§0: Crash course in sheaves

Let $\mathcal{C}$ be a small ∞-category

<u>Def:</u> A Groethendieck topology on $\mathcal{C}$ assigns to each $U \in \mathcal{C}$ a collection of families of maps $\{\varphi_i : U_i \to U\}_{i \in I}$, known as coverings, s.t.

- For any isomorphism φ, $\{\varphi\} \in J$
- If $\{U_i \to U\} \in J$ and $\{V_{ij} \to U_i\} \in J$ for all i, then $\{V_{i,j} \to U\} \in J$.
- If $\{U_i \to U\} \in J$ and $V \to U$ is a morphism, then $U_i \times_U V$ exists and $\{U_i \times_U V \to V\} \in J$.

A category equipped with a Groethendiek topology is called a ∞-site

<u>Ex:</u> $\mathcal{C} = \mathcal{O}(X) = \{$open subsets of topological space $X\}$. A covering family of $U \overset{\circ}{\subseteq} X$ is a collection of open subsets $V_i \subset U$ which covers U in the ordinary sense

<u>Def:</u> A presheaf is a contravariant functor

$$F : \mathcal{C}^{op} \to \mathcal{D}$$

Let $(\mathcal{C}, J)$ be a site. A presheaf $F : \mathcal{C}^{op} \to \mathcal{D}$ is called a sheaf with respect to J if

- For every covering family $\{\varphi_i : U_i \to U\}_{i \in I}$ in J
- and for every compatible family of elements given by tuples $(s_i \in F(U_i))_{i \in I}$ s.t. $\forall j, k \in I$ and all $U_j \xleftarrow{f} K \xrightarrow{g} U_k$ in $\mathcal{C}$ with $p_j \circ f = p_k \circ g$ we have $F(f)(s_j) = F(g)(s_k) \in F(K)$

Then there is a unique element $s \in F(U)$ s.t. $F(p_i)(s) = s_i$ $\forall i \in I$

<u>Construction:</u> There is a canonical way to obtain a sheaf from a presheaf: Let $(\mathcal{C}, J)$ be an ∞-site, then the inclusion

$$\begin{tikzcd} \mathrm{Sh}^{\mathcal{D}}_{J}(\mathcal{C}) \arrow[r, hook, bend right, "\mathrm{incl}"'] \arrow[r, phantom, "\perp"] & \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{D}) \arrow[l, bend right, "L"'] \end{tikzcd}$$

admits a left adjoint called the **sheafification**, hence $\mathrm{Sh}^{\mathcal{D}}_{J}(\mathcal{C})$ is a localizing of $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{D})$.

Note: sheafification is left exact.

§1 Bounded Derived Categories

Throughout:

- $\mathcal{C}$ (small) stable ∞-category
- $\mathcal{A}$ (small) abelian category with enough injectives.

We will write

product preserving

$$P_\Sigma(\mathcal{C}) := \mathrm{Fun}_\Sigma(\mathcal{C}^{op}, \mathcal{S})$$

Recall:

- Freyd envelope: $A(\mathcal{C}) \subseteq \mathrm{Fun}_\Sigma(\mathcal{C}^{op}, \mathrm{Ab})$ cokernels of representables
- Almost perfect presheaves: $A_\infty(\mathcal{C}) \subseteq P_\Sigma(\mathcal{C})$ full subcat. of objects which are geometric realization of repr.
- Perfect presheaves $A_\infty^\omega(\mathcal{C}) \subseteq P_\Sigma(\mathcal{C})$ full subcat of objects which are finite colimit of representables

which has the following correspondences:

- $A_\infty^\omega(\mathcal{C}) \subseteq A_\infty(\mathcal{C})$
- $A_\infty^\omega(\mathcal{C})^\heartsuit \simeq A_\infty(\mathcal{C})^\heartsuit \simeq A(\mathcal{C})$

Goal: Any reasonable abelian category $\mathcal{A}$ has an associated connective derived ∞-category, which is prestable with heart equivalent to $\mathcal{A}$, and is initial w.r.t. this property.

↳ We want to construct to such things, one for $\mathcal{A}$ and one associated to a stable ∞-category equipped with a class of epimorphism

Def: The epimorphism topology on an abelian category $\mathcal{A}$ is the Groethendieck (pre-)topology where covering families $\{a_i \to b\}$ consists of a single epimorphism.

It can be shown that there exists a sheafification functor (w.r.t. this topology)

$$L : P_\Sigma(\mathcal{A}) \to P_\Sigma(\mathcal{A})$$

and we wish to understand how this gives a sheafification of perfect presheaves. First we see that the needed sheafification exists on the heart:

Lem: 5.6 The yoneda embedding $y: \mathcal{A} \to A(\mathcal{A})$ identifies the source with the subcategory of those finitely presented presheaves which are also sheaves w.r.t. the epimorphism topology.

"Pf": It can be shown that yoneda factors through the subcategory of sheaves

$$\begin{tikzcd} & Sh(\mathcal{A}) \arrow[dr] & \\ \mathcal{A} \arrow[ur, "G"] \arrow[rr, "y"'] & & A(\mathcal{A}) \end{tikzcd}$$

where G is exact.

We know that the image $y(\mathcal{A})$ generates $A(\mathcal{A})$ under finite colimits, the same is true for the subcategory of sheaves. Therefor the map into the subcategory of finitely presented sheaves has to be an equivalence.

To extend this to perfect presheaves we need the following technical result:

Prop: 5.11 Let $a \in \mathcal{A}$ and consider the suspension $\Sigma^n y(a) \in A_\infty(\mathcal{A})$. Then

1) $L\Sigma^n y(a)$ is almost perfect
2) $\pi_k(L\Sigma^n y(a))(b) \simeq Ext^{n-k}_{\mathcal{A}}(b,a)$

Cor: 5.13 $X \in A^\omega_\infty(\mathcal{A}) \Rightarrow LX \in A^\omega_\infty(\mathcal{A})$

Pf: Since L is left exact (preserves limits) and being almost perfect is a property which is closed under limits, we get that the class of those almost perfect $X \in A_\infty(\mathcal{A})$ which satisfies that $LX \in A_\infty(\mathcal{A})$, is also closed under limits. By prop. 5.11 we get that this subcategory of $A_\infty(\mathcal{C})$ contains all those presheaves which has homotopy concentrated in a single degree, which can be shown to hold for any perfect presheaf, i.e. it contains $A_\infty^\omega(\mathcal{A})$. It then follows by 4.37 that LX is perfect. WHY?!

So we have a sheafification functor on perfect presheaves $A_\infty^\omega(\mathcal{A})$.

Def: The bounded (connective) derived ∞ category of $\mathcal{A}$, denoted by $\mathcal{D}^b(\mathcal{A})$ is the ∞-category of bounded almost perfect sheaves on $\mathcal{A}$ w.r.t. the epimorphism topology.

Q: Our localization is only defined on $A_\infty^\omega(\mathcal{C})$ which is only a subcat of $A_\infty(\mathcal{C})$

Prop 5.16: We have that

1) $L: A_\infty^\omega(\mathcal{A}) \longrightarrow \mathcal{D}^b(\mathcal{A})$ is an exact localization
2) $\mathcal{D}^b(\mathcal{A})$ is prestable and admits finite limits
3) $\mathcal{D}^b(\mathcal{A})^\heartsuit \simeq \mathcal{A}$
1) $\pi_0 \mathrm{Map}_{\mathcal{D}^b(\mathcal{A})}(a, \Sigma^n b) \simeq \mathrm{Ext}^n_{\mathcal{A}}(a,b)$ for all $a,b \in \mathcal{A}$

Thm 5.18 | Universal property of $\mathcal{D}^b(\mathcal{A})$ | Let $\mathcal{A}$ be an abelian category with enough injectives and $\mathcal{D}$ a prestable ∞-category with finite limits. Then left Kan extension gives an equivalence between the following two kinds of data:

(1) exact functors $\mathcal{A} \to \mathcal{D}^\heartsuit$ of abelian categories

(2) exact functors $\mathcal{D}^b(\mathcal{A}) \to \mathcal{D}$ of prestable ∞-categories with finite limits

Pf: | Sketch |

(2) => (1) follows by restricting a given functor to the hearts

$$\mathcal{D}^b(\mathcal{A})^\heartsuit \simeq \mathcal{A} \longrightarrow \mathcal{D}^\heartsuit$$

(1) => (2) is alot harder. The idea is that any additive functor $f: \mathcal{A} \to \mathcal{D}$ extends uniquely to a right exact functo[r] $F: A^{\omega}_{\infty}(\mathcal{A}) \to \mathcal{D}$, and when f is exact, this functor F uniquely through an exact functor out of $\mathcal{D}^b(\mathcal{A})$

$$\begin{tikzcd} & \mathcal{D}^b(\mathcal{A}) \arrow[dr, "\exists!"] & \\ A^{\omega}_{\infty}(\mathcal{A}) \arrow[ur, "L"] \arrow[rr, "F"'] & & \mathcal{D} \end{tikzcd}$$

§3 Perfect derived ∞-category

Now we want to construct a similar derived ∞-category in the case where we have a stable ∞-category equipped with a fixed adapted homology theory

$$H: \mathcal{C} \longrightarrow \mathcal{A}.$$

The reason why we need to fix a homology is because we need a good notion of epimorphisms to define derived categories, which we have in the abelian case but not in the stable case.

As when we defined $\mathcal{D}^b(\mathcal{A})$ we first need a topology:

Def: The H-epimorphism topology is a Grothendieck (pre)topology on $\mathcal{C}$ in which a family of maps $\{c_i \to d\}$ is a covering if it consists of a single map which is an H-epimorphism

We have a sheafification w.r.t. this H-epimorphism topology

$$L_{\mathcal{C}} : P_{\Sigma}(\mathcal{C}) \to P_{\Sigma}(\mathcal{C}).$$

We wish to show this restricts to perfect presheaves. We first consider the case of finitely presented discrete presheaves:

Prop: 5.24 A finitely presented discrete presheaf $x \in A(\mathcal{C})$ is a sheaf w.r.t. the H-epimorphism topology iff it is in the image of the right adjoint $\mathcal{A} \hookrightarrow A(\mathcal{C})$. Fully faithful right adjoint to the induced functor $A(\mathcal{C}) \to \mathcal{A}$.

To be able to say more we need a connection between presheaves on $\mathcal{C}$ and on $\mathcal{A}$ - since the abelian case is easier to work in.

Def: Using functoriality of left Kan extension along our fixed homology theory $H: \mathcal{C} \to \mathcal{A}$ we obtain a functor

$$H^*: P_{\Sigma}(\mathcal{C}) \longrightarrow P_{\Sigma}(\mathcal{A})$$

This preserves almost perfect presheaves and therefore induces an adjunction

$$\begin{tikzcd} A_\infty(\mathcal{C}) \arrow[r, shift left, "H^*"] \arrow[r, phantom, "\perp"] & A_\infty(\mathcal{A}) \arrow[l, shift left, "H_*"] \end{tikzcd}$$

$$(\mathcal{C}^{op} \xrightarrow{H^{op}} \mathcal{A}^{op} \to \mathcal{S}) \longleftarrow\!\shortmid (\mathcal{A}^{op} \to \mathcal{S})$$

which we call the homology adjunction.

Fact: $H_*: A_\infty(\mathcal{A}) \to A_\infty(\mathcal{C})$ commutes with sheafification

means for things $H_* Y \in A_\infty(\mathcal{C})$ we can calculate the sheafification in $A_\infty(\mathcal{A})$.

Lem 5.29: The restriction of H_* to the hearts

$$H_*: A(\mathcal{A}) \to A(\mathcal{C})$$

induces an adjoint equivalence between the categories of finitely presented sheaves. In particular, any discrete sheaf $X \in A(\mathcal{C})$ can uniquely be written as $H_* Y$ for some $Y \in A(\mathcal{A})$.

i.e. the images of $\mathcal{A} \hookrightarrow A(\mathcal{A})$

Pf: By 5.24 we know the subcategory of finitely presented sheaves of $A(\mathcal{C})$ can be identified with $\mathcal{A}$, and by 5.6 we have the same result for $A(\mathcal{A})$. One then chase through the constructions to show that H_* is compatible with these equivalences.

Cor 5.30: Let $X \in A_\infty^\omega(\mathcal{C})$ be a presheaf with homotopy groups concentrated in a single degree n. Then,

$$L_{\mathcal{C}} X \simeq H_*(L_{\mathcal{A}} Y)$$

for some $Y \in A_\infty(\mathcal{A})$ with homotopy groups concentrated in a single degree n.

Pf: Since $X \in A_\infty^\omega(\mathcal{C})$ has only non-zero homotopy group in degree n, and $A_\infty^\omega(\mathcal{C})^\heartsuit \cong A(\mathcal{C})$, we can write $X \simeq \Sigma^n x$ for some $x \in A(\mathcal{C})$. Since the natural map

$$\Sigma^n x \longrightarrow \Sigma^n Lx$$

sheafification on the heart of bounded presheaves

must be an isomorphism on homotopy groups we get that it is an equivalence after applying L:

$$L\Sigma^n x \simeq L\Sigma^n Lx,$$

HOW DOES THIS WORK ?

hence we can assume $X \simeq \Sigma^n x$ with x already a sheaf. By 5.29 we can write $x \simeq H_* y \rightsquigarrow \Sigma^n x \simeq H_* \Sigma^n y$. Using that H_* commutes with sheafification we get the desired

$$LX \simeq L\Sigma^n x \simeq LH_* \Sigma^n y \simeq H_* L\Sigma^n y$$

by putting $Y = \Sigma^n y$. $\square$

Prop: 5.31 Let $X \in A^\infty_\omega(\mathcal{C})$ be a perfect presheaf. Then, the sheafification LX is again perfect.

Pf: The proof is done in 3 steps:

X has bounded presheaf homotopy groups:

In this case, X belongs to the smallest subcategory of $A^\infty_\omega(\mathcal{C})$ which is closed under finite limits and contains all perfect presheaves with a single non-zero homotopy group. For any such perfect presheaf w. a single non-zero homotopy group, we know from 5.30 that we can write its sheafification as

$$H_*(LY)$$

for some $Y \in A_\infty(\mathcal{A})$, and in this abelian case we have already showed in 5.13 that LY is perfect. Using that the subcategory of those perfect presheaves for which

LX is again perfect, is closed under finite limits since L is left exact the desired follows

From this one prove it for $X \simeq \nu(c)$ for some $c \in \mathcal{C}$ and then a general X.

Def. Let $H: \mathcal{C} \to \mathcal{A}$ be an adapted homology theory. Then the perfect derived ∞-category of $\mathcal{C}$ relative to H is given by

$$\mathcal{D}^{\omega}(\mathcal{C}) := A_{\infty}^{\omega,sh}(\mathcal{C})$$

the ∞-category of perfect sheaves on $\mathcal{C}$ w.r.t. the H-epimorphism topology.

Proof 5.13

Why does LX have vanishing homotopy groups in almost all degrees when X is perfect?

Case 1: $X = \Sigma^n y(a)$

Then by 5.11 we have

$$\pi_k(L\Sigma^n y(a))(b) \cong \mathrm{Ext}^{n-k}_{\mathcal{A}}(b,a)$$

and negative Ext-groups vanish.

Case 2: X general almost perfect presheaf with only one non-trivial homotopy group in degree n.

In this case we can write $X \simeq \Sigma^n Y$ where Y is finitely presented presheaf and sits into an exact sequence using Hom is left exact

$$0 \to y(\ker(f)) \to y(a) \xrightarrow{y(f)} y(b) \to Y \to 0.$$

Short exact sequences in $A(\mathcal{A})$ gives fiber sequences in $A_\infty(\mathcal{A})$ and this gives two fiber sequences in $A_\infty(\mathcal{A})$:

$$\Sigma^n \ker(f) \to \Sigma^n y(a) \to \Sigma^n K$$

$$\Sigma^n K \to y(b) \longrightarrow \Sigma^n Y = X$$

where $K = \ker(y(b) \to Y)$. Now, L preserves fiber sequences, and the LES for both fiber sequences implies that LX has vanishing homotopy groups almost everywhere.

Case 3: X = General perfect presheaf

By 4.37 we get that $\pi_k X$ is finitely presented (for all $k \geq 0$), and vanishes for almost all of them, so we get a finite Postnikov tower $\rightsquigarrow$ Can write X as a finite limit of things from case 2:

$$m: X = X_m \to \cdots \to X_0$$

s.t. the fibers

$$F_s \to X_s \to X_{s-1}$$

have non-trivial homotopy in only one degree. We then apply L which will preserve both this tower and fibers and again LES for every

$$LF_s \to LX_s \to LX_{s-1}$$

will inductively imply that LX has vanishing homotopy almost everywhere.