{E}_\infty$-map

$$K(E_n) \to A$$

where height(A) = n+1.

- height$(E_{n+1}) = n+1$
- Blue shift: height$(E_{n+1}^{tC_p}) = n$ — trivial action
- $L_{T(n+1)} K(E_{n+1}^{tC_p}) \neq 0 \Rightarrow K(E_{n+1}^{tC_p}) = n+1$

Red blue shift "inverses"

Dream: Construct a map $E_n \to E_{n+1}^{tC_p}$ which would induce a map on K-theory, but sadly this does not work...

Instead: Construct a map "up to a sequence of Étale extensions"

$$f: E_n \to \underbrace{L_{K(n)}(E_{n+1}^{tC_p})^{Sh}_p}$$

K-theory of this has height n+1.

4 Nullstellensatz

Def: Let $\mathcal{C}$ be a presentable ∞-category. A non-terminal object $C \in \mathcal{C}$ is nullstellensatzian if every compact object in $\mathcal{C}_{C/}$ has a map to the initial object C.

∞-categorical analogue of "finitely generated"

Hilbert's Nullstellensatz: Let L be an algebraically closed field, $\mathcal{J}$ some ideal of the polynomial ring $L[x_1, \dots, x_n]$. Then for all common roots of polynomials in L, there exists an L-algebra map

$$L[x_1, \dots, x_n]/\mathcal{J} \longrightarrow L$$

Note that $L[x_1, \dots, x_n]/\mathcal{J}$ is a finitely generated commutative algebra over L

$\rightsquigarrow$ L is "nullstellensatzian"

Rem: Nullstellensatzian objects in $\mathrm{CAlg}(\mathrm{Ab})$ are exactly the algebraically closed fields L, since all of the 'compact objects' finitely generated in $\mathrm{CAlg}_{L/}$, have a map $A \to L$ by Hilbert's nullstellensatz

nullstellensatzian objects ~ those that behave like algebraically closed fields in the category of rings

Chromatic Nullstellensatz: Let $0 \neq R \in \mathrm{CAlg}(Sp_{T(n)})$. Then R is nullstellensatzian iff there exists some algebraically closed field L, such that

$$R \simeq E_n(L).$$

Object in $\mathrm{CAlg}(Sp_{T(n)})$ ~ $\mathbb{E}_\infty$-ring R + $\underbrace{L_{T(n)}R \neq 0}_{T(n)\text{-local}}$

Hence: Nullstellensatz $T(n)$-local $\mathbb{E}_\infty$-rings are exactly the Lubin-Tate theories over algebraically closed fields

"$E_n(L)$ = 'algebraically closed field objects' in $Sp_{T(n)}$"

An important result they use to prove this, is the existence of maps

$$A \longrightarrow E_n(L)$$

for any $T(n)$-local $\mathbb{E}_\infty$-ring A.

More generally: $R \in \mathrm{CAlg}(Sp_{T(n)})$ $\Rightarrow$ $\exists$ perfect algebra A of Krull dimension 0 and a nilpotence detecting map $R \to E_n(A)$

What have we gained? For any $\mathbb{E}_\infty$-ring spectrum R we have

or

- R is $T(n)$-acyclic
- There is a map of $\mathbb{E}_\infty$-ring spectra $R \to E_n(L)$

The map

$$R \longrightarrow E_n(L) \quad \rightsquigarrow \quad K(R) \longrightarrow K(E_n(L))$$

We know redshift holds for Lubin-Tate spectra, so

$$\mathrm{height}(K(E_n(L))) = \mathrm{height}(E_n(L)) + 1 = n+1$$

So we conclude

$$\boxed{\mathrm{height}(K(R)) = n+1}$$