§7: The known big theorems & the telescope conjecture

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Where are we?

- Elements v_n: $v_n \in MU_{(p^n-1)}$ is the coefficient of x^{p^n} in
  $$[p]_{F_{MU}}(x) = px + \ldots + v_1 x^p + \ldots + v_n x^{p^n} + \ldots$$
- Height: A FGL classified by $\varphi: MU_* \to R_*$ has height $\leq n$ if $\varphi(v_n)$ is a unit and $\varphi(v_i)=0$ for $0 \leq i < n$.
- Landweber Exact Functor theorem $\rightsquigarrow$ Morava E-theory
  $$E(n)_* \simeq \mathbb{Z}_{(p)}[v_1, \ldots, v_n, v_n^{\pm 1}] \qquad \text{height } n$$
  $$W(k)[[v_1, \ldots, v_{n-1}]][\beta^{\pm 1}]$$
- Morava K-theory $K(n)$ for $0 \leq n < \infty$
  - $K(n)_* \cong \mathbb{F}_p[v_n^{\pm 1}]$ height n
  - $K(n)_* K(m) = 0$ if $m \neq n$
  - X finite & $K(n)_* X = 0 \Rightarrow K(n-1)_* X = 0$.

Want to understand localizations with these things.

Need wedge of spectra

$$E \vee F = \mathbb{L}\left(E_n \vee F_n,\ E_n \vee F_n \xrightarrow{\omega_n^E \vee \omega_n^F} \Omega E_{n+1} \vee \Omega F_{n+1} \longrightarrow \Omega(E_{n+1} \vee F_{n+1}) \right)$$

(spectrification)

↪ Both product and coproduct in SH

Chromatic fracture square

Write $L_n X := L_{K(0) \vee \ldots \vee K(n)} X$

Intuition:

- L_n = inverting v_n
- $L_{K(n)}$ = inverting v_n and completing at $(p, v_1, \ldots, v_{n-1})$

Thm: $L_{E(n)} \cong L_n \cong L_{v_n^{-1} MU_{(p)}}$

There are clearly natural transformations $L_n \to L_{n-1}$ so we get

$$\cdots \to L_{E(n)} \to L_{E(n-1)} \to \cdots$$

Def: chromatic tower of $X \in Sp$

$$\cdots \to L_{E(n)} X \to L_{E(n-1)} X \to \cdots$$

Monochromatic layers = Fibers of the maps in this tower

The natural transformation $\eta_X : X \to L_{E(n)}X$ gives a map

$$X \to \operatorname{holim}_n (L_{E(n)} X)$$

If this is an equivalence, we say chromatically complete

Thm: |Chromatic convergence - Barthel| X connective spectrum w. finite projective dimension is chromatically complete

In particular

- S^0 p-locally is chromatically complete
- p-local finite spectra are chromatically complete

Thm: |Smash product theorem| $L_n X \simeq L_{E(n)} X \simeq L_{E(n)}(S^0) \wedge X \simeq (L_n S^0) \wedge X$ smashing

Thm: |Localization theorem| $BP \wedge L_{E(n)} X \simeq X \wedge L_{E(n)} BP$ can compute $BP_*(L_n X)$ in terms of $BP_* X$

$\Rightarrow$ If $v_{n-1}^{-1} BP(X) \simeq 0$ then $BP \wedge L_n X \simeq X \wedge v_n^{-1} BP$ $\Rightarrow BP_* L_n X \simeq v_n^{-1} BP_* X$

Want to understand these maps $L_{E(n)} \to L_{E(n-1)}$.

Thm |Hasse square|

Chromatic fracture square

$$\begin{array}{ccc} L_{E(n)}X & \longrightarrow & L_{K(n)} \\ \downarrow & \lrcorner & \downarrow \\ L_{E(n-1)}X & \longrightarrow & L_{E(n-1)} L_{K(n)}(X) \end{array}$$

(natural map $L_{E(n)} \simeq L_n \to L_{K(n)}$)

($L_{E(n-1)}$ on $X \to L_{K(n)}(X)$)

Chromatic splitting conjecture: This glueing process is as simple as possible without being trivial

Consider the following diagram

$$\begin{array}{ccc} L_n X & \longrightarrow & L_{K(n)} X \\ \downarrow & \swarrow^{\alpha_n} & \downarrow \\ L_{n-1} X & \underset{\gamma_n}{\overset{\longrightarrow}{\longleftarrow}} & L_{n-1} L_{K(n)} X \end{array}$$

Turns out that there exists such an α_n making the top triangle commutes exactly if there exists a map γ_n splitting $L_{n-1}X \to L_{n-1} L_{K(n)} X$

Weak CSC: X p-completion of a finite spectrum $\Rightarrow$ γ_n exists for all n

This would imply that taking the limit of

$$L_{K(n-1)} X_p \xrightarrow{\alpha_{n-1}} L_n X_p \longrightarrow L_{K(n)} X_p$$

gives an equivalence

$X \simeq_p \lim_n L_{K(n)} X$ — From chromatic convergence theorem by cofinality

"Finite spectrum X can be recovered from its monochromatic pieces $L_{K(n)}X$"

Another consequence: $f: X \to Y$ map between (finite) spectra and $L_{K(n)}f : L_{K(n)}X \to L_{K(n)}Y$ is null $\Rightarrow$ f is null

General version is known for

- $n=1, p\geq 2$: Adams-Bousfield-Baird-Ravenel
- $n=2, p>5$: Hopkins based on Shimomura-Yabe
- $n=2, p=3$: Goerss-Henn-Mahowald
- $n=2, p=2$: Beaudry-Goerss-Henn
- $n>2, p\geq 2$: Wide open

} Brutal computational — No tactics that can be generalised

There are two different approaches to consider a "filtration" of the chromatic tower. The first one:

Algebraic chromatic filtration of a p-local spectrum X is, for $n \geq 1$

$$C_n^a(X) := \ker(\pi_* X \to \pi_* L_{n-1} X) \qquad C_0^a(X) := \pi_* X$$

The other filtration will be a bit harder to construct, and relies on another localization.

Geometric chromatic filtration

Def A full subcategory $\mathcal{T}$ of the (homotopy) category of p-local spectra is **thick** if

- $0 \in \mathcal{T}$
- Closed under fibers and cofibers
- Closed under retracts

Def: A p-local finite spectrum X is of **type n** if

$$K(i)_* X \cong \begin{cases} \neq 0 & i=n \\ =0 & i<n \end{cases}$$

Ex: • $S^0_{(p)}$ type 0 since $K(0)_*(S^0_{(p)}) \neq 0$

• $S\mathbb{Z}/p$ type 1: $K(0)_* S\mathbb{Z}/p = 0$, $K(1)_* S\mathbb{Z}/p \neq 0$

Write

$$\mathcal{P}_n = \{\text{finite } p\text{-local spectra of type} \geq n\}$$

i.e. those s.t. $K(m)_* X \cong 0$, $m<n$

↳ since finite, $K(m)_* X = 0 \Rightarrow K(m-1)_* X = 0$, so enough to consider $n-1$

Note: Every such finite p-local spectrum is of type n for some n, and it can be shown that for all $n \geq 0$ there exists one of type n — so all these $\mathcal{P}_{\geq n}$'s are different

Prop: $\mathcal{P}_n$ is a thick subcategory — Actually "thick prime tensor ideals of $SH^c_{(p)}$"

The LES of $K(m)$-homology gives us that a cofiber sequence

$$X' \to X \to X''$$ satisfies 2-out-of-3 w.r.t. $\mathcal{P}_{\geq n}$

A retract of a type n spectrum is again type n.

Thm: | Thick subcategory theorem - Ravenel/Mitchell/Hopkins-Smith |

Put $\mathcal{P}_0$ = Category of p-local finite spectra $SH^c_{(p)}$. Then

$$\mathcal{P}_0 \supsetneq \mathcal{P}_1 \supsetneq \cdots \supsetneq \mathcal{P}_n \supsetneq \mathcal{P}_{n+1} \supsetneq \cdots \supsetneq *$$

If $\mathcal{C}$ is a thick subcategory, then $\mathcal{C} \simeq \mathcal{P}_n$ for some $n \geq 0$.

So $\mathcal{P}_n$ are all of the thick subcategories $\sim$ The thick subcategories are the kernels of $K(n)_*$

Cor: Let X be of type n, then $L_n X \simeq L_{K(n)} X$

Pf Follows by the chromatic fracture square

Being of 'type n' can equivalently be described as existence of some specific maps.

First we consider how to construct spectra of a specific type:

n=0: $H_*(X;\mathbb{Q}) \neq 0$ $\sim$ take e.g. $\mathbb{S}_{(p)}$

n=1: Define X to be the mod p more spectrum which is defined by the cofiber

$$S \xrightarrow{\cdot p} S \longrightarrow X$$

This has no rational homology. Furthermore, since multiplication by p annihilates $K(1)_* S \simeq \mathbb{F}_p[v_1^{\pm}]$, the map $K(1)_* S \to K(1)_* X$ is injective so in particular $K(1)_* X \neq 0$ $\rightsquigarrow$ X type 1

n>1 is much harder! We wish to proceed inductively.

Assume X is of type n. Then we wish to construct a self-map

$$f: \Sigma^k X \to X$$

So we can form the cofiber sequence

$$\Sigma^k X \to X \to X/f$$

such that X/f is of type n+1. — By looking at the LES on $K(m)_*$

Turns out this is exactly the case when

- f induces an isomorphism $K(n)_*X \to K(n)_*X$ — $K(n)$-homology of X/f vanish
- f does not induce an isomorphism $K(n-1)_*X \to K(n-1)_*X$ — $K(n-1)$-homology does not vanish

This motivates the following definition:

Def: A v_n-self map on a p-local finite spectrum X, is a map $f:\Sigma^k X \to X$ s.t.

- f induces an isomorphism $K(n)_*X \to K(n)_*X$
- For $m \neq n$, the induced map $K(m)_*X \to K(m)_*X$ is nilpotent.

This is equivalent to saying

$$K(m)_*f = \begin{cases} 0 & n \neq m \\ v_n^{N} & n = m \end{cases}$$

For a suitable power (N)

Can be done more generally (Nilpotence II, Hopkins-Smith)

Ex: If X has type $> n$, then $K(n)_*X$ vanishes, so the zero map $0: X \to X$ is a v_n-self map.

Thm: | Periodicity theorem |

Follows by showing $\mathcal{J} = \{p\text{-local finite spectra w. } v_n\text{-self map}\}$ is thick, followed by thick subcategory theorem.

- A spectrum X has type n iff it admits a v_n-self map
- Furthermore, if f, g both are v_n-self maps, then $\exists i, j \geq 0$ s.t.

$$f^i = g^j$$

Essentially unique!

$\rightsquigarrow$ Want to think of these as periodic operators — induces iso on $K(n)_*$-hom and iterating will give us the same back at some point

So, if we have a type n spectrum and a v_n-self map we can construct a spectrum of type $n+1$:

Ex:

- $S \xrightarrow{p} S \to S\mathbb{Z}/p$ type 1 $\sim$ sometimes denoted $M(1)$ (where $p = v_0^1$)
- p odd,

$$\alpha : \Sigma^{2(p-1)} M(1) \to M(1) \qquad \text{Adams map}$$

satisfies $K(1)_*(\alpha) = v_1^1$. The cofiber has type 2 and we write $M(1,1)$

In general: We inductively define a type $n+1$ spectrum as follows.

- cokernel of a v_0-self-map f_0 satisfying $K(1)_*(f_0) = v_0^{i_0}$

  $\rightsquigarrow$ $M(i_0)$ type 1

- cokernel of a v_1-self-map $f_1 : \Sigma^{2(p-1)i_1} M(i_0) \to M(i_0)$ s.t.
  $K(1)_*(f_1) = v_1^{i_1}$
  $\rightsquigarrow$ $M(i_0, i_1)$ type 2
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- ⋮
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$M(i_0, i_1, \dots, i_n)$ is the type $n+1$ spectrum defined as the cofiber of a v_n-self map

$$f_n : \Sigma^{2(p^n-1)i_n} M(i_0, \dots, i_{n-1}) \longrightarrow M(i_0, \dots, i_{n-1})$$

satisfying

$$K(n)_*(f_n) = v_n^{i_n}.$$

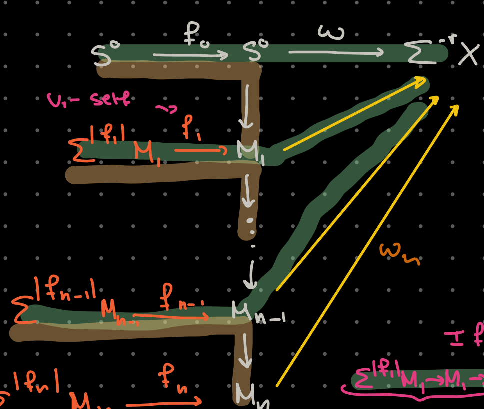
"Periodic families":

There is a lot of choices when constructing these $M(i_0, \dots, i_n)$!

<u>Construction:</u> Write $M_n := M(i_0, \dots, i_{n-1})$ type n $\quad K(n)_* f_n = v_n^{i_n}$

Let $w \in \pi_r X$ $\quad S^r \xrightarrow{w} X \rightsquigarrow S^0 \xrightarrow{w} \Sigma^{-r} X$

- w is v_{n-1}-torsion if there exists a diagram



If w is p-torsion $\rightsquigarrow f_0 = \underset{v_0}{p^{i_0}}$

so if the comp $w \circ f_0 \simeq 0$, we get that it extends, since $S^0 \xrightarrow{f_0} S^0 \longrightarrow M_1$ cofibre seq

If we can find a power of f_1 s.t. $\underbrace{\Sigma^{|f_1|} M_1 \to M_1 \to \Sigma^{-r} X}_{\simeq 0}$, then we can again extend

So we are assuming we can continue this process until a type n spectrum M_n.

- w is v_n-periodic if for any v_n-self map f_n of M_n, $w_n \circ f_n \neq 0$ so we can't continue the constr.

<u>Def</u> Geometric Chromatic Filtration

$C_0^g(X) = \pi_* X$

$C_n^g(X) = v_{n-1}$-torsion elements $\quad n \geq 1$

Decreasing filtration: $\quad C_0^g(X) \supseteq C_1^g(X) \supseteq C_2^g(X) \supseteq \cdots$

We now have two filtrations - when are they the same? Telescope conjecture

Telescope conjecture

Recall that by the periodicity theorem tells us that a v_n-self map $f: \Sigma^k X \to X$, for X a type n spectrum, is essentially unique, so the following colimit is independent of f.

Telescope of f $\quad X[f^{-1}] := \mathrm{colim}(X \xrightarrow{\Sigma^{-k} f} \Sigma^{-k} X \xrightarrow{\Sigma^{-2k} f} \Sigma^{-2k} X \to \cdots)$

<u>Def:</u> For $M_n = M(i_0, \dots, i_{n-1})$ w. v_n self map f_n, write $\mathrm{Tel}(n) := M_n[f_n^{-1}]$.

Telescopic localization

$$L_n^f X := L_{\mathrm{Tel}(0) \vee \cdots \vee \mathrm{Tel}(n)} X$$

sometime people write 'f' for finite - It's a finite localisation w. $\ker(L_n^f)$ generated by any (finite) $(n+1)$-type spectra

└ p-local spectrum

<u>Prop:</u> If X is of type $\geq n$ and f is a v_n-self map of X, then

$$L_n^t X \simeq X[f^{-1}].$$

<u>Prop:</u> L_n^t is a finite smashing localisation

This explains the name: It is the colimit of the telescope of a map

Using this we can redefine the geometric chromatic filtration

$$C_n^g X = \begin{cases} \pi_* X & n = 0 \\ \ker(\pi_* X \to \pi_* L_{n-1}^t X) & n \geq 1 \end{cases}$$

p-local spectrum ↑

This is very similar to the algebraic one now!

$$C_n^a X = \begin{cases} \pi_* X \\ \ker(\pi_* X \to L_{n-1} X) \end{cases}$$

There exist a natural transformation:

$$L_n^t X \to L_n X$$

which is known to be an equivalence if

- X is $E(m)$-local for some $m \geq 0$
- X is an MU-module spectrum localization theorem

<u>Telescope conjecture:</u> For every spectrum X, Ravenel made this conjecture

$$L_n^t X \xrightarrow{\sim} L_n X.$$

and the conjecture that it is false

Known to be true for $n = 0,\ p \geq 2$ – Bousfield (tautology $\mathrm{Tel}(0) = S\mathbb{Q} = H\mathbb{Q} = K(0)$)

$n = 1,\ p \geq 2$ $\quad$ >2 Miller, $=2$ Mahowald

⇝ completely open for $n \geq 1$, $p \geq 2$. But attempts to disprove

Prop: For $n \geq 1$ the following is equivalent.

- $L_{n-1}^{t} \simeq L_{n-1} \implies L_n^t \simeq L_n$
- There exists a type n spectrum X w. $X[f^{-1}] \not\simeq L_n X$

Using the thick subcategory theorem

so one example or counter example is enough to settle the passage from $n-1$ to n.

Periodic families

Let $w \in \pi_r X$ be v_n-periodic, and $M = M_n$ (CW-spectrum) as above w. v_n-self map s.t.

$$\Sigma^d M \xrightarrow{f_n} M \xrightarrow{w_n} \Sigma^{-r} X \quad \text{non-zero}$$

Let M^r = r-skeleton of M and cofiber sequences

$$M^{r-1} \to M^r \to M^r_r \,, \quad M^{r-1} \to M \to M_r = M_r^{\dim M}$$

take r-skeleton and quotient out w. (r-1)-skeleton

Then there exists an r s.t. we can form the following diagram

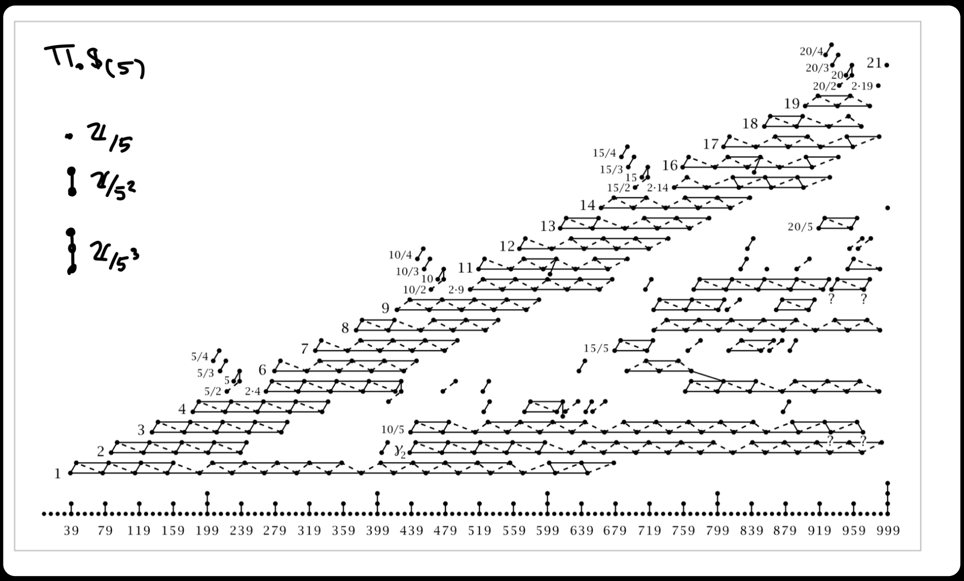
$$\begin{array}{ccccc} \Sigma^d M & \xrightarrow{f_n} & M & \xrightarrow{w_n} & \Sigma^{-r} X \\ & & \downarrow & & \\ S^k \simeq \Sigma^d M^r_r & \xrightarrow{\text{incl}} & \Sigma^d M_r & & \end{array}$$

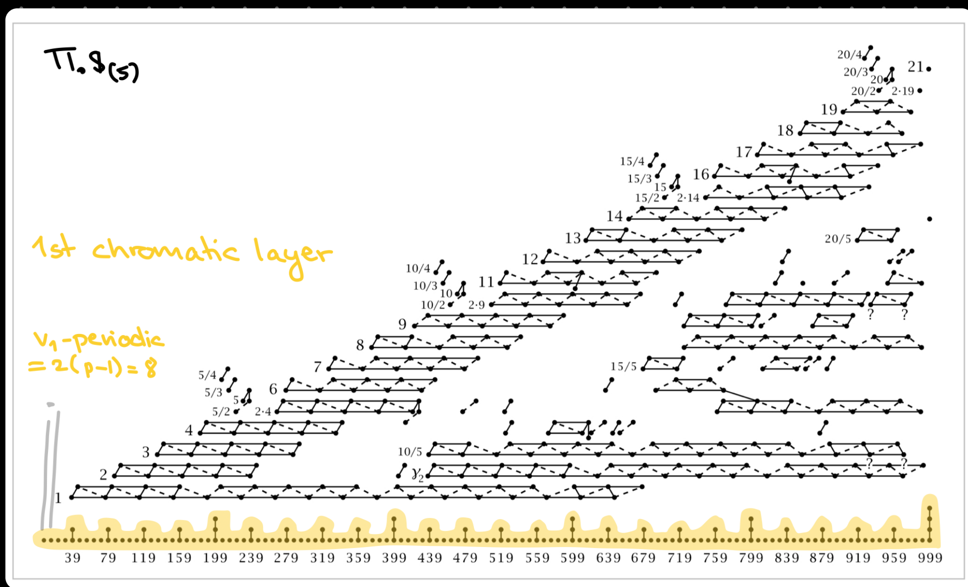
$\exists g$ s.t. $g \circ i$ non-trivial

⇝ i.e. $w_n \circ f_n$ is non-trivial on "some cell of M" – A cell that detects it

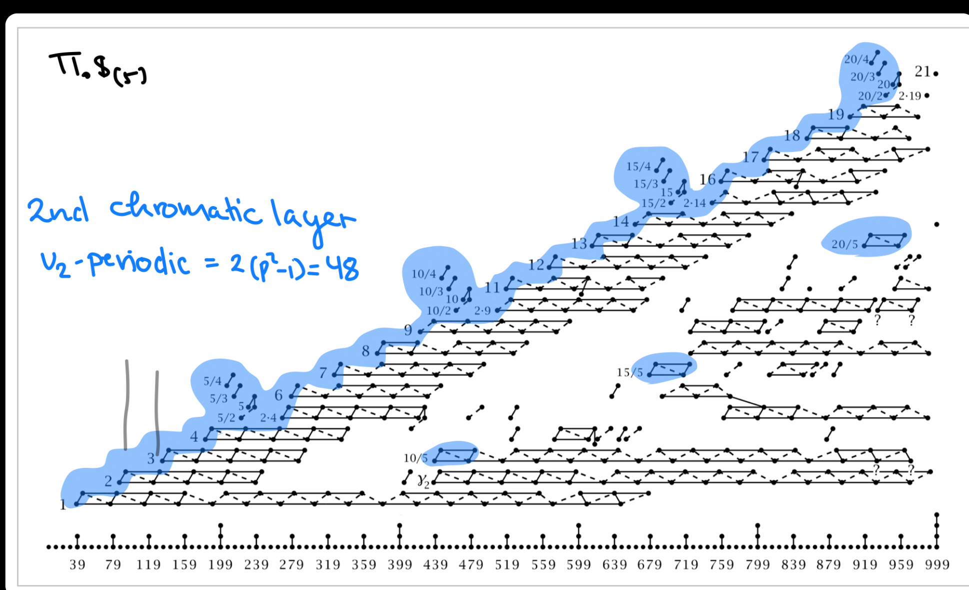
Such elements $g \circ i \in \pi_{k+r} X$ are part of the v_n-periodic family of w.

Thinking of f_n as "multiplication by v_n"





$\pi_0 S_{(5)}$
1st chromatic layer
v_1-periodic $= 2(p-1) = 8$
39 79 119 159 199 239 279 319 359 399 439 479 519 559 599 639 679 719 759 799 839 879 919 959 999



$\pi_0 S_{(5)}$
2nd chromatic layer
v_2-periodic $= 2(p^2-1) = 48$
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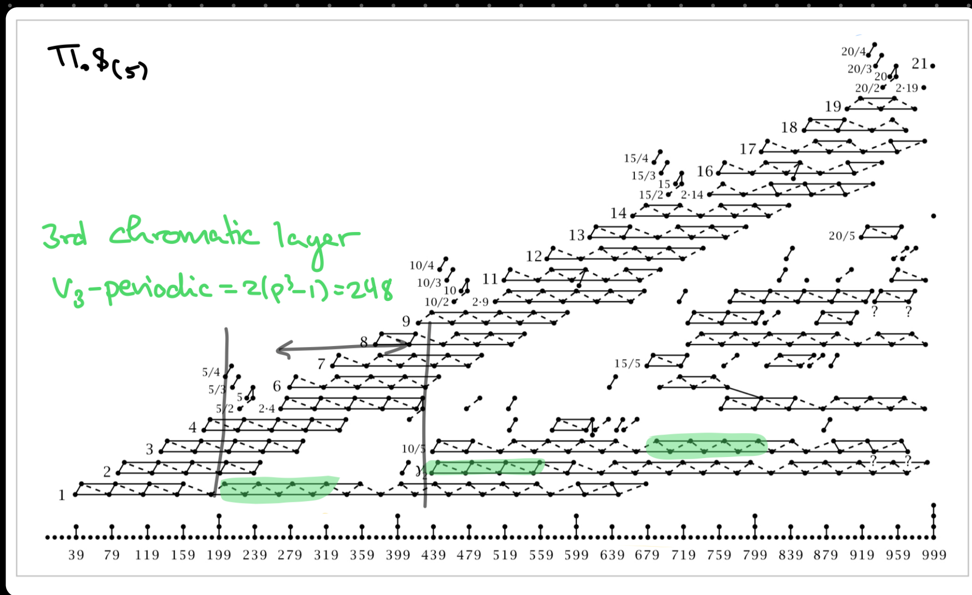
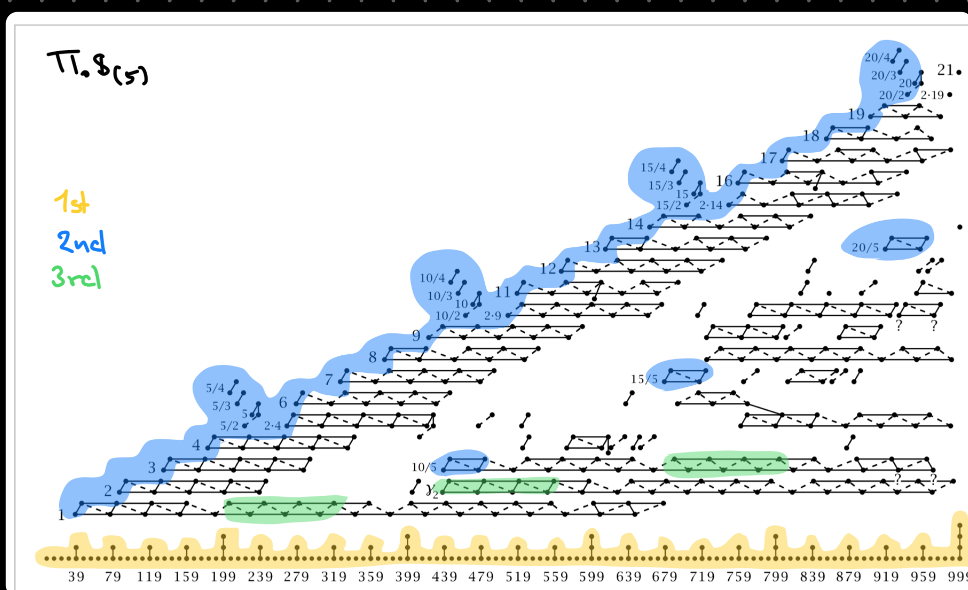

π.8(5)
3rd chromatic layer
V₃-periodic = 2(p²-1)=248
39 79 119 159 199 239 279 319 359 399 439 479 519 559 599 639 679 719 759 799 839 879 919 959 999


π.8(5)
1st
2nd
3rd
39 79 119 159 199 239 279 319 359 399 439 479 519 559 599 639 679 719 759 799 839 879 919 959 999